Optimality conditions

Lectures for PHD course on Numerical Optimization

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Notes

Outline

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Introduction

1. Optimality conditions attempt to constructs easily verifiable criteria that allows to classify points into optimal and non-optimal ones.
2. Unfortunately, this is impossible in practice, because such a universal criterion does not exists.
3. It is possible to construct either practical conditions, that admit some mistakes in the characterization, or perfect conditions which are impossible to use in the computations.
4. Practical conditions may further be classified into two distinct subgroups based on the type of mistakes allowed in the decision-making process, namely:
   1. necessary conditions
   2. sufficient conditions

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Outline

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Importance of Optimality Conditions (1/2)

- To illustrate the importance of optimality conditions we recall the controversial story of US Air Forces B-2 Stealth bomber program in the Reagan era of the 1980s.
- There were many design variables such as the various dimensions, the distribution of volume between the wing and the fuselage, flying speed, thrust, fuel consumption, drag, lift, air density, etc., the objective function was: maximize the distance the Stealth can fly starting with full tanks, without refueling.
- The problem was modeled as an Non Linear Programming (NLP) in a secret Air Force study going back to the 1940s.
- A solution to the necessary optimality conditions of this problem was found; it specified values for the design variables that put almost all of the total volume in the wing, leading to the flying wing design for the B-2 bomber.

Notes

Importance of Optimality Conditions (2/2)

- After spending billions of dollars it was found that the design solution implemented works, but that its range was too low in comparison with other bomber designs.
- A careful review of the model indicated that all the formulas used, and the model itself, are perfectly valid.
- The model was a nonconvex NLP and a second solution to the system of necessary conditions was found and implemented as a result of earlier studies.
- This new solution makes the wing volume much less than the total volume, and seems to maximize the range; while the old solution seems to actually minimize the range.
- The design implemented was the aerodynamically worst possible choice of configuration.

For an account, see the research news item Skeleton Alleged in the Stealth Bombers Closet, Science, vol. 244, 12 May 1989 issue, pages 650-651.

Notes

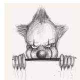
Geometric Optimality Condition

The problem we analyse is the following

$$\begin{aligned} &\text{minimize} && \mathsf{f}(\boldsymbol{x}) \\ &\text{subject to} && \boldsymbol{x} \in S \end{aligned}$$

where $S \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$.

- Since S is not described in terms of equality or inequality constraints, the optimality conditions will be based on purely geometrical ideas.
- The optimality conditions developed on purely geometrical ideas are almost useless, because they are also not very easy, even impossible, to verify for an optimization algorithm.
- When algebraic description of the set S is available, easily verifiable optimality conditions such as Fritz-John and Karush-Kuhn-Tucker can be used.



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Some definitions

Definition (Feasible point)

The set S is said the feasible set. A point $x \in S$ is said to be a feasible point.

Definition (Local and Global minimum)

Let $x_\star \in S$ we say that $x_\star$ is:

global minimum:	$\mathsf{f}(x) \geq \mathsf{f}(x_\star)$,	$\forall x \in S$
strict global minimum:	$\mathsf{f}(x) > \mathsf{f}(x_\star)$,	$\forall x \in S \setminus \{x_\star\}$
local minimum:	$\mathsf{f}(x) \geq \mathsf{f}(x_\star)$,	$\forall x \in S \cap B(x_\star; \delta)$
strict local minimum:	$\mathsf{f}(x) > \mathsf{f}(x_\star)$,	$\forall x \in S \cap B(x_\star; \delta) \setminus \{x_\star\}$

Notes

Feasible directios

- The previous definition of minimum are prefect but less practical.
- As in the case of unconstrained optimisation a test based on gradient of $\mathsf{f}(\boldsymbol{x})$ is more operative.
- To define conditions based on the gradient we must define some sets of feasible direction where to check the gradient of $\mathsf{f}(\boldsymbol{x})$.
- Such a sets will results to be cones i.e. sets that satisfy:

$$C \text{ is a cone if} \qquad \boldsymbol{x} \in C \;\Rightarrow\; \lambda\boldsymbol{x} \in C, \quad \forall\lambda > 0.$$

Notes

Radial cone

To define maxima and minima in S we need the definition of feasible direction search

Definition (Radial cone)

Let $S \subset \mathbb{R}^n$ be a nonempty closed set. The radial cone for S at $x \in \mathbb{R}^n$ is defined as:

$$\mathcal{R}_S(\boldsymbol{x}) = \Big\{ \boldsymbol{p} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} \mid \exists\, \delta > 0, \forall \alpha \in [0, \delta] \;:\; \boldsymbol{x} + \alpha \boldsymbol{p} \in S \Big\}$$

This cone is too small to develop useful optimality conditions.

Example

For the set $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^1 + x_2^2 = 1\}$ it is easy to show:

$$\mathcal{R}_S(\boldsymbol{x}) = \emptyset, \qquad \forall \boldsymbol{x} \in \mathbb{R}^2$$

Notes

Tangent cone

Radial cone cannot define feasible direction when S is curved.

Definition (Tangent Cone)

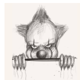
Let $S \subset \mathbb{R}^n$ be a nonempty closed set. The tangent cone for S at $x \in \mathbb{R}^n$, is defined as:

$$\mathcal{T}_S(\boldsymbol{x}) = \Big\{\boldsymbol{p} \in \mathbb{R}^n \mid \exists \, \{\boldsymbol{x}_k\} \subset S, \{\lambda_k\} \subset (0,\infty) : \\ \lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{x}, \qquad \lim_{k\to\infty} \lambda_k(\boldsymbol{x}_k - \boldsymbol{x}) = \boldsymbol{p}\Big\}$$

Example

For the set $S = \big\{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^1 + x_2^2 = 1\big\}$ it is easy to show:

$$\mathcal{T}_S(\boldsymbol{x}) = \begin{cases} \{\lambda(x_2, -x_1)^T | \lambda \in \mathbb{R}\} & \text{if } \boldsymbol{x} \in S. \\ \emptyset & \text{if } \boldsymbol{x} \notin S. \end{cases}$$



Notes

Radial and Tangent cone

Remark (Internal points)

Let $\boldsymbol{x}$ be an internal point of $S \subset \mathbb{R}^n$, i.e. there exists $\epsilon > 0$ such that $B(\epsilon; \boldsymbol{x}) \subset S$ where

$$B(\epsilon; \boldsymbol{x}) = \left\{ \boldsymbol{z} \mid \|\boldsymbol{z} - \boldsymbol{x}\| \leq \epsilon \right\}$$

Then the tangent cone and the radial cone coincide and are the whole space $\mathbb{R}^n$, i.e.

$$\mathcal{R}_S(\boldsymbol{x}) = \mathcal{T}_S(\boldsymbol{x}) = \mathbb{R}^n$$

Thus the concept of tangent and radial cone are important especially on the border of S.

Notes

Radial and Tangent cone

Proposition (closure of tangent cones)

The tangent cone $\mathcal{T}_S(\boldsymbol{x})$ is a closed set.

Proof: Consider a sequence $\{\boldsymbol{p}_k\} \subset \mathcal{T}_S(\boldsymbol{x})$ such that $\boldsymbol{p}_k \mapsto \boldsymbol{p}$. For every $\boldsymbol{p}_k \in \mathcal{T}_S(\boldsymbol{x})$, there exists $\boldsymbol{x}_k \in S$ and $\lambda_k > 0$, such that

$$\|\boldsymbol{x}_k - \boldsymbol{x}\| < \frac{1}{k} \quad \text{and} \quad \|\lambda_k(\boldsymbol{x}_k - \boldsymbol{x}) - \boldsymbol{p}_k\| < \frac{1}{k}$$

Then by the triangle inequality,

$$\|\lambda_k(\boldsymbol{x}_k - \boldsymbol{x}) - \boldsymbol{p}\| \leq \|\lambda_k(\boldsymbol{x}_k - \boldsymbol{x}) - \boldsymbol{p}_k\| + \|\boldsymbol{p}_k - \boldsymbol{p}\| \leq \frac{2}{k} \mapsto 0$$

which implies that $\boldsymbol{p} \in \mathcal{T}_S(\boldsymbol{x})$. $\square$

Notes

Radial and Tangent cone

Proposition

We have the following inclusion:

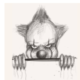
$$\overline{\mathcal{R}_S(\boldsymbol{x})} \subset \mathcal{T}_S(\boldsymbol{x})$$

for every $\boldsymbol{x} \in \mathbb{R}^n$*.*

Proof: Because $\overline{\mathcal{T}_S(\boldsymbol{x})} = \mathcal{T}_S(\boldsymbol{x})$ it is enough to show the inclusion $\mathcal{R}_S(\boldsymbol{x}) \subset \mathcal{T}_S(\boldsymbol{x})$. Let $\boldsymbol{p} \in \mathcal{R}_S(\boldsymbol{x})$, then, for all large integers k it holds that $\boldsymbol{x} + k^{-1}\boldsymbol{p} \in S$ and by setting

$$\boldsymbol{x}_k = \boldsymbol{x} + k^{-1}\boldsymbol{p} \qquad \text{and} \qquad \lambda_k = k$$

we see that $\boldsymbol{p} \in \mathcal{T}_S(\boldsymbol{x})$. $\square$



Notes

Examples of cones

(1/4)

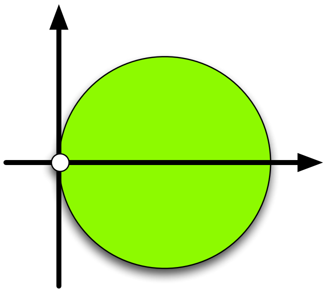
Example

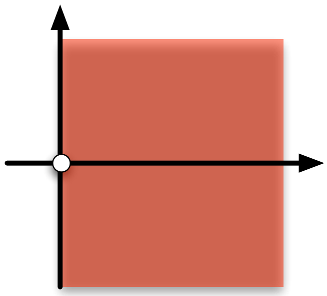
Consider the set

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 \leq 1 \right\}$$

then

$$\mathcal{R}_S(\mathbf{0}) = \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 > 0\} \setminus \{\mathbf{0}\} \qquad \mathcal{T}_S(\mathbf{0}) = \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 \geq 0\}$$





Notes

Examples of cones (2/4)

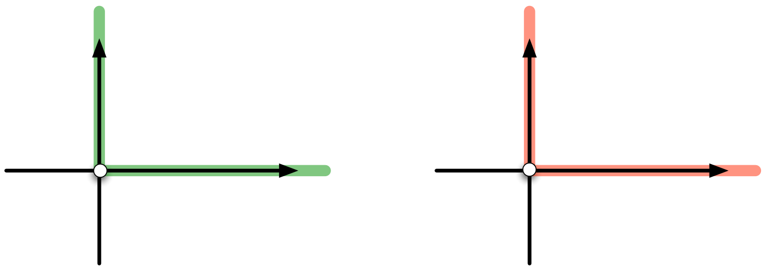
Example

Consider the set

$$S = \Big\{ \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 x_2 \leq 0 \Big\}$$
$$= \Big\{ \boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} = (\delta, 0) \text{ or } \boldsymbol{x} = (0, \delta) \text{ with } \delta \geq 0 \Big\}$$

then

$$\mathcal{R}_S(\mathbf{0}) = S \setminus \{\mathbf{0}\}, \qquad \mathcal{T}_S(\mathbf{0}) = S.$$



Notes

Examples of cones

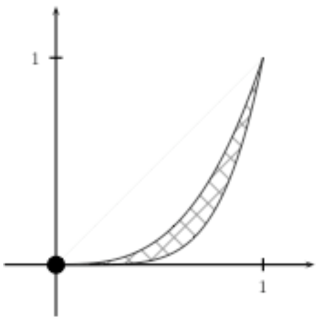
Example

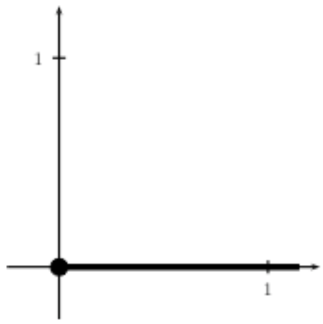
Consider the set

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid x_2 - x_1^3 \leq 0,\ x_1^5 - x_2 \leq 0,\ x_2 \geq 0 \right\}$$

then

$$\mathcal{R}_S(\mathbf{0}) = \emptyset \qquad \mathcal{T}_S(\mathbf{0}) = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \mid p_1 \geq 0,\ p_2 = 0 \right\}$$





Notes

Examples of cones (4/4)

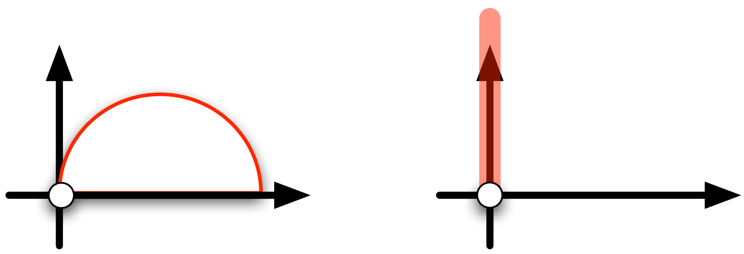
Example

Consider the set

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid x_2 \geq 0,\ (x_1 - 1)^2 + x_2^2 = 1 \right\}$$

then

$$\mathcal{R}_S(\mathbf{0}) = \emptyset \qquad \mathcal{T}_S(\mathbf{0}) = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \mid p_1 = 0,\ p_2 \geq 0 \right\}$$



Notes

Geometric necessary optimality conditions

Definition (cone of descent directions)

A cone of descent directions is the following set:

$$\mathring{\mathcal{D}}(\boldsymbol{x}_\star) = \Big\{ \boldsymbol{p} \in \mathbb{R}^n \mid \nabla \mathsf{f}(\boldsymbol{x}_\star)\boldsymbol{p} < 0 \Big\}$$

this cone may be empty if $\nabla \mathsf{f}(\boldsymbol{x}_\star)^T = \mathbf{0}$ (∇f is a row vector).

Theorem (geometric necessary optimality conditions)

Consider the optimization problem

$$\begin{aligned} &\textit{minimize} && \mathsf{f}(\boldsymbol{x}) \\ &\textit{subject to} && \boldsymbol{x} \in S \end{aligned}$$

If $\boldsymbol{x}_\star \in S$ is a local minimum then $\mathring{\mathcal{D}}(\boldsymbol{x}_\star) \cap \mathcal{T}_S(\boldsymbol{x}_\star) = \emptyset$.

Notes

Geometric necessary optimality conditions

Proof: Let be $\boldsymbol{p} \in T_s(\boldsymbol{x}_\star)$, then there exists $\{\boldsymbol{x}_k\} \subset S$, and $\{\lambda_k\} \subset (0,\infty)$ such that $\lim_{k\to\infty} = \boldsymbol{x}_\star$ and $\lim_{k\to\infty} \lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star) = \boldsymbol{p}$.
Using the first order Taylor expansion for large k we get:

$$\mathsf{f}(\boldsymbol{x}_k) - \mathsf{f}(\boldsymbol{x}_\star) = \nabla \mathsf{f}(\boldsymbol{x}_\star)(\boldsymbol{x}_k - \boldsymbol{x}_\star) + o(\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|) \geq 0$$

Multiplying by $\lambda_k > 0$ and taking limit we get

$$\begin{aligned} 0 \leq \lim_{k\to\infty} \left[\lambda_k \nabla \mathsf{f}(\boldsymbol{x}_\star)(\boldsymbol{x}_k - \boldsymbol{x}_\star) + \|\lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star)\| \frac{o(\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|)}{\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|}\right] \\ = \nabla \mathsf{f}(\boldsymbol{x}_\star)\boldsymbol{p} + \|\boldsymbol{p}\| \cdot 0 \end{aligned}$$

and thus $\boldsymbol{p} \notin \mathring{\mathcal{D}}(\boldsymbol{x}_\star)$. $\square$

Notes

Geometric necessary optimality conditions

Example (cone of descent directions)

Consider the differentiable (linear) function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $\mathsf{f}(\boldsymbol{x}) = x_1$. Then, $\nabla \mathsf{f}(\boldsymbol{x}) = (1, 0)$, and

$$\mathring{\mathcal{D}}(0, 2) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 < 0\}$$

It is easy to see from geometric considerations that $\boldsymbol{x}_\star = \mathbf{0}$ is a local (in fact, even global) minimum in all the previous examples. It is easy to check that the geometric necessary optimality condition

$$\mathring{\mathcal{D}}(\mathbf{0}) \cap \mathcal{T}_S(\mathbf{0}) = \emptyset$$

is satisfied in all these examples.

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First–order conditions

For the problem

$$\text{minimize} \quad \mathsf{f}(\boldsymbol{x}) \qquad\qquad \text{subject to} \quad \boldsymbol{x} \in S$$

where $S \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$.

- The cone $\mathcal{T}_S(\boldsymbol{x}_\star)$ is near impossible to compute for general S!
- Algebraic characterization of the set S are useful define easy to compute cones that approximate $\mathcal{T}_S(\boldsymbol{x}_\star)$.
- In practical problem S can be defined as the set of points $\boldsymbol{x}$ that satisfy:

$$\begin{aligned} \mathsf{c}_k(\boldsymbol{x}) &= 0, \qquad k \in \mathcal{E} \\ \mathsf{c}_k(\boldsymbol{x}) &\geq 0, \qquad k \in \mathcal{I} \end{aligned}$$

  where $\mathsf{c}_k : \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable constraints, $\mathcal{E}$ is the set of index of the equality constraints while $\mathcal{I}$ is the set of index of the inequality constraints.

Notes

First–order conditions

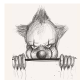
Assumption (Algebraic characterization of S)

We assume that the set S is defined as the solution set of a system of differentiable equality/inequality constraints defined by the functions $\mathsf{c}_k \in \mathtt{C}^1(\mathbb{R}^n)$:

$$S = \Big\{ \boldsymbol{x} \in \mathbb{R}^n \mid \quad \begin{aligned} &\mathsf{c}_k(\boldsymbol{x}) \geq 0, \; k \in \mathcal{I}, \\ &\mathsf{c}_k(\boldsymbol{x}) = 0, \; k \in \mathcal{E} \end{aligned} \Big\}$$

Remark

We may also assume that $\mathcal{E} = \emptyset$ because any equality constraint $h(\boldsymbol{x}) = 0$ may be written in the form $h(\boldsymbol{x}) \geq 0 \;\wedge\; -h(\boldsymbol{x}) \geq 0$. However this whill change the forward definition of $\mathring{\mathcal{C}}(\boldsymbol{x})$.



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Active constraints

Definition (Active constraints)

We will use the symbol $\mathcal{I}(\boldsymbol{x})$ and $\mathcal{A}(\boldsymbol{x})$ to denote, respectively, the index set of active inequality constraints and active constraints at $\boldsymbol{x} \in \mathbb{R}^n$, i.e.

$$\mathcal{I}(\boldsymbol{x}) = \big\{k \in \mathcal{I} \mid \mathsf{c}_k(\boldsymbol{x}) = 0\big\}, \qquad \mathcal{A}(\boldsymbol{x}) = \mathcal{E} \cup \mathcal{I}(\boldsymbol{x}).$$

moreover $|\mathcal{I}(\boldsymbol{x})|$ and $|\mathcal{A}(\boldsymbol{x})|$ denotes their cardinality.

Remark

If we define $S(\boldsymbol{x}) = \big\{\boldsymbol{x} \in \mathbb{R}^n \mid \; \mathsf{c}_k(\boldsymbol{x}) = 0,\; k \in \mathcal{A}(\boldsymbol{x})\;\big\}$ it is easy to show that if $\boldsymbol{x}_\star$ is a local minima of $\mathsf{f}(\boldsymbol{x})$ respect to S is also a local minima respect to $S(\boldsymbol{x}_\star)$.

Notes

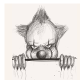
Algebraic approximation of $\mathcal{T}_S(\boldsymbol{x})$

Definition (Constraints cone)

Given a system of differentiable constraints $\mathsf{c}_k \in \mathtt{C}^1(\mathbb{R}^n)$*, the cone associated at a given point:*

$$
\begin{aligned}
\mathcal{C}(\boldsymbol{x}) = \Big\{ \boldsymbol{p} \in \mathbb{R}^n \mid\ & \nabla \mathsf{c}_k(\boldsymbol{x})\boldsymbol{p} \geq 0,\ k \in \mathcal{I}(\boldsymbol{x}), \\
& \nabla \mathsf{c}_k(\boldsymbol{x})\boldsymbol{p} = 0,\ k \in \mathcal{E} \Big\}
\end{aligned}
$$

The cone $\mathcal{C}(\boldsymbol{x})$ is a computable cone that can be used in numerical approximation. Next we prove that $\mathcal{T}_S(\boldsymbol{x}) \subseteq \mathcal{C}(\boldsymbol{x})$.



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Lemma (cone inclusion)

For every $\boldsymbol{x}_\star \in \mathbb{R}^n$ it holds that $\mathcal{T}_S(\boldsymbol{x}_\star) \subseteq \mathcal{C}(\boldsymbol{x}_\star)$.

Proof:(Proof. $(1/2)$) Let $\boldsymbol{p} \in \mathcal{T}_S(\boldsymbol{x}_\star)$, then there exists $\{\boldsymbol{x}_k\} \subset S$, and $\{\lambda_k\} \subset (0,\infty)$ such that $\lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{x}_\star$ and $\lim_{k\to\infty} \lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star) = \boldsymbol{p}$.
For any $\ell \in \mathcal{I}(\boldsymbol{x}_\star)$ by the feasibility of $\boldsymbol{x}_k$ we have:

$$0 \leq \mathsf{c}_\ell(\boldsymbol{x}_k) - \mathsf{c}_\ell(\boldsymbol{x}_\star) = \nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)(\boldsymbol{x}_k - \boldsymbol{x}_\star) + o(\boldsymbol{x}_k - \boldsymbol{x}_\star)$$

multiplying by $\lambda_k > 0$ and taking limit we get

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \left[\lambda_k \nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)(\boldsymbol{x}_k - \boldsymbol{x}_\star) + \|\lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star)\| \frac{o(\boldsymbol{x}_k - \boldsymbol{x}_\star)}{\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|} \right] \\ &= \nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)\boldsymbol{p} + \|\boldsymbol{p}\| \cdot 0 \end{aligned}$$

and thus $\nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)\boldsymbol{p} \geq 0$. $\square$

Notes

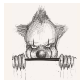
Proof:(Proof. $(2/2)$) For any $\ell \in \mathcal{E}$ by the feasibility of $\boldsymbol{x}_k$ we have:

$$
\begin{aligned}
0 &= \mathsf{c}_\ell(\boldsymbol{x}_k) = \mathsf{c}_\ell(\boldsymbol{x}_k) - \mathsf{c}_\ell(\boldsymbol{x}_\star) \\
&= \nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)(\boldsymbol{x}_k - \boldsymbol{x}_\star) + o(\boldsymbol{x}_k - \boldsymbol{x}_\star)
\end{aligned}
$$

multiplying by $\lambda_k > 0$ and taking limit we get

$$
\begin{aligned}
0 &= \lim_{k\to\infty} \left[\lambda_k \nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)(\boldsymbol{x}_k - \boldsymbol{x}_\star) + \|\lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star)\| \frac{o(\boldsymbol{x}_k - \boldsymbol{x}_\star)}{\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|} \right] \\
&= \nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)\boldsymbol{p} + \|\boldsymbol{p}\| \cdot 0
\end{aligned}
$$

and thus $\nabla \mathsf{c}_\ell(\boldsymbol{x}_\star)\boldsymbol{p} = 0$. So we can conclude that $\boldsymbol{p} \in \mathcal{C}(\boldsymbol{x}_\star)$. $\square$



Notes

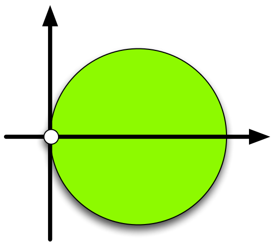
Examples of cone inclusions (1/4)

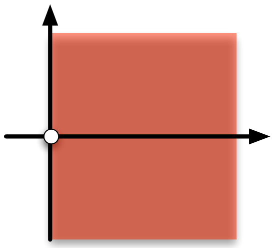
Example

Let be $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 \leq 1\}$ then

$$
\begin{aligned}
\mathcal{R}_S(\mathbf{0}) &= \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 > 0\} \\
\mathcal{T}_S(\mathbf{0}) &= \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 \geq 0\} \\
\mathcal{C}(\mathbf{0}) &= \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 \geq 0\}
\end{aligned}
$$

in this case $\quad \mathcal{R}_S(\mathbf{0}) \subsetneq \mathcal{T}_S(\mathbf{0}) = \mathcal{C}(\mathbf{0}).$





Notes

Examples of cone inclusions

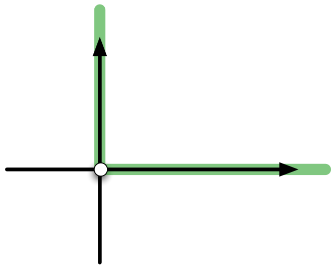
Example

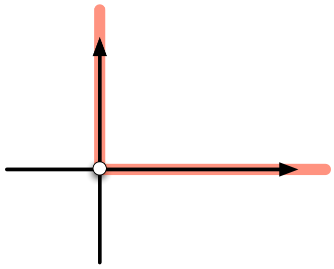
Let be $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 x_2 \leq 0\}$ then

$$\mathcal{R}_S(\mathbf{0}) = S \setminus \{\mathbf{0}\}, \qquad \mathcal{T}_S(\mathbf{0}) = S, \qquad \mathcal{C}(\mathbf{0}) = \mathbb{R}^2$$

in this case

$$\mathcal{R}_S(\mathbf{0}) \subsetneq \mathcal{T}_S(\mathbf{0}) \subsetneq \mathcal{C}(\mathbf{0})$$





Notes

Examples of cone inclusions (3/4)

Example

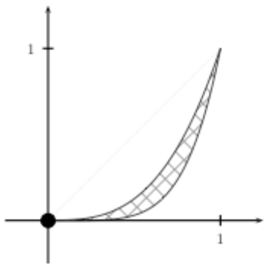
Let be $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 - x_1^3 \leq 0,\ x_1^5 - x_2 \leq 0,\ x_2 \geq 0\}$ then

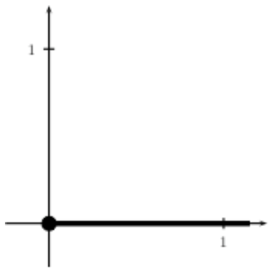
$$\mathcal{R}_S(\mathbf{0}) = \emptyset$$

$$\mathcal{T}_S(\mathbf{0}) = \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 \geq 0,\ p_2 = 0\}$$

$$\mathcal{C}(\mathbf{0}) = \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_2 = 0\}$$

in this case $\quad \mathcal{R}_S(\mathbf{0}) \subsetneq \mathcal{T}_S(\mathbf{0}) \subsetneq \mathcal{C}(\mathbf{0}).$





Notes

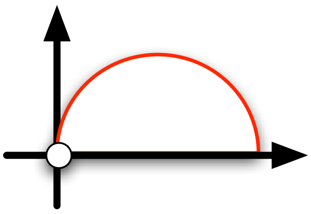
Examples of cone inclusions (4/4)

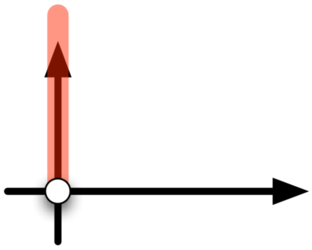
Example

Let be $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 \geq 0,\ (x_1-1)^2 + x_2^2 = 1\}$ then

$$
\begin{aligned}
\mathcal{R}_S(\mathbf{0}) &= \emptyset \\
\mathcal{T}_S(\mathbf{0}) &= \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 = 0,\ p_2 \geq 0\} \\
\mathcal{C}(\mathbf{0}) &= \{\boldsymbol{p} \in \mathbb{R}^2 \mid p_1 = 0,\ p_2 \geq 0\}
\end{aligned}
$$

in this case $\mathcal{R}_S(\mathbf{0}) \subsetneq \mathcal{T}_S(\mathbf{0}) = \mathcal{C}(\mathbf{0})$.





Notes

The Farkas's lemma

Lemma (Farkas's lemma)

Let $\boldsymbol{A} \in \mathbb{R}^{n\times m}$, $\boldsymbol{b} \in \mathbb{R}^{n}$ and consider the following two problems

- (I) *Find $\boldsymbol{x} \in \mathbb{R}^{m}$ such that:* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *and* $\boldsymbol{x} \geq 0$;
- (II) *Find $\boldsymbol{y} \in \mathbb{R}^{n}$ such that:* $\boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{0}$ *and* $\boldsymbol{b}^T\boldsymbol{y} > 0$;

then exactly only one of them has a solution.

Notes

The Karush-Kuhn-Tucker optimality conditions

Theorem (Karush-Kuhn-Tucker optimality conditions)

Let $\boldsymbol{x}_\star \in S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathsf{c}_k(\boldsymbol{x}) \geq 0,\; k \in \mathcal{I},\; \mathsf{c}_k(\boldsymbol{x}) = 0,\; k \in \mathcal{E}\}$ *be a local minimum of* $\mathsf{f}(\boldsymbol{x})$ *and*

$$\mathcal{T}_S(\boldsymbol{x}_\star) = \mathcal{C}(\boldsymbol{x}_\star),$$

then there exists λ_k*,* $k \in \mathcal{I} \cup \mathcal{E}$ *such that:*

$$\nabla \mathsf{f}(\boldsymbol{x}_\star) = \sum\nolimits_{k \in \mathcal{I} \cup \mathcal{E}} \lambda_k \nabla \mathsf{c}_k(\boldsymbol{x}_\star) \qquad \text{(a)}$$

$$\mathsf{c}_k(\boldsymbol{x}_\star) = 0 \qquad k \in \mathcal{E} \qquad \text{(b)}$$

$$\mathsf{c}_k(\boldsymbol{x}_\star) \geq 0 \qquad k \in \mathcal{I} \qquad \text{(c)}$$

$$\lambda_k \geq 0 \qquad k \in \mathcal{I} \qquad \text{(d)}$$

$$\lambda_k \mathsf{c}_k(\boldsymbol{x}_\star) = 0 \qquad k \in \mathcal{I} \qquad \text{(e)}$$

Notes

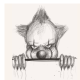
Proof: From the necessary geometric conditions:
$\mathring{\mathcal{D}}(\boldsymbol{x}_\star) \cap \mathcal{T}_S(\boldsymbol{x}_\star) = \emptyset$, from the property $\mathcal{T}_S(\boldsymbol{x}_\star) = \mathcal{C}(\boldsymbol{x}_\star)$ do not exists a $\boldsymbol{p} \in \mathbb{R}^n$ such that

$$\nabla \mathsf{f}(\boldsymbol{x}_\star)\boldsymbol{p} < 0 \qquad \text{and} \qquad \begin{aligned} \nabla \mathsf{c}_k(\boldsymbol{x}_\star)\boldsymbol{p} &\geq 0 \qquad \text{for } k \in \mathcal{I}(\boldsymbol{x}_\star) \\ \nabla \mathsf{c}_k(\boldsymbol{x}_\star)\boldsymbol{p} &= 0 \qquad \text{for } k \in \mathcal{E} \end{aligned}$$

Define the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ as follows

$$\boldsymbol{b} = -\nabla \mathsf{f}(\boldsymbol{x}_\star)^T, \qquad \boldsymbol{A}^T = \left[-\nabla \mathsf{c}_k(\boldsymbol{x}_\star)^T \; k \in \mathcal{A}(\boldsymbol{x}_\star) \right];$$

then the system $\boldsymbol{A}^T\boldsymbol{p} \leq \boldsymbol{0}$ with $\boldsymbol{b}^T\boldsymbol{p} > 0$ is unsolvable.
By Farkas' Lemma there exists a vector $\boldsymbol{x} \in \mathbb{R}^{|\mathcal{A}(\boldsymbol{x}_\star)|}$ with $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Now, let $(\lambda_k, k \in \mathcal{A}(\boldsymbol{x}_\star)) = \boldsymbol{x}^T$, and set $\lambda_i = 0$ for $i \in \mathcal{A}(\boldsymbol{x}_\star) \setminus \mathcal{I}(\boldsymbol{x}_\star)$. It is easy to verify that λ_i so defined satisfy the conditions $(a - e)$. $\square$



Notes

Constraint Qualifications

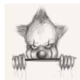
The Karush-Kuhn-Tucker theorem is valid assuming that $\mathcal{T}_S(\boldsymbol{x}_\star) = \mathcal{C}(\boldsymbol{x}_\star)$. This condition is practically impossible to check for generic constraint. However there are simpler condition to check that imply $\mathcal{T}_S(\boldsymbol{x}_\star) = \mathcal{C}(\boldsymbol{x}_\star)$. Such a condition are called Constraint Qualifications. A simple to check Constraint Qualifications is the following:

Definition (Linear Independence Constraint Qualification)

If the gradients $\mathsf{c}_k(\boldsymbol{x}_\star)$ for $k \in \mathcal{A}(\boldsymbol{x}_\star)$ are linear independent we say that the point $\boldsymbol{x}_\star$ has Linear Independence Constraint Qualification, LICQ.

Lemma

If point $\boldsymbol{x}_\star$ has LICQ then $\mathcal{T}_S(\boldsymbol{x}_\star) = \mathcal{C}(\boldsymbol{x}_\star)$.



Notes

Proof: To prove the lemma given a $\boldsymbol{p} \in \mathcal{C}(\boldsymbol{x}_\star)$ we must construct a feasible sequence $\{\boldsymbol{x}_k\}$ and a sequence $\{\lambda_k\} \subseteq (0, \infty)$ such that $\lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{x}_\star$ and $\lim_{k\to\infty} \lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star) = \boldsymbol{p}$.
Consider the non linear system $\boldsymbol{r}(\boldsymbol{x}, \theta) = \boldsymbol{0}$ where:

$$
\mathrm{r}_i(\boldsymbol{x}, \theta) = \begin{cases} \mathrm{c}_i(\boldsymbol{x}) - \theta \nabla \mathrm{c}_i(\boldsymbol{x}_\star)\boldsymbol{p} & \text{for } i \in \mathcal{A}(\boldsymbol{x}_\star) \\ (\boldsymbol{x} - \boldsymbol{x}_\star)^T \boldsymbol{b}_i - \theta \boldsymbol{p}^T \boldsymbol{b}_i & \text{otherwise} \end{cases}
$$

defining the matrices

$$
\boldsymbol{A}(\boldsymbol{x}) = \big[\nabla \mathrm{c}_i(\boldsymbol{x}) \quad i \in \mathcal{A}(\boldsymbol{x}_\star)\big], \qquad \boldsymbol{B} = \big[\boldsymbol{b}_i^T \quad i \in \mathcal{I} \setminus \mathcal{I}(\boldsymbol{x}_\star)\big]
$$

it easy to verify

$$
\frac{\partial \boldsymbol{r}(\boldsymbol{x}, \theta)}{\partial \boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x}, \theta) = \begin{pmatrix} \boldsymbol{A}(\boldsymbol{x}) \\ \boldsymbol{B} \end{pmatrix}, \qquad \frac{\partial \boldsymbol{r}(\boldsymbol{x}, \theta)}{\partial \theta} = -\boldsymbol{J}(\boldsymbol{x}_\star, 0)\boldsymbol{p}.
$$

□

Notes

Proof: For $\boldsymbol{x} = \boldsymbol{x}_\star$ the rows of $\boldsymbol{A}(\boldsymbol{x}_\star)$ are linearly independent (by assumption), so we can choose $\boldsymbol{b}_i$ such that $\boldsymbol{A}(\boldsymbol{x}_\star)$ and $\boldsymbol{B}$ are linearly independent. This means that $\boldsymbol{J}(\boldsymbol{x}_\star, \theta)$ is non singular. By continuity there exists $\delta > 0$ such that $\boldsymbol{J}(\boldsymbol{x}, \theta)$ is non singular for $\boldsymbol{x} \in B(\boldsymbol{x}_\star; \delta)$. We can apply implicit function theorem to conclude that there exists a function $\boldsymbol{x}(\theta) \in \mathtt{C}^1((-\epsilon, \epsilon), \mathbb{R}^n)$ such that

$$\boldsymbol{x}(0) = \boldsymbol{x}_\star, \qquad \boldsymbol{r}(\boldsymbol{x}(\theta), \theta) = \boldsymbol{0}, \qquad \theta \in (-\epsilon, \epsilon).$$

and by differencing we obtain the system of ordinary differential equations:

$$\begin{aligned} \boldsymbol{x}(0) &= \boldsymbol{x}_\star, \\ \boldsymbol{J}(\boldsymbol{x}(\theta), \theta)\boldsymbol{x}'(\theta) &= \boldsymbol{J}(\boldsymbol{x}_\star, 0)\boldsymbol{p}, \qquad \theta \in (-\epsilon, \epsilon). \end{aligned}$$

$\square$

Notes

Proof: The solution of the differential equation $\boldsymbol{x}(\theta)$ has the property $\boldsymbol{x}'(0) = \boldsymbol{p}$. Giving the sequences

$$\lambda_k = k, \qquad \boldsymbol{x}_k = \boldsymbol{x}(1/k)$$

we have

$$\lim_{k\to\infty} \boldsymbol{x}_k = \lim_{k\to\infty} \boldsymbol{x}(1/k) = \boldsymbol{x}_\star.$$

and

$$\lim_{k\to\infty} \lambda_k(\boldsymbol{x}_k - \boldsymbol{x}_\star) = \lim_{k\to\infty} \frac{\boldsymbol{x}(1/k) - \boldsymbol{x}(0)}{1/k} = \boldsymbol{x}'(0) = \boldsymbol{p}.$$

$\square$

Notes

The Fritz—John necessary optimality conditions

Theorem (Fritz—John necessary optimality conditions)

Let $\boldsymbol{x}_\star \in S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathsf{c}_k(\boldsymbol{x}) \geq 0,\ k \in \mathcal{I},\ \mathsf{c}_k(\boldsymbol{x}) = 0,\ k \in \mathcal{E}\}$ *be a local minimum of* $\mathsf{f}(\boldsymbol{x})$ *where then it is necessary that exists* μ_0 *and* μ_k*,* $k \in \mathcal{I} \cup \mathcal{E}$ *such that:*

$$\mu_0 \nabla \mathsf{f}(\boldsymbol{x}_\star)^T - \sum_{k \in \mathcal{I} \cup \mathcal{E}} \mu_k \nabla \mathsf{c}_k(\boldsymbol{x}_\star)^T = \mathbf{0} \qquad \text{(A)}$$

$$\mathsf{c}_k(\boldsymbol{x}_\star) = 0 \qquad k \in \mathcal{E} \qquad \text{(B)}$$

$$\mathsf{c}_k(\boldsymbol{x}_\star) \geq 0 \qquad k \in \mathcal{I} \qquad \text{(C)}$$

$$\mu_k \mathsf{c}_k(\boldsymbol{x}_\star) = 0 \qquad k \in \mathcal{I} \qquad \text{(D)}$$

$$\mu_k \geq 0 \qquad k \in \mathcal{I} \qquad \text{(E)}$$

$$\mu_0^2 + \sum_{k \in \mathcal{I} \cup \mathcal{E}} \mu_k^2 > 0 \qquad \text{(F)}$$

Notes

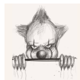
Proof: Consider the matrix

$$\boldsymbol{A}(\boldsymbol{x}) = \big[\nabla \mathsf{c}_i(\boldsymbol{x}_\star) \quad i \in \mathcal{A}(\boldsymbol{x}_\star)\big]$$

if the rows of $\boldsymbol{A}(\boldsymbol{x}_\star)$ are linearly dependent we can find μ_i with $i \in \mathcal{A}(\boldsymbol{x}_\star)$ such that

$$\sum_{i \in \mathcal{A}(\boldsymbol{x}_\star)} \mu_i \nabla \mathsf{c}_i(\boldsymbol{x}_\star) = \boldsymbol{0}^T$$

(by assumption), so we can choose $\boldsymbol{b}_i$ such that $\boldsymbol{A}(\boldsymbol{x}_\star)$ and $\boldsymbol{B}$ are linearly independent. This means that $\boldsymbol{J}(\boldsymbol{x}_\star, \theta)$ is non singular. By continuity there exists $\delta > 0$ such that $\boldsymbol{J}(\boldsymbol{x}, \theta)$ $\square$



Notes