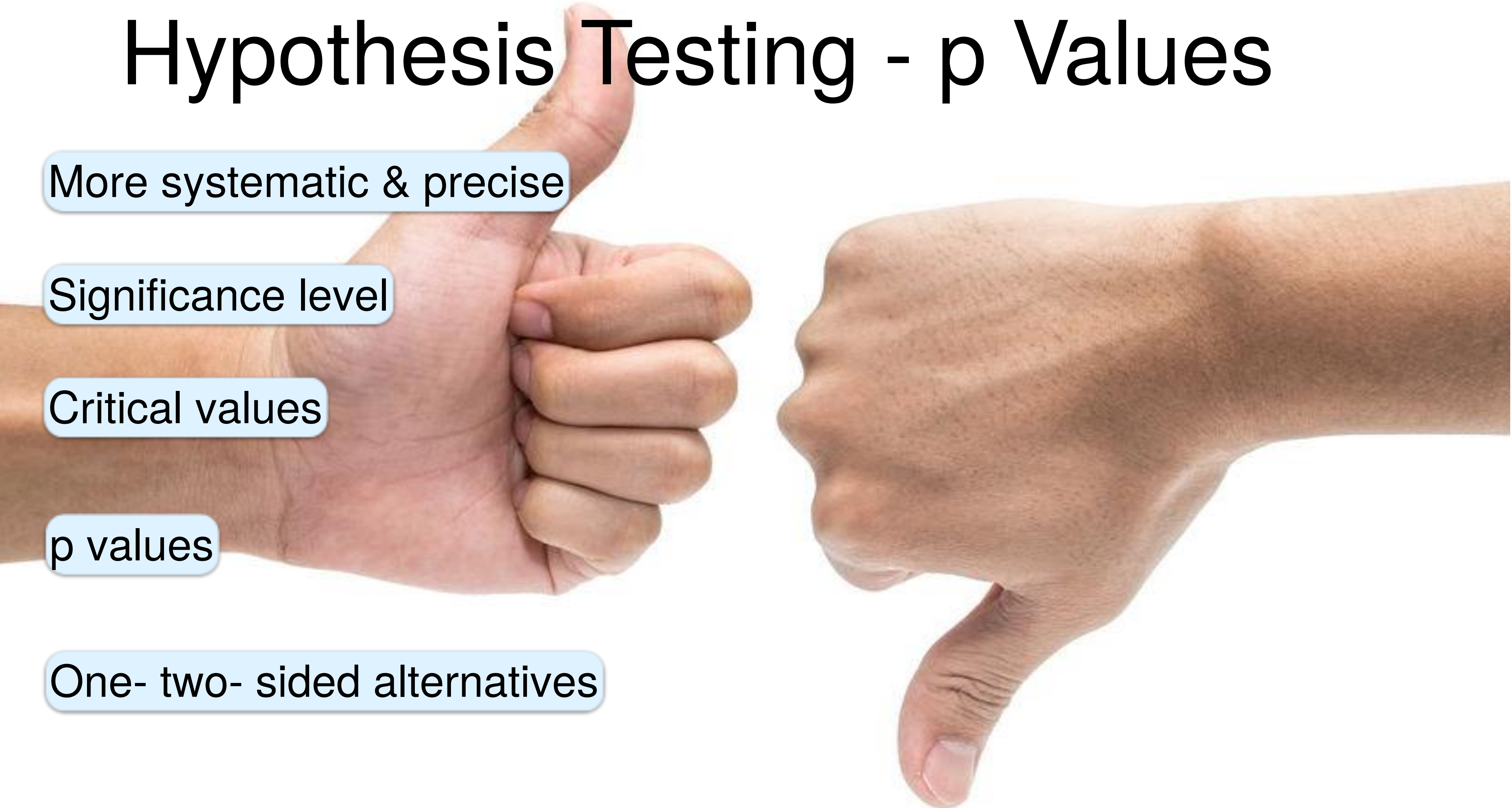
Hypothesis Testing - p Values



More systematic & precise

Significance level

Critical values

p values

One- two- sided alternatives


Found something we like
Put a number on it
5%
Just need

Coin Bias

$H_0 : p_h = 0.5$

Unbiased

$H_A : p_h > 0.5$

Heads more likely

Data 20

X : #

Test Statistic

Type-I Error

H_0 : Coin unbiased

We declare H_A : heads more likely

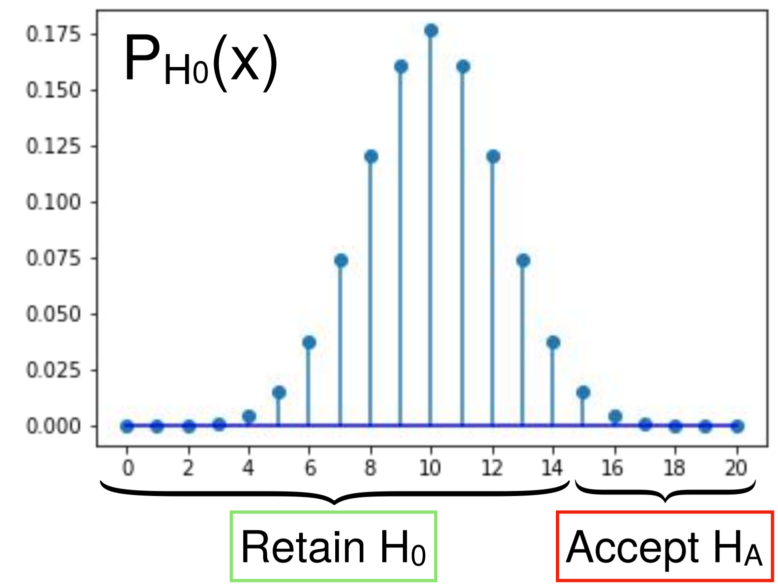
Under H_0

$X \sim B_{0.5,20}$

$P_{H_0}(x)$

← Eyeballed

→ More systematic



Nomenclature

Spiro Agnew

We have more than our share of nattering nabobs of negativism.

Complicated word for terminology | Apropos | Appropriate

H_A | Reject H_0 in favor of H_A | Reject H_0 | Accept H_A

H_0 | Do not reject H_0 | Data not significant | H_0 plausible | ~~Accept H_0~~ | Retain H_0

"Reluctantly retain"

Why verbal gymnastics? | Test asymmetric

X=12 | Accept H_0?

What if p_h = 0.6? | 0.55? | Better explain 12 than p_H = 0.5

Don't know that H_0 true | Just not enough data to reject it

Significance Level

Reject null (status quo) hypothesis H_0 only if strong evidence for alternative H_A

Precise probabilistic formulation

Significance level α Typically 5% 1%

Tiny

If H_0 is true, accept H_A with probability $\leq \alpha$

$P_{H_0}(\text{ accept } H_A) \leq \alpha$

Type-I error

Two methods | Critical Values | p Values

Critical-Value

$H_0 : p_h = 0.5$ $H_A : p_h > 0.5$ Data

X : #

H_0 $X \approx 10$ If accept H_A when X = 16 → 17 → 18 → 19 → 20

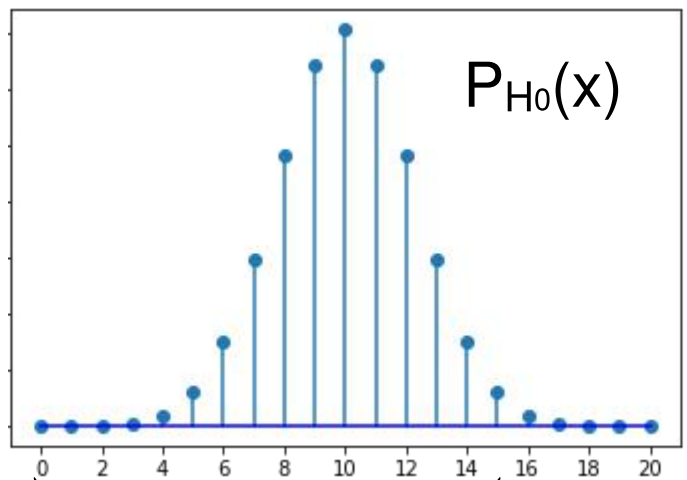
Threshold Critical value x_α

X $\geq x_\alpha$ → Accept H_A

X $< x_\alpha$ → Retain H_0

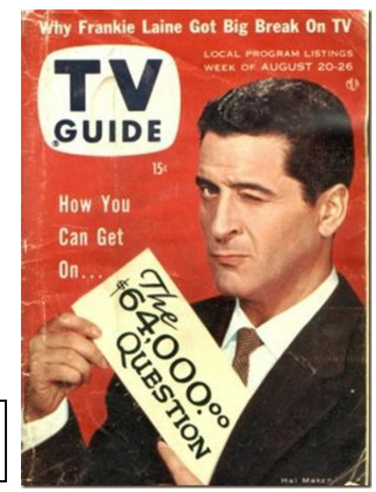
x_α ← Significance level α

Upper bound on $P_{H_0}(\text{accept } H_A)$



Retain H_0 Accept H_A

critical value x_α



What is x_α ?

Finding x_α

$H_0 : p_h = 0.5$

$H_A : p_h > 0.5$

Data 

X : # 

Critical value x_α

$X < x_\alpha \rightarrow$ Retain H_0

$X \geq x_\alpha \rightarrow$ accept H_A

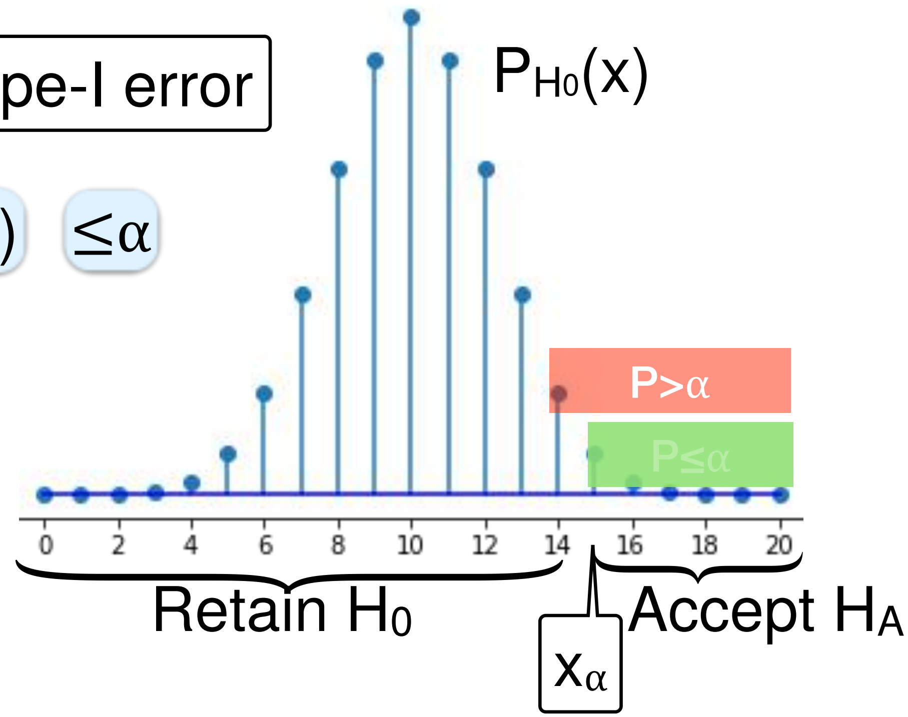
α Significance level 5% 1% Type-I error

$P_{H_0}(X \geq x_\alpha) =$ P_{H_0}(falsely accept H_A) $\leq \alpha$

Need $P_{H_0}(X \geq x_\alpha) \leq \alpha$

x_α large — Almost never declare H_A

Smallest x such that $P_{H_0}(X \geq x) \leq \alpha$



Example: $x_{5\%}$ and $x_{1\%}$

$H_0 : p_h = 0.5$ $H_A : p_h > 0.5$ Data

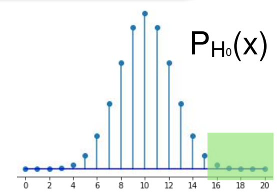
X : #

Significance $\alpha = 5\%$ **Critical value** $x_{5\%}$ Smallest x $P_{H_0}(X \geq x) \leq 5\%$

#heads x	$P_{H_0}(x)$	%	$P_{H_0}(X \geq x)$
20	$\binom{20}{20}\frac{1}{2^{20}}$	0.0001%	0.0001%
19	$\binom{20}{19}\frac{1}{2^{20}}$	0.0019%	0.002%
18	$\binom{20}{18}\frac{1}{2^{20}}$	0.0181%	0.0201%
17	$\binom{20}{17}\frac{1}{2^{20}}$	0.1087%	0.1288%
16	$\binom{20}{16}\frac{1}{2^{20}}$	0.4621%	0.5909%
15	$\binom{20}{15}\frac{1}{2^{20}}$	1.4786%	2.0695%
14	$\binom{20}{14}\frac{1}{2^{20}}$	3.6964%	5.7659%

P ≤ 5%

P > 5%



$P_{H_0}(X \geq 15) \approx 2.07\% \leq 5\%$

$P_{H_0}(X \geq 14) \approx 5.77\% > 5\%$

Accept H_A

$x_{1\%} = 16$

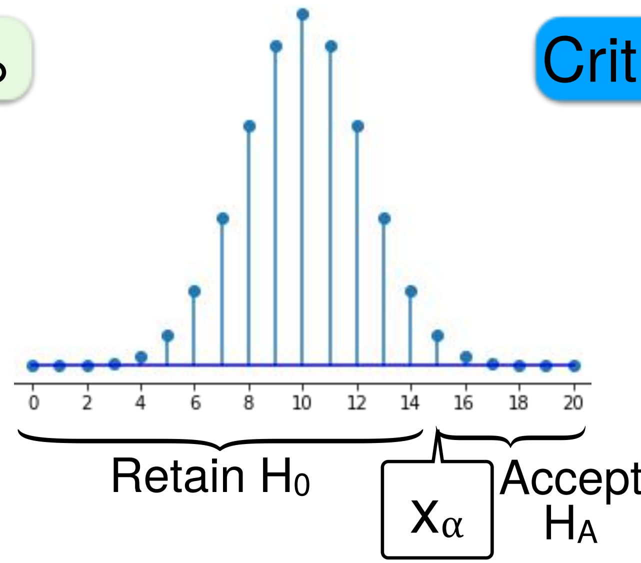
$x_{5\%} = 15$

Retain H_0

Room for Improvement

Significance level | α | 5%

Critical value | x_α | 15



X $< x_\alpha$ Retain H_0

X $\geq x_\alpha$ Accept H_A

Practically | Observe one value x of X | $X = 13$ | Retain H_0 or accept H_A

Can we..

Get around finding smallest x for accepting H_A?

Find a rule just for X itself?

p (probability) Values

Critical value $x_{5\%}$ 15 Smallest x $P_{H_0}(X \geq x) \leq 5\%$

x

$\geq x_{5\%}$ Accept H_A $\quad x \geq x_{5\%} \leftrightarrow P_{H_0}(X \geq x) \leq P_{H_0}(X \geq x_{5\%}) \leq 5\%$

$< x_{5\%}$ Retain H_0 $\quad x < x_{5\%} \leftrightarrow P_{H_0}(X \geq x) > 5\%$

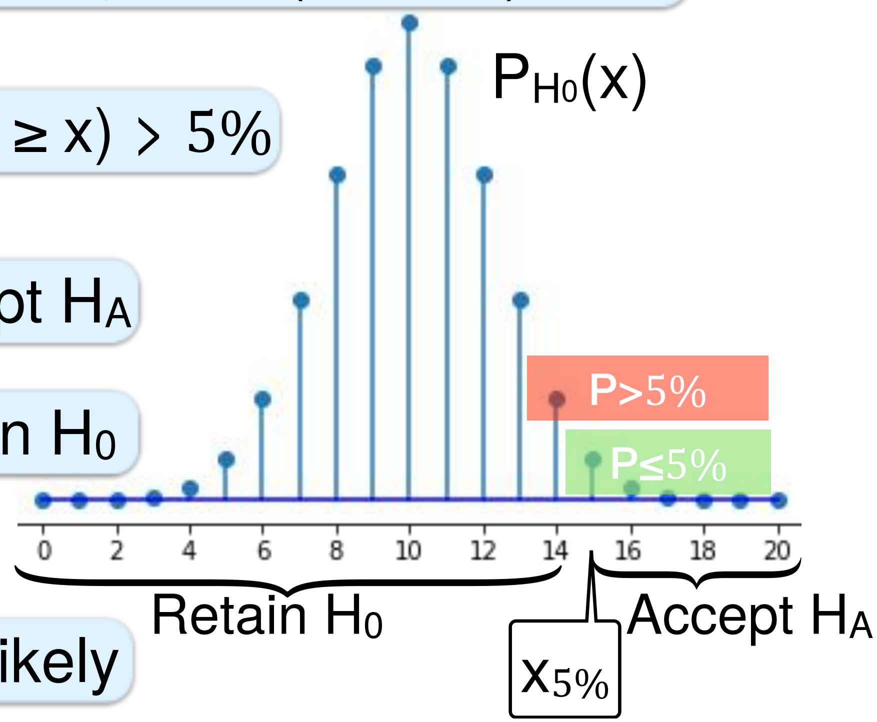
p value of x $P_{H_0}(X \geq x)$

$\leq 5\%$ $\quad x \geq x_{5\%}$ $\quad$ Accept H_A

$> 5\%$ $\quad x < x_{5\%}$ $\quad$ Retain H_0

Same H_0 and H_A regions as before!

Intuitively P under H_0 small $\rightarrow$ H_A more likely



p Values Example

$H_0 : p_h = 0.5$

$H_A : p_h > 0.5$

Data

X : # 

Significance $\alpha = 5\%$

Critical value $x_{5\%} = 15$

$X \geq 15 \rightarrow H_A$

$X < 15 \rightarrow H_0$

# heads x	$P_{H_0}(x)$	%	$P_{H_0}(X \geq x)$	
20	$\binom{20}{20}\frac{1}{2^{20}}$	0.0001%	0.0001%	H_A
19	$\binom{20}{19}\frac{1}{2^{20}}$	0.0019%	0.002%	•
18	$\binom{20}{18}\frac{1}{2^{20}}$	0.0181%	0.0201%	•
17	$\binom{20}{17}\frac{1}{2^{20}}$	0.1087%	0.1288%	•
16	$\binom{20}{16}\frac{1}{2^{20}}$	0.4621%	0.5909%	•
15	$\binom{20}{15}\frac{1}{2^{20}}$	1.4786%	2.0695%	H_A
14	$\binom{20}{14}\frac{1}{2^{20}}$	3.6964%	5.7659%	H_0

$\alpha = 1\%$

H_0

p value

$\leq 5\%$	$x \geq x_{5\%}$	Accept H_A
$> 5\%$	$x < x_{5\%}$	Retain H_0

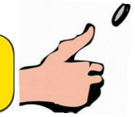
p value of x

Probability $X \geq x$

$\leq \alpha$ accept H_A

$> \alpha$ retain H_0

Opposite Alternative

$H_0 : p_h = 0.5$ $H_A : p_h < 0.5$ Data 20    X : # 

Significance level α 5%   1% P_{H_0} (falsely accept H_A) $\leq \alpha$

ensures

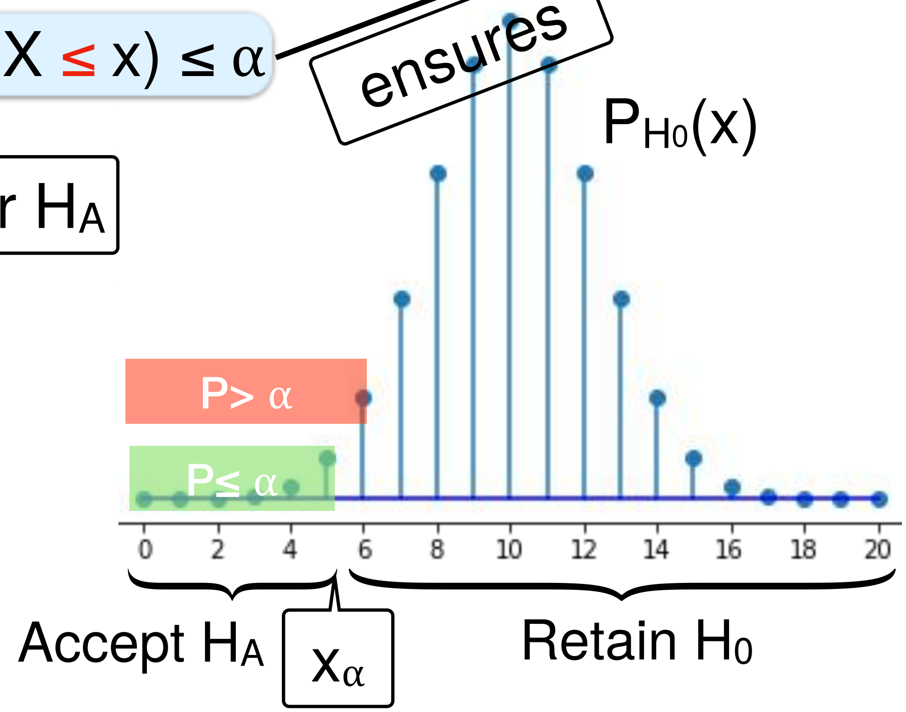
Critical value x_α Largest x s.t. $P_{H_0}(X \leq x) \leq \alpha$

Small error under H_A

X $\leq x_\alpha$ Accept H_A

X $> x_\alpha$ Retain H_0

$\alpha = 5\%$ By symmetry $x_{5\%} = 5$



Using p Values

$H_0 : p_h = 0.5$ | $H_A : p_h < 0.5$ | Data 20 | X : #

Significance level $\alpha = 5\%$

# heads x	$P_{H_0}(x)$	%	$P_{H_0}(X \leq x)$
0	$\binom{20}{20}\frac{1}{2^{20}}$	0.0001%	0.0001%
1	$\binom{20}{19}\frac{1}{2^{20}}$	0.0019%	0.002%
2	$\binom{20}{18}\frac{1}{2^{20}}$	0.0181%	0.0201%
3	$\binom{20}{17}\frac{1}{2^{20}}$	0.1087%	0.1288%
4	$\binom{20}{16}\frac{1}{2^{20}}$	0.4621%	0.5909%
5	$\binom{20}{15}\frac{1}{2^{20}}$	1.4786%	2.0695%
6	$\binom{20}{14}\frac{1}{2^{20}}$	3.6964%	5.7659%

p value

≤ 5%	$x \leq x_{5\%}$	Accept H_A
> 5%	$x > x_{5\%}$	Retain H_0

H_A (x = 0)

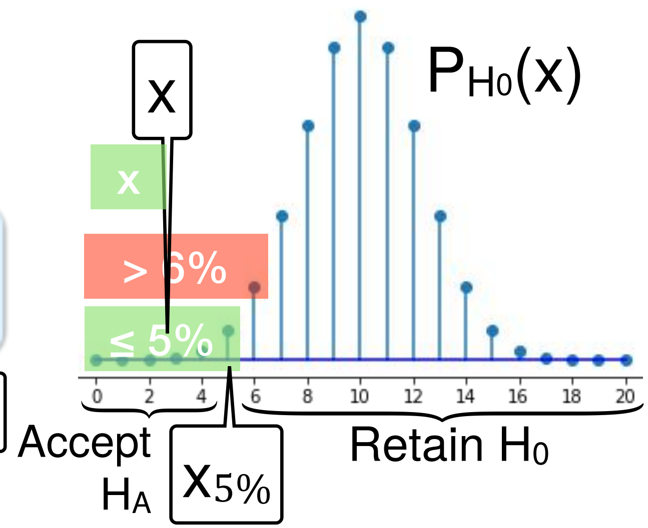
•
•
•
•

H_A (x = 5)

H_0 (x = 6)

$\alpha =$ 1%

H_0 (x = 5)



Two-Sided Alternative

$H_0 : p_h = 0.5$ — Unbiased

$H_A : p_h \neq 0.5$ — Two-sided

Data 

X : # 

H_0 Mean — $\mu_X = 10$

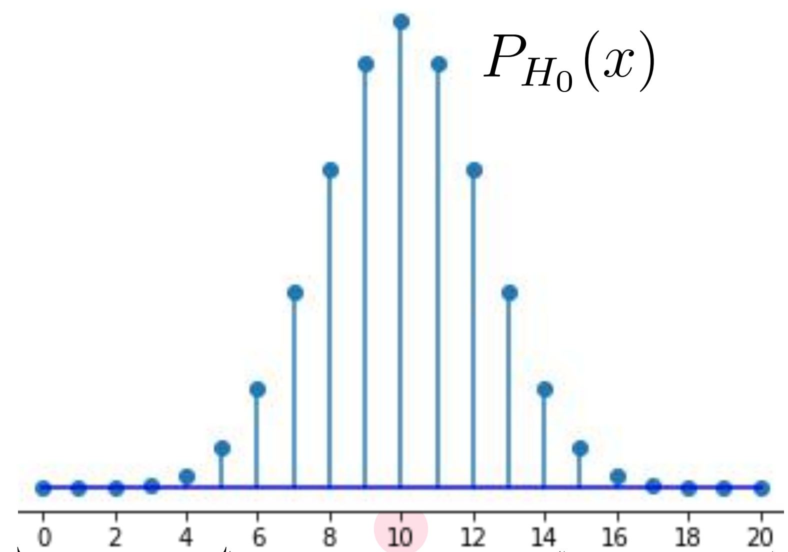
Intuition

Under H_0: X close to 10

Under H_A: X far from 10

$|X - 10|$ small: Retain H_0

$|X - 10|$ large: Accept H_A



Accept H_A — Retain H_0 — Accept H_A

Critical Value

$H_0 : p_h = 0.5$ $H_A : p_h \neq 0.5$ Data

X : #

Significance level α Upper bound on type-I error

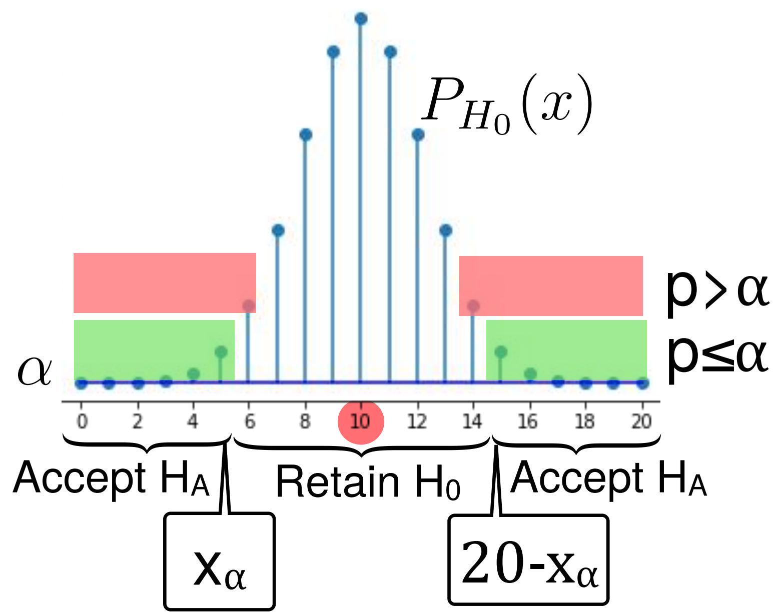
H_0 Mean $\mu_X = 10$

Critical value x_α x closest to 10 s.t. $P_{H_0}(|X-10| \geq |x-10|) \leq \alpha$

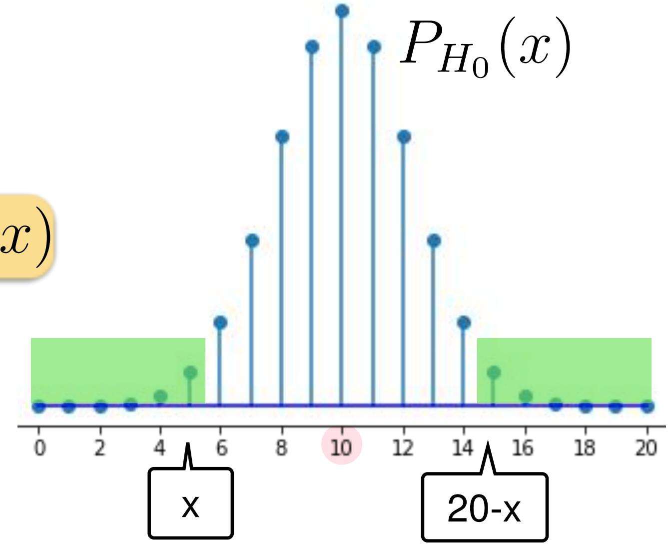
$$|X-10| \begin{cases} \geq |x_\alpha - 10| & \text{Accept } H_A \\ < |x_\alpha - 10| & \text{Retain } H_0 \end{cases}$$

$P_{H_0}(\text{type-I error}) = P_{H_0}(|X-10| \geq |x_\alpha - 10|) \leq \alpha$

x_α closest to 10 minimizes type-II error



p Values



p value of x

$P_{H_0}(X \text{ is at least as far from 10 as } x)$

$P_{H_0}(|X - 10| \geq |x - 10|)$

Low p value

x far from mean

Low H_0 prob of outcome x or further towards H_A

x less likely to be generated under H_0

High p value

x far from mean

High H_0 prob of outcome x or further towards H_A

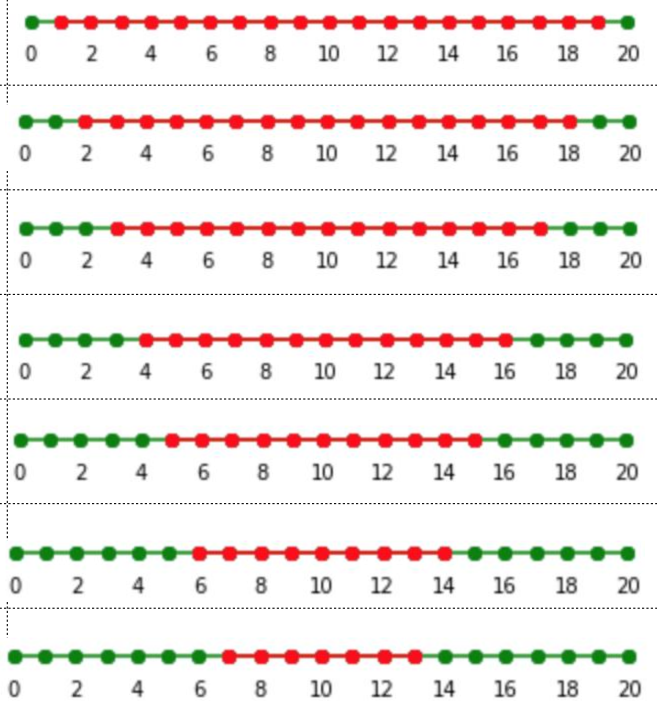
x more likely to be generated under H_0

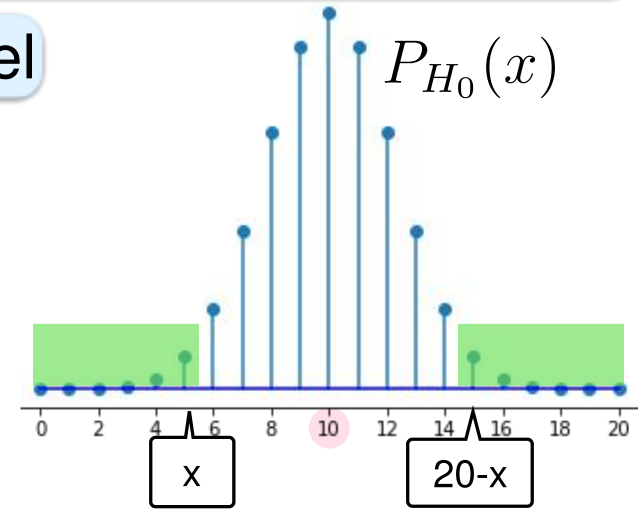
p Values → Hypotheses

p value of x: $P_{H_0}(X \text{ is at least as far from 10 as } x)$ = $P_{H_0}(|X-10| \geq |x-10|)$

5% Significance level

x	|X-10| ≥ |x-10|	p value of x
0,20		0.0002% — H_A
1,19		0.0040% •
2,18		0.0402% •
3,17		0.2577% •
4,16		1.1818% •
5,15		4.1389% — H_A
6,14		11.5318% — H_0





p value

≤ 5% → Accept H_A

> 5% → Retain H_0

General p value

p value of statistic t of T | P_{H_0}(T is t or further towards H_A)

Significance level | α

p value	$\leq \alpha$	Accept H_A
	$> \alpha$	Retain H_0

Hypothesis Testing - p Values

- More systematic & precise
- Significance level
- Critical values
- p values
- One- two- sided alternatives