ST 705 Linear models and variance components
Homework problem set 1

December 31, 2021

1. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be sequences of real numbers. Show that

$$\min\{a_i\} + \min\{b_i\} \leq \min\{a_i + b_i\} \leq \min\{a_i\} + \max\{b_i\}.$$

2. Use Jensen's inequality to establish the arithmetic-geometric mean inequality. That is, show that if $a_1, \ldots, a_n$ are positive constants, then

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}.$$

3. Let $x = (x_1, \ldots, x_p)' \in \mathbb{R}^p$. Show that for $i \in \{1, \ldots, p\}$,

$$|x_i| \leq \|x\|_2 \leq \|x\|_1,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the l_1 and l_2 vector norms, respectively.

4. Prove that all norms on a finite-dimensional vector space V over $\mathbb{C}$ are *equivalent*. That is, show that for any two norms, say $\|\cdot\|_a$ and $\|\cdot\|_b$, defined on V, there exists real-valued positive constants c_1 and c_2 such that for every $x \in V$,

$$c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b.$$

   (a) First, show that it is without loss of generality to consider $\|\cdot\|_b = \|\cdot\|_1$.

   (b) Second, demonstrate that it suffices to only consider $x \in V$ with $\|x\|_1 = 1$.

   (c) Next, prove that any norm $\|\cdot\|_a$ is a continuous function under $\|\cdot\|_1$-distance.

   (d) Finally, apply a result from calculus such as the Bolzano-Weierstrass theorem or the extreme value theorem to finish your argument that all norms on a finite-dimensional vector space are *equivalent*.

   This notion of *equivalence* is in reference to the fact that if a sequence is convergent in *some* norm, then it is convergent in *all* norms. Note the assumption of a *finite*-dimensional vector space.

5. Let U and V be random variables. Establish the following inequalities.

   (a) $P(|U+V| > a+b) \leq P(|U| > a) + P(|V| > b)$ for every $a, b \geq 0$.

   (b) $P(|UV| > a) \leq P(|U| > a/b) + P(|V| > b)$ for every $a \geq 0$ and $b > 0$.

6. Prove the following theorem. Let V be a vector space and $B = \{u_1, \ldots, u_n\}$ be a subset of V. Then B is a basis if and only if each $v \in V$ can be expressed *uniquely* as

$$v = a_1 u_1 + \cdots + a_n u_n$$

   for some set of scalars $\{a_1, \ldots, a_n\}$.