ST 705 Linear models and variance components
Homework problem set 2

January 19, 2022

1. Show that every eigenvalue of a real symmetric matrix is real.

2. Prove that the eigenvalues of an upper triangular matrix M are the diagonal components of M.

3. Prove that a (square) matrix that is both orthogonal and upper triangular must be a diagonal matrix.

4. Show that the covariance function defined for $X, Y \in \mathbb{R}^p$ by

$$\operatorname{Cov}(X, Y) := E[(X - E(X))(Y - E(Y))']$$

satisfies the following properties. For random variables $X, Y, Z \in \mathbb{R}^p$ with finite covariance, and any $c \in \mathbb{R}$,

(a) $\operatorname{Cov}(X + Y, Z) = \operatorname{Cov}(X, Z) + \operatorname{Cov}(Y, Z)$

(b) $\operatorname{Cov}(cX, Y) = c \cdot \operatorname{Cov}(X, Y)$

(c) $\operatorname{Cov}(X, Y)^* = \operatorname{Cov}(Y, X)$

(d) $\operatorname{Cov}(X, X) \geq 0$ for all X, and $\operatorname{Cov}(X, X) = 0$ implies that X is constant a.s.

Then, deduce that if $p = 1$ the covariance is an inner product over some (quotient) vector space, and if $p > 1$ the the function $f(X, Y) := \operatorname{tr}\big(\operatorname{Cov}(X, Y)\big)$ is an inner product.

5. Let $A \in \mathbb{R}^{p \times p}$ be symmetric. Use the spectral decomposition of A to show that

$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x'Ax}{x'x} = \lambda_{\max},$$

where $\lambda_{\max}$ is the largest eigenvalue of A. Observe that this is a special case of the Courant-Fischer theorem (see `https://en.wikipedia.org/wiki/Min-max_theorem`).

6. Construct an $n \times n$ matrix A such that $\lambda_{\max}(A) \neq \sup_{v \neq 0} \left\{ \frac{v'Av}{v'v} \right\}$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its argument. Why does your counter example not violate the Courant-Fischer theorem?

7. Let V be a convex subset of some vector space. Recall that a function $f : V \to \mathbb{R}$ is said to be *convex* if for every $x, y \in V$ and every $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

Show, by definition, that the sum of squared errors function

$$Q(\beta) := \|Y - X\beta\|_2^2$$

is convex.