Finance in the Frequency Domain

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E. B. Legrand

Literature Survey

Delft University of Technology

Delft Center for Systems and Control

Finance in the Frequency Domain
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LITERATURE SURVEY

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July 5, 2021

Faculty of Mechanical, Maritime and Materials Engineering (3mE) · Delft University of Technology

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Abstract

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Acknowledgements

I would like to thank my supervisor for his assistance during the writing of this thesis. . .

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology
July 5, 2021

E. B. Legrand

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Chapter 1

Introduction

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Chapter 2

Introductory actuarial concepts

Chapter structure

1. Simple interest & compound interest
2. Discounting & (Net) Present value
3. Internal rate of return
4. Bond valuations
5. Duration and convexity, immunization
6. Term structure of interest rates

Time value of money? Introduce better difference continuous/discrete.

(... `Chapter intro ...`)

2-1 The concept of interest

(... `Intro with some history ...`)

Usually, two types of interest are distinguished: simple interest and compound interest. The key difference is that in the case of compound interest, the money earned (or due) on an interest-bearing instrument is subject to interest itself. That is, if somebody takes out a loan, the corresponding interest payments are also considered as a contribution to the principal and therefore result in higher future interest payments. In contrast, the case of simple interest does not consider the compounding effect; the interest payments remain constant over time.

The compounding of interest is always associated with a certain period; this is the period over which the interest earned is calculated and added to the amount due (or 'reinvested' in case of an investment). The choice of this period is part of the agreement between the lender and the borrower, common compounding periods are daily, weekly, monthly or yearly. From a mathematical perspective, one can view the choice of compounding period as a limiting process: clearly, the choice in practice is rather arbitrary. Indeed, the 'ideal' compounding period is continuous, but before the advent of computers this was not achievable in practice; in banking applications it is still not very commonly used. [1]

2-1-1 Interest terminology

Many interest calculations are subject to conventions, which is why there are several terms associated 'interest' which must be clearly distinguished. Usually, an interest-bearing instrument is associated with some amount that is initially borrowed or invested, which is called the *principal*, denoted by K. Secondly, one can define the *accumulation function* a which yields the accumulated value of a unit investment (or loan).[1] Similarly, the *amount function* A measures the accumulation of any principal value K: $A(k) = Ka(k)$. [2]

Concerning the period k, some additional remarks are in order. In most banking applications, the interest process is discrete, i.e. the compounding effect occurs over discrete intervals. Apart from the compounding process, one can distinguish a 'measurement' interval which can, but does not necessarily have to, coincide with the compounding period. For now, the interest process is assumed to be calculated on a discrete basis; the transition to continuous time will be made after that.

> The *effective rate of interest* r is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period. In terms of the accumulation function:
>
> $$i(k) = \frac{a(k) - a(k-1)}{a(k-1)} \tag{2-1}$$
>
> where a is in this case assumed to take discrete values.

Simple interest

For the case of simple interest, the accumulation function has the form

$$a(k) = ik + 1$$

where $i = i(0)$ denotes the constant simple interest rate, which turns out to be identical to the effective rate of interest over the first period. An important fact one has to bear in mind

[1] In this discussion, the distinction between credit or debit (i.e. investments or loans) is quite irrelevant for the principles at hand, they just differ in sign on the balance sheet of the respective counterparties. Hence, these terms will be used interchangeably. Minus signs are there to indicate cash flows in the 'opposite' direction, regardless whether this is on the credit or debit side of the balance sheet.

is that for the case of simple interest the effective rate of interest is *not* constant over time (this, in fact, motivates the existence of compound interest, as will become clear later). Using eq. (2-1):

$$i(k) = \frac{(ki+1) - [(k-1)i+1]}{(k-1)i+1} = \frac{i}{1+i(n-1)}$$

which means that simple interest results in a decreasing effective rate of interest over time. [2]

Compound interest

As mentioned before, compound interest relies on the reinvestment of the interest already earned. At the end of the period the accumulation function is a factor $1+i$ larger than the period before. One can therefore say that

$$a(k) = (1+i)a(k-1) \quad \text{or} \quad a(k) = (1+i)^k$$

assuming that the compounding period coincides with the measurement period.

Similarly, eq. (2-1) can be used to compute the effective rate of interest for an arbitrary period:

$$i(k) = \frac{(1+i)^k - (1+i)^{k-1}}{(1+i)^{k-1}} = i$$

which means that for compound interest, the effective rate of interest is *constant*. This is the reason why compound interest plays an ubiquitous role in modern finance; otherwise it would become less and less profitable for investors to keep their money in a certain investment (they would rather just stay a single period and then turn to a new investment — this effectively emulates compound interest!). Only for short periods (less than one year), simple interest is sometimes used because the difference with compound interest is negligible, which can be motivated by means of the Taylor series of a for compound interest:

$$(1+i)^k = 1 + i + \binom{k}{2} i^2 + \binom{k}{3} i^3 + \ldots \approx 1 + i \quad \text{for} \quad i \ll 1,\ k \text{ small}$$

In fact, from this equation it is clear that simple interest and compound interest yield the same result after the first compounding period, after which compound interest will take the upper hand.

For now, it was assumed that one compounding period is equal to one measurement period. However, this does not necessarily have to be the case. More generally, one can write the accumulation function as:

$$a(k) = \left(1 + \frac{i}{n}\right)^{nk}$$

where n is the number of compounding periods per measurement periods (which is assumed to be an integer number). In actuarial sciences, the measurement period is normally equal to one year; meaning that e.g. weekly compounding amounts to $n = 52$.

Here, i is called the *nominal interest rate*, which is usually denoted by $i^{(n)}$, indicating that the nominal rate i is compounded n times at a rate $i^{(n)}/n$. Hence, a nominal interest rate of $i^{(m)}$ per measurement period is equivalent to an effective interest rate of $i^{(m)}$ per nth part of that measurement period.

Continuous time

Now, the discrete accumulation (and amount) functions will be converted to their continuous counterparts, since that will be the most convenient form from a mathematical perspective. To do so, consider the nominal interest over single measurement period, but subdivided into an ever growing number of compounding intervals. Because the number of compounding intervals grows to infinity, the choice of measurement interval becomes immaterial which is why it can simply be replaced by 't' denoting continuous time.

In case of simple interest, the result is trivial:

$$a(t) = \lim_{n\to+\infty} \left(\frac{i}{n}\right)(tn) + 1 = it + t$$

which basically means that the nominal interest remains the same no matter the 'compounding period'.

For compound interest, the result is more interesting:[2]

$$a(t) = \lim_{n\to\infty} \left(1 + \frac{i}{n}\right)^{nt} = \mathrm{e}^{it}$$

2-1-2 Discounting, Net Present Value and the Internal Rate of Return

Internal Rate of Return (IRR) Net Present Value (NPV)

2-1-3 Lorentz structure from the compounding process

Figure 2-1 exemplifies the compounding process with 5% interest on an initial investment of \$1000. For now, it is immaterial what the compounding frequency is. The total value of the investment is decomposed into two parts, which will be called the 'capital' and 'yield' fractions. Every time the investment is 'compounded', the following happens:

- The interest rate acts on the amount of capital outstanding, the result of which adds to the current total yield.
- The interest rate acts on the current total yield, the result of which is added to the capital outstanding. This is sometimes called *interest on interest* and lies at the very core of the compounding principle. If this action would not be present, the process reduces to simple interest.

Clearly, apart from the obvious symmetry, this decomposition is motivated by its intuitive interpretability. Much of what is to come in the dissertation hinges on this principle, which is why this example, obvious as it may seem, is important. Two approaches will be discussed that elegantly capture this process from a computational standpoint: discrete LTI systems and hyperbolic-complex numbers. These methods are also closely related and essentially represent two sides of the same coin. The results that follow should therefore not come as surprising, though looking at them from different perspectives is certainly instructive.

[2]There exist a few equivalent definitions of the exponential function, one of which is this limit. As such, some may argue that this statement is true *by definition*.

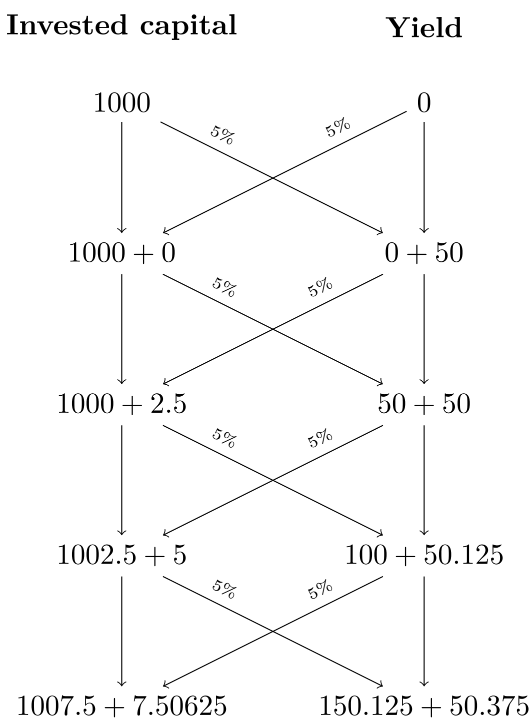


Figure 2-1: Caption

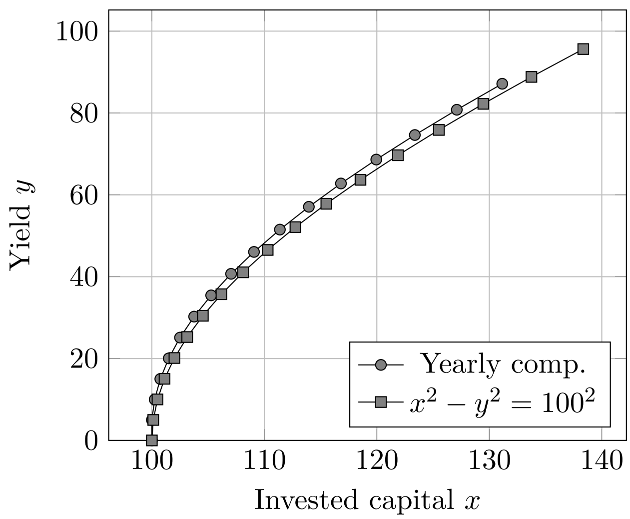


Figure 2-2: blabla

Chapter 3

The Lorentz-Minkowski Plane

In the previous chapter, it has been established that interest processes can be represented on hyperbolic curves by disregarding the time dimension and 'decomposing' the exponential in a specific way. This result is almost trivially encapsulated in the identity

$$\exp(\zeta) = \cosh(\zeta) + \sinh(\zeta) \tag{3-1}$$

The most intuitive way to interpret this identity is by inspection of the Taylor series of the functions appearing in eq. (3-1):

$$\begin{aligned} \exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \cosh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\ \sinh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \end{aligned} \tag{3-2}$$

Or, alternatively, via the definition of the hyperbolic functions

$$\begin{aligned} \cosh(x) &\triangleq \frac{\exp(x) + \exp(-x)}{2} \\ \sinh(x) &\triangleq \frac{\exp(x) - \exp(-x)}{2} \end{aligned} \tag{3-3}$$

Clearly, the cosh and sinh functions are constructed by isolating the even or the odd powers respectively from the Taylor expansion of the exponential. The astute reader may notice that eq. (2-1) bears some resemblance to Euler's formula $\exp(\mathrm{i}x) = \cos(x) + \mathrm{i}\sin(x)$. As described by Needham [3], this connection can be generalized by recognizing that

$$\cos(\mathrm{i}x) = \cosh(x) \qquad \sin(\mathrm{i}x) = \mathrm{i}\sinh(x)$$

As such, both the hyperbolic functions sinh, cosh and the trigonometric functions cos and sin can all be represented by looking at the modular surface of $|\sin(z)|$, visualized in fig. 3-1: of course, sin and cos only differ by a shift of $\pi/2$ along the real line, and cosh and sinh exist at cross-sections into the complex at integer multiples of $\pi/2$ and π respectively.

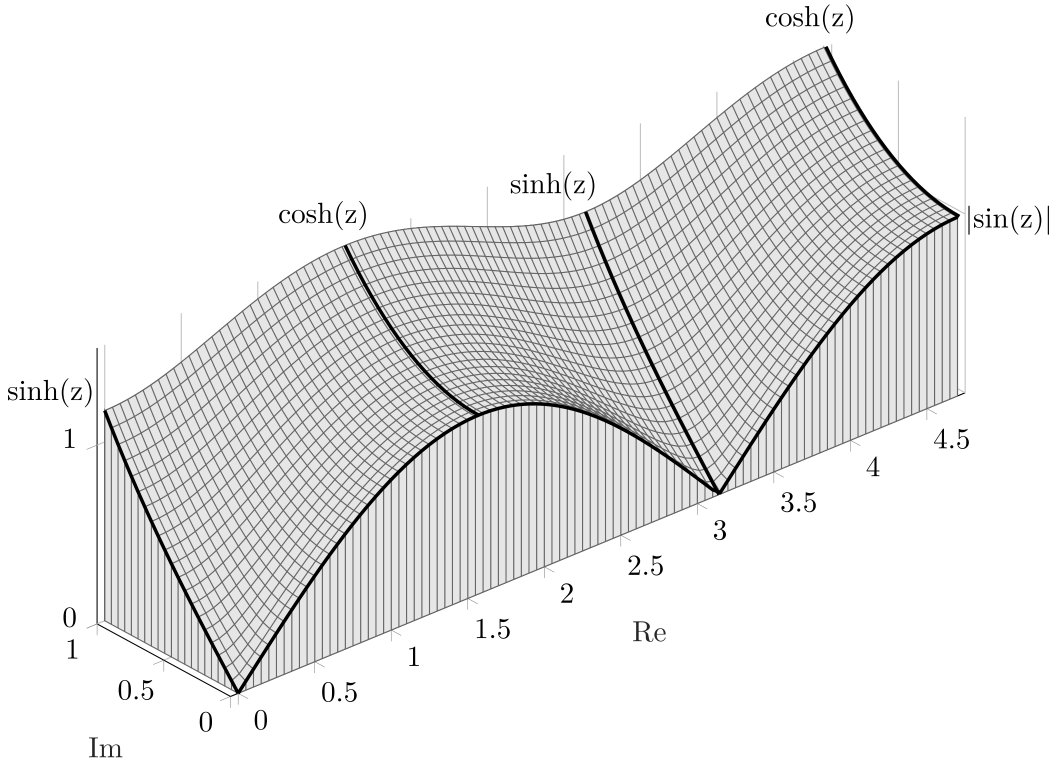


Figure 3-1: Modular surface of the sine over the complex plane, embedding all the trigonometric and hyperbolic functions at specific cross-sections.

3-1 Conic sections

There are a variety of different (representations) of hyperbolae; in this case the discussion will be limited to so-called *rectangular hyperbolae*, which are basically rotations, translations or a combination thereof of the scaled reciprocal function $y = K/y$. The term 'rectangular' refers to the fact that there is no squeeze or stretch along any particular direction. Although the term 'rectangular hyperbola' therefore corresponds to a whole range of shapes, including the unit hyperbola which is to be defined later, it will henceforth be used here as a *totum pro parte* to refer to the function $y = K/y$ in particular in order accentuate the distinction with the unit hyperbola. An implicit parameterization of this rectangular hyperbola is

$$\begin{cases} x = \pm K\mathrm{e}^{t} \\ y = \pm K\mathrm{e}^{-t} \end{cases} \quad t \in \mathbb{R},\ K \in \mathbb{R}_*^+$$

Clearly, this hyperbola has two asymptotes along the x and y axes. The line $x = y$ is called the *major axis* of the hyperbola. By rotating the hyperbola over a 45° angle in clockwise direction, the major axis will coincide with the x-axis, and a subsequent 'squeeze' by factor $\sqrt{2}/2$, one arrives at the so-called *unit hyperbola* defined by the implicit equations $x^2 - y^2 = K^2$. Applying this linear transformation (scale with $\sqrt{2}/2$ and rotate by $-\pi/4$) as a matrix operation to the aforementioned parametric description, one arrives at

$$\frac{\sqrt{2}}{2}\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} \pm K\mathrm{e}^{t} \\ \pm K\mathrm{e}^{-t} \end{pmatrix} = \pm\frac{K}{2}\begin{pmatrix} \mathrm{e}^{t}+\mathrm{e}^{-t} \\ \mathrm{e}^{t}-\mathrm{e}^{-t} \end{pmatrix} = K\begin{pmatrix} \pm\cosh(t) \\ \sinh(t) \end{pmatrix}$$

which is the common parameterizaton of the unit hyperbola, this will be the standard representation throughout this text. Please note that the '$\pm$' in front of the sinh can be disregarded because the sinh is an odd function. One can now also recognize the asymptotes as an alternative axis system, which recovers the original hyperbola; this axis system will be referred to as the *idempotent axis system*. The hyperbolae and the corresponding axis system are shown in fig. 3-2.

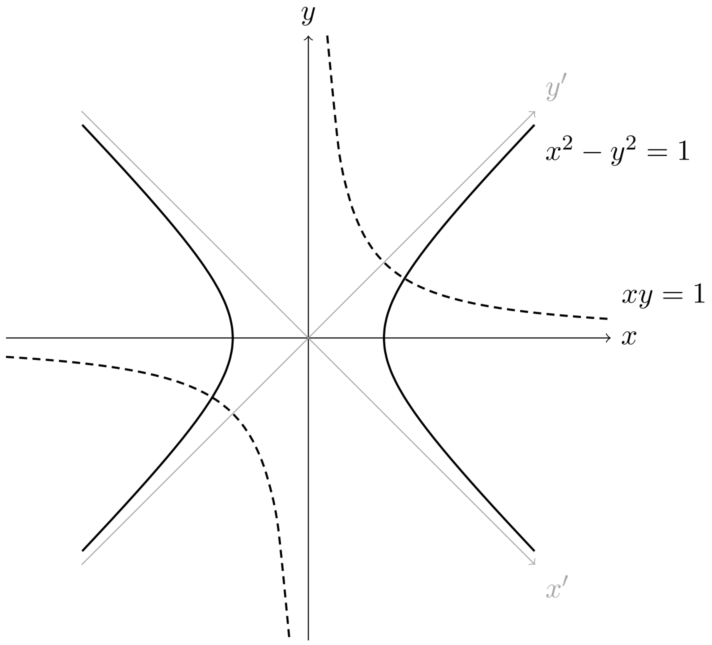


Figure 3-2: Comparison between the reciprocal function $(x, \frac{1}{x})$ and the unit hyperbola $(\cosh(t), \sinh(t))$ (implicit equation $x^2 - y^2 = 1$). The idempotent axis system (denoted by the gray lines) coincides with the asymptotes of the unit hyperbola; from this axis system the unit hyperbola again satisfies the equation $x'y' = 1$.

3-2 Hyperbolic angles

Until now, the argument of the sinh/cosh parameterization has been t as is the familiar notation for a parametric curve. However, the duality between circles and hyperbolas can be exploited further by defining the arguments of the hyperbolic functions as *hyperbolic angles*, like one does for the trigonometric functions sin and cos, which could be named 'circular' angles to further highlight the distinction. In both cases, an angle refers to a certain region bounded by the curve (hyperbola/circle) at issue. The standard notation for the hyperbolic angle will be ζ, in accordance to the rapidity from special relativity which can also be viewed as a hyperbolic angle as will be discussed later on.

Hyperbolic sector

A hyperbolic sector is the region bounded by two lines extending from the origin to each to a point on the (unit) hyperbola, and the graph of the hyperbola itself.

Clearly, hyperbolic sectors are entirely analogous to their 'traditional' circular cousins. Fixing one of the rays to the x-axis, one can define the corresponding hyperbolic angle:

Hyperbolic angle

A hyperbolic angle corresponding to a point A is defined as twice the area of the hyperbolic sector based on the point A and the intersection point of the unit hyperbola and the x-axis $(K, 0)$.

Clearly, any point on a hyperbola with radius K can be parameterized using

$$(K \cosh(\zeta), K \sinh(\zeta))$$

By allowing the radius K to be negative, all the points in the disconnected open set bounded by $y = x$ and $y = -x$ can be identified with a unique radius K and hyperbolic angle ζ. This is somewhat similar to polar coordinates, which is why these coordinates will be referred to as 'hyperbolic polar coordinates'.

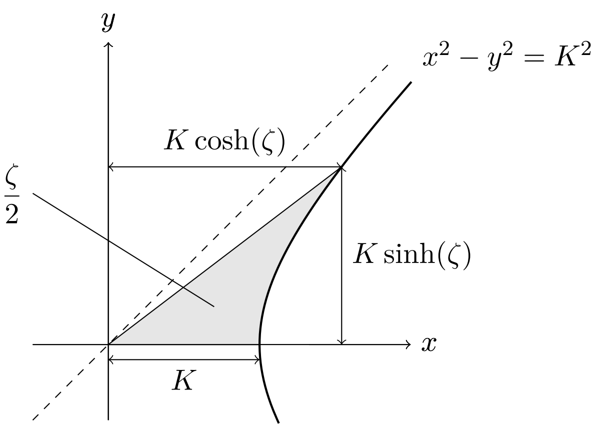


Figure 3-3: Illustration of a hyperbolic angle along the hyperbola with semi-major axis K.

It may already be clear that hyperbolic polar coordinates do *not* provide coordinates for the entire plane like regular polar coordinates do. Indeed, the mapping defined by the coordinate functions from the $K - \zeta$ space to the x-y space is neither injective nor surjective: its image is the disconnected open set bounded by the lines $y = x$ and $y = -x$ (not surjective), and the entire line $K = 0$ in the $K - \zeta$ plane is mapped to the origin in the $x - y$ plane (not injective). As such, one can obtain a bijection by disregarding the degenerate cases for which $K = 0$ and restricting the codomain of the mapping to the set $\{(x, y) \in \mathbb{R}^2 : |x| > |y|\}$. The action of the mapping is illustrated by fig. 3-4.

A possible workaround for this problem can be found in the so-called *generalized trigonometry* as described by Harkin and Harkin [4]: in this case, the notion of the hyperbolic angle itself is

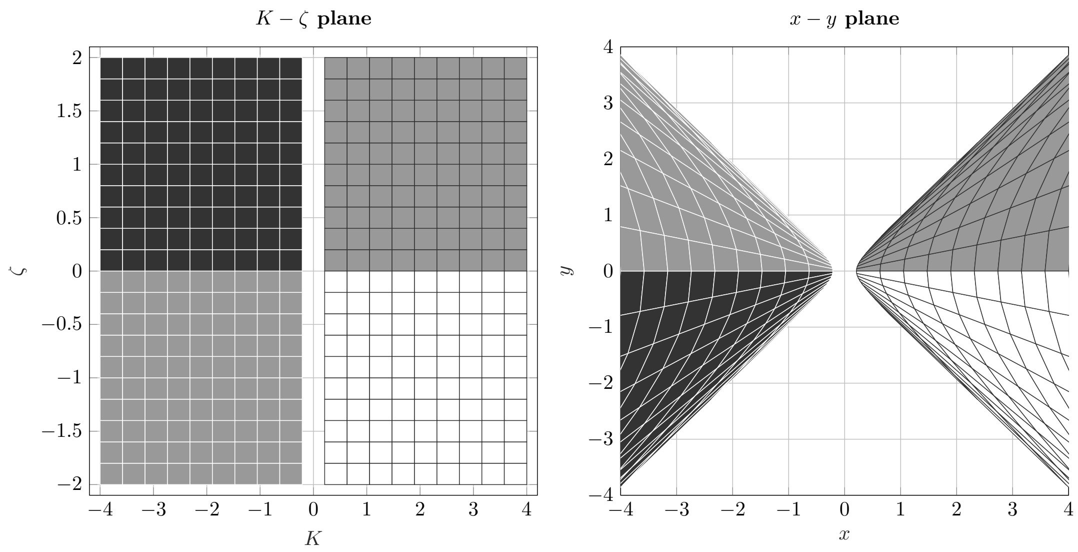


Figure 3-4

considered ambiguous and must be accompanied the branch of the hyperbola that is associated with the angle. In this case four hyperbolic branches are considered, so both branches of $x^2 - y^2 = K^2$ combined with $y^2 - x^2 = K^2$ which were disregarded up till now.[1]

$$\zeta = \begin{cases} \tanh^{-1}(x/y) & \text{for branches I and III} \\ \tanh^{-1}(y/x) = \coth^{-1}(x/y) & \text{for branches II and IV} \end{cases} \tag{3-4}$$

Although this extended definition brings the other two quadrants of the hyperbolic plane within reach of the polar form as well, there is still a particular set that even now cannot be represented this way: $\{(x, y) \mid |x| = |y|\}$, or the 'light cone' from special relativity.

3-3 Special relativity

3-4 The Lorentz metric

Central in the discussion of interest movements as hyperbolic motions will be the so-called Lorentzian inner (or Lorentz) product[2].

[1] In fact, Harkin and Harkin provide an even more general treatment, covering so-called generalized complex numbers of the form $z = x + iy \quad x, y \in \mathbb{R}$ with $i^2 = iq + p \quad p, q \in \mathbb{R}$. However, in this case $p = 1$ and $q = 0$, as will become clear later in the discussion about hyperbolic numbers.

[2] In literature (both physics and mathematics) many variations on the so-called 'metric signature' of the Lorentz product. In terms of $\mathbb{R}^4$, these variations come down to $(+,-,-,-)$, $(-,-,-,+)$, $(-,+,+,+)$ and $(+,+,+,-)$, where the first one coincides with the definition used here, in accordance with the magnificent work of Landau and Lifshitz [5]. Luckily, this is just a matter of convention and its influence on the mathematical machinery at hand is limited to the switching of some signs.

Lorentz product

Let $\boldsymbol{u}, \boldsymbol{v}$ be vectors in $\mathbb{R}^n$. The *Lorentzian inner product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined to be the real number

$$\boldsymbol{u} \diamond \boldsymbol{v} = u_1 v_1 - u_2 v_2 - \ldots - u_n v_n$$

The Lorentz product is an inner product, which means that it takes two elements from a vector space and returns a scalar value, while satisfying two (or three) conditions: (1) bilinearity, (2) symmetry and (3) nondegeneracy [6].

In some literature, the condition of nondegeneracy is replaced by a stronger notion of positive definiteness, which means that an inner product of a vector with itself is always nonnegative and zero if and only if the vector is the zero vector. However, this condition does not apply to the Lorentz product — this fact has rather far-reaching ramifications, as will become clear with the next definition.

Based on the Lorentz inner product it is natural to define also a corresponding *Lorentzian norm* $\|\cdot\|_L$:

$$\|\boldsymbol{v}\|_L = (\boldsymbol{v} \diamond \boldsymbol{v})^{\frac{1}{2}}$$

Because the Lorentz product is not positive definite but rather indefinite, the result of $\boldsymbol{v} \diamond \boldsymbol{v}$ is not guaranteed to be a positive number. As such, $\|\boldsymbol{v}\|_L \in \mathbb{C}$, in stark contrast with the familiar Euclidean norm which *is* positive definite and will therefore always return a nonnegative real number.

The norm provides a notion of length for a vector. Based on this length, a *metric* yields a distance between two points, i.e. the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to

$$d_L(\boldsymbol{v}, \boldsymbol{u}) \triangleq \|\boldsymbol{v} - \boldsymbol{u}\|_L$$

Because the norm is not positive definite but 'only' nondegenerate, it is called a *pseudo-Euclidean metric*, and the vector space it is associated with a *pseudo-Euclidean space*. A vector space equipped with this pseudometric (in more general terms, a bilinear nondegenerate form) is called a Lorentz space. The particular case for $\mathbb{R}^4$ sets the stage for the theory of special relativity (with one 'special' dimension for time and three spatial dimensions) and is called the Minkowski space, after the German physicist Hermann Minkowski [7]. Sometimes the two-dimensional plane that is discovered here is also called the Minkowski plane, because this is what he used to explain his ideas, being unable to draw anything like four-dimensional space. However, this text will adhere to the more mathematically inclined tradition and call it 'Lorentz(ian) space' which applies for any dimension larger than one [6].

The sign of the Lorentz norm gives rise to an equivalence relation $\sim_L$ defined to be $a \sim_L b \iff \operatorname{sgn}\|a\| = \operatorname{sgn}\|b\|$.[3] Therefore, the *quotient set* of all the points in the plane $\mathbb{R}/\sim_L$ contains three elements $\{-1, 0, 1\}$, these equivalence classes are given the respective names $\{\texttt{spacelike}, \texttt{lightlike}, \texttt{timelike}\}$ based on the terminology from special relativity [5].

[3] An equivalence relation is a binary relation that is symmetric, reflexive and transitive.

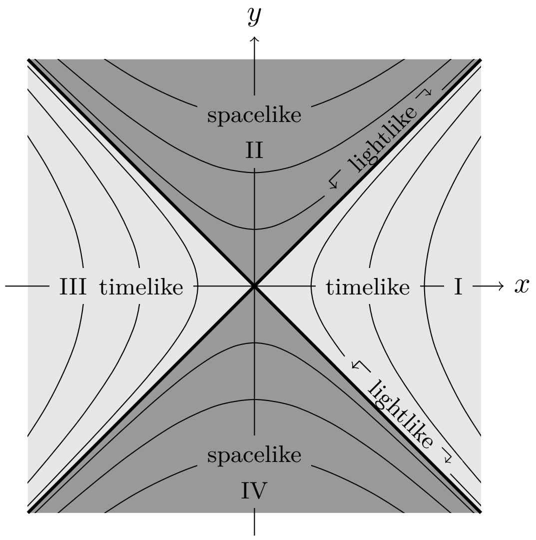


Figure 3-5: Overview of the three 'types' of vectors in the Lorentz-Minkowski plane: spacelike ($\|\boldsymbol{v}\|_L < 0$), lightlike ($\|\boldsymbol{v}\|_L = 0$) and timelike ($\|\boldsymbol{v}\|_L > 0$). The lines $y = x$ and $y = -x$ containing all the lightlike vectors form the so-called light cone or null cone. The hyperbola of in the spacelike region (dark gray) obey the equation $y^2 - x^2 = K^2$, they will be referred to the hyperbolic branches II ($y > 0$) and IV ($y < 0$). In contrast, the timelike hyperbolic branches with equation $x^2 - y^2 = K^2$ (light gray region) are referred to as I ($x > 0$) and III ($x < 0$).

3-5 The Lorentz group

3-6 Hyperbolic numbers

Perhaps the most convenient way to view the hyperbolic motions is in terms of so-called hyperbolic numbers[4]. These form an alternative number system similar to complex numbers, based on the 'hyperbolic' unit j with the defining unipotence property $\mathrm{j}^2 = 1$, where obviously $\mathrm{j} \notin \mathbb{R}$. Like complex numbers, the hyperbolic numbers can have a real and a hyperbolic part

$$z = x + y\mathrm{j} \qquad x, y \in \mathbb{R}$$

Combined with addition and multiplication defined on them, the hyperbolic numbers form a commutative ring. Each hyperbolic number z is associated with its *hyperbolic conjugate* $z^* = x - y\mathrm{j}$. The product of a hyperbolic number with its own hyperbolic conjugate creates a quadratic form $zz^* = x^2 - y^2$, which always returns a real number and is equivalent Lorentzian metric. Again, three cases can be distinguished for z:

- *timelike* when $zz^* > 0$

[4]The hyperbolic number system has been assigned a myriad of names, with different terminology and mathematical notation for almost every influential paper that has been published about them. Among others, hyperbolic numbers are referred to as split-complex numbers, double numbers, perplex numbers, algebraic motors, etc. In this text, 'hyperbolic numbers' is chosen to highlight their connections with hyperbolae. For the choice of the hyperbolic unit, j will be used, though in literature also u (for unipotent) and h (for hallucinatory or hyperbolic) make their appearance [8, 9, 10, 4].

- *lightlike* when $zz^* = 0$
- *spacelike* when $zz^* < 0$

However, in contrast to complex numbers (whose quadratic form would be $x^2 + y^2$), this quadratic form is *isotropic*, which means that there exists a $z \neq 0$ such that $zz^* = 0$ — this is precisely the case for all the z on the light cone in the hyperbolic plane. The *hyperbolic modulus* therefore requires an absolute value in order like so

$$|z| \triangleq \sqrt{|zz^*|}$$

which considered to be the hyperbolic distance from z to the origin [9].

Again, in correspondence with complex numbers, the hyperbolic numbers also have a *polar form*, i.e.

$$z = K \exp(\zeta \mathrm{j})$$

which can be evaluated using the Taylor expansion of the exponential exp:

$$z = K \exp(\zeta \mathrm{j}) = K \sum_{k=0}^{\infty} \frac{(\zeta \mathrm{j})^k}{k!} = K \sum_{k=0}^{\infty} \frac{(\zeta \mathrm{j})^{2k}}{(2k)!} + K \sum_{k=0}^{\infty} \frac{(\zeta \mathrm{j})^{2k+1}}{(2k+1)!} = K \cosh(\zeta) + K \mathrm{j} \sinh(\zeta)$$

when the hyperbolic angle ζ is associated with hyperbolic branches I and III. For branches II and IV, one essentially has to consider $z = K\mathrm{j} \exp(\zeta \mathrm{j})$ or

$$z = K \sinh(\zeta) + K \mathrm{j} \cosh(\zeta)$$

Furthermore, consider the *conjugate product* of two hyperbolic numbers $z_1 = x_1 + y_1 \mathrm{j}$ and $z_2 = x_2 + y_2 \mathrm{j}$:

$$z_1^* z_2 = \underbrace{(x_1 x_2 - y_1 y_2)}_{\text{inner product}} + \underbrace{(x_1 y_2 - x_2 y_1)}_{\text{outer product}} \mathrm{j}$$

of the resulting expression, the real part is called the inner product and the hyperbolic part the outer product. The inner product is equivalent to the Lorentz product while the outer product yields the directed area of the parallelogram spanned by z_1 and z_2 (or the determinant of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$. The outer product is the same as for regular complex numbers, while the inner product recovers the Lorentz inner product instead of the Euclidean inner product. This is why one can say that the hyperbolic number plane and the complex plane have an identical notion of area. Based on the inner product, one can however recognize a different notion of orthogonality, i.e. z_1 and z_2 are *hyperbolically orthogonal* if their hyperbolic inner product equals zero — this will yield a different notion of orthogonality than their Euclidean counterpart [3, 9].

3-6-1 Matrix representation

The hyperbolic numbers are ring-isomorphic to the matrix ring

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \quad x, y \in \mathbb{R}$$

under matrix addition and matrix multiplication. The determinant of said matrix then recovers the Lorentz metric

$$\det \begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2$$

3-6-2 The idempotent basis

Previously it was already mentioned that the hyperbola can be viewed conveniently in two particular axis systems, the standard axis system and the idempotent axis system, as illustrated in fig. 3-2. This basis can also be defined in terms of hyperbolic numbers; in terms of the standard basis $\{1, \mathrm{j}\}$ the idempotent basis is $\{\mathrm{j}_+, \mathrm{j}_-\}$ with

$$\mathrm{j}_+ = \frac{1}{2}(1 + \mathrm{j}) \qquad \mathrm{j}_- = \frac{1}{2}(1 - \mathrm{j})$$

The term 'idempotent' is a testament to the fact that $\mathrm{j}_+^2 = \mathrm{j}_+$ and $\mathrm{j}_+^2 = \mathrm{j}_+$. As such, a given hyperbolic number $z = x + y\mathrm{j}$ in idempotent coordinates is

$$\underbrace{(x + y)}_{z_+}\mathrm{j}_+ + \underbrace{(x - y)}_{z_-}\mathrm{j}_+$$

Furthermore, the idempotent basis is mutually annihilating, i.e. $\mathrm{j}_+\mathrm{j}_- = 0$, which is why they possess a projective property: [9]

$$z\mathrm{j}_+ = z_+\mathrm{j}_+ \quad \text{and} \quad z\mathrm{j}_- = z_-\mathrm{j}_-$$

The idempotent axis system is interesting because it returns the total amount compounded or discounted over time; in contrast to the standard yield-capital decomposition.

3-6-3 Clifford algebra

Chapter 4

Hyperbolic geometry

4-1 Basic facts

Hyperbolic geometry, sometimes called Lobachevskian geometry[1] is the geometry on the hyperbolic plane. This type of geometry is called non-Euclidean, because it does not satisfy all of Euclid's axioms that form the building blocks of traditional geometry. More specifically, the fifth axiom, called the parallel axiom, states that [3]

> Through any point p not on the line L there exists precisely one line L' that does not meet L.

Logic dictates that if this axiom is not true, two possible alternatives arise. Given again the point p and the line L,

- there exists *no* line through p that does not meet L, or
- there exist *at least two* lines through p that do not meet L.

The first statement corresponds to what is called *spherical geometry*, while the latter is the defining axiom for *hyperbolic geometry*. It does not take too much imagination to realize that the parallel axiom is equivalent to another basic fact in Euclidean geometry: the sum of the angles of a triangle is equal to π. As such, for both types of non-Euclidean geometry this will not be the case; let $\mathcal{E}(T)$ denote the *angular excess* of a triangle T in a certain geometry.

- Naturally, for Euclidean geometry $\mathcal{E}(T) = 0$,
- in spherical geometry, $\mathcal{E}(T) > 0$,
- in hyperbolic geometry, $\mathcal{E}(T) < 0$.

[1] After the Russian mathematician Nikolai Lobachevsky (1832) who first published about this subject [3].

It may come as a surprise that the angular excess can be related to the size of the triangle (more specifically, its area) like so [3]

$$\mathcal{E}(T) = k\mathcal{A}(T) \tag{4-1}$$

where $k = 0$ for Euclidean geometry, $k < 0$ for hyperbolic geometry and $k > 0$ for spherical geometry. This hints at the fact that spherical geometry and hyperbolic geometry are somehow 'larger' classes of geometry than Euclidean geometry, since they exist for a whole range of values for k, be it positive or negative. There is indeed a 'different' geometry for every value of k, and only one of those values corresponds to the traditional concept of Euclidean geometry that most people are familiar with. Another consequence of eq. (4-1) is that similar triangles cannot exist in non-Euclidean geometry; since apart from the trivial case where the triangles are also congruent, they must differ in area, which yields a different angular excess. The value of k is, as it turns out, equal to the *Gaussian curvature* of the surface: a negatively curved surface has an angular deficiency while a positively curved surface has an angular excess; only for surface with zero curvature the sum of the angles of a triangle will be precisely equal to π.

4-1-1 Surface curvature

Perhaps one of the most remarkable results attributed to Carl Friedrich Gauss is his *Theorema Egregium* about the (Gaussian) curvature of surfaces. The theorem states that curvature is an *intrinsic* property, which means that it remains preserved when the surface is transformed by 'bending without stretching'. Curvature can be positive, negative or zero. Positively curved surfaces 'bend away' from their tangent plane in any direction (local convexity) and negatively curved surfaces intersect their tangent plane like a saddle. For surfaces with zero curvature, there is always at least one straight line that lies in the tangent plane; examples are a cylinder (always one straight line) and a plane (straight lines in two directions). If a coordinate system is defined such that the point at issue is at the origin and the tangent plane to the surface described by $f(x, y)$ coincides with the horizontal, the Gaussian curvature can be computed by means of the determinant of the Hessian [11, 12]

$$k = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Surface curvature can be approached more rigorously from the perspective of Riemannian geometry. The core concept behind Riemannian geometry are the eponymous manifolds: these are manifolds equipped with an positive definite inner product[2] on their tangent space, hence providing a *metric (tensor)* that allows to determine lengths, angles and curvature. The metric on a parametric surface $\boldsymbol{r}(u, v)$ can be written in terms of the so-called *first fundamental form* I, which is defined as the inner product of two tangent vectors to the

[2]In contrast to section 3-4, the positive definiteness of the inner product is essential for Riemannian geometry. When the inner product is not positive definite but nondegenerate, as the Lorentz product, the manifold is called *pseudo-Riemannian*.

surface and is characterised by coefficients E, F and G.

$$\begin{aligned}
E &= \boldsymbol{r}_u \cdot \boldsymbol{r}_u \\
F &= \boldsymbol{r}_u \cdot \boldsymbol{r}_v \\
G &= \boldsymbol{r}_v \cdot \boldsymbol{r}_v \\
\text{with } \boldsymbol{r}_u &= \frac{\partial \boldsymbol{r}}{\partial u}, \ \boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v}.
\end{aligned} \tag{4-2}$$

The components of the metric tensor $\boldsymbol{g}_{ij}$ then coincide with the components of the first fundamental form: $\boldsymbol{g}_{11} = E$, $\boldsymbol{g}_{12} = \boldsymbol{g}_{21} = F$ and $\boldsymbol{g}_{22} = G$. Therefore, the first fundamental form determines the length of curves lying in the surface.

In contrast, the *second fundamental form* II provides information on the curvature or shape of the embedded surface, more specifically the rate of change of the tangent planes in any direction. Again, it is characterised by three coefficients e, f and g like so

$$e\,\mathrm{d}u^2 + 2f\,\mathrm{d}u\,\mathrm{d}v + g\,\mathrm{d}v^2\,.$$

The coefficients of the second fundamental form can be computed using the surface unit normal vector

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}, \tag{4-3}$$

the coefficients are then given by

$$\begin{aligned}
e &= \boldsymbol{r}_{uu} \cdot \boldsymbol{n} \\
f &= \boldsymbol{r}_{uv} \cdot \boldsymbol{n} \\
g &= \boldsymbol{r}_{vv} \cdot \boldsymbol{n} \\
\text{with } \ \boldsymbol{r}_{uu} &= \frac{\partial^2 \boldsymbol{r}}{\partial u^2} \ \text{ etc.}
\end{aligned} \tag{4-4}$$

Finally, the first and second fundamental form can then be used to determine the Gaussian curvature of the parametric surface: [12]

$$k = \frac{\det(\mathrm{II})}{\det(\mathrm{I})} = \frac{eg - f^2}{EG - F^2}. \tag{4-5}$$

To summarize, the connection between hyperbolic geometry and Gaussian curvature is concisely stated again below.

1. Hyperbolic geometry is the geometry with an alternative parallel axiom, where there are always at least two parallel lines for a given line through a point.

2. The former fact is equivalent to stating that the angular excess $\mathcal{E}$ is characterised by the negative constant k (constant over the entire hyperbolic plane).

3. The Gauss-Bonnet theorem states that the constant k is given by the Gaussian curvature.

Therefore, in order to represent hyperbolic geometry, one whishes to find a surface that exhibits constant negative Gaussian curvature. Some candidates exist for this criterion, the simplest of which is the so-called *pseudosphere*.

4-1-2 The pseudosphere

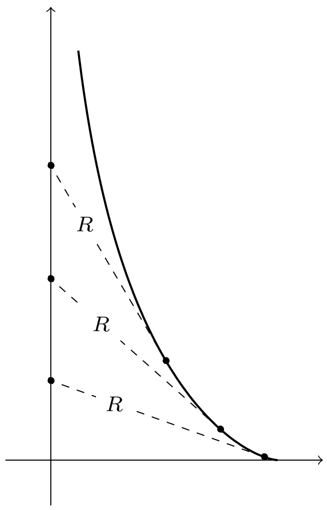
Pseudospherical surfaces are surfaces of constant negative curvature, in that sense exhibiting a certain duality to a sphere which is a surface of constant positive curvature. The most notable example is the *pseudosphere*; which is the surface of revolution of a tractrix — another name for the pseudosphere is a *tractricoid*. A tractrix is a curve defined by the property that the segment of the tangent from the point to the axis has constant length p. Figure 4-1a shows an example of such a curve. A parametric representation of the tractrix γ is given by

$$\gamma : u \mapsto (p\,\mathrm{sech}(u),\ p(u - \tanh(u))) \quad u \in \mathbb{R}^+ \tag{4-6}$$

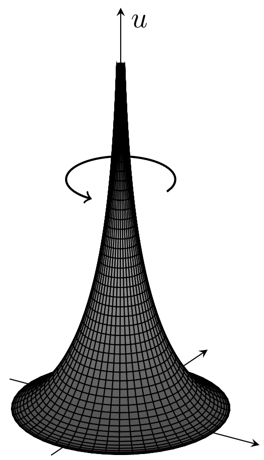
The a parameterization of the pseudosphere in terms of u and v is then easily obtained by taking the surface of revolution along the second axis

$$\boldsymbol{r}(u,v) = \begin{pmatrix} p\,\mathrm{sech}(u)\cos(v) \\ p\,\mathrm{sech}(u)\sin(v) \\ p(u - \tanh(u)) \end{pmatrix} \quad u \in \mathbb{R}^+,\ v \in [0, 2\pi). \tag{4-7}$$

A pseudosphere is visualized by fig. 4-1b; the mesh on the surface are isoparametric curves for u and v.



(a) The tractrix - the line segments defined on the tangent to the curve between the curve and the intersection with the vertical axis all have the same length R.



(b) The pseudosphere

Figure 4-1

Curvature of the pseudosphere

As explained in section 4-1-1, the Gaussian curvature of a parametric surface may be calculated using the first and second fundamental form. It was already alluded that the pseudosphere has a constant negative Gaussian curvature, which will be formally shown in this section. Let the pseudosphere be parameterized as eq. (4-7). Then, by virtue of eq. (4-2), the coefficients of the first fundamental form E, F and G are

$$\begin{aligned} E &= p^4 \tanh^2(u) \\ F &= 0 \\ G &= p^4 \operatorname{sech}^2(u) \end{aligned}$$

From eq. (4-3), the unit normal vector is

$$\boldsymbol{n} = \begin{pmatrix} -\cos(v)\tanh^{-1}(u) \\ -\sin(v)\tanh(u) \\ -\cosh^{-1}(u) \end{pmatrix}$$

and, consequently, the coefficients of the second fundamental form are

$$\begin{aligned} e &= -p \operatorname{sech}(u)\tanh(u) \\ f &= 0 \\ g &= p \operatorname{sech}(u)\tanh(u) \end{aligned}$$

using eq. (4-4). The Gaussian curvature can then be computed to be [12]

$$k = \frac{\det(\mathrm{II})}{\det(\mathrm{I})} = \frac{eg - f^2}{EG - F^2} = \frac{-p^2 \operatorname{sech}^2(u)\tanh^2(u)}{p^4 \operatorname{sech}^2(u)\tanh^2(u)} = -\frac{1}{p^2}. \tag{4-8}$$

As such, the pseudosphere is shown to have a constant negative curvature everywhere but on the rim (where differentiability is lost).

Because the pseudosphere has constant negative curvature, it shares many properties of the hyperbolic plane. For example, the angular excess equation eq. (4-1) holds on the pseudosphere [3]. But, unlike a sphere can serve as a 'globe' for spherical geometry, the pseudosphere cannot be used as a representative for the entire hyperbolic plane. First of all, they are not homeomorphic (the pseudosphere is homeomorphic to a cylinder, which has a different fundamental group than the hyperbolic plane[3]). Secondly, the pseudosphere has a 'rim' at the bottom which prevents any line segment from extending further downwards — naturally, no such thing exists in the hyperbolic plane. As such, the pseudosphere can only serve as a model for finite regions of the hyperbolic plane, but not in its entirety. In technical terms, this means that the pseudosphere is *locally isometric* to the hyperbolic plane [14]. This is why one must resort to models of the hyperbolic plane, they will be discussed in the next section.

4-2 Models of the hyperbolic plane

The pseudosphere cannot serve as a model for the hyperbolic plane. In fact, a theorem attributed to David Hilbert shows that, *in Euclidean three-space, there can be no complete*

[3]Intuitively, one can see this as the number of different ways it is possible to draw loops on the surface which are distinct up to a homotopy. On the pseudosphere and the cylinder, loops can 'wrap' around the vertical axis any integer number of times, while on the (hyperbolic) plane all loops are homotopic [13].

smooth surface with the intrinsic geometry of the pseudosphere [11]. This is the reason why there can only be models of the hyperbolic plane. Several have been devised in the past, and they will provide a lot more mileage than the pseudosphere alone. The most popular models will be discussed in the following sections: there is a natural way to map the pseudosphere and the Poincaré half plane, which will be discussed first. Then, via the so-called *Cayley transformation*, the half plane can be mapped to the Poincaré disk, perhaps the most illustrous model of them all. Using the disk model, one can naturally arive at the Cayley-Klein disk and the hyperboloid model — the latter will be the starting point in the financial analogy.

4-2-1 Poincaré half plane

It has been pointed out that there are essentially two incompatibility problems with the pseudosphere that prevent devise a mapping between it and the complete hyperbolic plane (also, Hilbert's theorem immediately shatters any aspiration to find one): first, it is homeomorphic to the cylinder and secondly, it has an edge. This section will describe a conformal mapping between the surface of the pseudosphere and a 'half plane', which can serve as a global model for the hyperbolic plane. However, the theorem by Hilbert already hints at the fact that this plane will have some 'weird' properties, more specifically, a different notion of distance.

As described by Needham [3], one can imagine the pseudosphere to be cut open along any tractrix; the resulting surface can then be 'unfolded' in the horizontal direction as if it were a treasure map on a table. The edges that were cut can be extended towards infinity by recognizing the periodicity that exists naturally on the pseudosphere itself; a particle traveling horizontally would wrap once around the pseudosphere every distance of 2π traveled. Consequently, the circles on the pseudosphere that arise for constant values of z will be mapped to horizontal lines in the half plane. Formally, this simply means that the horizontal axis of the half plane $\tilde{x}$ is equal to the angle u. It has already been mentioned that the mapping between the pseudosphere and the half plane is *conformal*, i.e. it locally preserves angles. Therefore, since the tractrix lines are everywhere perpendicular on the pseudosphere to the circles for constant z, they must map to vertical lines in order to maintain this orthogonality. A horizontal movement in the half plane $\mathrm{d}\tilde{x}$ therefore corresponds on the pseudosphere with a traveled distance of $\mathrm{d}\tilde{s} = \operatorname{sech}(v) = \operatorname{sech}(v)\,\mathrm{d}v$, because of the radius of the circle at that particular height. Because the mapping is conformal, a movement along a tractrix $\mathrm{d}\sigma$ must be scaled by the same factor:

$$\mathrm{d}\sigma = \operatorname{sech}(u)\,\mathrm{d}y\,.$$

Subsequently, the movement along the tractrix can be written in terms of u and v using the parameterization:

$$\mathrm{d}\sigma = \left\| \frac{\partial \boldsymbol{r}}{\partial u}\,\mathrm{d}u \right\| = \sqrt{\mathrm{d}u^2 - \operatorname{sech}^2(u)\,\mathrm{d}u^2}$$

Combining this with the previous expression found for $\mathrm{d}\sigma$, one arrives at

$$\mathrm{d}y = \sinh(u)\,\mathrm{d}u \implies y = \cosh(u) + C. \tag{4-9}$$

Thus, the conformality of the mapping imposes a restriction on the mapping for y up to a constant C, which is usually taken to be 0 [3]. Using this information, the metric of the

Poincaré half plane can also easily be deduced by *pushing forward* the Euclidean metric of the pseudosphere:

$$\mathrm{d}s = \left\| \frac{\partial \boldsymbol{r}}{\partial u} \, \mathrm{d}u + \frac{\partial \boldsymbol{r}}{\partial v} \, \mathrm{d}v \right\| = \operatorname{sech}(u) \sqrt{\sinh^2(u) \, \mathrm{d}u^2 + \mathrm{d}v^2} = \frac{\sqrt{x^2 + y^2}}{y}$$

This metric is also called the *Poincaré metric*. Equation (4-9) already states that for $u = 0$, i.e. the rim of the pseudosphere, maps to the horizontal line $y = 1$. This suggests that the pseudosphere is covered by the region of the half plane for which $y \geq 1$.

The Poincaré metric suggests that distances get larger and larger when travelling downwards in y-direction. At the line $y = 0$ they even become infinitely large! For someone living in the half plane, this line would never be reachable, as they would have to travel for an infinite amount of time. It is not part of the half plane itself, which is why it is called the *horizon*; points on the horizon are named *ideal points* [3].

Geodesics in the Poincaré half plane connecting two points are either straight vertical lines if the two points have the same x-coordinate. Otherwise, the geodesic is the arc defined by the portion of the circle going through each of the two points that also intersects the x-axis at a right angle [15]. Apart from the horizon, two other peculiar curves exist in the half plane that will also show up in later models:

- *horocycles*: circles tangent to the x-axis or horizontal lines,
- and *hypercycles*: circular arcs that intersect the x-axis at non-right angles or straight lines that intersect the x-axis at a non-right angle.

`(... Figure of the half plane and pseudosphere ...)`

4-2-2 Poincaré disk

The Poincaré disk arises naturally from the Poincaré half plane by virtue of the Cayley transform

4-2-3 Hyperboloid model

4-2-4 Cayley-Klein disk

4-2-5 Horizontal strip

4-3 Möbius transformations

Chapter 5

Economic engineering

Chapter 6

Analytical Mechanics

6-1 Lagrangian mechanics

Lagrangian system

Let M be a differentiable manifold, TM its tangent bundle, and $\mathcal{L} : TM \to \mathbb{R}$ a differentiable function. A map $\gamma : \mathbb{R} \to M$ is called a motion in the lagrangian system with configuration manifold M and lagrangian function $\mathcal{L}$ if γ is an extremal of the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(\dot{\gamma})\,\mathrm{d}t$$

where $\dot{\gamma}$ is the velocity vector $\dot{\gamma}(t) \in TM_{\gamma(t)}$ [16].

6-2 Hamiltonian mechanics

Appendix A

Some math

A topology is a collection of elements from the power set of a given set. Roughly speaking, a topology gives rise to a notion of 'neighborhoods', which further allows for the concepts of convergence and continuity to be properly defined on a set.

Definitions from [13]

Definition 1 (Inner product)**.** An *inner product* on a real vector space V is a function from $V \times V$ to $\mathbb{R}$, denoted by $(\mathbf{u}, \mathbf{v}) \mapsto \langle \mathbf{u}, \mathbf{v} \rangle$ such that for all $\mathbf{u}, \mathbf{v}$ in V

1. $\langle \mathbf{u}, \rangle$ and $\langle , \mathbf{v} \rangle$ are linear functions from V to $\mathbb{R}$ (bilinearity),
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ (symmetry),
3. If $\mathbf{u} \neq 0$, then there is a $\mathbf{v} \neq 0$ such that $\rangle \mathbf{u}, \mathbf{v} \rangle \neq 0$ (nondegeneracy).

Definition 2 (Topology)**.** A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X called *open sets*, satisfying the following properties:

i X and $\varnothing$ are elements of $\mathcal{T}$.

ii $\mathcal{T}$ is closed under finite intersections; if $U_1, \ldots, U_n \in \mathcal{T}$, then their intersection $U_1 \cap \ldots \cap U_n$ is in $\mathcal{T}$.

iii $\mathcal{T}$ is closed under arbitrary unions: if $U_{\alpha \in A}$ is any (finite or infinite) collection of elements of $\mathcal{T}$, then their union $\bigcup\limits_{\alpha \in A} U_\alpha$ is in $\mathcal{T}$.

Definition 3 (Homeomorphism)**.** If X and Y are topological spaces, a *homeomorphism* from X to Y is defined to be a continuous bijective map $\varphi : X \to Y$ with continuous inverse. If there exists a homeomorphism between X and Y, they are called homeomorphic or topologically equivalent.

Appendix B

Differential geometry and topology

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Glossary

List of Acronyms

IRR	Internal Rate of Return
NPV	Net Present Value

Index