Towards a consistent interpretation of financial instruments in economic engineering using Noether's theorem.

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Literature Survey

Delft University of Technology

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Abstract

Two approaches have been used in economic engineering to tackle financial problems up till now: action-angle coordinates and the rotational analogy. However, both methods present some crucial issues. Action-angle coordinates apply exclusively to conservative systems; moreover, their applicability to financial systems is still a matter of debate. Likewise, the rotational analogy has not been applied consistently in past research and some of its aspects remain yet unclear. Hence, the aim of this research is to develop a consistent interpretation of financial systems that is rigorous and applicable to concrete modeling problems. The belief is that this goal can only be achieved by starting from the elementary fundamentals of economic engineering, i.e. the analytical mechanics approach. Therefore, a thorough study of the foundations of the current methods is required to successfully address their shortcomings.

An economic system can be modeled as a Lagrangian system. Hamilton's principle of least action dictates that economic agents attempt to maximize their convenience yield by sacrificing the least amount of economic surplus as possible. This gives rise to the Euler-Lagrange equations that describe the evolution of the economic system. Using the Legendre transform, the Lagrangian economic system can be ported to a Hamiltonian system, which in turn allows for the usage of action-angle coordinates. Noether's theorem links the symmetry properties of the Lagrangian function to the presence of conservation laws. These conservation laws have a concrete economic interpretation that one can exploit to understand complex systems.

The rotational analogy is inspired by the fact that compound interest and exponential returns can be modeled by a hyperbolic rotation in the capital-yield plane. Hyperbolic rotations can be converted to circular rotations through the usage of complex angles. The capital-yield plane belongs to the class of Lorentzian vector spaces. These spaces are inherently connected to special relativity (because the spacetime interval is a Lorentzian distance) and hyperbolic geometry by virtue of the hyperboloid model. Both special relativity, the hyperbolic motions, and hyperbolic geometry are subsumed by the Möbius transformations. Analysis of the group structure of these transformations reveals connections with the aforementioned subjects, but also with the special linear group and the two-dimensional symplectic group.

These closely intertwined mathematical subjects combined with fundamental principles like Noether's theorem will be employed to understand the implications of the choices that have to be made in the pursuit of a consistent interpretation of financial instruments.

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Chapter 1

Introduction

On the eighth page of the October 16, 1929 edition of the New York Times, a small article headed [5]

> *"Fisher Sees Stocks Permanently High; Yale Economist Tells Purchasing Agents Increased Earnings Justify Rise."*

Responsible for this bold claim was Irving Fisher, a very prominent American economist who pioneered the subject of monetary economics. Unfortunately for him, on Thursday the 29th of October — merely nine days later — the Dow Jones Industrial Average dropped by 11 percent, only to lower by another 13 percent on Monday, and then by 12 percent on Tuesday. The result was the Great Depression, an unprecedented financial crisis with dramatic repercussions: in the three years that followed, unemployment rose to 20 percent and industrial production almost halved [6].

Of course, this anecdote is not meant to discredit Irving Fischer as an economist, for he has been a major contributor to modern economic theory, but rather to illustrate that even specialists tend to misinterpret the present state of financial markets. This thesis does not aim to solve this issue right away, but to provide a new approach to the interpretation of the financial system; this may help in the understanding of this vital component of modern-day society. This new approach is founded in the recent theoretical framework of economic engineering, developed by prof. em. M. Mendel [7].

Economic engineering An introduction to relevant aspects of economic engineering will be provided in chapter 2, but the core idea is to include economic systems in the traditional multi-domain modeling framework that (control) engineers use to study systems of widely varying nature (mechanical, electrical, hydraulic, etc.), possibly with the intention of developing a suitable control strategy. Of course, control strategies are vital for economic systems as well — although this practice would probably be called 'policy making' — either on small scale, such as firms managing their stock levels and revenue in a changing market, or on a macroeconomic scale, e.g. a central bank deciding whether to lower the interest rate or not.

Economic engineering research has been concerned with financial systems before: an economic analogy to the usage of *action-angle coordinates* was exploited by Vos [8] to improve the models used for monetary policy. Kruimer [9] developed a macroeconomic model of the U.S. economy, where he included bond and equity markets as vital components of the economic machinery by means of the so-called *rotational analogy*. Apart from the fact that both interpretations are manifestly different ways to describe the same concept, they appear to be flawed in some fundamental ways. As such, the necessity arises to reconcile these methods and address their problems in order to find a unifying economic engineering approach to incorporate financial systems; primarily concerning markets for equity and debt and perhaps their related derivatives such as futures and options.

Current approach As mentioned, economic engineering currently recognizes two ways to deal with 'money problems'. Firstly, money is considered analogous to *action*, as (will be explained in chapter 2); action is a quantity that represents the integral of energy over time, with units $[\mathrm{J\,s}]$. Action-angle coordinates are a choice of coordinates in the phase space that consist of (constant) action coordinates and dimensionless 'angles', indicating a *periodic* motion. Secondly, there is the rotational analogy described in chapter 3; which is arguably a bit more flexible than the action-angle coordinates. Again, based on dimensional analysis, an analogy can be made between 'money' and angular momentum (in physics, angular momentum and action have the same dimension). Here the role of the angle represents a return or an accumulated interest, and the 'arm' of the rotation is the principal of the investment/debt instrument.

Research goal The core idea is that, instead of *ad hoc* applying the existing theories, to return to the fundamentals of economic engineering and its ties with analytical mechanics to build a more rigorous foundation for future work. Energy and its analogy to utility in economics play a crucial role here, because (i) it lies at the foundation of analytical mechanics as well and (ii) it allows to make the connection with the existing (not strictly financial) theory of economic engineering, just like the multi-domain modeling techniques in engineering are connected through the universal notions of energy and power. Within analytical mechanics, Noether's theorem describes precisely how mathematical symmetries in the general nature of the system expressed in terms of energy (encoded in a special state function called the *Lagrangian*) dictate *conservation laws* that the system must obey. Naturally, perfect conservation of energy and momentum is equally unlikely in both physics and economics, but one strives nevertheless to construct a 'platonic ideal', a conserved and isolated financial system to form the basis of the modeling framework.

The goal of this research can therefore be stated as follows:

Research goal

To develop a new, consistent, and unified framework to interpret debt and equity instruments in economic engineering, using the formal methods of analytical mechanics; in particular Noether's theorem.

Structure of the literature study That being said, it is clear that the scope of this research is *theoretical*, as its purpose is to expand and refine the current economic engineering framework. Hence, this literature study contains an overview of some related subjects that will hopefully play a role in the development of the theory.

This literature study consists of two parts:

- **Part I** dives into the very fundamentals of economic engineering and the current approach to financial systems. Chapter 2 goes in-depth in the fundamentals of analytical mechanics; for all the concepts that are introduced (Lagrangian, kinetic energy, potential energy, etc.) the economic engineering analog is discussed and motivated as well. In this chapter, the action-angle coordinates and Noether's theorem as well. However, it is by no means meant to be an introduction to action-angle coordinates only; for the entire theory of analytical mechanics is bound to play a vital role in the application to financial theory. Secondly, chapter 3 argues why the rotational analogy works; i.e. how it can be seen as a hyperbolic rotation. Some closely related concepts such as Lorentzian vector spaces and the algebra of hyperbolic numbers are introduced as well.
- **Part II** contains subjects that are not directly related to economic engineering nor finance, but they have some important connections with the theoretical background of the subjects discussed in chapters 2 and 3. Chapter 4 provides a quick introduction to the theory of special relativity, for it is the primary physical application and *raison d'être* of the Lorentzian vector spaces. Subsequently, chapter 5 deals with hyperbolic geometry, which is a natural extension of the hyperbolic motions arising from compound interest and the Lorentzian vector spaces. Finally, chapter 6 shows how the group of Möbius transformations (complex fractional linear transformations) subsumes both special relativity, hyperbolic geometry, and even some aspects of analytical mechanics. These connections are most clearly explained using the tools from group theory. Although chapters 4 to 6 are not directly required for the discussion in part I, they are presumed to be of great use in future developments.

Part I

Finance in economic engineering

Chapter 2

Economic engineering with analytical mechanics

Economic engineering is built on analogies between (macro)economic theory and common engineering disciplines such as thermodynamics, circuit theory, and (classical) mechanics[1]. Especially for the latter, a rich variety of useful analogs can be devised. There are two common classes of interpretations mechanics: *analytical mechanics* including the formulations by Joseph-Louis Lagrange and Sir William Rowan Hamilton, and *vectorial mechanics*, better known as *Newtonian mechanics*. The former variant is usually preferred in the field of engineering whereas the analytical mechanics are more common in physics due to their mathematical elegance and powerful theoretical foundation. Likewise, the theory of economic engineering can be approached in two similar ways. Usually, the 'Newtonian' approach is given the most attention, but in the case of this work, the energy-based approach will prove to be more useful, which is why it is the subject of this discussion. This discussion will mostly pertain to 'regular' economic systems, but it is the underlying aspiration of this research that these formal methods will allow establishing a sound framework for financial systems as well.

Analytical mechanics, more specifically *Lagrangian* and *Hamiltonian* mechanics, are established around the definition of special state functions, respectively called the Lagrangian $\mathcal{L}$ and the Hamiltonian $\mathcal{H}$. As per usual, Lagrangian mechanics will be introduced first, for it has the most intuitive explanation. Then, a more formal approach allows (Legendre) transform the discussion into one of Hamiltonian mechanics.

In this chapter, frequent analogies will be made between mechanics and economics; as such, the idea is to have the 'normal text' pertain mostly to the discussion of classical mechanics, and to provide the analogies in special 'boxes' like so:

Example

An analogy between economics and classical mechanics.

[1] as opposed to more recent theories in relativistic mechanics and quantum mechanics

The reason for this particular choice of layout is twofold: first, it allows to make a sharp distinction between the older, extremely rigorous theory of classical mechanics and the novel approach of economic engineering: many ideas and propositions are still tentative (especially in the realm of classical mechanics) as the field of economic engineering matures. Secondly, it allows for easier reference as to not obscure the economic analogies (which are arguably the most important aspects of this chapter) with the highly theoretical discussion of analytical mechanics.

2-1 Economic engineering

The discipline of 'economic engineering' is a very new one. The theoretical foundations have been developed over the past years at the Delft Center of Systems and Control primarily by prof. em. dr. ir. Mendel, combined with the contributions of several theses that have been recently written about the subject. The purpose of economic engineering is to use tools from various engineering disciplines and physics to improve the predictive power of (macro)economic models. The core idea is to extend 'domain neutral modeling techniques such as bond graph modeling [4] that are built on analogiesween mechanical, electrical,,,,, hydraulic, ... systems to economic systems as well. Hence, one attempts to give an economic interpretation to a generalized mass (I-element), generalized spring (C-element) and generalized damper (R-element). The idea is that this consistent engineering approach leads to actual *predictive* models that provide much richer insights than the 'stylized facts' from macroeconomy or, on the other end of the spectrum, the 'black-box' econometric models that have lost all interpretative value. Indeed, applied economic engineering pursues *gray-box* modeling instead, as is most common in traditional engineering models [9]. Some analogies between mechanical, electrical and economic systems are listed in table 2-1 to merely for the sake of illustration; a thorough motivation for each of them will be given in the following section. Table 2-1 provides some examples of the analogies that are used within economic engineering. In general, their meaning is not as specific as given here and this table applies only to the most elementary cases. The main theory behind economic engineering is outlined by Mendel [7]. Some other notable results in the field have been achieved in recent years

Table 2-1: Some examples of the analogies that are used in the application of economic engineering. The theory behind bond-graph modeling defines a generalization of the mechanical concepts of displacement, velocity, momentum, and effort and applies these to electrical, thermodynamic, hydraulic, ... systems too [4].

General	**Mechanical**	**Electrical**	**Economic**
Displacement	Displacement	Charge	Stock level
Flow	Velocity	Current	Flow of goods
Momentum	Momentum	Flux linkage	Price
Effort	Force	Voltage	Economic want
I-element	Mass	Inductor	Market
C-element	Spring	Capacitor	Storage of goods
R-element	Damper	Resistor	Depreciation / Consumption

as well. Hutters and Mendel [10] demonstrated the application of Hamiltonian mechanics to dissipative systems as to include the mechanics of consumption in port-Hamiltonian systems. A formal approach inspired by the theory of thermodynamics was used by Manders [11] to explain economic growth and productivity. Kruimer [9] and Van Ardenne [12] used economic engineering bond graph techniques to build extensive models for the U.S. economy and a 'generalized' firm (as to improve business valuation techniques).

2-2 Lagrangian mechanics

2-2-1 The configuration manifold

Central to the concept of Lagrangian mechanics is the so-called *configuration space* M, which is an n-dimensional manifold provided with some parameterization called *generalized coordinates* assembled in the vector $\boldsymbol{q}$. The configuration manifold may just be equal to $\mathbb{R}^n$, but in more interesting and realistic cases it is often some manifold embedded in $\mathbb{R}^n$. This is often the result of holonomic constraints, which are constraints that impose a restriction only on the configuration space, but not, for example, on the allowable velocities. By using the generalized coordinates, one can parameterize all the allowable positions of a system whose motions may occur in a higher-dimensional space with a smaller coordinate set (e.g. the two-dimensional motion of a pendulum can be expressed in a single coordinate due to the constraint imposed by the rigid link). The crucial insight here is that the constraint forces will never perform any work on the system[2]; as such, they act always orthogonally to the configuration manifold. This is why, provided that one is successful in correctly describing this manifold with a suitable coordinate set, the constraint forces need not be taken into account: a major advantage over Newtonian mechanics.

Unfortunately, the pursuit of finding a set of these all-encompassing generalized coordinates is fruitless for some systems, for they cannot possibly be represented in this simple fashion. Luckily, for a certain class of constraints, they may be included to constrict the motion of the system nevertheless by means of the *Lagrange multipliers*. They are applicable to holonomic constraints for which it is impractical or non-intuitive to take them into account directly in the parameterization of the configuration manifold, and a restricted class of nonholonomic constraints that can be written in the so-called *Pfaffian* form [13].

The configuration manifold and the economic system

In economic engineering, $\boldsymbol{q}$ has a similar intuitive notion, namely the stock values of various products, which denote the 'position' of some economic system. The name 'position' may be misleading when intuitively ported to the familiar three-dimensional space; the configuration space usually has less structure such as the absence of a metric or inner product.

Just like in mechanics, the simplest shape the configuration manifold M can take is a simple n-dimensional vector space, but more sophisticated cases exist as well. As mentioned, a nontrivial configuration manifold is usually the result of holonomic

[2] This is known as *D'Alembert's principle*.

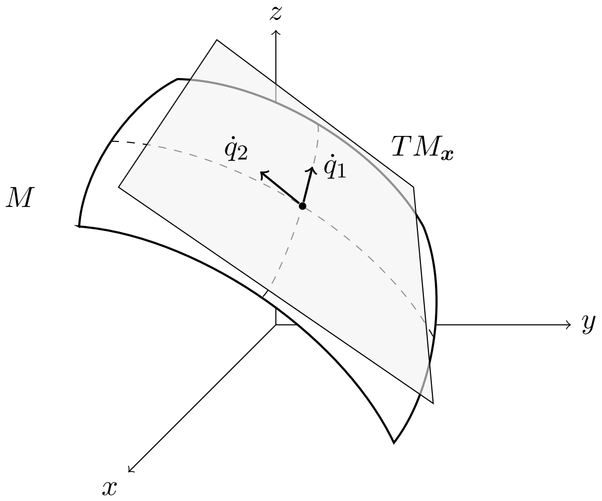


Figure 2-1: Schematic of a two-dimensional (configuration) manifold M embedded in $\mathbb{R}^3$; the local generalized velocities associated with a point $\boldsymbol{x}$ are vectors that live in the tangent space $TM_{\boldsymbol{x}}$.

constraints applied to the system; these constraints have their meaning in economics too. For example, when Lagrangian analysis is applied to the analysis of electrical circuits, the generalized coordinates naturally reflect Kirchoff's current laws (what comes in must go out); as they provide simple constraints to the current in each part of the circuit. An intuitive extension can be made to economics, where the flow of goods is often subject to a Kirchoff-type law as well, especially when considering supply chains or transport.

A simple example in economic engineering might be a market with a supplier, a trader, and a demander, each with their own storage. Because the total amount of product is limited by the endowment of the system (the so-called Edgeworth Box), a constraint is imposed on the allowable configurations of the system.

2-2-2 Hamilton's principle of stationary action

The configuration manifold is described in the previous section is the first step in the Lagrangian approach, for it defines a single mathematical space containing all the possible motions of the system. For all intents and purposes, it usually pertains to the very nature of the system itself; e.g. the wiring of the electrical circuit, how the mechanical parts are connected to each other or which goods are present in an economic system, and whether their quantities are fundamentally related. Of course, the configuration manifold itself does not provide any information about the behavior of the system: this is where Hamilton's principle comes in — the second, crucial puzzle piece that makes Lagrangian mechanics work.

Hamilton's principle (also referred to as the principle of least or stationary action) concerns the existence of a special state function, the Lagrangian $\mathcal{L}$, determines the system's behavior [14]:

Hamilton's principle

Motions $\gamma : \mathbb{R} \to M$ of a mechanical system coincide with extremals of the action functional

$$S(\gamma) = \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t \,, \tag{2-1}$$

where $\mathcal{L}$ is the *Lagrangian function* of the system.

The Lagrangian $\mathcal{L}$ is a mapping from the *tangent bundle* TM of the configuration manifold M, optionally paired with a time argument for time-varying problems to the reals,

$$\mathcal{L} : TM \times \mathbb{R}^+ \to \mathbb{R},$$

i.e. it takes a generalized position $\boldsymbol{q}$ and a generalized velocity $\dot{\boldsymbol{q}}$ (which live in the tangent space of M), and a time instance to some scalar values.

The aforementioned principle only is for now only helpful on a conceptual level, because it does not how to arrive at any solutions. For this, a branch of mathematics called the calculus of variations comes to aid, which is concerned with finding extremals of *functionals*[3], in this case, S. A necessary condition for S to attain an extremum is that

$$\delta S = 0,$$

where δS is called the *first variation* of S. Just like a regular differential can be seen as an infinitesimal perturbation of a function value, the variation is a very small perturbation of a functional by means of a trajectory $h(t)$. The resulting perturbed functional can then in general be decomposed in a part that varies linearly with h, and a nonlinear part. The requirement for the extremal is that the *linear part vanishes for any* h [14].

Landau and Lifshitz [15] use the the tools of the calculus of variations to show that the solution of eq. (2-1) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0.$$

This yields a total of n second-order differential equations, or equivalently a system of $2n$ first-order equations. A special significance is assigned to the vector $\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}$, defined as the *generalized momentum* $\boldsymbol{p}$.

Hamilton's principle and utility maximization

The generalization of Hamilton's principle to economics is not far-fetched; it is generally accepted that elementary economic agents act to maximize their own utility; this is known as the *utility maximization problem*[a].

Thus, the formulation of economic motions as an extremal problem is quite straightforward. However, in order to descend from a purely philosophical debate a formalism that is actually useful, the Lagrangian must be assigned with concrete meaning. As such, our aim is to translate the concept of energy to economics. In mechanics, the *energy* of the system is, bluntly speaking, its ability to perform *work*. For now, a

[3] A functional is a real-valued function on a vector space (of functions.)

purely mechanical interpretation is pursued, neglecting the connection between temperature and energy — the application of economic engineering and thermodynamics is described in the thesis of Manders [11]. The economic engineering interpretation of work is the fulfilling of wants. As such, the energy of an economic system is its ability to fulfill wants. A natural dichotomy arises when viewing the economic system in terms of its configuration manifold and generalized 'velocities' (product flows), akin to the of forms energy in mechanics.

- *Kinetic energy* is related to the utility due to market (trading) activity, it, therefore, depends in the first place on the *flow of goods*; it is the surplus of the economic agent [16].

- *Potential energy* gives significance to utility obtained from the possession of goods; it must therefore depend on stock levels. One can see this as a sort of 'convenience yield' (a term used in futures pricing): the benefits of actually possessing the good.

A more rigorous definition of these concepts in economics will be given later in this section. The intuition behind the principle of stationary action can be found in Feynman [17]:

> *"[...] the solution is some kind of balance between trying to get more potential energy with the least amount of extra kinetic energy—trying to get the difference, kinetic minus the potential, as small as possible."*

Likewise, this can be restated as the basic least action principle in economic engineering:

> **Hamilton's principle in economic engineering**
> Economic agents try to maximize their convenience yield (the utility of the goods they possess) by sacrificing as little economic surplus as possible

As described by Feynman [17], this is not only a global trajectory, but also a local one at every (infinitesimal) piece of the trajectory: one can see this as a formal restatement of the rational behavior of economic agents, which is a crucial assumption in may economic theories [16].

[a] It is important to realize that the term 'extremal' does not necessarily refer to a minimum, as is often incorrectly stated when explaining Hamilton's theorem — indeed, the least action principle is sometimes called the principle of *stationary* action, which is more in line with its mathematical definition.

2-2-3 Kinetic energy

In classical mechanics, the Lagrangian is defined by convention

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}; t) = T^*(\boldsymbol{q}, \dot{\boldsymbol{q}}; t) - U(\boldsymbol{q}; t),$$

where T^* is the kinetic co-energy[4] of the system and U the potential energy. If M is a Riemannian manifold and its Lagrangian has the aforementioned form, the system is called *natural* [14].

In the most general terms, the kinetic energy of the system is *defined* as a quadratic form on the tangent space of the configuration manifold. Assuming that m is a Riemannian manifold (i.e. it is equipped with a Riemannian metric $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle$), one can define the kinetic co-energy as

$$T^* = \frac{m}{2}\langle \boldsymbol{v}, \boldsymbol{v} \rangle \quad \text{with} \quad \boldsymbol{v} \in TM_{\boldsymbol{x}}. \tag{2-2}$$

The usage of T^* in the Lagrangian formulation is only useful if it is expressed in the generalized coordinates and generalized velocities; in general, T^* will be of the form

$$T^*(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} m_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j, \tag{2-3}$$

observing the Einstein summation convention. This interpretation of kinetic co-energy as a Riemannian metric on the configuration manifold must not be overlooked; indeed, a free particle (i.e. in the absence of potential forces) will follow a trajectory along the *geodesic* dictated by the 'kinetic co-energy metric'; this is called the Maupertuis-Jacobi principle [14].

The distinction between energy and co-energy is not very common in literature, although thoroughly discussed by Jeltsema and Scherpen [18]. Energy is the ability to do work, while co-energy is the *complement of energy.* Because energy is defined in terms of work, kinetic energy T should be defined in terms of momentum – the integral of an applied force over time, instead of a velocity. The linear relation behind the change of variables from momentum to velocity by means of the mass makes the distinction between energy and co-energy moot at first glance, but it is nevertheless important to consider. The kinetic energy and co-energy are not always equal after all, e.g. in the relativistic case where the mass will depend on the velocity as well. In the simple non-relativistic case, for a single particle with mass m, the following relations hold:

$$\text{kinetic energy } T = \int \frac{p}{m}\, \mathrm{d}p = \frac{p^2}{2m} \qquad \text{kinetic co-energy } T^* = \int mv\, \mathrm{d}v = \frac{mv^2}{2}.$$

Kinetic energy and market surplus

The interpretation of kinetic energy in economic engineering is a big leap forward, perhaps one of the most fundamental aspects of the entire theoretical framework. However, the formulation in eq. (2-2) obscures the intuition behind it. This is why it is more instructive to look at the simple scalar case, where kinetic energy is a notion of the amount of work it takes to accelerate a particle from rest to a certain velocity.

To explain the significance of kinetic energy as surplus, the example given by Marshall [19, chap. 6] about consumer's surplus will be recycled here in order to illustrate the point.

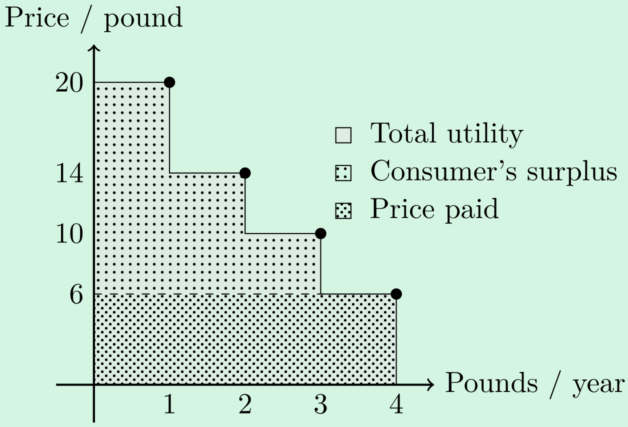
Imagine a woman (Jane), who likes to drink tea. When buying tea, she continuously makes the unconscious deliberations between (i) how much she likes to drink tea and (ii)

[4] As will become clear later, the term 'co-energy' is used to distinguish between the definition of kinetic energy in terms of the generalized momenta, which is the perspective of Hamiltonian mechanics.

how much she likes to pay for tea — this is a consequence of the assumption that Jane is a rational market participant. Marshall quantifies Jane's unconscious deliberation process by means of the following table:

Price / pound	20	14	10	6
Pounds bought / year	1	2	3	4

It is rather straightforward that Jane is willing to buy more tea as it gets cheaper and vice versa. Marshall explains this by means of the concept of utility: if the price of tea, say 20$, drops to 14$, Jane obtains her additional pound of tea for 14$ for instead of the 20$ she was willing to pay for the first pound. As such, she gains a surplus satisfaction (or consumer's surplus) of 6$ — this is, as Marshall states, *precisely* the additional utility of the second pound of tea, i.e. how much Jane values it on top of the first one. The total utility of the two pounds of tea per year is then 20$ + 14$ = 34$. More simply stated, the consumer's surplus is the total utility of the products bought minus the price actually paid. In this example, the total utility is *at least* 34$, and the price paid is 2×14$ = 28$: therefore, Jane's consumer's surplus is 6$.



Indeed, this example illustrates that *surplus* and *utility* are closely related; indeed, they only differ by the choice of a 'set point' (the price at which Jane is buying tea).
Both have units of $/yr, a consequence of the demand being in quantity/yr (of course 'yr' is an arbitrary choice for a time unit for the sake of this example) – this is an important distinction from other texts in economics, which tend to be rather vague as to whether the demand is a flow or an absolute quantity of goods. Based on this discussion, the following relations can be obtained:

$$\text{total surplus} = \sum_{i=1}^{i^m} p_i \Delta v \qquad \text{consumer's surplus} = \sum p_i \Delta v - \underbrace{\sum p_m \Delta v}_{\text{amount paid}} \tag{2-4}$$

with p the price and v the amount of tea sold per year. Hence, the summations happen over a set of prices between two points: a 'reference' price (in this case, $20), and the market price p_m; of course, the value of the consumer's surplus depends on the choice for these prices. Naturally, the market price may be quite undisputed, but the reference price is just that, and its choice is arbitrary. In this simple example, it happens to coincide with the reservation price for the first pound of tea, but that need not be the case at all.

By virtue of the foregoing discussion, it is established that *the trade utility measured at a given price is the (consumer's) surplus.* The Lagrangian represents an exchange of 'trade utility' (dependent on the flow of goods) and the 'product utility', dependent on stock levels. To establish the analogy with mechanics, observe that the part of the mechanical Lagrangian that depends on the generalized velocities is the kinetic energy. This connection is the foundation of the following economic engineering principle:

Kinetic (co-)energy is analogous to market surplus.

In mechanics, the calculation of kinetic energy is *dependent on the frame of reference*, and this is analogous to the reference price used to calculate the consumer's surplus. There is one additional loose end: in the example of Jane, there was extensive mention of price, while the Lagrangian and the kinetic co-energy are only dependent on the flow of goods. One can observe from the example that the price determines the additional increase of surplus for every increase in the amount of tea bought per year, or otherwise

$$\frac{\Delta(\text{total utility})}{\Delta v} = p_i$$

To generalize, one can assume that v and p are continuous variables instead, related to each other by the bijective 'reservation price mapping' that is denoted by $m : v \mapsto p$. The summations in the previous examples can then all be replaced by integrals:

$$\text{total utility} = \int p \, \mathrm{d}v \qquad \text{consumer's surplus} = \int p \, \mathrm{d}v - p_1 \int \mathrm{d}v$$

i.e. the total utility and consumer's surplus only differ by a choice of reference frame. With the consumer's surplus being analogous to kinetic energy, one can say that

Price is analogous to momentum.

In this analogy, the 'mass' plays the role of the market (in)elasticity, or the slope of the supply/demand line. Some generalizations are in order with respect to the classical notion of mass in mechanics. First of all, the elasticity is allowed to depend on the stock level, which is very common in economics (scarcity affects the supply and demand lines). Secondly, the product space (configuration manifold) is generally far from isotropic, i.e. mass is direction-dependent. This seems far from realistic for mechanical systems, but when one generalizes 'mass' to the mass matrix found in the linearization of nonlinear mechanical systems, both position dependence and anisotropy are frequently encountered (e.g. double pendulum).

2-2-4 Potential energy

In mechanics, the potential energy arises due to the presence of a conservative force vector field $\boldsymbol{F}$. A vector field is conservative if the work done along any path only depends on the

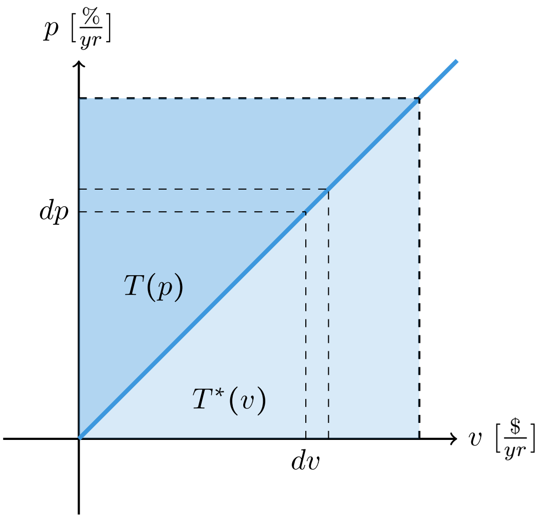


Figure 2-2: Kinetic energy and co-energy in terms of the relation between momentum and velocity. Figure courtesy of B. Krabbenborg [1].

endpoints of the path and not on the intermediate shape. If that is the case, then it is always true that one can find a function such that[5]

$$\boldsymbol{F} = -\frac{\partial U}{\partial \boldsymbol{x}}.$$

In the Lagrangian context, the position vector $\boldsymbol{x}$ can be expressed in terms of the generalized coordinates, such that (with a slight abuse of notation)

$$\boldsymbol{F} = -\frac{\partial U}{\partial \boldsymbol{q}}.$$

Potential energy

While in ‘regular physics’ potential energy arises due to the presence of elastic, electric, gravitational ... forces, its interpretation in economic engineering is related to the utility gained from the possession of goods; i.e. it is a function that has units of cash flow (just like surplus) that depends only on the stock levels (possibly in terms of generalized coordinates).

The simplest example is the economic analogue of a spring: just like a spring, a restoring ‘force’ (i.e. an economic want) occurs whenever the spring is elongated or stretched with respect to a certain reference distance. For economics agents, this means either being long or short on stock levels, again with respect to a certain reference q_0 (although important for practical interpretations, this reference does not really contribute on a conceptual level, which is why it is omitted in the theoretical discussion, just like the reference momentum for the kinetic energy). Hence, there is a ‘potential utility in either being long or short, in the sense that it can be exchanged for ‘market utility’ or

[5] This is a consequence of the fundamental theorem for line integrals [20].

economic surplus, because it requires an exchange of goods. The continuous reciprocity between kinetic energy and potential energy is a central theme in mechanics, and it is henceforth been given a succinct economic interpretation as well.

The stereotypical example in mechanics that contains a single storage element of kinetic energy (a mass) and a single storage element of potential energy (a spring) is the mass-spring system, which also arises as the linearization of a multitude of undamped nonlinear systems (i.e. the pendulum); the result is a second-order scalar autonomous differential equation:

$$m\ddot{q} + \frac{\partial U}{\partial q} = 0, \quad \text{with} U = \frac{kq^2}{2} \tag{2-5}$$

with k being the spring constant; this immediately results in Newton's second law of motion. A simple mass-spring system is shown in fig. 2-3.

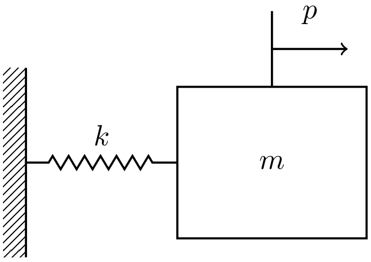


Figure 2-3: A simple mass-spring system. Figure courtesy of B. Krabbenborg [1].

The distinction between energy and co-energy exists for potential energy too. Again, the potential energy itself is defined in terms of work: the integral of force over a distance. Comparable between the bijective simple linear relation between velocity and momentum, for simple linear springs the force F and displacement q are related by $F = kq$. Again, in this case, the distinction may seem a bit pointless; but, when this relation is nonlinear (as it is the real world often is), the potential energy and co-energy are no longer the same.

$$\text{potential energy } U = \int F \, \mathrm{d}q \qquad \text{potential co-energy } U^* = \int q \, \mathrm{d}F$$

which for in the case of the simple spring turn out to be

$$U = \frac{kq^2}{2} \qquad U^* = \frac{f^2}{2k}.$$

In the case of potential energy, the familiar equation really *is* the one that corresponds to energy (and not to co-energy, as for kinetic energy). The second equation seems really odd, intuitively turning the causal relation around, but one should bear in mind that this equation is, from a fundamental perspective, equally 'odd' as the extremely familiar $\frac{mv^2}{2}$. Figure 2-4 shows the relation between potential energy and co-energy and why they are equal for a linear relation between force and displacement. The solution of the simple mass-spring system is a sinusoid that continues indefinitely in time. These 'perfect' sinusoids are not often encountered in real economic systems. One must, however, bear in mind that (i) the perfect mass-spring system does not exist in mechanics either, and (ii) that economic systems are much harder to

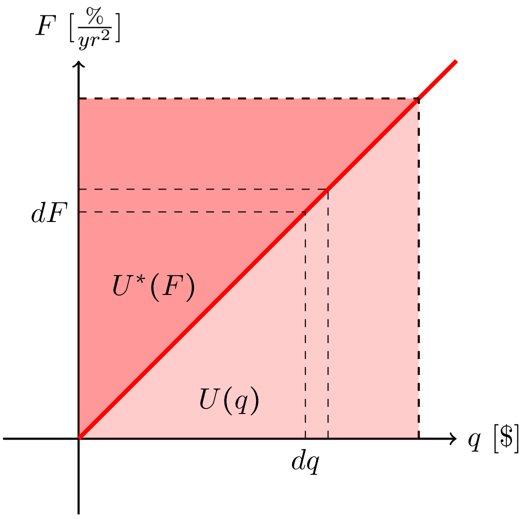


Figure 2-4: The distinction between potential energy and co-energy in terms of the relation between force and displacement. Figure courtesy of B. Krabbenborg [1].

'decouple' from their surroundings due to the complex interconnected nature of society. This makes the theoretical achievement that are the basic foundations of economic engineering perhaps even more admirable. Because of these external factors, eq. (2-5) will in practice usually contain external inputs in the form of forcing terms in lieu of the 0 after the equality sign, and *dissipative terms* which indicate path-dependent forces in the system.

2-3 Hamiltonian mechanics

2-3-1 The Legendre transform

Whereas the Lagrangian of a system is a function of its generalized positions and generalized velocities, the Hamiltonian is a function of the generalized position and generalized momentum. Recall from the definition that the generalized (or *conjugate*) momenta are defined in terms of the Lagrangian; $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$. This change of variables is performed in a particular fashion called the *Legendre transform.*

The total differential of the Lagrangian is

$$\mathrm{d}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\,\mathrm{d}\boldsymbol{q} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\,\mathrm{d}\dot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t\,.$$

Two observations can be made: first, by definition, $\boldsymbol{p} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}$; and secondly, the Euler-Lagrange equations dictate that $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}$. As such, the total differential of the Lagrangian may be rewritten as: [15]

$$\mathrm{d}\mathcal{L} = \dot{\boldsymbol{p}}\,\mathrm{d}\boldsymbol{q} + \boldsymbol{p}\,\mathrm{d}\dot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t\,.$$

By means of the product rule $\mathrm{d}(\boldsymbol{p}\dot{\boldsymbol{q}}) = \boldsymbol{p}\,\mathrm{d}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}\,\mathrm{d}\boldsymbol{p}$, the above can be rewritten as:

$$\mathrm{d}\underbrace{(\boldsymbol{p}\dot{\boldsymbol{q}} - \mathcal{L})}_{\mathcal{H}} = \dot{\boldsymbol{q}}\,\mathrm{d}\boldsymbol{p} - \dot{\boldsymbol{p}}\,\mathrm{d}\boldsymbol{q} - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t\,. \tag{2-6}$$

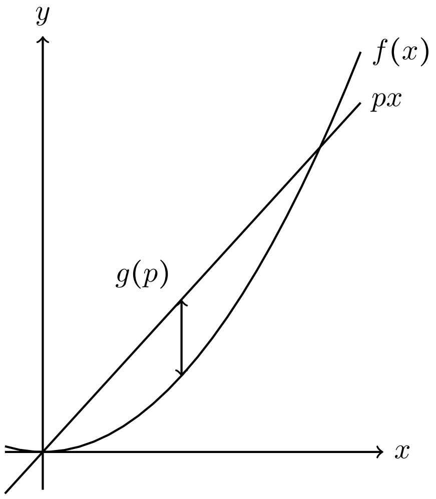


Figure 2-5: Illustration of the Legendre transform. g is the Legendre transform of the function f, as it maximizes the $px - f(x)$; this optimality is attained when $p = \frac{\mathrm{d}f}{\mathrm{d}x}$. In classical mechanics, this is extended to the multivariable case where $\mathcal{L}$, $\mathcal{H}$ and $\dot{\boldsymbol{q}}$ take the role of f, g and x respectively.

The argument of the differential on the left-hand side is the *Hamiltonian* $\mathcal{H}$ of the system, i.e.

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{p}, t) \triangleq \boldsymbol{p}\dot{\boldsymbol{q}} - \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t).$$

This coincides with the Hamiltonian being the Legendre transform of the Lagrangian, given that the Lagrangian is a convex function (in $\dot{\boldsymbol{q}}$). Roughly speaking, the Legendre transform of a function $f(\dot{\boldsymbol{q}})$ finds $g(\boldsymbol{p}) = \boldsymbol{p}\dot{\boldsymbol{q}} - f(\dot{\boldsymbol{q}})$ is maximal, i.e. $\dot{\boldsymbol{q}}(\boldsymbol{p})$ is chosen such that $g(\boldsymbol{p})$ is maximal. Clearly, $\boldsymbol{p}$ must be chosen such that $\boldsymbol{p} = \frac{\partial f}{\partial \dot{\boldsymbol{q}}}$. This is precisely how the generalized momentum is defined in terms of the Lagrangian, which automatically implies that the Hamiltonian is indeed the Legendre transform of the Lagrangian. The convexity of the function guarantees uniqueness and maximality.

The Legendre transform is involutive; that is, the $\mathcal{H}$ and $\mathcal{L}$ are the Legendre transform of each other; this is why they are called *dual in the sense of Young* [14]. For the scalar case, a graphical explanation of the Legendre transform exists is provided by fig. 2-5. Here, the function $g(p) = \max_x\{px - f(x)\}$; this optimality condition is met when $p = \frac{\mathrm{d}f}{\mathrm{d}x}$, i.e. where px and $f(x)$ are locally parallel.

The Legendre transform in economics

In economics, the Lagrangian measures the exchange between market surplus and convenience yield. This exchange has the units of $\left[\frac{\$}{\mathrm{yr}}\right]$, and is in economic engineering sometimes named 'cost' for short, although a definition in terms of utility would probably be more appropriate. The term $\boldsymbol{p}\dot{\boldsymbol{q}}$ has a very clear interpretation in economics: it is the *revenue* associated with the flow of goods and their prices. The Legendre transform then maximizes the difference between revenue and the Lagrangian 'cost' with respect to the number of goods exchanged; $\dot{\boldsymbol{q}}$. The optimality is guaranteed by the price representing the change of utility with respect to a change in the flow of goods. Because the Hamiltonian is equal to a difference between revenue and costs, its said to be analogous to earnings.

2-3-2 Hamilton's equations

Equation (2-6) provides an expression for the total differential of the Hamiltonian, $\mathrm{d}\mathcal{H}$. By definition, this differential is equal to:

$$\mathrm{d}\mathcal{H}\left(\boldsymbol{q},\boldsymbol{p},t\right) = \frac{\partial\mathcal{H}}{\partial\boldsymbol{q}}\,\mathrm{d}\boldsymbol{q} + \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}}\,\mathrm{d}\boldsymbol{p} + \frac{\partial\mathcal{H}}{\partial t}\,\mathrm{d}t\,.$$

Then, eq. (2-6) gives rise to the following equalities:

$$\dot{\boldsymbol{q}} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}} \qquad \dot{\boldsymbol{p}} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{q}} \qquad \frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t}, \tag{2-7}$$

which are known as *Hamilton's equations*. The Hamiltonian combined with Hamilton's equations gives an equivalent representation of the system dynamics as the Lagrangian with the Euler-Lagrange equations.

In mechanics, the Hamiltonian corresponding to $\mathcal{L} = T^* - U$, and where T^* is the quadratic form given in eq. (2-3), the Hamiltonian is equal to:

$$\mathcal{H} = T + U, \tag{2-8}$$

which means that $\mathcal{H}$ is equal to the *total mechanical energy in the system*. A vital property of the Hamiltonian is that of its time (in)dependence, namely:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \underbrace{\frac{\partial\mathcal{H}}{\partial\boldsymbol{q}}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}}_{=0} + \frac{\partial\mathcal{H}}{\partial t} \quad\Longrightarrow\quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial t},$$

that is, the Hamiltonian is only time-varying if it features an explicit time-dependence. The vanishing of the first two terms is a direct consequence of Hamilton's equations. This property leads directly to the conservation of (mechanical) energy in time-invariant systems.

Whereas Lagrangian mechanics acts on the tangent bundle of the configuration manifold, Hamiltonian mechanics takes place in the *cotangent bundle* T^*M, for it is expressed in terms of the momenta rather than the generalized velocities. The generalized momenta live in the cotangent space at a certaint point of the configuration manifold because they are the differential of a scalar function (the Lagrangian) on the manifold. The cotangent bundle is a vector bundle and thus a manifold in its own right; here the base manifold of the cotangent bundle is the configuration manifold M, the fibers are the cotangent space for each point. A natural 1-form can be defined on this cotangent bundle that exploits the distinction between the base manifold and the fibers; namely a 1-form on the co-tangent bundle that is 'equivalent' to the covector in the cotangent space (with respect to their action on the tangent vector to the cotangent bundle and configuration manifold respectively). The action of this natural 1-form, which is called the *tautological 1-form*, on a tangent vector to the cotangent bundle is equal to the action of the covector on this tangent vector *under the pushforward of the* *projection map*. That is, if $\pi : T^*M \to M$ is the projection map, $\boldsymbol{\xi} \in T(T^*M)$ is a tangent vector to the cotangent bundle, and m is an element of the cotangent bundle, the tautological 1-form ω^1 is defined by [14, 21]

$$\omega^1 = m \circ \pi_* \quad \text{or} \quad \omega^1(\boldsymbol{\xi}) = m(\pi_*(\boldsymbol{\xi})).$$

In classical mechanics, ω^1 is also named the *Poincaré 1-form* or *canonical 1-form*. The Poincaré 1-form can be expressed in canonical coordinates as follows:

$$\omega^1 = \boldsymbol{p}\, \mathrm{d}\boldsymbol{q}\,,$$

it is therefore a mathematical object that associates each component of the generalized momentum with the generalized position, and thus provides a fundamental connection between Hamiltonian and Lagrangian mechanics.

The exterior derivative of the Poincaré 1-form is a closed 2-form that is called the *symplectic form*;

$$\omega^2 = \mathrm{d}\omega^1\,.$$

In the given coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$, the symplectic form is

$$\omega^2 = \mathrm{d}\boldsymbol{p} \wedge \mathrm{d}\boldsymbol{q}\,;$$

which makes the cotangent bundle T^*M a symplectic manifold. The symplectic 2-form plays a similar role in symplectic geometry as the Riemannian metric does in Riemannian geometry; for example, by providing an isomorphism between the tangent and cotangent vectors. This isomorphism can be used to define the *Hamiltonian vector field* [14].

Hamiltonian vector field

Let $\mathcal{H}$ be a function on a $2n$-dimensional symplectic manifold (Q, ω^2). The symplectic 2-form defined on Q provides an isomorphism I between its tangent and cotangent vectors, such that $\omega^2(\boldsymbol{\eta}, \boldsymbol{\xi}) = \omega^1_{\boldsymbol{\xi}}(\boldsymbol{\eta})$. The vector field $I\,\mathrm{d}\mathcal{H}$ is called a Hamiltonian vector field, with $\mathcal{H}$ being the hamiltonian function.

A crucial property of the Hamiltonian vector field (and the corresponding Hamiltonian flow) is that it preserves the symplectic structure: [14]

$$(g^t)^*\omega^2 = \omega^2,$$

i.e. where $(g^t)^*$ represents the pullback of the one-parameter group of diffeomorphisms of Q that are governed by the Hamiltonian flow. Hence, the symplectic form is an integral invariant of the phase flow of the Hamiltonian system.

The symplectic nature of economics

The preceding discussion presents the existence symplectic 2-form as a consequence of the phase space being the cotangent bundle of the configuration manifold. In economics, the existence of the symplectic structure is even more intuitive — one could even argue that economics is evidently symplectic in the first place, and that the connection with classical mechanics is a consequence of this idea. If one asserts that the economy as a system 'takes place' in the necessarily even-dimensional space of products and their prices, then there is only one, natural structure in economics that ties them together: the concept of money. More specifically, multiplying a number of products with their respective price allows one to 'compare': this is precisely the role of money as the *unit of account*, as stated by Mankiw and Taylor [16]. Said otherwise, the area of the phase plane gives the economic space its defining structure, which in turn gives rise to the canonical 2-form $\mathrm{d}\boldsymbol{p} \wedge \mathrm{d}\boldsymbol{q}$. The area is signed because money is also 'signed' from the perspective of an economic agent.

The canonical 2-form is only symplectic if it obeys two additional requirements: it mus be closed and nondegenerate. Both requirements translate easily to economics:

- *Closed* means that the exterior derivative of the canonical 2-form is equal to zero. Crudely speaking, one could say that the exterior derivative is the 'divergence' of the economic phase flow. If this divergence is zero, than the economic phase flow is incompressible with money being the notion of volume. Intuitively, this means that the 'value' of money is consistent throughout the phase flow. This makes sense from an economic perspective, for money wouldn't be very good as a unit of account if its value were inconsistent. Of course, some objections can be made, such as the existence of different currencies and inflation, but it is argued that these are not fundamental enough to violate this elementary principle.
- *Nondegenerate* means that there is no product for which any price is necessarily 0, i.e. nothing is inherently 'free' in economics. Again, this is a reasonable (and even fundamental) assumption for an economic system.

Hence, the economic space of prices and products naturally exhibits a symplectic structure. This can be further supported by the fact that another type of structure for this space, say a Riemannian structure (e.g. something like $\mathrm{d}p^2 + \mathrm{d}q^2$), would make much less sense as it does not capture the role of money as means to compare and exchange goods of different nature.

The reason why one can make the additional connection with classical mechanics is that the price-product space is a cotangent bundle. This in turn is motivated by the additional observation that the prices are the partial derivative of a products' utility with respect to its transaction rate, i.e. $\boldsymbol{p} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}$, i.e. the prices live in the fibers attached to the configuration manifold. Finally, there are the *additional* assumptions about the form of the Lagrangian and Hamiltonian; i.e. that they can be used in the same form as those in classical mechanics ($\mathcal{L} = T^* - U$, $\mathcal{H} = T + U$). Although the previous sections motivate that this is indeed a sensible thing to do, they are essentially additional assumptions on top of the symplectic structure of the $\boldsymbol{p}$-$\boldsymbol{q}$ space and the nature of price.

The canonical formalism

One of the great advantages of Hamiltonian mechanics is that it admits a much wider range of coordinate transformations. The Lagrangian formalism already allowed point-transformations, which simply apply to the configuration manifold (i.e. a change in generalized coordinates), where the conjugate momenta transform according to the Legendre transform. However, in the phase space the transformations may apply both to the momenta as the positions. Due to this large variety of transformations, the coordinates for $\boldsymbol{q}$ and $\boldsymbol{p}$ may no longer just pertain to a 'spatial' component and a 'momentum' component, this distinction will then just be a matter of definition [15]. The 'allowed' class of coordinate transformations, called the *canonical transformations*, preserve the symplectic form ω^2 and thus the form of Hamilton's equations [14]; they are *symplectomorphisms* of the phase space. There is a rich variety in canonical transformations that can be helpful to better understand or integrate Hamiltonian systems; but this text will focus on one particular type that has been studied in economic engineering in the past: the action-angle coordinates.

Action-angle coordinates A special choice of canonical coordinates arises for conservative systems, called the *action-angle coordinates*. These are pairs of coordinates of the form $(\boldsymbol{I}, \boldsymbol{\phi})$, where $\boldsymbol{I}$ consists of the action coordinate and $\boldsymbol{\phi}$ are the angle coordinates. The reason why this set of coordinates exist in the first place is found in Liouville's theorem, which states that when a system with n degrees of freedom has n independent first integrals $\boldsymbol{F}$, then the level set of these first integrals is diffeomorphic to the the n-dimensional torus $\mathbb{T}^n$. This torus is then completely parameterized by the n action variables, while the angle coordinates determine the position on the torus. Figure 2-6 visualizes the action-angle coordinates for the simple case of a system with one degree of freedom.

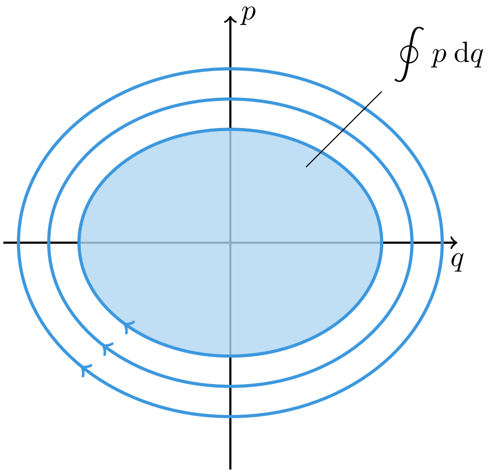


Figure 2-6: Illustration of the action-angle coordinates for a simple system where the phase space is $\mathbb{R}^2$. Then, the trajectories are diffeomorphic to $\mathbb{T}^1$, or a circle for a given energy level. The area enclosed by the trajectories is the area enclosed by the trajectory divided by 2π; the angle coordinate determines the position in this trajectory.

The action coordinates are functions of these first integrals as to ensure that the action-angle coordinate set is symplectic, i.e. [14]

$$\omega^2 = \sum \mathrm{d}I_i \wedge \mathrm{d}\phi_i \,.$$

The major advantage of this particular choice of coordinates is that the corresponding differential equations are very easy to integrate. The action coordinates are constant over time, while the angle coordinates are characterized by a constant 'angular velocity' — another consequence of Liouville's theorem is that these systems are necessarily periodic. The angular velocity vector $\boldsymbol{\omega}$ depends on the value of the first integrals $\boldsymbol{f}$. That is, [14]

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \boldsymbol{\omega}(\boldsymbol{f}) \implies \boldsymbol{\phi}(t) = \boldsymbol{\phi}(0) + \boldsymbol{\omega}t.$$

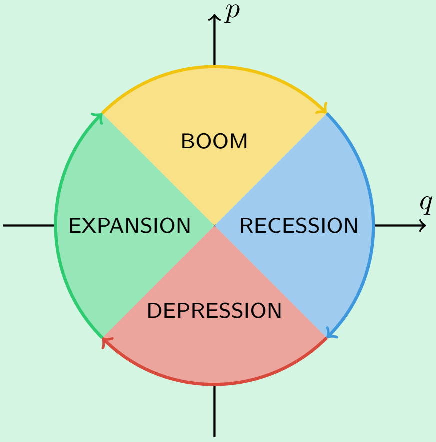
In the case where the cotangent bundle is $\mathbb{R}^{2n}$ (which is often the case for the linearized systems in economic engineering), the angle coordinates may constructed as follows:

$$I_i(\boldsymbol{f}) = \frac{1}{2\pi} \oint_{\gamma_i} \boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,, \tag{2-9}$$

where $\gamma_1, \ldots, \gamma_n$ form a *basis* for the one-dimensional cycles on the torus (that is, they each define a cycle in terms of one of the angle coordinates). This coordinate transformation can be shown to preserve the symplectic form.

Action-angle coordinates and the business cycle

Action-angle coordinates have been applied in past economic engineering research, most notably in the works of Van Ardenne [12] and Vos [8]. Based on dimensional analysis, they have been used to model financial markets; however, it is the opinion of the author that this application is a bit *ad hoc* and does not entirely respect the precise formal meaning of action-angle coordinates as outlined in this section. Rather, these coordinates can be used to express an economic system in terms of the 'magnitude' of the business cycle, where the 'angle' measures the phase of the business cycle (in Dutch, one would call this '*conjunctuur*'). Equation (2-9) essentially measures a total revenue over an entire business cycle, which has a characteristic frequency depending on the system parameters.



The above figure gives a typical business cycle with the associated terminology for the four 'phases'; it can be interpreted directly from the perspective of the phase space.

2-3-3 Noether's theorem

Conservation principles play a vital role in physics and mechanics. Notable examples are the conservation of energy, linear momentum, angular momentum and charge. A surprising result in physics is that all of the mentioned conservation laws are a consequence of the same theorem, attributed to Emmy Noether. This theorem essentially looks at the nature of the Lagrangian of the system, and sees whether it is 'ignorant' with respect to a certain group of transformations. As defined by Arnol'd [14], the theorem is as follows:

Noether's theorem

If the Lagrangian system $(M, \mathcal{L})$ admits the one-parameter group of transformations $h^s : M \to M, s \in \mathbb{R}$, then the Lagrangian system of equations corresponding to $\mathcal{L}$ has a first integral: $I : TM \to \mathbb{R}$. In local coordinates q on M the integral I is written in teh form

$$I(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \frac{\mathrm{d}h^s(\boldsymbol{q})}{\mathrm{d}s}\bigg|_{s=0}$$

The word 'admits' here means that the Lagrangian is invariant with respect to the transformation h; that is, the $\mathcal{L}$ exhibits a *symmetry* in that regard. A perspicuous consequence of this theorem is that when the Lagrangian has a cyclic coordinate (that is, it does not depend on some q_i), then the conjugate momentum of that coordinate is conserved, for the conjugate momentum of a coordinate q_i is $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$.

Energy conservation When the Lagrangian is time-invariant, one can define $h^s : (\boldsymbol{q}, t) \mapsto (\boldsymbol{q}, t+s)$ as translation in time. To find the conserved quantity, the Lagrangian has to be extended to consider the time t as an explicit generalized coordinate, i.e. define

$$\tilde{\mathcal{L}}\left(\boldsymbol{q}, t, \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau}, \frac{\mathrm{d}t}{\mathrm{d}\tau}\right) \triangleq \mathcal{L}\left(\boldsymbol{q}, \frac{\mathrm{d}\boldsymbol{q}/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}, t\right)\frac{\mathrm{d}t}{\mathrm{d}\tau},$$

where τ now plays the role of the independent variable. This definition ensures that the action integral for $\tilde{\mathcal{L}}$ and $\mathcal{L}$ are identical, which is why Noether's theorem can be applied to $\tilde{\mathcal{L}}$ instead. Of course, $\frac{\mathrm{d}h}{\mathrm{d}s}$ equal to 1 for the time coordinate; so there is one first integral

$$\begin{aligned} I = \frac{\partial \tilde{\mathcal{L}}}{\partial (t')} &= \mathcal{L} + \frac{\partial}{\partial t'}\left(\mathcal{L}\left(\boldsymbol{q}, \frac{\boldsymbol{q}'}{t'}, t\right)\right)t' \\ &= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{q}'}{t'}\right)t' = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\frac{\boldsymbol{q}'}{t'} \\ &= \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} \end{aligned}$$

with $t' = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ and $\boldsymbol{q}' = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau}$. This is the negative of the Legendre transform of the Lagrangian, or the negative of the Hamiltonian; $I = -H$. Therefore, the total energy of the system is conserved.

(Angular) momentum conservation The invariance of the Lagrangian in one or more coordinate directions implies the conservation of the associated conjugate momentum.

For example, a common assumption on Earth is that the gravitational force is constant and acts downwards. Therefore, only one coordinate appears in the potential energy expression, which is why the other coordinate directions (that is, the two ‘horizontal’ directions, with proper choice of the coordinate system) are cyclic. Hence, by virtue of Noether’s theorem, the linear momentum in those directions is conserved, for it is unaffected by any potential force.

Conservation of angular momentum often appears in the presence of so-called *central* force fields, which means that the potential energy is a function of the *distance* from a certain point: notable examples are Coulomb forces or gravitational forces in celestial mechanics. Because only distance matters, the orientation of the displacement vector between the origin of the force field and the body is of no importance; it will not appear in the Lagrangian either. Consequently, Noether’s theorem dictates that these central force fields preserve angular momentum.

Noether’s theorem and conservation in economics

Noether’s theorem can be applied to economic systems as well. The most simple application is the conservation of prices in the absence of scarcity. When the utility Lagrangian does not depend on a certain stock level q_i, that essentially means that there is no utility associated with the possession (or shortage) of it; in other words, economic agents are not driven by scarcity (or abundance) of that particular product. The ‘conjugate momentum’ of an economic good is its price; so *in the absence of scarcity, price is conserved.*

A similar comment about cyclic coordinates can be made in the rotational analogy (which is to be discussed in chapter 3). If the utility Lagrangian is independent on the return or interest ζ, then borrowers or lenders will be indifferent about the realized returns of their investments, and never capitalize them (or in the case of borrowers, never repay their debts).

Finally, the conservation of income in the economic system is related to the time-invariance of its Lagrangian. The Lagrangian does not depend on time if (i) there are no external influences or forcing terms and (ii) all parameters in the system remain constant over time. If that is the case, the Hamiltonian (or total income) in the economic system is conserved; or volume in the phase space (i.e. total amount of money) is conserved.

Admittedly, all of these interpretations are arguably rather straightforward, since they mostly rely on either the presence of cyclic coordinates or the direct correspondence of total energy and energy of the system. However, the discussion does prove that Noether’s theorem has very tangible interpretations in economics. Its real power will, though, appear with more sophisticated symmetries that are present in complex economic systems. The idea is that the Lagrangian and Noether’s theorem permit the extraction of ‘stylized facts’ from these types of systems that would otherwise be concealed by their complexity.

Chapter 3

Investment as a hyperbolic rotation

In this chapter,

3-1 Terminology

The aim of this research is to refine the way that economic engineering currently deals with financial instruments, primarily fixed-interest assets, but also marketable equity (stocks), derivatives, etc. All of these can be regarded as 'investments', for lack of a better word, because they require a *principal*, that is the amount invested, and a certain return that is proportional to the principal. This is why the return is usually expressed as a percentage of the principal, which in the case of debt is the interest rate. The latter contains the word 'rate', because it has an inherent timely aspect: interest rates (and returns of investments in general) are associated with a certain 'term' or duration over which the return is realized. As such, any type of investment is characterized by three components:

- a *principal*, something that is originally invested;
- a *return* proportional to the principal;
- a *period* over which the return is realized.

In the world of finance, the principal and return often refer to some amount of money, but that does not necessarily have to be the case. For example, some stocks pay *stock dividends*, which means that dividends take the form of additional stock. Alternatively, one could look outside finance to other factors of production, such as land: a farmer may 'invest' a certain amount of land to gain the earnings (crops that grow on it) after the period of one year; of course, the amount of crops grown is proportional to the surface area of the land that the farmer has cultivated.

3-2 Reinvestment of earnings

After the end of the investment period, the investor has two options: either he can spend his earnings, or he can reinvest them; essentially, he starts the next investment period with a higher principal than the first one. This reinvestment of capital is a crucial concept in the world of finance and is the reason for the existence of compound interest. Indeed, compound interest assumes that one's earnings after one period also bear interest; the so-called *interest on interest*. This can be illustrated by means of the example shown in fig. 3-1. An investor holds a fixed-interest investment with a principal of \$1000 and an interest rate of 5% annually. After one compounding period, he naturally earns \$50 directly on the principal. As such, his direct earnings, which will be called 'yield', now amount to \$50 while the invested 'capital' is still \$1000. The next compounding period, however, provided that the investor does not withdraw his earnings from the account, the \$50 generates \$2.5 of interest as well, which are *reinvested*; i.e. they take the form of capital instead of yield. Simultaneously, the investor also keeps the additional benefits from the \$1000 of capital in the form of another \$50 dollars of direct earnings. This reinvestment of earnings is the driving force behind compound interest:

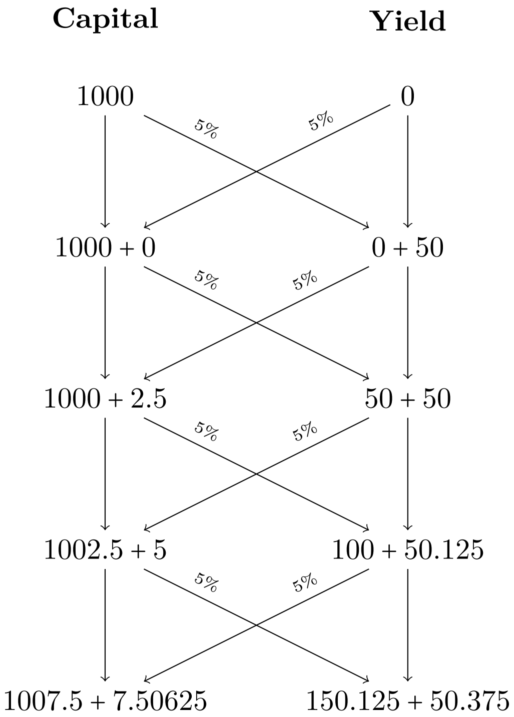


Figure 3-1: A simple example of compound interest with a principal of \$1000 and an interest of 5%. The direct earnings on the capital are called 'yield', while the (re)invested amount is called 'capital'.

without this step, the investor would end up with *simple interest*, which sees its relative returns essentially decreasing over time. One should, however, be cautious to associate this process exclusively with financial investments; for example, a farmer could use the earnings from his land to buy more land, increase his earnings, use those again to buy even more land, and so forth — this will again give rise to exponential growth. The farming example makes it quite clear that the object that 'generates' the return (capital) is of a very different nature than the return itself (yield). They can be connected through the usage of money as both a unit of account and a medium of exchange, which makes the whole process work [16].

Financial investments are arguably the culmination of this concept, since money itself earns money that can be reinvested again instantaneously. Still, the capital-yield distinction remains relevant: for example, companies pay dividends to their shareholders in proportion to the amount of stock they own; the shareholders could then choose to reinvest these earnings in more stock of that (or another) company, increasing their 'capital'. An additional example here is stock dividends, which are stocks with dividends that are paid as additional stock — essentially taking the role of money out of the equation.

3-3 The hyperbolic shape of compound interest

If the numbers from the example in fig. 3-1 are plotted against each other, with capital on the horizontal axis and yield on the vertical axis, a hyperbolic shape appears, shown in fig. 3-2. Hyperbolae are characterized by the implicit equation

$$X^2 - Y^2 = K^2 \tag{3-1}$$

where K represents the 'radius' of the hyperbola, which coincides with the principal amount of the investment. The amount of capital and yield accumulated at a certain time is then on the X and Y-axis respectively. However, as fig. 3-2 demonstrates, the shape arising from the example is not quite hyperbolic, since it essentially lags behind the real hyperbola of that radius. The reason why the 'perfect' hyperbola is not recovered in the example is

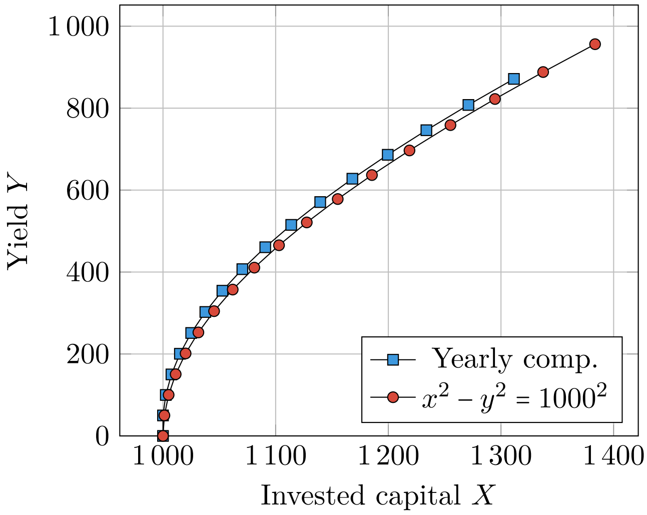


Figure 3-2: Plot of the numbers from the example in fig. 3-1. The 'ideal' hyperbola with the same radius, given by the equation $X^2 - Y^2 = K^2$ is shown as well.

demonstrated as follows. The capital-yield evolution can be modeled as a second-dimensional difference equation like so:

$$\begin{pmatrix} X(k+1) \\ Y(k+1) \end{pmatrix} = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \begin{pmatrix} X(k) \\ Y(k) \end{pmatrix} \quad \text{with} \quad X(0) = K, Y(0) = 0$$

where X denotes capital, Y denotes yield and r is the interest rate. The solution of this autonomous system is given by:

$$\begin{pmatrix} X(k) \\ Y(k) \end{pmatrix} = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^k \begin{pmatrix} K \\ 0 \end{pmatrix}.$$

One can then write $X(k)^2 - Y(k)^2$ in terms of this solution:

$$X(k)^2 - Y(k)^2 = \begin{pmatrix} K & 0 \end{pmatrix} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^k \begin{pmatrix} K \\ 0 \end{pmatrix}.$$

In order to easily evaluate the matrix powers, one can use the eigenvalue decomposition: with

$$\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^k = V\Lambda^k V^{-1} \quad \text{with} \quad V = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 1-r & 0 \\ 0 & 1+r \end{pmatrix}.$$

Straightforward computations then yield:

$$X(k)^2 - Y(k)^2 = K^2\left(1-r^2\right)^k,$$

which suggests that for reasonably small values of r, the compounding process approximates the equation for the hyperbola. One way to achieve this is to compound faster, i.e. to use more periods with a smaller interest, such that $r \mapsto r/n$ and $k \mapsto kn$ for $n \to \infty$. Indeed,

$$\lim_{n\to\infty} K^2\left(1 - \left(\frac{r}{n}\right)^2\right)^{kn} = K^2.$$

That is, for infinitely fast compounding, the capital-yield decomposition of compound interest follows a hyperbolic shape.

Another way to view this is to see the difference equation as a sampled version of an underlying continuous system. One can find the continuous-time state-transition matrix by applying the matrix logarithm to its discrete-time counterpart:

$$\log\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \log(r+1) + \log(r-1) & \log(r+1) - \log(r-1) \\ \log(r+1) - \log(r-1) & \log(r+1) + \log(r-1) \end{pmatrix}.$$

This continuous-time state-transition matrix has eigenvalues $\log(1+r)$ and $\log(1-r)$, the former of which is the continuous-time equivalent interest rate, or the *force of interest*, r_c [22]. From this discussion, one can infer that the discrete compounding process is essentially an approximation of the underlying 'ideal' continuous-time process, which is why the rotational analogy will be developed for a continuous compounding scenario; since the discrete-time equivalent may always be obtained by means of sampling.

3-4 Conic sections and hyperbolic rotations

The implicit equation for a hyperbola eq. (3-1) is remarkably similar to the one of a circle: the only difference being the minus sign. Indeed, circles and hyperbola are both members of a larger family called conic sections, which play a prominent role in geometry. Conic sections are usually classified based on their eccentricity e, for which three general cases can be distinguished:

- $0 \le e < 1$ are *ellipses*, the special case for which $e = 0$ is a circle;
- $e = 1$ holds for *parabolae*;
- $e > 1$ are hyperbolae.

The elliptic-parabolic-hyperbolic (EPH) distinction is a common theme within mathematics and in this literature study, as it also applies to surface curvature, which gives rise to elliptic, parabolic, or hyperbolic geometry (the subject of chapter 5), and the classification of Möbius transformations discussed in chapter 6.

Harkin and Harkin [23] use the conic sections to extend the 'classic' trigonometry based on circles (i.e. elliptic curves) to hyperbolic and parabolic trigonometry as well. Associated with each of these types of geometry is also a special number system akin to complex numbers, called double numbers (or hyperbolic numbers) and dual numbers (or parabolic numbers) — hyperbolic numbers will briefly be explored in section 3-7. The results of the previous section suggest that hyperbolic trigonometry can be used to describe investment problems: this will be the basis for the *rotational analogy* in economic engineering, which will be introduced in section 3-5.

3-4-1 Hyperbolic angles

For both circles and hyperbolae, an angle refers to a certain region bounded by the curve at issue. The standard notation for the hyperbolic angle will be ζ, in accordance with the rapidity from special relativity, which can also be viewed as a hyperbolic angle (cf. chapter 4).

Hyperbolic sector

A hyperbolic sector is a region bounded by two lines extending from the origin to each to a point on the (unit) hyperbola, and the graph of the hyperbola itself.

Clearly, hyperbolic sectors are entirely analogous to their 'traditional' circular cousins. Fixing one of the rays to the x-axis, one can define the corresponding hyperbolic angle:

Hyperbolic angle

A hyperbolic angle corresponding to a point A is defined as twice the area of the hyperbolic sector based on the point A and the intersection point of the unit hyperbola and the x-axis $(K, 0)$.

Figure 3-3 visualizes the concept of a hyperbolic angle: the angle ζ is defined in terms of the unit hyperbola $X^2 - Y^2 = 1$, or the hyperbola with 'radius' 1. Just like for circles, the hyperbolic angle is defined on the unit hyperbola, but using a special (polar) coordinate set points can be expressed in terms of a radius K and an angle ζ. By allowing the radius K to be negative, all the points in the disconnected open set bounded by $y = x$ and $y = -x$ can be identified with a unique radius K and hyperbolic angle ζ. This is somewhat similar to polar coordinates, which is why these coordinates will be referred to as 'hyperbolic polar coordinates'. These coordinates are interesting because they allow to express investments in a very natural way in terms of their principal and realized return.

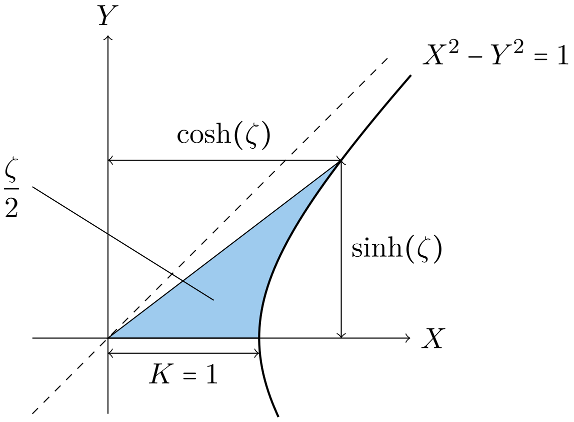


Figure 3-3: Illustration of a hyperbolic angle along the unit hyperbola, with projection on the axes using the hyperbolic sine and cosine.

The preceding discussion suggests that hyperbolic polar coordinates do *not* provide coordinates for the entire plane like regular polar coordinates do. Indeed, the mapping defined by the coordinate functions from the $K - \zeta$ space to the x-y space is neither injective nor surjective: its image is the disconnected open set bounded by the lines $y = x$ and $y = -x$ (not surjective), and the entire line $K = 0$ in the $K - \zeta$ plane is mapped to the origin in the $x - y$ plane (not injective). As such, one can obtain a bijection by disregarding the degenerate cases for which $K = 0$ and restricting the co-domain of the mapping to the set $\{(x, y) \in \mathbb{R}^2 : |x| > |y|\}$. The action of the mapping is illustrated by fig. 3-4. From a financial perspective, this limit signifies the maximal yield one can obtain for a given amount of capital, i.e. one can never obtain more yield than (re)invested capital at a given moment in time.

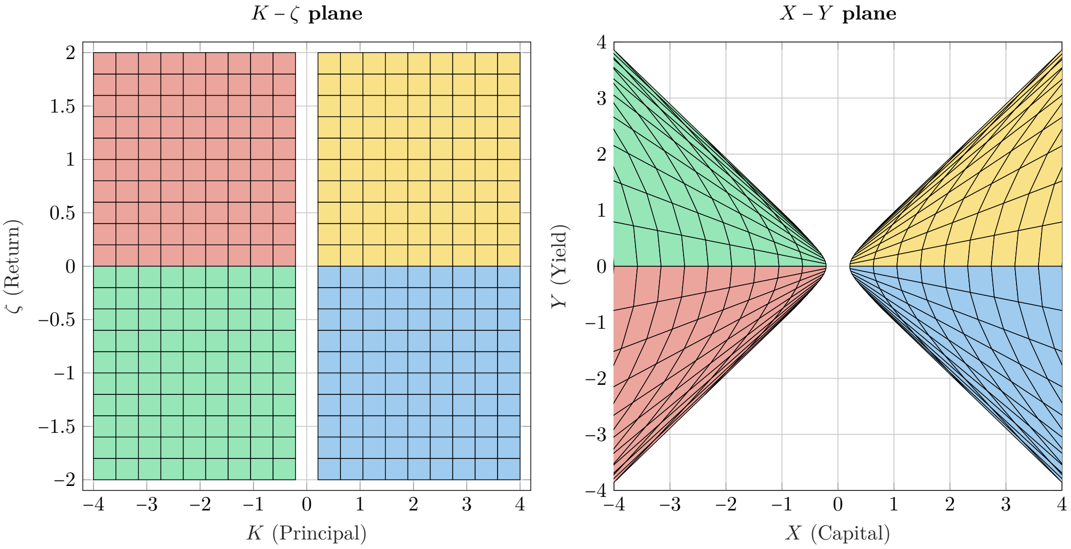


Figure 3-4: Visualization of the usage of hyperbolic polar coordinates as a mapping form the $K - \zeta$ (principal-return) space and the $X - Y$ (capital-yield) plane.

Remark A possible workaround for this problem from a mathematical perspective can be found in the so-called *generalized trigonometry* as described by Harkin and Harkin [23]: in this case, the notion of the hyperbolic angle itself is considered ambiguous and must be accompanied by the branch of the hyperbola that is associated with the angle. In this case, four hyperbolic branches are considered, so both branches of $x^2 - y^2 = K^2$ combined with $y^2 - x^2 = K^2$ which were disregarded up till now.[1]

$$\zeta = \begin{cases} \tanh^{-1}(x/y) & \text{for branches I and III} \\ \tanh^{-1}(y/x) = \coth^{-1}(x/y) & \text{for branches II and IV} \end{cases} \tag{3-2}$$

Although this extended definition brings the other two quadrants of the hyperbolic plane within reach of the polar form as well, there is still a particular set that even now cannot be represented this way: $\{(x, y) \mid |x| = |y|\}$, or the 'light cone' from special relativity.

3-4-2 Hyperbolic functions

Figure 3-3 also shows that any point on a hyperbola may be projected on the capital and yield axis using a special set of functions, called *hyperbolic functions*. These functions are akin to the trigonometric functions sin and cos for 'circular' angles.

This decomposition of an investment in capital and yield in terms of these functions is almost trivially encapsulated in the mathematical identity:

$$\exp(\zeta) = \cosh(\zeta) + \sinh(\zeta). \tag{3-3}$$

The most intuitive way to interpret this identity is by inspection of the Taylor series of the functions appearing in eq. (3-3):

$$\begin{aligned} \exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \cosh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\ \sinh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \end{aligned} \tag{3-4}$$

Or, alternatively, via the definition of the hyperbolic functions

$$\begin{aligned} \cosh(x) &\triangleq \frac{\exp(x) + \exp(-x)}{2} \\ \sinh(x) &\triangleq \frac{\exp(x) - \exp(-x)}{2} \end{aligned} \tag{3-5}$$

As such, the hyperbolic functions allow finding the associated capital and yield for a given accumulated interest ζ and initial investment K.

[1] In fact, Harkin and Harkin provide an even more general treatment, covering so-called generalized complex numbers of the form $z = x + \iota y \quad x, y \in \mathbb{R}$ with $\iota^2 = \iota q + p \quad p, q \in \mathbb{R}$. However, in this case $p = 1$ and $q = 0$, as will become clear later in the discussion about hyperbolic numbers.

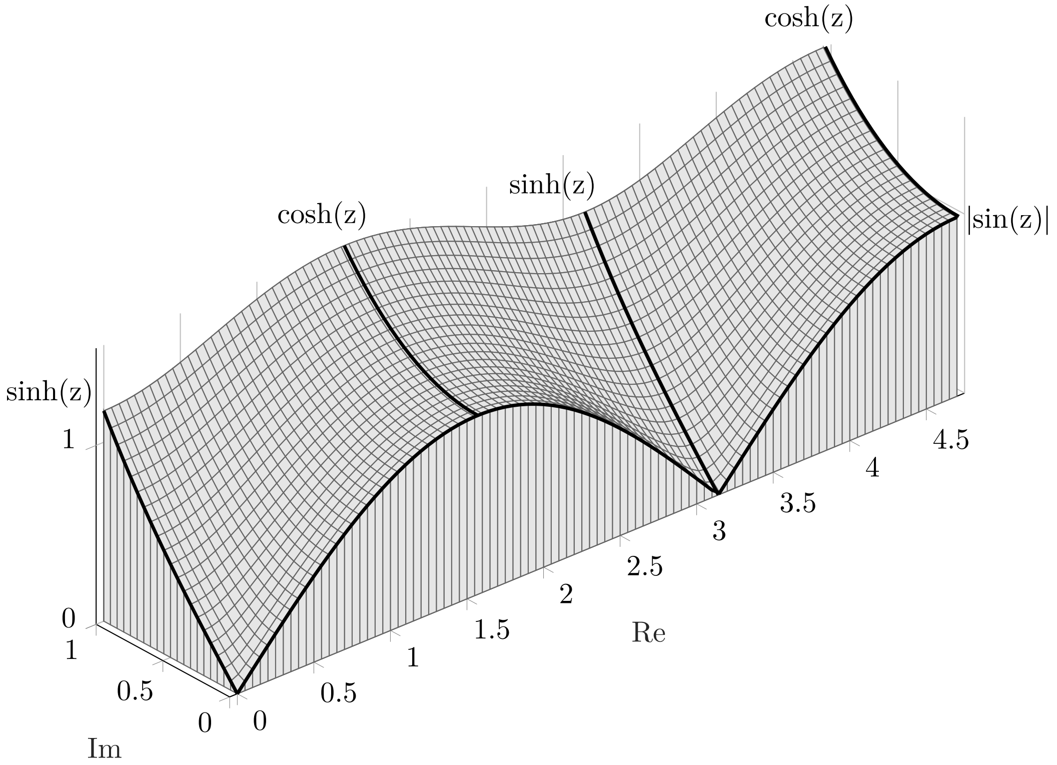


Figure 3-5: The modular surface of the sine over the complex plane, embedding all the trigonometric and hyperbolic functions at specific cross-sections.

Clearly, the cosh and sinh functions are constructed by isolating the even or the odd powers respectively from the Taylor expansion of the exponential. Equation (3-3) bears some resemblance to Euler's formula $\exp(\mathrm{i}x) = \cos(x) + \mathrm{i}\sin(x)$; as described by Needham [2], this connection can be generalized by recognizing that

$$\cos(\mathrm{i}x) = \cosh(x) \qquad \sin(\mathrm{i}x) = \mathrm{i}\sinh(x). \tag{3-6}$$

Hence, the *hyperbolic rotation may be expressed as a 'normal' circular rotation through the usage of complex angles*, which makes it possible to connect the hyperbolic rotations for investments to normal 'circular' rotations in mechanics, as is the purpose of economic engineering.

To make eq. (3-6) more visual, the hyperbolic and trigonometric functions can all be represented by looking at the modular surface of $|\sin(z)|$, shown in fig. 3-5: sin and cos only differ by a shift of $\pi/2$ along the real line, and cosh and sinh exist at cross-sections into the complex at integer multiples of $\pi/2$ and π respectively [2].

3-5 The rotational analogy for economic engineering

It has now been established that investments can be represented using (hyperbolic) angles. Since economic engineering attempts to find analogies between traditional engineering disciplines and economics, the rotational nature of investments is a strong motivation to make the connection with *rotational mechanics* and the associated concepts of rotational kinetic energy, angular momentum, angular velocity, etc.

Some aspects of the analogy are already apparent from the foregoing discussion: the *angle* corresponds to the accumulated interest or return gained on the investment, while the *arm* or radius is analogous to the principal of the investment. Likewise, the time derivative of the angle $\dot{\zeta}$ is the angular velocity, i.e. the interest rate, which may itself be time-varying.

The current interpretation in economic engineering is that credit corresponds to angular momentum. One reason for this is that the units of angular momentum are [N m s], in economic engineering, this would be units of currency [\$]. Furthermore, the analogy can be supported by means of the Lagrangian as well, similar to the reasoning in chapter 2 connecting prices and momentum. If one assumes that the rotational kinetic energy is the 'market surplus' gained from borrowing or lending money (or investing in general) based on a certain return rate $\dot{\zeta}$, the conjugate momentum of this form of kinetic energy is necessarily the angular momentum. In the case of investments, the market surplus is associated with a new type of elasticity that relates the amount invested L with the return rate $\dot{\zeta}$, similar to the supply/demand curves for 'regular' goods and services. In economics, this is called the *curve of loanable funds.* Therefore, the 'borrower' or 'investor surplus' is well-defined: it is related to how much more the current return (or how much less the current interest) is than the minimal interest for which one is willing to invest or borrow. From this definition, the angular momentum must be analogous to the number of funds invested (or borrowed); being the partial derivative of this 'market surplus' with respect to the return.

Table 3-1: Overview of the analogies between rotational mechanics and economic engineering.

Rotational mechanics	**Economic engineering**
Arm	Principal
Angle	Return / accumulated interest
Angular velocity	Return rate / interest rate
Rotational kinetic energy	Borrower / lender surplus
Angular momentum	Credit

Issues with the rotational analogy Although the rotational analogy has been applied successfully in the past (see Kruimer [9] and Van Ardenne [12]), there appear to be some inconsistencies in its present interpretation.

- Currently, the difference between the arm (initial investment) and the angular momentum (credit) is not entirely clear.
- The sign of the interest rate does not determine the sign (i.e. direction) of the angular momentum; the former has to do with either discounting/appreciation while the latter denotes the direction of the contract (e.g. debit vs. credit).
- The interpretation of the elasticity as the mass moment of inertia (because it determines the relation between the amount of credit and the interest rate) has not been consistently

applied. In the past, the mass moment of inertia has been put on equal footing with the *duration* of a financial contract, but more recent works have pointed out that duration is a part of the valuation and is therefore associated with frequency domain analysis [1].

The resolution of these problems is one of the main goals of this research.

3-6 Lorentzian vector spaces

If investments are represented in the capital-yield plane, they are accompanied by a special notion of 'distance', which determines the radius of the hyperbola that they are positioned on. This distance is different from the traditional Euclidean distance; rather, it is a *Lorentzian* distance. This makes the capital-yield plane a *Lorentzian vector space.* In order to make this formal, the definition of the Lorentzian inner (or Lorentz) product is required:[2]

Lorentz product

Let $\boldsymbol{u}, \boldsymbol{v}$ be vectors in $\mathbb{R}^n$. The *Lorentzian inner product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined to be the real number

$$\boldsymbol{u} \bullet \boldsymbol{v} = u_1 v_1 - u_2 v_2 - \ldots - u_n v_n. \tag{3-7}$$

The Lorentz product is an inner product, which means that it takes two elements from a vector space and returns a scalar value, while satisfying two (or three) conditions: (i) bilinearity, (ii) symmetry, and (iii) nondegeneracy [25].

In some literature, the condition of nondegeneracy is replaced by a stronger notion of positive definiteness, which means that an inner product of a vector with itself is always nonnegative and zero if and only if the vector is the zero vector. However, this condition does not apply to the Lorentz product — this fact has rather far-reaching ramifications, as will become clear with the next definition.

Based on the Lorentz inner product it is natural to define also a corresponding *Lorentzian norm* $\|\cdot\|_L$:

$$\|\boldsymbol{v}\|_L = (\boldsymbol{v} \bullet \boldsymbol{v})^{\frac{1}{2}}$$

Because the Lorentz product is not positive definite but rather indefinite, the result of $\boldsymbol{v} \bullet \boldsymbol{v}$ is not guaranteed to be a positive number. As such, $\|\boldsymbol{v}\|_L \in \mathbb{C}$, in stark contrast with the familiar Euclidean norm which *is* positive definite and will therefore always return a non-negative real number.

The norm provides a notion of length for a vector. Based on this length, a *metric* yields a distance between two points, i.e. the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to

$$d_L(\boldsymbol{v}, \boldsymbol{u}) \triangleq \|\boldsymbol{v} - \boldsymbol{u}\|_L$$

[2] In literature (both physics and mathematics) many variations on the so-called 'metric signature' of the Lorentz product make their appearance. In terms of $\mathbb{R}^4$, these variations come down to $(+,-,-,-)$, $(-,-,-,+)$, $(-,+,+,+)$ and $(+,+,+,-)$, where the first one coincides with the definition used here, in accordance with the magnificent work of Landau and Lifshitz [24]. Luckily, this is just a matter of convention and its influence on the mathematical machinery at hand is limited to the switching of some signs.

Because the norm is not positive definite but 'only' nondegenerate, it is called a *pseudo-Euclidean metric*, and the vector space it is associated with a *pseudo-Euclidean space*. A vector space equipped with this pseudometric (in more general terms, a bilinear nondegenerate form) is called a Lorentz space. The particular case for $\mathbb{R}^4$ sets the stage for the theory of special relativity (with one 'special' dimension for time and three spatial dimensions) and is called the Minkowski space, after the German physicist Hermann Minkowski [26]. Sometimes the two-dimensional plane that is discovered here is also called the Minkowski plane, because this is what he used to explain his ideas, being unable to draw anything like four-dimensional space. However, this text will adhere to the more mathematically inclined tradition and call it 'Lorentz(ian) space' which applies to any dimension larger than one [25]. The theory of special relativity, and especially the role of the Lorentz metric will be discussed in chapter 4.

The sign of the Lorentz norm gives rise to an equivalence relation $\sim_L$ defined to be $a \sim_L b \iff \operatorname{sgn}\|a\| = \operatorname{sgn}\|b\|$.[3] Therefore, the quotient set of all the points in the plane $\mathbb{R}/\sim_L$ contains three elements $\{-1, 0, 1\}$, these equivalence classes are given the respective names $\{$`spacelike`, `lightlike`, `timelike`$\}$ based on the terminology from special relativity [24]. The difference between timelike, spacelike and lightlike vectors is illustrated in fig. 3-6.

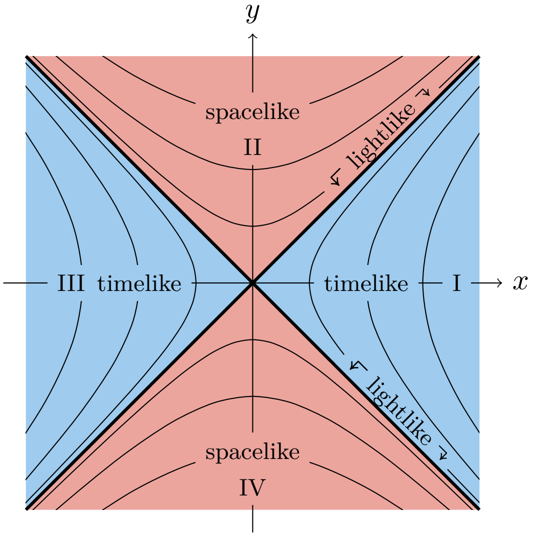


Figure 3-6: Overview of the three 'types' of vectors in the Lorentz-Minkowski plane: spacelike ($\|\boldsymbol{v}\|_L < 0$), lightlike ($\|\boldsymbol{v}\|_L = 0$) and timelike ($\|\boldsymbol{v}\|_L > 0$). The lines $y = x$ and $y = -x$ containing all the lightlike vectors form the so-called light cone or null cone. The hyperbola of in the spacelike region (dark gray) obey the equation $y^2 - x^2 = K^2$, they will be referred to the hyperbolic branches II ($y > 0$) and IV ($y < 0$). In contrast, the timelike hyperbolic branches with equation $x^2 - y^2 = K^2$ (light gray region) are referred to as I ($x > 0$) and III ($x < 0$).

The financial interpretation of the lightlike vectors is all the capital-yield combinations that are attainable, which is why spacelike vectors are fictitious in this regard if only the action of compound interest is considered. However, when one looks at velocities in this vector space; i.e. how fast a position vector is changing, the velocity vector can be spacelike due to capital injections that change the 'arm' of the investment.

[3] An equivalence relation is a binary relation that is symmetric, reflexive and transitive.

3-7 Hyperbolic numbers

An alternative way to view hyperbolic motions is in terms of so-called hyperbolic numbers[4]. These form an alternative number system similar to complex numbers, based on the 'hyperbolic' unit j with the defining unipotence property $\mathrm{j}^2 = 1$, where $\mathrm{j} \notin \mathbb{R}$. Like complex numbers, hyperbolic numbers can have a real and a hyperbolic part

$$z = x + y\mathrm{j} \quad x, y \in \mathbb{R}.$$

Combined with addition and multiplication defined on them, the hyperbolic numbers form a commutative ring. Each hyperbolic number z is associated with its *hyperbolic conjugate* $z^* = x - y\mathrm{j}$. The product of a hyperbolic number with its own hyperbolic conjugate creates a quadratic form $zz^* = x^2 - y^2$, which always returns a real number and is equivalent Lorentzian metric. Again, three cases can be distinguished for z:

- *timelike* when $zz^* > 0$
- *lightlike* when $zz^* = 0$
- *spacelike* when $zz^* < 0$

However, in contrast to complex numbers (whose quadratic form would be $x^2 + y^2$), this quadratic form is *isotropic*, which means that there exists a $z \neq 0$ such that $zz^* = 0$ — this is precisely the case for all the z on the light cone in the hyperbolic plane. The *hyperbolic modulus* therefore requires an absolute value in order like so

$$|z| \triangleq \sqrt{|zz^*|}$$

which is considered to be the hyperbolic distance from z to the origin [28].

Again, in correspondence with complex numbers, the hyperbolic numbers also have a *polar form*, i.e.

$$z = K \exp(\zeta\mathrm{j})$$

which can be evaluated using the Taylor expansion of the exponential exp:

$$z = K \exp(\zeta\mathrm{j}) = K \sum_{k=0}^{\infty} \frac{(\zeta\mathrm{j})^k}{k!} = K \sum_{k=0}^{\infty} \frac{(\zeta\mathrm{j})^{2k}}{(2k)!} + K \sum_{k=0}^{\infty} \frac{(\zeta\mathrm{j})^{2k+1}}{(2k+1)!} = K \cosh(\zeta) + K\mathrm{j} \sinh(\zeta)$$

when the hyperbolic angle ζ is associated with hyperbolic branches I and III. Clearly, the polar form of the hyperbolic numbers reflects precisely the two different representations for investments: using principal and return (polar form), or capital and yield ('cartesian' form). For the spacelike branches II and IV, one essentially has to consider $z = K\mathrm{j} \exp(\zeta\mathrm{j})$ or

$$z = K \sinh(\zeta) + K\mathrm{j} \cosh(\zeta)$$

[4] The hyperbolic number system has been assigned a myriad of names, with different terminology and mathematical notation for almost every influential paper that has been published about them. Among others, hyperbolic numbers are referred to as split-complex numbers, double numbers, perplex numbers, algebraic motors, etc. In this text, 'hyperbolic numbers' is chosen to highlight their connections with hyperbolae. For the choice of the hyperbolic unit, j will be used, though in literature also u (for unipotent) and h (for hallucinatory or hyperbolic) make their appearance [27, 28, 29, 23].

Furthermore, consider the *conjugate product* of two hyperbolic numbers $z_1 = x_1 + y_1\mathrm{j}$ and $z_2 = x_2 + y_2\mathrm{j}$:

$$z_1^* z_2 = \underbrace{(x_1x_2 - y_1y_2)}_{\text{inner product}} + \underbrace{(x_1y_2 - x_2y_1)}_{\text{outer product}}\,\mathrm{j}$$

of the resulting expression, the real part is called the inner product and the hyperbolic part the outer product. The inner product is equivalent to the Lorentz product while the outer product yields the directed area of the parallelogram spanned by z_1 and z_2 (or the determinant of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$. The outer product is the same as for regular complex numbers, while the inner product recovers the Lorentz inner product instead of the Euclidean inner product. This is why one can say that the hyperbolic number plane and the complex plane have an identical notion of area. Based on the inner product, one can recognize a different notion of orthogonality, i.e. z_1 and z_2 are *hyperbolically orthogonal* if their hyperbolic inner product equals zero [2, 28].

The inner and outer product both have their financial interpretations: the inner product is related to the hyperbolic distance between two hyperbolic vectors, just like the Lorentz product in the previous section. The outer product is akin to the cross product, and is therefore used to rigorously define angular momentum in the Lorentzian space, which is an important component of the economic engineering analogy.

3-7-1 Matrix representation

For many algebraic structures (groups, rings, fields, ...) an isomorphism can be found in the realm of linear algebra, by identifying a specific class of matrices. For example, the complex number system $a + b\mathrm{i}$ is isomorphic to the matrices

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad a, b \in \mathbb{R}.$$

Likewise, one can establish an isomorphism between the hyperbolic numbers and a certain class of matrices. The hyperbolic number system defined with addition and multiplication defined on it is ring-isomorphic to the matrix ring

$$\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \quad X, Y \in \mathbb{R},$$

under matrix addition and matrix multiplication. The determinant of the said matrix then recovers the Lorentz metric

$$\det\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = X^2 - Y^2.$$

3-7-2 The idempotent basis

Previously it was already mentioned that the hyperbola can be viewed conveniently in two particular axis systems, the standard axis system, and the idempotent axis system, differing by a 45° rotation. This basis can also be defined in terms of hyperbolic numbers; in terms of the standard basis $\{1, \mathrm{j}\}$ the idempotent basis is $\{\mathrm{j}_+, \mathrm{j}_-\}$ with

$$\mathrm{j}_+ = \frac{1}{2}(1+\mathrm{j}) \qquad \mathrm{j}_- = \frac{1}{2}(1-\mathrm{j})$$

The term 'idempotent' is a testament to the fact that $\mathrm{j}_+^2 = \mathrm{j}_+$ and $\mathrm{j}_+^2 = \mathrm{j}_+$. As such, a given hyperbolic number $z = x + y\mathrm{j}$ in idempotent coordinates is

$$\underbrace{(x+y)}_{z_+}\mathrm{j}_+ + \underbrace{(x-y)}_{z_-}\mathrm{j}_+$$

Furthermore, the idempotent basis is mutually annihilating, i.e. $\mathrm{j}_+\mathrm{j}_- = 0$, which is why they possess a projective property: [28]

$$z\mathrm{j}_+ = z_+\mathrm{j}_+ \quad \text{and} \quad z\mathrm{j}_- = z_-\mathrm{j}_-$$

The idempotent axis system is interesting because it returns the total amount compounded or discounted over time; in contrast to the standard yield-capital decomposition.

To summarize, it has been established that the hyperbolic numbers can be used to represent the rotational investment analogy, and that all the important concepts attached to it (inner product, cross product, total value, etc) are naturally incorporated in their algebra.

Part II

Theoretical background

Chapter 4

Special Relativity

This chapter gives a basic outline of the theory of special relativity. It is by no means meant to be a comprehensive overview, for which many other excellent resources exist such as Misner et al. [30], Taylor and Wheeler [31], Landau and Lifshitz [24], or Penrose [32] for a shorter, less technical introduction. Instead, the goal of this section is to give a physical context for the concept of Lorentz geometry and the associated Lorentz space and metric, as well as their connection with hyperbolic geometry and Möbius transforms which will be introduced in chapter 5 and chapter 6.

Before the advent of the theory of special relativity, developed by i.a. Poincaré, Minkowski, Lorentz and Einstein, so-called 'Galilean relativity' was the norm. Galilean relativity entails the definition of Galilean transformations which links reference frames that move relative to each other at a constant linear speed. The laws of physics should be invariant between these inertial reference frames. However, a problem presented itself in the form of the famous Maxwell equations that pose the governing laws in electromagnetism. A consequence from Maxwell's laws is the finite propagation speed of light and therefore all possible interactions between particles in the universe. It is not hard to imagine that the Galilean invariance breaks down as a result of the introduction of a 'special' speed - indeed, the Galilean transform proclaims a complete independence between the physical laws and the constant velocity of the frame in which they are applied. But, as a direct consequence from Maxwell's equations, the laws of physics in a moving reference frame *will* depend on the speed of that particular reference frame: a major inconsistency with the traditional train of thought.

The new principle of relativity that brought reconciliation with Maxwell's ideas not only required space to be relative (i.e. dependent on a frame of reference), but also views *time as a relative concept* whereas it had always been assumed to be absolute in classical mechanics. As such, the notion of time is dependent on the choice of reference frame too. This has the immediate consequence that the traditional three-dimensional setting of classical mechanics (often with Cartesian coordinates x, y, z) will not suffice for the description of special relativity: a fourth coordinate four time is indispensable to incorporate the relativity of time. Points in the four-dimensional *spacetime* are called *world points*, their associated trajectories are *world lines* [24].

4-1 Spacetime intervals

Overwhelming experimental evidence has pointed out that the propagation of light is completely independent of its direction. This can be encapsulated in spacetime by means of *spacetime intervals* which provide a notion of distance between two world points. If a signal travels at the speed of light c, the distance between two world points along its trajectory should be zero. The spatial distance squared between two points is equal to

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

whereas the distance squared covered by a signal traveling at the speed of light is equal to

$$c^2(t_2 - t_1)^2.$$

Therefore, the spacetime interval between two world points is

$$s_{12} = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}$$

which will amount to zero for the world lines corresponding to a signal traveling at the speed of light. For an infinitesimal distance $\mathrm{d}s$, the spacetime interval can be expressed as

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2\,.$$

The spacetime interval is the same in any inertial reference frame. This is the mathematical translation of the invariance of the speed of light in the universe [24]. Based on the sign of the spacetime interval, three classes can be distinguished.

- If $s_{12}^2 > 0$, the interval is *timelike* and there exists a frame of reference in which both events occurred *at the same location* in space, they are simply separated by the passage of time $t_{12} = \frac{s_{12}}{c}$;
- in contrast, when $s_{12}^2 < 0$, the interval is *spacelike* and the events are 'too far apart' to reach within the limits of the speed of light — the events must therefore be at different locations (absolutely remote), and there exists a reference frame in which the events occur *simultaneously* at distance $l_{12} = \mathrm{i}s_{12}$;
- intervals for which $s_{12}^2 = 0$ are called *lightlike*, because only light can travel between these events.

Now, one can ask the question what the actual time is that an observer would experience in uniform motion, i.e. what difference in time is there on clocks which have been traveling at different velocities? The time experienced by an observer is called *proper time*, and it can be computed by evaluating the following path integral:

$$t_2' - t_1' = \int_{t_1}^{t_2} \mathrm{d}t \sqrt{1 - \left(\frac{v}{c}\right)^2}\,. \tag{4-1}$$

Clearly, if the velocity makes up a larger fraction of the speed of light, the proper time is lower; that is: moving clocks run slower than a clock at rest (hypothetical clocks traveling at the speed of light do not register the passage of time whatsoever).

4-2 Lorentz transformations

As mentioned, special relativity corrects the flaw of the Galilean transforms, which represent the classical view of inertial reference systems: for example, if one coordinate system moves at constant velocity with respect to the other in x-direction (the coordinate directions are assumed to coincide for simplicity), the Galilean transform takes the form:

$$x = x' + Vt, \quad y = y', \quad z = z', \quad t = t'. \tag{4-2}$$

The statement $t = t'$ encodes the traditional assumption in mechanics that time has an absolute character. Of course, it is precisely this statement that is refuted by special relativity. As such, one could devise a new type of transformation that takes this (and the invariance of spacetime intervals between events, as discussed in the previous section) into account. These transformations are called *Lorentz transformations*.

As described by Landau and Lifshitz [24], these transformations comprise the rotations in four-dimensional space: since there are six ways to pick a plane (or two coordinates) from a set of four axes, every rotation in four-space can be decomposed into six successive rotations. Of these six rotations, three are purely spatial: they are the familiar rotations that can be parameterized by e.g. Euler angles. On the other hand, the three other rotations involve time as well, and they are of a different nature. Whereas the spatial rotations are circular, the time-rotations are hyperbolic (they are represented by hyperbolic functions rather than trigonometric functions). For example, a rotation in the tx-plane would take the following form:

$$x = ct' \sinh(\zeta) \qquad ct = ct' \cosh(\zeta); \tag{4-3}$$

or, using the fact that $V = x/t$:

$$x = \frac{x' + Vt'}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \qquad t = \frac{t' + \frac{V}{c^2}x'}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \qquad y = y' \qquad z = z'; \tag{4-4}$$

which also indicates that the hyperbolic (boost) angle ζ can be written in terms of the velocity V of one frame with respect to the other

$$\tanh(\zeta) = \frac{V}{c},$$

which means that the argument of the hyperbolic rotation purely depends on the relative velocity between the two reference frames as a fraction of the speed of light. A few observations can be be made based on these equations:

- Clearly, V cannot be larger than c; there is no real ζ for which this could be true. This again reaffirms the statement that there can be no motions with velocities larger than the speed of light.
- Secondly, this transform keeps $c^2t^2 - x^2$ unaffected (of course, z and y keep their value for obvious reasons); all points in the tx-plane that remain invariant under this type of transformations lie on the same hyperbola. This underlines the connection with the capital-yield plane discussed in the previous sections: in that analogy, the accumulated interest corresponds to the Lorentz boost ζ.

- In the limit for $c \to \infty$, the original Galilean transform is recovered; as such, the original laws still function as an approximation when V is of negligible size with respect to c.
- Due to the multiplication factor in the transform, two points x_1 and x_2 are closer together when travelling at speed than when they are at rest. A length measured in a rest frame are called *proper* , and contracts when in a moving frame: this phenomenon is called *Lorentz contraction* [24].
- In contrast to Galilean transforms, Lorentz transforms are generally not commutative: just like regular three-dimensional rotations, they depend on the order in which they are applied.

Velocity transform The Lorentz transform described by eqs. (4-3) and (4-4) shows how to transform coordinates from one frame to another. However, because the transform affects both x and t, a velocity measured in the frame (not to be confused with the relative velocity between the frames V) $\boldsymbol{v}$ with components will see not only its x-component affected, but the other two components v_x and v_z as well. The transformation of $\boldsymbol{v}$ to $\boldsymbol{v}'$ is then: [24]

$$v_x = \frac{v'_x + V}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \qquad v_y = \frac{v'_y \sqrt{1 - \left(\frac{V}{c}\right)^2}}{1 + v'_x \left(\frac{V}{c}\right)} \qquad v_z = \frac{v'_z \sqrt{1 - \left(\frac{V}{c}\right)^2}}{1 + v'_x \left(\frac{V}{c}\right)}. \tag{4-5}$$

Four-vectors Instead of the usual three-vectors that are common in classical mechanics, the points in four-dimensional spacetime may be regarded as elements in a four-dimensional vector space instead:

$$A^0 = ct \qquad A^1 = x \qquad A^2 = y \qquad A^3 = z; \tag{4-6}$$

where the superscript indices indicate *contravariant* (vector) components. These can be converted to covariant indices by virtue of the metric tensor $\boldsymbol{g}$

$$g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Using $\boldsymbol{g}$, indices can then be lowered (or raised) like so

$$A^0 = A_0 \qquad A^1 = -A_1 \qquad A^2 = -A_2 \qquad A^3 = -A_3;$$

such that the spacetime interval may be expressed in tensor notation (observing the Einstein summation convention) as

$$s^2 = A^i A_i = c^2 t^2 - x^2 - y^2 - z^2.$$

Much like position, velocities have a four-dimensional spacetime counterpart as well; these objects are called *four-velocities*, they are defined as [24]

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}s} \qquad \text{with } \mathrm{d}s = c\,\mathrm{d}t \sqrt{1 - \left(\frac{v}{c}\right)},$$

v being the three-dimensional velocity of the particle. The components of the four-velocity are then

$$u^0 = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \qquad u^1 = \frac{v_x}{c\sqrt{1 - \left(\frac{v}{c}\right)^2}} \qquad u^2 = \frac{v_y}{c\sqrt{1 - \left(\frac{v}{c}\right)^2}} \qquad u^3 = \frac{v_z}{c\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

which are all dimensionless quantities. Clearly, any four-velocity squared amounts to one; or $u^i u_i = 1$. This is analogous to the statement that all four-velocities live on a four-dimensional *unit hyperboloid* (due to the nature of the metric tensor); this means that the four-velocities *do not* form a vector space; the sum of two four-velocities does not generally yield another four-velocity. Instead, four-velocities exhibit a special type of geometry called *hyperbolic geometry*, an important concept to which chapter 5 is entirely devoted.

The Lorentz group All the Lorentz transformations form a Lie group called the Lorentz group Lor. When applied to the four-dimensional Lorentzian vector space, i.e. $\mathbb{R}^4$ equipped with the Lorentz product, they can be regarded as the special orthogonal group for this space.

Chapter 5

Hyperbolic geometry

5-1 Basic facts

Hyperbolic geometry, sometimes called Lobachevskian geometry[1] is the geometry on the hyperbolic plane. This type of geometry is called non-Euclidean, because it does not satisfy all of Euclid's axioms that form the building blocks of traditional geometry. More specifically, the fifth axiom, called the parallel axiom, states that [2]

> Through any point p not on the line L there exists precisely one line L' that does not meet L.

Logic dictates that if this axiom is not true, two possible alternatives arise. Given again the point p and the line L,

- there exists *no* line through p that does not meet L, or
- there exist *at least two* lines through p that do not meet L.

The first statement corresponds to what is called *spherical geometry*, while the latter is the defining axiom for *hyperbolic geometry.* It does not take too much imagination to realize that the parallel axiom is equivalent to another basic fact in Euclidean geometry: the sum of the angles of a triangle is equal to π. As such, for both types of non-Euclidean geometry this will not be the case; let $\mathcal{E}(T)$ denote the *angular excess* of a triangle T in a certain geometry.

- Naturally, for Euclidean geometry $\mathcal{E}(T) = 0$,
- in spherical geometry, $\mathcal{E}(T) > 0$,
- in hyperbolic geometry, $\mathcal{E}(T) < 0$.

[1] After the Russian mathematician Nikolai Lobachevsky (1832) who first published about this subject [2].

It may come as a surprise that the angular excess can be related to the size of the triangle (more specifically, its area) like so [2]

$$\mathcal{E}(T) = k\mathcal{A}(T) \tag{5-1}$$

k where $k = 0$ for Euclidean geometry, $k < 0$ for hyperbolic geometry and $\mathcal{K} > 0$ for spherical geometry. This hints at the fact that spherical geometry and hyperbolic geometry are somehow 'larger' classes of geometry than Euclidean geometry, since they exist for a whole range of values for k, be it positive or negative. There is indeed a 'different' geometry for every value of k, and only one of those values corresponds to the traditional concept of Euclidean geometry that most people are familiar with. Another consequence of eq. (5-1) is that similar triangles cannot exist in non-Euclidean geometry; since apart from the trivial case where the triangles are also congruent, they must differ in area, which yields a different angular excess. The value of k is, as it turns out, equal to the *Gaussian curvature* of the surface: a negatively curved surface has an angular deficiency while a positively curved surface has an angular excess; only for surface with zero curvature the sum of the angles of a triangle will be precisely equal to π.

5-1-1 Surface curvature

Perhaps one of the most remarkable results attributed to Carl Friedrich Gauss is his *Theorema Egregium*[2] about the (Gaussian) curvature of surfaces. The theorem states that curvature is an *intrinsic* property, which means that it remains preserved when the surface is transformed by 'bending without stretching'. Curvature can be positive, negative or zero. Positively curved surfaces 'bend away' from their tangent plane in any direction (local convexity) and negatively curved surfaces intersect their tangent plane like a saddle. For surfaces with zero curvature, there is always at least one straight line that lies in the tangent plane; examples are a cylinder (always one straight line) and a plane (straight lines in two directions). If a coordinate system is defined such that the point at issue is at the origin and the tangent plane to the surface described by $f(x, y)$ coincides with the horizontal, the Gaussian curvature can be computed by means of the determinant of the Hessian [34, 35]

$$k = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Surface curvature can be approached more rigorously from the perspective of Riemannian geometry. The core concept behind Riemannian geometry are the eponymous manifolds: these are manifolds equipped with an positive definite inner product[3] on their tangent space, hence providing a *metric (tensor)* that allows to determine lengths, angles and curvature. The metric on a parametric surface $\boldsymbol{r}(u, v)$ can be written in terms of the so-called *first fundamental form*[4] $\mathbf{I}$, which is defined as the inner product of two tangent vectors to the

[2] Loosely translated by John M. Lee [33] as the *Totally Awesome Theorem.*

[3] In contrast to section 3-6, the positive definiteness of the inner product is essential for Riemannian geometry. When the inner product is not positive definite but nondegenerate, as the Lorentz product, the manifold is called *pseudo-Riemannian.*

[4] Unfortunately, the first (and second) fundamental forms are not forms in the modern mathematical sense of the word; this terminology is merely inherited from older works [36].

surface and is characterised by coefficients E, F and G.

$$\begin{aligned} E &= \boldsymbol{r}_u \cdot \boldsymbol{r}_u \\ F &= \boldsymbol{r}_u \cdot \boldsymbol{r}_v \\ G &= \boldsymbol{r}_v \cdot \boldsymbol{r}_v \\ \text{with } \boldsymbol{r}_u &= \frac{\partial \boldsymbol{r}}{\partial u},\ \boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v}. \end{aligned} \tag{5-2}$$

The components of the metric tensor $\boldsymbol{g}$ then coincide with the components of the first fundamental form: $g_{11} = E$, $g_{12} = g_{21} = F$ and $g_{22} = G$. Therefore, the first fundamental form determines the length of curves lying in the surface.

In contrast, the *second fundamental form* $\mathbf{II}$ provides information on the curvature or shape of the embedded surface, more specifically the rate of change of the tangent planes in any direction. Again, it is characterised by three coefficients e, f and g like so

$$e\,\mathrm{d}u^2 + 2f\,\mathrm{d}u\,\mathrm{d}v + g\,\mathrm{d}v^2\,.$$

The coefficients of the second fundamental form can be computed using the surface unit normal vector

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}, \tag{5-3}$$

the coefficients are then given by

$$\begin{aligned} e &= \boldsymbol{r}_{uu} \cdot \boldsymbol{n} \\ f &= \boldsymbol{r}_{uv} \cdot \boldsymbol{n} \\ g &= \boldsymbol{r}_{vv} \cdot \boldsymbol{n} \\ \text{with } \boldsymbol{r}_{uu} &= \frac{\partial^2 \boldsymbol{r}}{\partial u^2} \text{ etc.} \end{aligned} \tag{5-4}$$

Finally, the first and second fundamental form can then be used to determine the Gaussian curvature of the parametric surface: [35, 36]

$$k = \frac{\det(\mathbf{II})}{\det(\mathbf{I})} = \frac{eg - f^2}{EG - F^2}. \tag{5-5}$$

To summarize, the connection between hyperbolic geometry and Gaussian curvature is concisely stated again below.

1. Hyperbolic geometry is the geometry with an alternative parallel axiom, where there are always at least two parallel lines for a given line through a point.
2. The former fact is equivalent to stating that the angular excess $\mathcal{E}$ is characterized by the negative constant k (constant over the entire hyperbolic plane).
3. The Gauss-Bonnet theorem states that the constant k is given by the Gaussian curvature.

Therefore, in order to represent hyperbolic geometry, one whishes to find a surface that exhibits constant negative Gaussian curvature. Some candidates exist for this criterion, the simplest of which is the so-called *pseudosphere*.

5-1-2 The pseudosphere

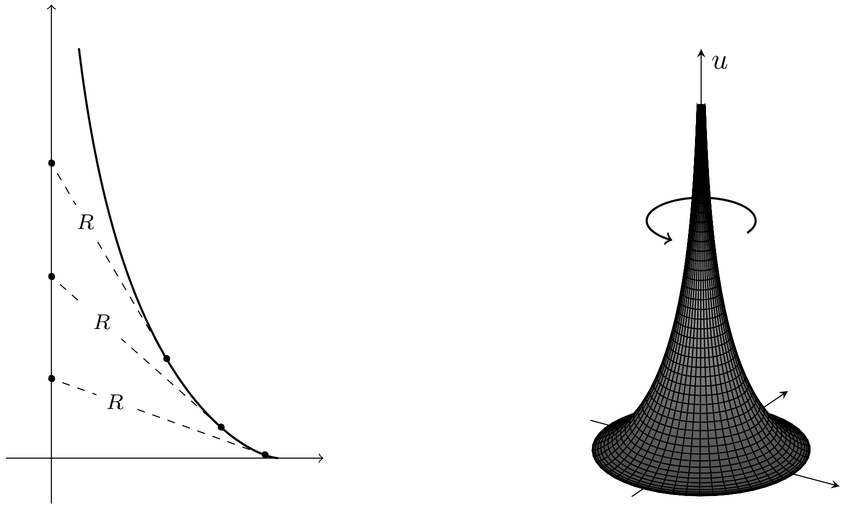
Pseudospherical surfaces are surfaces of constant negative curvature, in that sense exhibiting a certain duality to a sphere which is a surface of constant positive curvature. The most notable example is the eponymous *pseudosphere*; which is the surface of revolution of a tractrix — another name for the pseudosphere is a *tractricoid*. A tractrix is a curve defined by the property that the segment of the tangent from the point to the axis has constant length p. Figure 5-1a shows an example of such a curve. A parametric representation of the tractrix γ is given by

$$\gamma : u \mapsto (p\,\mathrm{sech}(u),\ p(u - \tanh(u))) \quad u \in \mathbb{R}^+. \tag{5-6}$$

A parameterization of the pseudosphere in terms of u and v is then easily obtained by introducing a second parameter v that indicates the angle of rotation around the vertical axis:

$$\boldsymbol{r}(u,v) = \begin{pmatrix} p\,\mathrm{sech}(u)\cos(v) \\ p\,\mathrm{sech}(u)\sin(v) \\ p(u - \tanh(u)) \end{pmatrix} \quad u \in \mathbb{R}^+,\ v \in [0, 2\pi). \tag{5-7}$$

A pseudosphere is visualized by fig. 5-1b; the mesh on the surface are isoparametric curves for u and v.



(a) The tractrix - the line segments defined on the tangent to the curve between the curve and the intersection with the vertical axis all have the same length R.

(b) The pseudosphere - the horizontal lines are maintain a constant value for v, while the vertical lines are for a constant u.

Figure 5-1

Curvature of the pseudosphere

As explained in section 5-1-1, the Gaussian curvature of a parametric surface may be calculated using the first and second fundamental form. It was already alluded that the pseudosphere has a constant negative Gaussian curvature, which will be formally shown in this section. Let the pseudosphere be parameterized as eq. (5-7). Then, by virtue of eq. (5-2), the coefficients of the first fundamental form E, F and G are

$$\begin{aligned} E &= p^4 \tanh^2(u) \\ F &= 0 \\ G &= p^4 \operatorname{sech}^2(u) \end{aligned}$$

From eq. (5-3), the unit normal vector is

$$\boldsymbol{n} = \begin{pmatrix} -\cos(v)\tanh^{-1}(u) \\ -\sin(v)\tanh(u) \\ -\cosh^{-1}(u) \end{pmatrix}$$

and, consequently, the coefficients of the second fundamental form are

$$\begin{aligned} e &= -p \operatorname{sech}(u)\tanh(u) \\ f &= 0 \\ g &= p \operatorname{sech}(u)\tanh(u) \end{aligned}$$

using eq. (5-4). The Gaussian curvature can then be computed to be [35]

$$k = \frac{\det(\mathbf{II})}{\det(\mathbf{I})} = \frac{eg - f^2}{EG - F^2} = \frac{-p^2 \operatorname{sech}^2(u)\tanh^2(u)}{p^4 \operatorname{sech}^2(u)\tanh^2(u)} = -\frac{1}{p^2}. \tag{5-8}$$

As such, the pseudosphere is shown to have a constant negative curvature everywhere but on the rim (where differentiability is lost).

Because the pseudosphere has constant negative curvature, it shares many properties of the hyperbolic plane. For example, the angular excess equation eq. (5-1) holds on the pseudosphere [2]. But, whereas a sphere can serve as a 'globe' for spherical geometry, the pseudosphere cannot be used as a representative for the entire hyperbolic plane. First of all, they are not homeomorphic (the pseudosphere is homeomorphic to a cylinder, which has a different fundamental group than the hyperbolic plane[5]). Secondly, the pseudosphere has a 'rim' at the bottom which prevents any line segment from extending further downwards — naturally, no such thing exists in the hyperbolic plane. As such, the pseudosphere can only serve as a model for finite regions of the hyperbolic plane, but not in its entirety. In technical terms, this means that the pseudosphere is *locally isometric* to the hyperbolic plane [39]. This is why one must resort to models of the hyperbolic plane, they will be discussed in the next section.

5-2 Models of the hyperbolic plane

The pseudosphere cannot serve as a model for the hyperbolic plane. In fact, a theorem attributed to David Hilbert shows that, *in Euclidean three-space, there can be no complete*

[5] Intuitively, one can see this as the number of different ways it is possible to draw loops on the surface which are distinct up to a homotopy. On the pseudosphere and the cylinder, loops can 'wrap' around the vertical axis any integer number of times, while on the (hyperbolic) plane all loops are homotopic [37]. This concept is defined more rigorously in terms of the *fundamental group* π_1, a heavily used invariant of topological spaces [38].

smooth surface with the intrinsic geometry of the pseudosphere [34]. This is the reason why there can only be models of the hyperbolic plane. One could say that this means 'double trouble' for make-believe cartographers of the hyperbolic plane. Of course, there is the traditional problem that normal cartographers also face, in that the Earth can never be completely mapped on a flat surface, which is why either (i) a suitable projection method must be used or (ii) they must resort to an additional dimension an make a globe to obtain a completely faithful representation. Hilbert's theorem implies that even the additional dimension will be to no avail for the hyperbolic plane, as illustrated by the pseudosphere.

Several models for the hyperbolic plane have been devised in the past, and they will provide a lot more mileage than the pseudosphere alone. The most popular models will be discussed in the following sections: there is a natural way to map the pseudosphere and the Poincaré half-plane, which will be discussed first. Then, via the so-called *Cayley transformation*, the half-plane can be mapped to the Poincaré disk, perhaps the most illustrous model of them all. Using the disk model, one can naturally arive at the Cayley-Klein disk and the hyperboloid model — the latter will be the starting point in the financial analogy.

5-2-1 Poincaré half-plane

It has been pointed out that there are essentially two incompatibility problems with the pseudosphere that prevent devise a mapping between it and the complete hyperbolic plane (also, Hilbert's theorem immediately shatters any aspiration to find one): first, it is homeomorphic to the cylinder and secondly, it has an edge. This section will describe a conformal mapping between the surface of the pseudosphere and a 'half-plane', which can serve as a global model for the hyperbolic plane. However, the theorem by Hilbert already hints at the fact that this plane will have some 'weird' properties, more specifically, a different notion of distance.

As described by Needham [2], one can imagine the pseudosphere to be cut open along any tractrix; the resulting surface can then be 'unfolded' in the horizontal direction as if it were a treasure map on a table. The edges that were cut can be extended towards infinity by recognizing the periodicity that exists naturally on the pseudosphere itself; a particle traveling horizontally would wrap once around the pseudosphere every distance of 2π traveled. Consequently, the circles on the pseudosphere that arise for constant values of z will be mapped to horizontal lines in the half-plane. Formally, this simply means that the horizontal axis of the half-plane $\tilde{x}$ is equal to the angle u. It has already been mentioned that the mapping between the pseudosphere and the half-plane is *conformal*, i.e. it locally preserves angles. Therefore, since the tractrix lines are everywhere perpendicular on the pseudosphere to the circles for constant z, they must map to vertical lines in order to maintain this orthogonality. A horizontal movement in the half-plane $\mathrm{d}\tilde{x}$ therefore corresponds on the pseudosphere with a traveled distance of $\mathrm{d}\tilde{s} = \operatorname{sech}(v) = \operatorname{sech}(v)\,\mathrm{d}v$, because of the radius of the circle at that particular height. Because the mapping is conformal, a movement along a tractrix $\mathrm{d}\sigma$ must be scaled by the same factor:

$$\mathrm{d}\sigma = \operatorname{sech}(u)\,\mathrm{d}y\,.$$

Subsequently, the movement along the tractrix can be written in terms of u and v using the parameterization:

$$\mathrm{d}\sigma = \left\| \frac{\partial \boldsymbol{r}}{\partial u}\,\mathrm{d}u \right\| = \sqrt{\mathrm{d}u^2 - \operatorname{sech}^2(u)\,\mathrm{d}u^2}$$

Combining this with the previous expression found for $\mathrm{d}\sigma$, one arrives at

$$\mathrm{d}y = \sinh(u)\,\mathrm{d}u \implies y = \cosh(u) + C. \tag{5-9}$$

Thus, the conformality of the mapping imposes a restriction on the mapping for y up to a constant C, which is usually taken to be 0 [2]. Using this information, the metric of the Poincaré half-plane can also easily be deduced by *pushing forward*[6] the Euclidean metric of the pseudosphere:

$$\mathrm{d}s = \left\| \frac{\partial \boldsymbol{r}}{\partial u}\,\mathrm{d}u + \frac{\partial \boldsymbol{r}}{\partial v}\,\mathrm{d}v \right\| = \operatorname{sech}(u)\sqrt{\sinh^2(u)\,\mathrm{d}u^2 + \mathrm{d}v^2} = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y} \tag{5-10}$$

This metric is called the *Poincaré metric.* Equation (5-9) already states that for $u = 0$, i.e. the rim of the pseudosphere, maps to the horizontal line $y = 1$. This suggests that the pseudosphere is covered by the region of the half-plane for which $y \geq 1$.

The Poincaré metric suggests that distances get larger and larger when travelling downwards in y-direction. At the line $y = 0$ they even become infinitely large. For someone living in the half-plane, this line would never be reachable, because they would have to travel for an infinite amount of time. It is not part of the half-plane itself, which is why it is called the *horizon*; points on the horizon are named *ideal points* [2].

Geodesics in the Poincaré half-plane connecting two points are either straight vertical lines if the two points have the same x-coordinate. Otherwise, the geodesic is the arc defined by the portion of the circle going through each of the two points that also intersects the x-axis at a right angle [40]. Apart from the horizon, two other peculiar curves exist in the half-plane that will also show up in later models:

- *horocycles*: circles tangent to the x-axis or horizontal lines,
- and *hypercycles*: circular arcs that intersect the x-axis at non-right angles or straight lines that intersect the x-axis at a non-right angle.

5-2-2 Poincaré disk

The Poincaré disk arises naturally from the Poincaré half-plane by virtue of the Cayley transform. When the Poincaré half-plane is viewed 'on top' of the complex plane, the Cayley transform is defined as

$$D(z) = \frac{\mathrm{i}z + 1}{z + \mathrm{i}} \quad z \in \mathbb{C}. \tag{5-11}$$

which maps the entire half-plane to the unit disk. Furthermore,

[6] It must be clearly noted that the term *pushforward metric* is not at all in place, and its usage here would in fact be quite pathological. The reason is as follows: the relation sketched here is not even strictly a mapping, as any point on the pseudosphere maps to an infinite amount of points on the half-plane. Beyond this problem of the unique images there is also the obvious fact that their is no preimage for the part of the half-plane where $y < 1$ (non-surjective). As such, it would definitely make more sense to define the Poincaré metric first and then define the *pullback metric* on the pseudosphere. However, since the pseudosphere exists in the familiar Euclidean space, it is more natural to introduce the pseudosphere first [38].

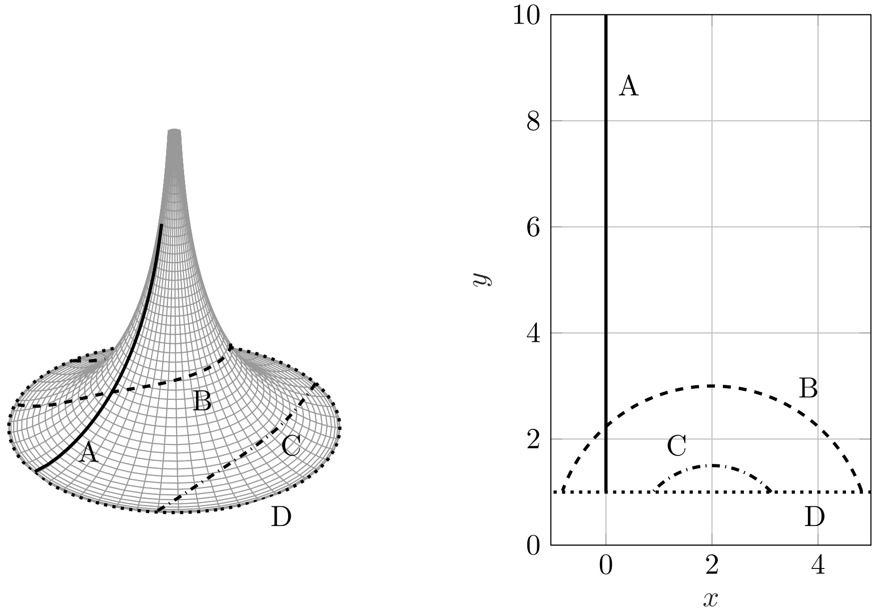


Figure 5-2: Comparison of various trajectories on the pseudosphere (left) and the Poincaré half-plane (right). Lines A, B and C are geodesics. The endpoints of B are closer together because it covers a wider range on the x-axis — if it would be larger than 2π, the curve would show an entire encirclement of the pseudosphere. Line D corresponds to the rim of the pseudosphere and the line $y = 1$ in the pseudosphere.

- the horizon coincides with the rim of the disk,
- the points ±1 are invariant,
- i maps to the origin,
- the origin corresponds to −i

As will become clear in chapter 6, eq. (5-11) belongs to the class of Möbius transforms, which means that it also must be conformal. Therefore, angles on the pseudosphere, half-plane and disk will all look alike. Since the horizon containing the ideal points is now the rim of the unit disk, it makes sense that the pullback metric based on eq. (5-10) will have the y replaced by the distance from the rim of the unit disk, which is $1 - x^2 - y^2$. Indeed, the half-plane metric can be written as in terms of the complex variable z, using the mapping eq. (5-11) the metric in the disk turns out to be [2]

$$\mathrm{d}s = \frac{|\mathrm{d}z|}{\mathfrak{Im}(z)} = \frac{2|\mathrm{d}\tilde{z}|}{1 - |z|^2} \qquad \text{with} \quad \tilde{z} = D(z).$$

In $x - y$ coordinates, this metric is

$$\mathrm{d}s = \frac{2\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{1 - x^2 - y^2}. \tag{5-12}$$

It has been pointed out that the Cayley transform is a member of a larger class called Möbius transforms, which will be elaborated upon in chapter 6. Apart from being conformal, Möbius transforms also preserve circles (i.e. the result of a a Möbius transform applied to a circle will again be a circle). The geodesics in the half-plane have the either the shape of straight lines extending from the horizontal axis, or circles that cross the horizontal axis at a right angle. Consequently, it is straightforward to deduce that the geodesics in the Poincaré disk are also circles that 'start' orthogonally to the rim of the unit disk. Additionally, all diameters of the Poincaré disk are geodesics as well: they can be interpreted as circles with infinite radius that are obviously also orthogonal to the rim of the disk.

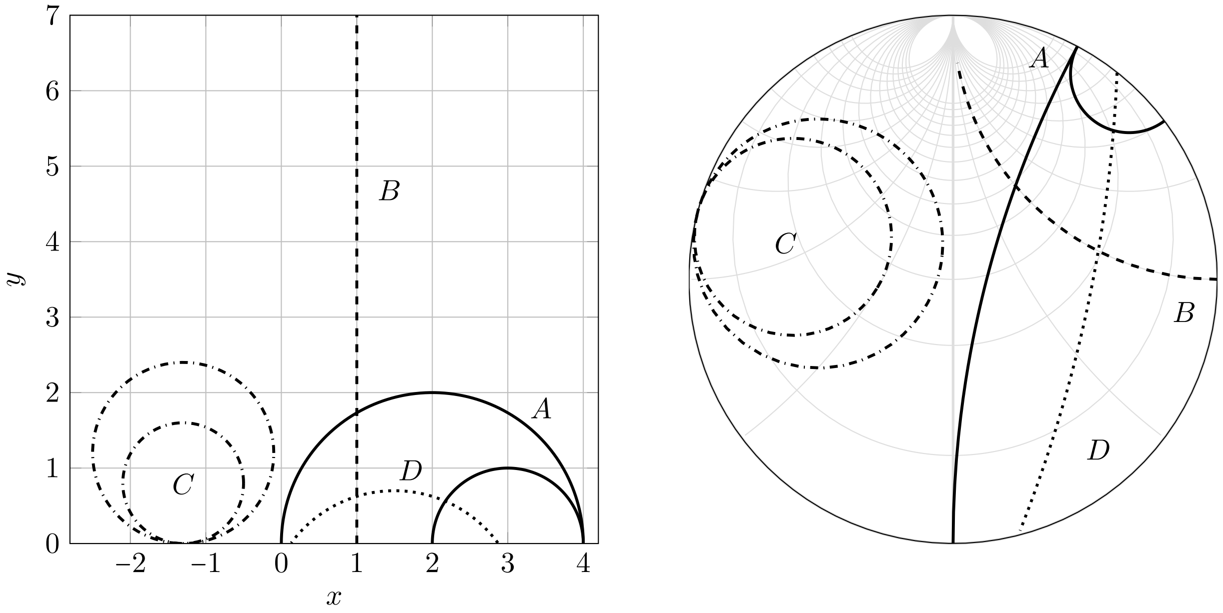


Figure 5-3: Comparison between trajectories in the Poincaré half-plane (left) and the Poincaré disk (right). Several types of trajectories are shown: the solid lines A are 'typical' geodesics, i.e. circles with finite radius in the half-plane. The dotted line B is also a geodesic but never reaches the horizon at its endpoint (which would take an infinite distance), this is also clearly visible in the disk. Clearly, the origin in the half-plane maps to $-\mathrm{i}$. The dashdotted lines C are horocycles; they preserve their shape under the action of the Cayley transform. Dotted line D is a hypercycle; a circle that crosses the horizon at an oblique or acute angle.

5-2-3 Hyperboloid model

Hilbert's theorem about the embedding of a hyperbolic surface in Euclidean space eradicates any hope to find a 'globe' for the hyperbolic space, like a regular sphere would be for an elliptic space. However, as strong as it may seem, Hilbert's theorem still leaves some room for other possibilities because it is very explicit about the nature of the space in question: it must be Euclidean. Chapter 3 already introduced a notorious alternative: the Lorentz space. It will probably not come as a surprise that the Lorentz space does allow for the embedding of a hyperbolic surface, which is known as the *hyperboloid model*.

Now, why would one find any solace in the Lorentz plane specifically? Recall that the curvature of a surface can be expressed in terms of its local radius, i.e. $1/p^2$. A negative curvature therefore implies that p is somehow not a real but imaginary number. Obviously, this can never make any sense in an Euclidean space, but the Lorentzian space perfectly allows this kind of odd witchcraft, as illustrated by fig. 3-6 [25]. The complete hyperboloid consists of two sheets, a positive sheet and a negative sheet, implicitly described by the following set:

$$H^2 = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x}\|_L = p \right\}.$$

However, on account of the positive and the negative sheet, this set is clearly *disconnected*[7]. It is therefore common to discard the negative sheet of H^2, the hyperboloid model of the hyperbolic 2-space is then [25]

$$H^2_+ = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x}\|_L = p,\ x_1 > 0 \right\}.$$

A parametric representation of H^2_+ is the following

$$\boldsymbol{r} = \begin{pmatrix} p\cosh(u) \\ p\cos(v)\sinh(u) \\ p\sin(v)\sinh(u) \end{pmatrix}. \tag{5-13}$$

To show that this surface has indeed a constant negative Gaussian curvature, a slightly more general approach must be followed than for the pseudosphere, since the calculations with the second fundamental form are restricted to a Euclidean space. As described by O'Neill [41]

$$E = \partial_u \diamond \partial_v \quad F = \partial_u \diamond \partial_v \quad G = \partial_u \diamond \partial_v$$

which correspond to the components of the first fundamental form, generalized to the Lorentz product. The coordinate system u, v is called orthogonal if F vanishes, which is indeed the case for the parametrization given by eq. (5-13). The Gaussian curvature of the surface is then given by [41]

$$k = \frac{-1}{eg}\left[\varepsilon_1 \left(\frac{g_u}{e} \right)_u + \varepsilon_2 \left(\frac{e_v}{g} \right)_v \right] \tag{5-14}$$

with $\varepsilon_1 = \operatorname{sgn} E$, $\varepsilon_2 = \operatorname{sgn} G$, $e = \sqrt{|E|}$ and $G = \sqrt{|G|}$. Performing these calculations on eq. (5-13) yields indeed that $k = -1/p^2$, confirming that the one-sheet hyperboloid embedded in three-dimensional *Lorentz* space indeed has the same curvature as the pseudosphere[8].

Relation with Poincaré disk

(... `TODO` ...)

H^2 H^2_+ $\mathbb{R}$ $\mathbb{C}$

[7] A space is called connected if it is impossible to find two open sets whose union is equal to that space; which in this case clearly is not the case (the open sets are the positive and negative sheet of the hyperboloid).

[8] Some authors even refer to the hyperboloid as 'pseudosphere' as well, e.g. Balazs and Voros [42]. However, this is deemed confusing in the context of the larger discussion presented here, the author refrained from using it here.

Chapter 6

Möbius transforms

6-1 Definition and basic properties

Many of the transforms and mappings discussed in the preceding chapters may be expressed as a Möbius transformation. These are mappings of the form [2]

$$\mathfrak{M}(z) = \frac{az + b}{cz + d} \tag{6-1}$$

where $a, b, c, d \in \mathbb{C}$ are constants. Möbius transformations carry a deep connection with hyperbolic geometry (and non-Euclidean in general) and Einstein's theory of relativity. As such, the connection with the financial interpretation of the Lorentz plane can be readily made. As stated by Needham [2], the complex mappings that correspond to a Lorentz transformation are Möbius transformations and vice versa — every Möbius transform corresponds to a unique Lorentz transformation.

The Möbius transformation $\mathfrak{M}$ is called *singular* if $ad - bc = 0$, which maps every point to the same image a/c. In general, any Möbius transformation can be decomposed into four elementary transformations:

1. A translation $z \mapsto z + \frac{d}{c}$
2. A *complex inversion* $z \mapsto \frac{1}{z}$
3. An expansion and rotation $z \mapsto -\frac{ad-bc}{c^2}z$
4. A translation $z + \frac{a}{c}$

Each of these transformations is conformal and preserves circles which is why general Möbius transformations inherit these vital properties as well.

It is quite clear from eq. (6-1) that multiplication of both the denominator and the numerator by the same constant k will not affect the result of the mapping. Therefore, a Möbius transformation is uniquely determined by only three quantities $a/b, b/c$ and c/d. This ambiguity allows for the notion of *normalized transformations*, for which $ad - bc = 1$.

If $\mathfrak{M}$ is nonsingular, the transformation is bijective; the inverse transformation is then given by [2]

$$\mathfrak{M}^{-1}(z) = \frac{dz - b}{-cz + a}$$

6-2 Group structure

It is neither surprising nor hard to show that the nonsingular Möbius transformations form a group under composition; the identity mapping is a Möbius transformation, the inverse transformation is also a Möbius transformation as illustrated by the expression for $\mathfrak{M}^{-1}$ stated above and that the composition of two transformations $\mathfrak{M}_2 \circ \mathfrak{M}_1$ again yields a member of the class of Möbius transformations [2].

6-2-1 The Riemann sphere

The Möbius group Möb is the automorphism group of the Riemann sphere. Being the simplest compact Riemann surface, the Riemann sphere is a special representation of the extended complex plane[1] $\hat{\mathbb{C}}$ in three-dimensional space: it can be visualized by placing the complex plane horizontally and considering a unit sphere centered around the origin, such that its intersection coincides with the unit disk of the complex plane. To map the plane to the sphere, a stereographic projection is used from the 'north pole' of the sphere. As such, anything inside the unit disk is mapped to the southern hemisphere, everything on the unit disk is mapped onto itself (since it lies at the intersection) and everything outside the unit disk lies on the northern hemisphere. The north pole of the Riemann sphere then coincides with the distinctive feature of the extended complex plane, namely the point at infinity ∞ [2]. Stated more formally:

$$\text{Möb} = \text{Aut}(\hat{\mathbb{C}}) \quad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

In order to turn towards the computational aspect of the Riemann sphere, one must consider a different coordinate system for the complex plane; namely the *homogeneous* or *projective* coordinates. These consist of an ordered pair of complex numbers[2] , i.e. an element of $\mathbb{C}^2/\{[0,0]\}$, denoted as $[\mathfrak{z}_1, \mathfrak{z}_2]$. These two complex numbers are subject to an equivalence relation that makes them projective coordinates, in that

$$[\mathfrak{z}_1, \mathfrak{z}_2] \sim [\mathfrak{w}_1, \mathfrak{w}_2] \iff \mathfrak{w}_1 = \lambda \mathfrak{z}_1 \text{ and } \mathfrak{w}_2 = \lambda \mathfrak{z}_2$$

with λ any nonzero complex number. The projective space is then formed by the quotient set of the complex 2-space (excluding the origin) and this equivalence relation. Every equivalence class can be uniquely identified with a point $[1, \mathfrak{z}_2/\mathfrak{z}_1]$, which then corresponds to a point on the Riemann sphere (or equivalently on the extended complex plane). The only point where this mapping breaks down is the point $[0,1]$, which can be associated with the north pole on the Riemann sphere, or the infinity point. In technical terms, this is called a one-point

[1] The complex plane combined with a value for infinity '∞'.

[2] In physics, these objects are also called *2-spinors* which are also fundamentally connected with the theory of relativity, as demonstrated by [43].

compactification of a plane into a sphere, with the desirable property that the infinity point *has no special meaning on the sphere*, which it inevitably has in a plane. The Riemann sphere is therefore equal to the one-dimensional complex projective space $\mathbb{CP}^1$ [34].

6-2-2 Matrix representation

A very powerful property of Möbius transforms is that they can each be associated to a 2×2 complex matrix like so:

$$\frac{az+b}{cz+d} \quad \leftrightsquigarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Because the transformations are only defined up to the multiplication of a constant, so is the associated matrix. However, one can assume to have the transformation normalized, i.e. $ad - bc = 1$ which is equivalent to restricting the matrix to have a unit determinant. When normalized, there are precisely two matrices corresponding to every Möbius transform, since multiplying the entire matrix with -1 would still yield the same result. Now, how can one actually effect a Möbius transformation using this matrix representation? This is where the projective coordinates come into play. As it turns out,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{pmatrix} = \begin{pmatrix} a\mathfrak{z}_1 + b\mathfrak{z}_2 \\ c\mathfrak{z}_1 + d\mathfrak{z}_2 \end{pmatrix} = \begin{pmatrix} a(\mathfrak{z}_1/\mathfrak{z}_2) + b \\ c(\mathfrak{z}_1/\mathfrak{z}_2) + d \end{pmatrix}$$

which are exactly the homogeneous coordinates of the image of z under this Möbius transform. This allows calculating the result of any Möbius transformation also by means of matrix multiplication.

The correspondence with matrices goes even deeper than this simple computational trick: the composition of two Möbius transforms can simply be obtained by multiplying their corresponding matrices (denoted by M_1 and M_2), that is

$$\mathfrak{M}_1 \circ \mathfrak{M}_2 \to M_1 M_2,$$

and furthermore, the inverse Möbius transformation $\mathfrak{M}^{-1}$ corresponds to the inverse of the matrix M^{-1} [3].

6-2-3 The Möbius group

The link with the matrices already hints at a very important fact about Möbius transforms: they form a group under composition — the Möbius group Möb. This fact is actually immediately clear from the correspondence between matrix multiplication and composition of the transformation: a composition of two transforms is again a transform. Additionally, matrix multiplication is associative and they are assumed to be nonsingular, so inverses always exist. As such, all the group axioms are satisfied (of course, all these results could have been obtained directly from the definition of the Möbius transform, but the reasoning based on the matrices is particularly straightforward). Perhaps not very surprising is that the identity Möbius transformation $E(z)$ corresponds to the 2×2 identity matrix.

Based on the matrix analogy, one can state that Möb is the group of linear[3] transformations on vector space $\mathbb{C}^2$ — as projective coordinates. This is the same as saying that the Möbius group is isomorphic to the group of linear transformations *modulo* the nonzero scaling operation on $\mathbb{C}^2$: the resulting quotient group is the *projective linear group* $\mathrm{PGL}(2,\mathbb{C})$.

An interesting fact from group theory arises here as well: it has already been established that Möbius transforms are only unique up to multiplication by a scalar — as such, they can be normalized (with unit determinant) without loss of generalization. This suggests that Möb is really the action of the *special* linear group modulo scalar multiplication, resulting in the *projective special linear group.* Luckily, this fact is also reconciled within group theory: the groups $\mathrm{PGL}(n,\mathbb{F})$ and $\mathrm{PSL}(n,\mathbb{F})$ over the field $\mathbb{F}$ are isomorphic as long as every element of $\mathbb{F}$ has an nth root within $\mathbb{F}$. The fact that it is true for $\mathbb{F}=\mathbb{C}$ is probably the most fundamental property of the complex numbers. To summarize, the Möbius group Möb is equal to the projective linear group and the special projective group (over the field of complex numbers), which are isomorphic to each other in this particular case.

6-3 Classification of Möbius transforms

In the preceding discussion about the matrix representation of Möbius transforms, one important aspect has not yet been addressed: what about the eigenvalues and eigenvectors of M? Eigenvectors are vectors that remain invariant (up to scaling) under the multiplication of a particular matrix. Of course, one should bear in mind that the vector, in this case, contains projective coordinates, so that even when scaled, its coordinate representation remains identical. As such, the eigenvectors of the matrix M are the *fixed points* of the Möbius transform $\mathfrak{M}$; any Möbius transform has two at most. This is even more perspicuous by solving the equation $z_0 = \mathfrak{M}(z^*)$, which has solutions

$$z_0 = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

When converted to projective coordinates, the two solutions for z^* then coincide with the eigenvectors of M. In the degenerate case for which $a+d=\pm 2$, the argument of the square root amounts to zero, which means that there is only one unique fixed point. These transforms are called *parabolic*, more will become clear about them later [2].

Now, it remains to analyse the significance of the eigenvalues. Of course, since eigenvalues are sensitive to scaling of the matrix, it is important to stress again that M must be normalized (have a unit determinant) in order for the following to hold. A well known fact in linear algebra states that if λ_1, λ_2 are eigenvalues of M, then

$$\operatorname{tr} M = \lambda_1 + \lambda_2 \quad \text{and} \quad \det M = \lambda_1 \lambda_2.$$

Because the matrix is normalized, these two results can be combined into:

$$\lambda + \frac{1}{\lambda} = a + d = \operatorname{tr} M. \tag{6-2}$$

[3] It is misleading to call the Möbius transforms 'linear' in general — they are definitely nonlinear in the complex plane! However, when using the homogeneous coordinates, they become linear transforms.

It can be shown that every non-parabolic Möbius transform is conjugate[4] to a Möbius transform that has fixed points 0 and ∞, and is therefore of the form $\mathfrak{J}(z) = kz$, where k is called the *multiplier* of this Möbius transform, and consequently all the transforms that are conjugate to it. The matrix J that coincides with this transform necessarily must have the form (the letter J is used to denote this transform because it is equal to the Jordan form of the transformation matrix M) [2]

$$J = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix},$$

because then of course $\mathfrak{J}(z) = \frac{\sqrt{k}z}{1/\sqrt{k}} = kz$. Because conjugacy translates to a similarity transform in the matrix analogy, it leaves the eigenvalues of the matrix unaffected. But, for J is a diagonal matrix, its eigenvalues are exactly on the main diagonal. As a result, *the multiplier of a Möbius transform is equal to the square of its eigenvalue*, or $k = \lambda^2$. Strictly speaking, every Möbius transform has of course two eigenvalues and two multipliers, but since they are both each other reciprocal, they do not have to be considered separately. With this result, eq. (6-2) can then also be restated in terms of the multiplier k instead of the eigenvalues:

$$\sqrt{k} + \frac{1}{\sqrt{k}} = a + d = \operatorname{tr} M.$$

Solving eq. (6-2), one obtains

$$\lambda^2 - (a+d)\lambda + 1 = 0$$

which is a quadratic equation with discriminant $\Delta = (a+d)^2 - 4$: from the sign of Δ one can then distinguish three possible cases:

1. $\Delta < 0$ or $(a+d)^2 < 4$: there are two complex solutions for λ. It is easy to show that the solution will then be equal to

   $$\frac{a+d}{2} \pm \frac{\mathrm{i}}{2}\sqrt{4 - (a+d)^2}.$$

   By inspection of this expression, it comes natural to make a further categorization:

   (a) if $(a+d)^2 \in [0,4)$, the argument of the square root is positive: consequently, the solutions for λ are both located on the unit circle (evidently, the unit circle as a whole is invariant under complex inversion). Any number on the unit circle can, by virtue of Euler's formula, be written as $\lambda = \mathrm{e}^{\frac{\mathrm{i}\theta}{2}} = \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\sin\left(\frac{\theta}{2}\right) \neq 1$, such that the multiplier $k = \lambda^2 = \mathrm{e}^{\theta \mathrm{i}}$ — the factor of one half is only there to identify the multiplier with the actual angle θ, which is the most meaningful from a geometric standpoint. The associated matrix archetype or Jordan form for transformations of this type is then

   $$J = \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\theta}{2}} & 0 \\ 0 & \mathrm{e}^{\frac{-\mathrm{i}\theta}{2}} \end{pmatrix} = \exp\left(\begin{pmatrix} \frac{\mathrm{i}\theta}{2} & 0 \\ 0 & -\frac{\mathrm{i}\theta}{2} \end{pmatrix}\right) \quad \theta \in \mathbb{R} \setminus \{k \in \mathbb{Z} \mid 2k\pi\}.$$

[4] Two group elements a and b are called *conjugate* if there exists another group element g such that $b = g^{-1}ag$. This is analogous to similarity transforms (and therefore the notion of similar matrices) in linear algebra.

It is more instructive to look at the *real Jordan form* of this complex diagonal matrix,

$$J = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

which is a rotation matrix[5]. Therefore, *matrices associated with elliptic transforms are rotation matrices.* These transforms are called *elliptic.* The edge case for which $(a+d)^2 = 0$ with a multiplier is equal to -1, is denoted as a *circular* transform (which is still an elliptic transform).

(b) Conversely, if $(a+d)^2 < 0$, the solutions will generally be complex (and not conjugate). These transforms are part of a larger class called *loxodromic.* As already stated, the loxodromic transforms also include the hyperbolic ones. Needham [2] reckons the elliptic transforms among the loxodromic transforms as well, but this is not general. In any case, the term 'loxodromic' usually refers as a pars pro toto to the non-hyperbolic transforms in particular to make the distinction.

2. $\Delta = 0$ or $(a+d)^2 = 4$: there is one solution for λ, either $1^{(2)}$ or $-1^{(2)}$, corresponding to a trace of -2 and 2 respectively (the superscript between parentheses indicates the algebraic multiplicity of the eigenvalues), because a normalized Möbius transform is only unique up to a sign. The multiplier for both cases is the same though, as $k = 1$. Möbius transformations of this kind are called *parabolic.* Because they have only one eigenvalue, there will also be one fixed point: the infinity point. The parabolic transforms give rise to translations in the complex plane of the form $z \mapsto z + b$ with matrix representation

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

which is a *unipotent matrix*[6]; these matrices form an abelian subgroup on their own (translations in the plane do indeed commute). The Jordan form of this matrix is then

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which corresponds to the degenerate case where the geometric multiplicity of the eigenvalue is lower than its algebraic multiplicity.

3. $\Delta > 0$ or $(a+d)^2 > 4$: there are two real solutions for λ, given by

$$\lambda = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4}.$$

The resulting solutions for λ are then always real and positive; they can then be expressed as the image of the exponential function: $\lambda = \mathrm{e}^{\frac{\zeta}{2}}$ such that $k = \mathrm{e}^{\zeta}$. The usage of

[5] One should be mindful that the conversion to the real Jordan form is only possible when the eigenvalues of the matrix are complex conjugate. In general, M is complex, which means that this is not necessarily the case. However, on the unit circle, a complex inversion results in a reflection along the real axis, which means that the eigenvalues are in the elliptic case indeed complex conjugate.

[6] In general, a unipotent (ring) element is an element that, when the unit element is subtracted from it, yields a nilpotent element. For matrices, this means that a matrix A is unipotent if $A - I$ is nilpotent, so $(A - I)^n = 0$ for some integer n.

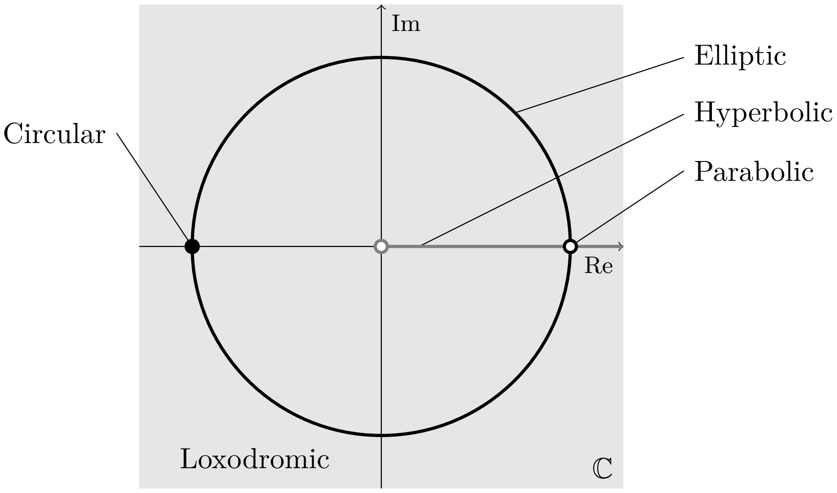


Figure 6-1: Classification of Möbius transform in terms of the location of the multiplier k in the complex plane. Any point that is not on the unit circle yields a loxodromic transform; a particular subclass are the hyperbolic transforms, which are on the real axis except at -1 and 1 where it intersects with the unit circle, and at the origin, where the transform becomes singular. If the multiplier lies on the unit circle except 1 are elliptic transforms, a special case is the circular transform for $k = -1$. Finally, the parabolic transforms have a multiplier of 1 [2].

ζ is not at all coincidental: indeed, the argument here represents a *hyperbolic angle*, as discussed in chapter 3. The corresponding Jordan form is a *squeeze mapping*:

$$J = \begin{pmatrix} e^{\frac{\zeta}{2}} & 0 \\ 0 & e^{\frac{-\zeta}{2}} \end{pmatrix} \quad \zeta \in \mathbb{R} \setminus \{0\}.$$

Similarly to the elliptic case, this matrix can also be expressed in terms of hyperbolic functions, completing the analogy that was already established in chapter 3:

$$M = \begin{pmatrix} \cosh(\zeta/2) & -\sinh(\zeta/2) \\ \sinh(\zeta/2) & \cosh(\zeta/2) \end{pmatrix}.$$

As such, these transforms are akin to hyperbolic rotations, which is why they are also classified as *hyperbolic*. The hyperbolic transforms are also part of the class of loxodromic transforms, together with the aforementioned class where λ is complex. They do however deserve their own subclass because, apart from frequently encountered, they also represent a particular part of the special linear group over the reals $\mathrm{SL}(n, \mathbb{R})$, as will be discussed later.

To summarize, there are five different classes of Möbius transform: circular, elliptic, hyperbolic, loxodromic and parabolic. Circular transforms are part of the elliptic transforms and hyperbolic transforms are a subclass of loxodromic transforms. The class to which a Möbius transform belongs is determined completely by its trace $a + d$, or equivalently, the value of the multiplier k. For the multiplier, one can distinguish several 'regions' in the complex plane that are each associated with a class of Möbius transforms, this is visualized in fig. 6-1. The nature of the Jordan form of the matrix associated to a Möbius transform also clearly give away to which class a transform belongs.

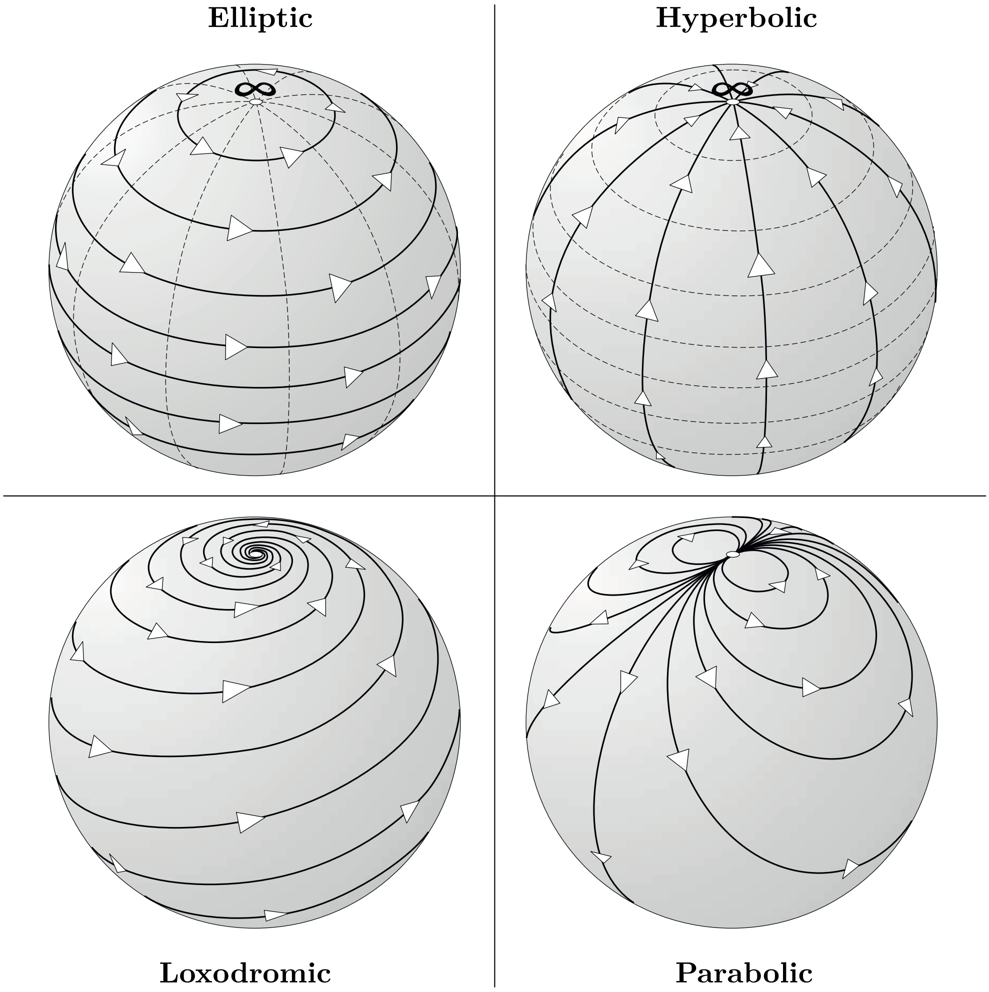


Figure 6-2: Overview of the four classes of Möbius transform and their typical action on the Riemann sphere. The curves shown on the Riemann sphere are the *invariant curves* of the transformation, i.e. these curves as a whole remain invariant under the transformation. Clearly, the elliptic, hyperbolic and loxodromic transformations have the North and South pole, or ∞ and 0 as fixed points, whereas the parabolic transformation only has a single fixed point at the North pole. The loxodromic transformations borrow their name from loxodromes, which are spiral-like trajectories on the Earth with constant bearing — a ship that taking a loxodromic path would maintain a constant angle with respect to true North. Illustration reprinted from Needham [3, p. 78].

Table 6-1: Overview of the five classes of Möbius transforms and the corresponding values for the trace squared of the matrix ($\mathrm{tr}\, M = a + d$), the multiplier of the transform and the Jordan form.

Class	Multiplier	$(a+d)^2$	Jordan form	Parent class
Circular	$\{-1\}$	$[0, 4)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	Elliptic
Elliptic	$\{k \in \mathbb{C} \mid \lvert k\rvert = 1, k \neq 1\}$	$[0, 4)$	$\begin{pmatrix}\mathrm{e}^{\theta\mathrm{i}/2} & 0\\ 0 & \mathrm{e}^{-\theta\mathrm{i}/2}\end{pmatrix}$	—
Parabolic	$\{1\}$	$\{4\}$	$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$	—
Hyperbolic	$\mathbb{R}_0^+ \setminus \{1\}$	$(4, +\infty)$	$\begin{pmatrix}\mathrm{e}^{\zeta/2} & 0\\ 0 & \mathrm{e}^{-\zeta/2}\end{pmatrix}$	Loxodromic
Loxodromic	$\{k \in \mathbb{C} \mid \lvert k\rvert \neq 1\}$	$\mathbb{C} \setminus [0, 4]$	$\begin{pmatrix}k & 0\\ 0 & k^{-1}\end{pmatrix}$	—

Table 6-1 provides an overview of the five Möbius classes, together with the values for the matrix trace, the multiplier and the Jordan form. Finally, fig. 6-2 visualizes the effect of each of the transform classes on points on the Riemann sphere. Elliptic transforms push points along circles of constant latitude, while hyperbolic transforms move points orthogonally, along the meridians of the Riemann sphere.

6-4 Subgroups

Apart from the classification considered in the preceding chapter, there are also several subgroups of the Möbius group that can be distinguished. It is compelling to observe that each of these subgroups can each be identified with one of the 'geometry types' that were considered in chapter 5: those of positive (spherical), zero (Euclidean) and negative (hyperbolic) geometry. For every geometry type, a specific subgroup of the Möbius group will play the role of the direct isometry group in a particular 'map' [3].

6-4-1 Euclidean geometry

The direct isometries in the Euclidean plane consist simply of translations and rotations. Clearly, a rotation in the complex plane (around the origin) can be represented simply by $z \mapsto e^{\mathrm{i}\theta}$, while a translation is $z \mapsto z + b$ with $b \in \mathbb{C}$. Hence, the entire direct isometry group of the Euclidean plane is given by

$$\mathfrak{E}(z) = e^{\mathrm{i}\theta} + b.$$

This group is also called the Euclidean group of dimension two.

6-4-2 Spherical geometry

One of the most prevalent applications of group theory is are the representation of rotations in three-dimensional Euclidean space. Of course, the group associated with these rotations is the special orthogonal group $\mathrm{SO}(3,\mathbb{R})$. This group is diffeomorphic to the three-dimensional real projective space $\mathbb{RP}^3$ (intuitively, this space consists of three 'directions'). The rotations in $\mathbb{R}^3$ can also be parameterized by *unit quaternions* (also called *versors*): these represent points on the 3-sphere $\mathbb{S}^3$. The difference between the 3-sphere and the real projective space is that the latter identifies the *antipodal* parts that are present on the sphere. As such, any rotation in $\mathbb{R}^3$ corresponds precisely to two points on the 3-sphere (or two unit quaternions) — the group of unit quaternions therefore is a double cover of $\mathrm{SO}(3,\mathbb{R})$.

Quaternions can also be represented as complex matrices: [44]

$$q = m + n\mathbf{i} + o\mathbf{j} + p\mathbf{k} \in \mathbb{H} \quad \leftrightsquigarrow \quad Q = \begin{pmatrix} m + \mathrm{i}p & -n - \mathrm{i}o \\ n - \mathrm{i}o & m - \mathrm{i}p \end{pmatrix},$$

which is a general represention of a 2 × 2 *unitary*[7] matrix. Likewise, the unit quaternions then translate to matrices of the above type with an additional restriction: they must have a determinant of 1: these matrices are members of the *special unitary group* $\mathrm{SU}(2)$. As such, the group of unit quaternions is isomorphic to the $\mathrm{SU}(2)$ which is therefore also a double cover of $\mathrm{SO}(3,\mathbb{R})$.

Since the members of $\mathrm{SU}(2)$ can be identified with a Möbius transform (inspection of the matrix above makes this immediately apparent), a specific subgroup of Möb can be used to represent the rotations in $\mathbb{R}^3$ — recall that every normalized Möbius transform also corresponds to two matrices, differing by a factor -1. Observing the matrix above, one can see that the entries on the main diagonal are each others' conjugate, while the entries on the antidiagonal are conjugate opposite. Therefore, the general expression of a rotation of the Riemann sphere as a Möbius transform can be written as: [3]

$$\mathfrak{S} = \frac{az + b}{-\bar{b}z + \bar{a}} \quad \text{where } |a|^2 + |b|^2 = 1;$$

the latter equivalent then enforces that $\det Q = 1$. There are always two quaternions representing the same rotation; they are antipodal and differ by a factor of -1. As a result, there are two possibilities for Q as well, again identical but with opposite entries. Recall that the same applies to the matrix representations of Möbius transforms: as such, the ambiguities are eliminated, and the group of transforms of type $\mathfrak{S}$ is isomorphic to $\mathrm{SO}(3,\mathbb{R})$. In the complex plane, these transformations represent the isometries of spherical geometry in the stereographic map [2].

6-4-3 Hyperbolic geometry

As described by Rovenski [45], the isometries of the Poincaré half-plane are any (composition) of the following types of transformations, i.e. they leave distances according to the Poincaré metric unaffected:

[7] A unitary matrix is a matrix whose inverse is its conjugate transpose — it is the complex counterpart of orthogonal matrices.

- horizontal translations: $(x, y) \mapsto (x + a, y)$ where $x, y, a \in real$;
- reflection around the vertical axis: $(x, y) \mapsto (-x, y)$;
- dilations centered around the origin: $(x, y) \mapsto (ax, ay)$;
- inversions with respect to the unit circle $(x, y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$.

The group of all these transformations is precisely $\mathrm{PSL}(2, \mathbb{R})$, or the Möbius tranforms with real parameters:

$$\mathfrak{H}(z) = \frac{az + b}{cz + d} \qquad a, b, c, d \in \mathbb{R} \quad ad - bc = 1;$$

these are the isometries of the Poincaré half-plane [3]. Recall from section 5-2-1 that there are two types of geodesics in the half-plane: semicircles centered at the origin and straight (from Euclidean perspective) vertical lines. These are precisely invariant curves for the transformations listed — isometries take geodesics to geodesics [33].

6-5 Relation with other groups

Another classic Lie group is the special linear group $\mathrm{SL}(n, \mathbb{F})$, which are the volume-preserving transformations on a vector space — this property turns out to be quite important. The special linear group over the complex numbers (2-dimensional) $\mathrm{SL}(2, \mathbb{C})$ can be represented as the group of all the complex 2×2 matrices with unit determinant. It has been mentioned previously that any such matrix coincides with a Möbius transform, albeit surjectively: for every Möbius transform, there are two such matrices. As such, $\mathrm{SL}(2, \mathbb{C})$ is a *double cover* of Möb.

Arguably more interesting than $\mathrm{SL}(2, \mathbb{C})$ is the special linear group over the reals $\mathrm{SL}(2, \mathbb{R})$, i.e. every invertible 2×2 matrix with real entries and a unit determinant; they form a subgroup of $\mathrm{SL}(2, \mathbb{C})$ as well. It has been mentioned before that for the normalized Möbius transforms, the eigenvalues should be complex inverses of each other. For matrices with real entries, the eigenvalues are either both real or complex conjugates of each other. When not on the real axis, the only way eigenvalues can be complex inverses and complex conjugate is to be located on the unit circle. As can be seen on fig. 6-1, that means that the matrices with real entries contain exactly the hyperbolic (both real), elliptic (unit circle), and parabolic classes. The remaining part consists of the loxodromic transforms which have a nonreal trace.

(... `TODO` ...)

’s

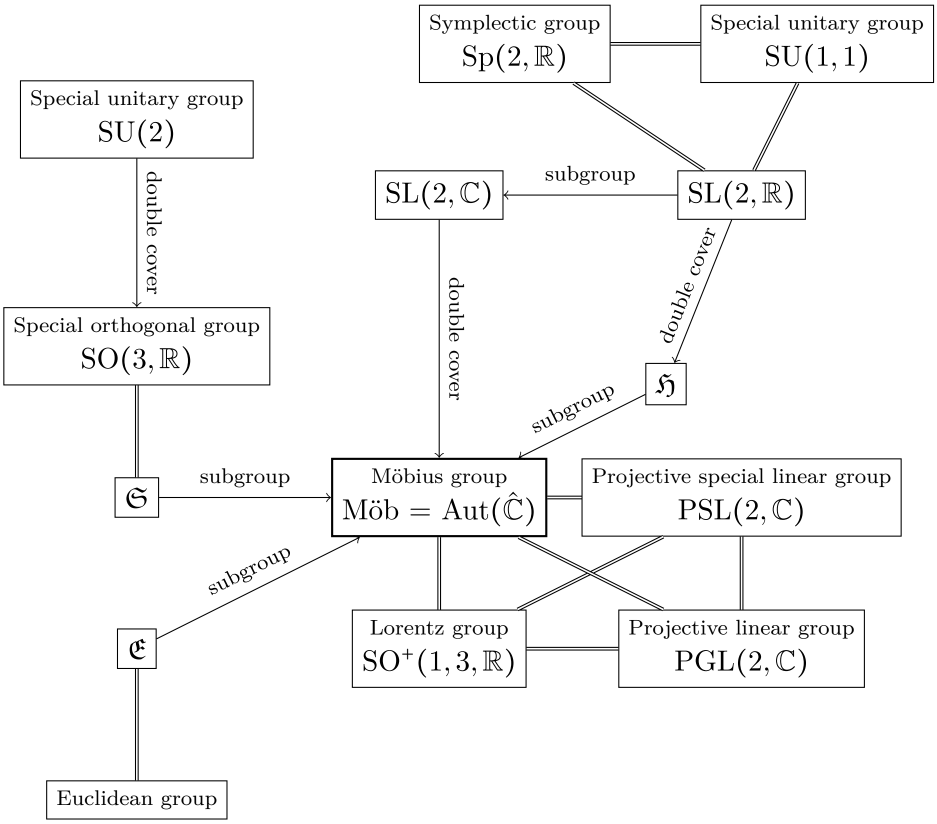


Figure 6-3: Schematic of the relation of the Möbius group with other important groups such as the special orthogonal group, special unitary group, Lorentz group etc.

Chapter 7

Research proposal

Chapter 8

Summary and conclusion

Bibliography

[1] B. N. M. Krabbenborg, "Duration Matching in the Frequency Domain - An Economic Engineering Approach to Asset and Liability Management," Literature Survey, Delft University of Technology, 2021.

[2] T. Needham, *Visual Complex Analysis.* New York: Oxford University Press, 1997.

[3] ——, *Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts*, 1st ed. Princeton, NJ: Princeton University Press, 2021.

[4] D. C. Karnopp, D. L. Margolis, and R. C. Rosenberg, *Modeling, Simulation, and Control of Mechatronic Systems*, 5th ed. Hoboken, NJ: John Wiley and Sons, 2012.

[5] "Fisher sees stocks permanently high; Yale economist tells purchasing agents increased earnings justify rise. Says trust and aid sales finds special knowledge, applied to diversify holdings, shifts risks for clients." *The New York Times*, oct 1929.

[6] C. D. Romer, "Great Depression: Definition, History, Dates, Causes, Effects, and Facts," 2021. [Online]. Available: https://www.britannica.com/event/Great-Depression

[7] M. Mendel, "Economic Engineering - Lecture Notes," Delft, 2019.

[8] M. A. Vos, "The Application of System and Control Theory to Monetary Policy: Development of a First-Principles LTI Model of the Economy," Master's thesis, Delft University of Technology, 2019.

[9] G. J. L. Kruimer, "An Engineering Approach to Macroeconomic Scenario Modelling," Master's thesis, Delft University of Technology, 2021.

[10] C. Hutters and M. Mendel, "Overcoming the dissipation obstacle with Bicomplex Port-Hamiltonian Mechanics," 2020.

[11] N. Manders, "The Thermodynamics of Economic Engineering with Applications to Economic Growth," Master's thesis, Delft University of Technology, 2019.

[12] X. A. Van Ardenne, "Business Valuation in the Frequency Domain," 2020.

[13] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems - Modeling, Analysis and Design for Simple Mechanical Control Systems*, 1st ed., ser. Texts in Applied Mathematics. New York, NY: Springer Science+Business Media, 2005.

[14] V. Arnol'd, *Mathematical Methods of Classical Mechanics*, 2nd ed., J. Ewing, F. Gehring, and P. Halmos, Eds. New York: Springer-Verlag, 1989.

[15] L. D. Landau and E. M. Lifshitz, *Mechanics*, 2nd ed., ser. Course of Theoretical Physics. Oxford: Pergamon Press, 1960, vol. 1.

[16] N. G. Mankiw and M. P. Taylor, *Economics*, 4th ed. Andover, Hampshire, UK: Cengage Learning, 2017.

[17] R. Feynman, *The Feynman Lectures on Physics*. New York, NY: Basic Books, 2010, vol. 1, 2, 3.

[18] D. Jeltsema and J. M. Scherpen, "Multidomain Modeling of Nonlinear Networks and Systems: Energy- and power-based perspectives," *IEEE Control Systems*, vol. 29, no. 4, pp. 28–59, 2009.

[19] A. Marshall, *Principles of Economics*, 8th ed., ser. Palgrave Classics in Economics. Macmillan Publishers, 1920.

[20] J. Stewart, *Calculus: Early Transcendentals*, 7th ed. Brooks/Cole Cengage Learning, 2012.

[21] R. Abraham and J. E. Marsden, *Foundations of Mechanics*. Addison-Wesley Publishing Company, 1978.

[22] S. G. Kellison, *The Theory of Interest*, 2nd ed. Richard D. Irwin, 1991.

[23] A. A. Harkin and J. B. Harkin, "Geometry of Generalized Complex Numbers," *Mathematics Magazine*, vol. 77, no. 2, pp. 118–129, 2004.

[24] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, ser. Course of Theoretical Physics. Oxford: Pergamon Press, 1971, vol. 2.

[25] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, ser. Graduate Texts in Mathematics. Cham: Springer International Publishing, 2019, vol. 149. [Online]. Available: http://link.springer.com/10.1007/978-3-030-31597-9

[26] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, and P. Zampetti, *The Mathematics of Minkowski Space-Time*, ser. Frontiers in Mathematics. Basel, Switzerland: Birkhaüser Verlag, 2008.

[27] P. Fjelstad, "Extending special relativity via the perplex numbers," *American Journal of Physics*, vol. 54, no. 5, pp. 416–422, 1986.

[28] G. Sobczyk, "The Hyperbolic Number Plane," *The College Mathematics Journal*, vol. 26, no. 4, pp. 268–280, 1995.

[29] A. E. Motter and M. A. F. Rosa, "Hyperbolic Calculus," *Advances in Applied Clifford Algebras*, vol. 8, no. 1, pp. 109–128, 1998.

[30] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* San Francisco, CA: W. H. Freeman and Company, 1970.

[31] E. F. Taylor and J. A. Wheeler, *Spacetime Physics - An Introduction to Special Relativity*, 2nd ed. New York, NY: W. H. Freeman and Company, 1992.

[32] R. Penrose, "The Geometry of the Universe," in *Mathematics Today: Twelve Informal Essays*, L. A. Steen, Ed. Springer-Verlag, 1978, ch. 4, pp. 83–125.

[33] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, ser. Graduate Texts in Mathematics. New York, NY: Springer-Verlag, 1997.

[34] W. P. Thurston, *Three-Dimensional Geometry and Topology*, S. Levy, Ed. Princeton, NJ: Princeton University Press, 1997, vol. 1.

[35] B. O'Neill, *Elementary Differential Geometry*, 2nd ed. New York: Academic Press, 2006.

[36] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, 3rd ed. Houston, TX: Publish or Perisch, Inc., 1999, vol. 2.

[37] J. M. Lee, *Introduction to Topological Manifolds*, ser. Graduate Texts in Mathematics. New York, NY: Springer-Verlag, 2000.

[38] F. P. Schuller, "The Geometric Anatomy of Theoretical Physics," Friedrich-Alexander Universität Erlangen-Nürnberg - Institut für Theoretische Physik II, Tech. Rep., 2014.

[39] E. Ghys, "Poincaré and His Disk," in *The Scientific Legacy of Poincaré*, ser. History of Mathematics. The American Mathematical Society, 2010, vol. 36, ch. 1, pp. 17–45.

[40] A. Ramsay and R. D. Richtmyer, *Introduction to Hyperbolic Geometry.* New York: Springer Science+Business Media, 1995.

[41] B. O'Neill, *Semi-Riemannian Manifolds with Applications to Relativity.* San Diego, CA: Academic Press, 1983.

[42] N. L. Balazs and A. Voros, "Chaos on the Pseudosphere," *Physics Reports*, vol. 143, no. 3, pp. 109–240, 1986.

[43] R. Penrose and W. Rindler, *Spinors and Space-Time - Two-spinor Calculus and Relativistic Fields.* Cambridge: Cambridge University Press, 1984, vol. 1.

[44] J. Stillwell, *Naive Lie Theory*, ser. Undergratuate Texts in Mathematics. New York: Springer Science+Business Media, 2008.

[45] V. Rovenski, *Modeling of Curves and Surfaces with MATLAB.* New York: Springer Science+Business Media, 2010.

Glossary

List of Acronyms

List of Symbols

K	Original investment
X	Capital
Y	Yield
$\mathbb{C}$	Complex numbers
$\mathbb{R}$	Real numbers
E	First component of the first fundamental form
e	First component of the second fundamental form
F	Second component of the first fundamental form
f	Second component of the second fundamental form
G	Third component of the first fundamental form
g	Third component of the second fundamental form
H^2	Two-sheeted hyperboloid embedded in three-dimensional Lorentz space.
H^2_+	Positive hyperboloid sheet embedded in three-dimensional Lorentz space.
k	Gaussian curvature