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The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

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E. B. Legrand

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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis...

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology
July 12, 2022

E. B. Legrand

Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*

Chapter 1

Introduction

Idea: introduction using Arnol'ds thermodynamics quote.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\Gamma\Delta\Upsilon\Omega\Theta$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\Gamma\Delta\Upsilon\Omega\Theta}$
Tensor	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\Gamma\Delta\Upsilon\Omega\Theta}$

Table 1-1: Caption

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$
$\Gamma(\mathrm{T}M)$
$\mathfrak{X}(\mathrm{T}M)$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by Abraham and Marsden [1] and Cannas da Silva [2] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.
- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

Symplectic and Contact Geometry in Economic Engineering

Chapter 3

Liouville and Contact Geometry for Dissipative Systems

3-1 Introduction

The traditional theory of Hamiltonian mechanics based on symplectic manifolds does not include energy dissipation without an explicit time dependence. A famous example is the time-dependent Hamiltonian and Lagrangian functions modeling the linearly damped harmonic oscillator, which is commonly attributed to Caldirola [3] and Kanai [4][1]. These systems are nonautonomous, which is to say that the phenomenon of dissipation is essentially treated to be *exogeneous* from the perspective of the system.

To the engineer, this does not seem at all natural. Every physical system, be it mechanical, electrical, economic, or otherwise, includes dissipative elements in some shape or form. These elements are inherent to the system: exogeneous effects are typically reserved for externally controlled inputs or disturbances. Hence, from an engineering point of view we seek to include the dissipative elements as phenomena that are *endogeneous* to the system itself.

Some efforts have been made in the past to resolve this issue by extending the Hamiltonian formalism: the celebrated report by Dekker [5] provides an excellent overview of the advances in this field. Roughly speaking, three approaches exist:

- The *Bateman approach*, doubles the number of degrees of freedom in the system to create a 'mirror system' that runs in the opposite time direction [6]. In the overall system the irreversible (i.e. dependent on the direction of time) effects, cancel out due to the two time directions, which is why it admits a symplectic structure.
- The family of *complex dissipative Hamiltonians* proposed in various forms by Bopp [7], Dekker [8], Dedene [9], Rajeev [10], and more recently, Hutters and Mendel [11] in the research group of the author. Originally, this method was developed to facilitate

[1] We will hence refer to it as the Caldirola-Kanai Hamiltonian (or Lagrangian).

the quantization of dissipative mechanics for applications in quantum mechanics. The relation of the work in this thesis with the complex dissipative Hamiltonians is given in Section 4-4-1.

- Some purely *mathematical Hamiltonians* have been devised as well, e.g. by Havas [12] solving the so-called Helmholtz conditions. These are Hamiltonians that produce the correct equations of motion but bear no connection to the energy in the system.

Although all of these methods work, they have some limitations for practical applications. The Bateman approach results in a system with twice the number of dimensions, but most (at least half) of these are redundant. This is because dissipation is in essence a first-order effect: as will be shown in this chapter, at most one additional dimension is required to model the dissipation. The complex Hamiltonians are mathematically elegant but lack physical interpretation of the canonical coordinates and the Hamiltonian function. Finally, the mathematical Hamiltonians do not offer insights from a physical standpoint. Furthermore, they rely on singularities of the Hamiltonian function to circumvent the inevitable limitations imposed by de Rham's theorem (this point is discussed in greater detail later in Section 3-3).

In addition, we remark that it is common practice in engineering applications to include the *Rayleigh damping function* in the Lagrangian function [13] to represent linearly damped elements. Although this damping function produces the correct equations of motion for linear damping, it does not conform to the deeper symplectic structure underlying Hamiltonian (and also Lagrangian) mechanics. Hence, we consider this an *ad hoc* method, and we we will not be further concerned with it in this thesis.

The purpose of this chapter is to propose a different framework that is both mathematically elegant *and* physically interpretable, to make it suitable for practical applications. Instead of trying to fit the dissipation into the symplectic framework, we propose that *contact manifolds* (instead of symplectic manifolds), are the natural setting for problems that include dissipation. Contact manifolds are necessarily odd-dimensional: next to the canonical pairs of generalized position of momenta, an extra coordinate is included that absorbs the dissipated energy. The Hamiltonian formalism can be extended from symplectic manifolds to contact manifolds as well. We will use this *contact Hamiltonian formalism* to describe dissipative mechanical systems.

This chapter takes two different paths to arrive at a single contact Hamiltonian representation of the damped harmonic oscillator, which serves as an exemplary dissipative system. These two paths are based on two parallel mathematical representations of a contact manifold: one is the contact structure itself, while the other is its *symplectification*. This is a procedure to cast the contact Hamiltonian system on a symplectic manifold with an additional *Liouville structure*. This procedure goes at the expense of yet another dimension added to the manifold, making it even-dimensional again.

Firstly, in Section 3-2, we use the principles of thermodynamics to derive the contact Hamiltonian representation of the damped harmonic oscillator. Secondly, in Section 3-3, we look at the symplectification of the manifold and derive an equivalent symplectified Hamiltonian system argued motivated by the form of the Caldirola-Kanai Hamiltonian.

We then proceed by transforming the contact Hamiltonian representation of the damped harmonic oscillator to the a contact Lagrangian representation in Section 3-4. Finally, Section 3-5

shows how the same principles can be used to model more complicated mechanical systems, including those with exogeneous inputs.

For the reader unfamiliar with contact geometry and contact Hamiltonian systems, a concise introduction to contact geometry and contact Hamiltonian systems is given in Appendix B. A more extensive overview is given by Geiges [14] and Libermann and Marle [15].

3-2 Contact structure based on thermodynamic principles

It has already been argued in the past by several authors that contact geometry is the natural framework for thermodynamics by i.a. Arnol'd [16, 17, 18, 19], Bamberg and Sternberg [20], Burke [21] and Hermann [22], ultimately leading back to the seminal work of Gibbs [23]. It is commonly seen as a testament to the brilliance of Gibbs' work that he managed to recognize and describe the correct geometric framework well before the required mathematical infrastructure came to invention [24]. In recent years, the contact Hamiltonian formalism has been succesfully applied to thermodynamic theory by e.g. Mrugała et al. [25], Mrugała [26, 27, 28, 29, 30], Balian and Valentin [31], van der Schaft [32], van der Schaft and Maschke [33], Maschke and van der Schaft [34], Bravetti et al. [35], and Simoes et al. [36].

Regarding mechanical systems, the applicability of the contact Hamiltonian formalism for dissipation has been described already by Bravetti et al. [37], in particular for the damped harmonic oscillator. The argument in this and the following section leads to similar results, but we propose a modification of the equations of motion. This modification allows us to assign a direct physical interpretation of the contact variables, which makes the method suitable for practical problems.

Contact geometry arises in thermodynamics as a consequence of the First Law, which asserts that the change in internal energy of the system is equal to the difference between the heat added *to* the system and the work performed *by* the system. Formally, this is stated as

$$\mathrm{d}U = \eta - \beta, \tag{3-1}$$

where U is the internal energy of the system, η the heat added to the system and β the work done by the system on its environment [20, 38]. Both η and β are 1-forms that are not exact. This is why, for the purposes of this thesis, we cannot denote them by $\mathrm{d}Q$ and $\mathrm{d}W$, for this would falsely indicate that these forms are exact. Likewise, there is no need for using the 'inexact' surrogate notation đ or δ, since the notion of inexactness already built into the theory of exterior calculus.

Equation (3-1) essentially states that the difference of the heat and work 1-form is *closed*. Locally, this difference can then be written as the gradient of a function, called the *internal energy* U of the system. As a result, the form

$$\alpha = \mathrm{d}U - \eta + \beta$$

should pull back to zero over the 'allowable' trajectories of the systems.

Classical thermodynamics is usually geared towards systems containing expanding gases or chemical mixtures, and not purely mechanical systems. The content of the next section

is not strictly required to understand the rest of the chapter but it is meant to elucidate the significance of contact geometry for a classical thermodynamical system. Afterwards, in Section 3-2-2, we will make the jump towards mechanical systems in the form of the damped harmonic oscillator.

3-2-1 Contact geometry in classical thermodynamics

The thermodynamic state of a system is given by a collection of thermodynamic state variables (or 'properties'). For example, for an ideal gas in a piston, we may consider its volume V, temperature T and pressure P. We also know that, for an ideal gas, the following relation must hold:

$$P\,V = n_{\mathrm{s}}\,R_{\mathrm{g}}\,T, \tag{3-2}$$

with n is the amount of substance (measured in mol), and $R_{\mathrm{g}} = 8.314\,\mathrm{J/(mol\,K)}$ is the ideal gas constant. Hence, for constant n_{s} the actual state of the system is dictated only by two state properties, for the ideal gas law allows us to find the other if two out of three are given. The states of the system that are thermodynamically meaningful therefore live on a *two-dimensional submanifold* of $\mathbb{R}^3$. Any selection of two state properties may serve as coordinates for this submanifold. However, due to this ambiguity, it is usually more convenient to consider the larger three-dimensional manifold together with the constraint as opposed to 'spending' it [31, 39].

If we are allowed to add heat to the gas in the piston, or we use its expansion to perform work on the environment, Equation (3-2) will not be sufficient, because it does not contain *all* the thermodynamic information of the system. We need a thermodynamic potential, in this case *internal energy*, to keep track of the full state. For the internal energy, there is another equation of state [40]

$$U = c\,n_{\mathrm{s}}\,R_{\mathrm{g}}\,T, \tag{3-3}$$

where c is a constant depending on the type of gas (for monatomic gases, $c = \frac{3}{2}$). A *fundamental thermodynamic relation*, expresses a thermodynamic potential, such as the internal energy U, in terms of the extensive variables in the system. For the ideal gas, the fundamental relation is of the form $U = U(S,V)$, for the entropy S and the volume V are the extensive state properties of the system.

The choice of the internal energy as the thermodynamic potential is certainly not unique; we may also invert the relation in favor of the entropy or use other potentials obtained through a Legendre transform, such as the Gibbs free energy, Helmholtz free energy, enthalpy etc. In particular, we refer to the specification of a system in terms of internal energy as the *energy representation*, and to a specification in terms of entropy as the *entropy representation* [32].

Using the fundamental thermodynamic relation, we now use a five-dimensional space to describe the complete thermodynamic state of the system. Coordinates for this space are the internal energy and entropy in addition to the pressure, volume and temperature considered earlier. This space is referred to as the *thermodynamic phase space*. Again, we are not to choose these variables completely independent from each other, since they are subject to constraints given by Equation (3-3) and Equation (3-2). In addition, the First Law of thermodynamics states that

$$\mathrm{d}U = \eta - \beta,$$

where we have that $\beta = P\,\mathrm{d}V$ and, according to the Second Law of Thermodynamics, $\eta = T\,\mathrm{d}S$. As such, the First Law states that the form

$$\alpha_{\mathrm{G}} \coloneqq \mathrm{d}U - T\,\mathrm{d}S + P\,\mathrm{d}V \tag{3-4}$$

should pull back to zero on the allowable states. This is known as Gibbs' fundamental relation.

The form α_{G} given by Gibbs' relation is a contact form on the thermodynamic phase space, since

$$\alpha_G \wedge (\mathrm{d}\alpha_G)^2 = 2\,\mathrm{d}U\,\mathrm{d}S \wedge \mathrm{d}T \wedge \mathrm{d}P \wedge \mathrm{d}V\,,$$

which is a top (or volume) form on the thermodynamic phase space.[2]

By definition, given that the internal energy is a function of the extensive state variables, we have

$$\mathrm{d}U \coloneqq \frac{\partial U}{\partial V}\,\mathrm{d}V + \frac{\partial U}{\partial S}\,\mathrm{d}S\,.$$

This knowledge, combined with Equations (3-2) and (3-3), allows us to integrate the Gibbs relation. We can use these equations to express T and P in terms of the extensive variables like so:

$$T = \frac{U}{cn_{\mathrm{s}}R_{\mathrm{g}}} \qquad P = \frac{U}{cV}.$$

The Gibbs form can then be integrated as follows:

$$\frac{\mathrm{d}U}{U} = \frac{\mathrm{d}S}{cn_{\mathrm{s}}R_{\mathrm{g}}} - \frac{\mathrm{d}V}{cV},$$

$$\log U = \frac{S}{cn_{\mathrm{s}}R_{\mathrm{g}}} - \log cV + C_0,$$

$$U = \log C_0\, \mathrm{e}^{\frac{S}{cn_{\mathrm{s}}R_{\mathrm{g}}}}\, V^{\frac{-1}{c}},$$

where C_0 is an integration constant.

From the above expression we can observe that the Gibbs relation and the equations of state specify a *two-dimensional submanifold* of the larger five-dimensional space determined by the conditions

$$T = \frac{\partial U}{\partial S}, \qquad P = -\frac{\partial U}{\partial V}. \tag{3-5}$$

Submanifolds of the thermodynamic phase space on which α_{G} pulls back to zero are called integral submanifolds of α_{G}. Due to the nondegeneracy of the contact form, integral submanifolds of an $(2n+1)$-dimensional contact manifold have at most dimension n; the *maximal* integral submanifolds are called *Legendre submanifolds*. In this case, the dimension of the

[2]Because

$$\begin{aligned}(\mathrm{d}\alpha_{\mathrm{G}})^2 &= (\mathrm{d}S \wedge \mathrm{d}T + \mathrm{d}P \wedge \mathrm{d}V)^2,\\ &= \mathrm{d}S \wedge \mathrm{d}T \wedge \mathrm{d}P \wedge \mathrm{d}V + \mathrm{d}P \wedge \mathrm{d}V \wedge \mathrm{d}S \wedge \mathrm{d}T\,,\\ &= 2\,\mathrm{d}S \wedge \mathrm{d}T \wedge \mathrm{d}P \wedge \mathrm{d}V\,,\end{aligned}$$

since the permutation $(S,T,P,V) \mapsto (P,V,S,T)$ is even.

contact manifold if five, which is why Legendre submanifolds are two-dimensional. Clearly, Legendre submanifolds play a pivotal role in this framework because they define the thermodynamically allowable states (Balian and Valentin [31] call them *thermodynamic manifolds*). Trajectories in the thermodynamic phase space are only physically meaningful if they lie in Legendre submanifolds.

3-2-2 Contact geometry in dissipative mechanics

In this and following sections, the damped harmonic oscillator system shown in Figure 3-1 serves an illustration for the application of the contact Hamiltonian formalism to dissipative system. The choice of the damped harmonic oscillator as examplary system is rather perspicuous, since it is arguably the simplest dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of choice in the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [5, 11]. The governing second-order differential equation of the damper harmonic oscillator is

$$m\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + b\frac{\mathrm{d}q}{\mathrm{d}t} + kq = 0, \tag{3-6}$$

where m is the mass of the object, b the damping coefficient, and k the spring constant. Some other relevant system parameters (e.g. damping coefficient, frequency) that are used throughout the chapter, are defined in Table 3-1.

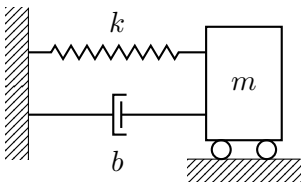


Figure 3-1: Schematic of the mass-spring-damper system.

Table 3-1: Parameter conventions of the damped harmonic oscillator. To avoid confusion with the symplectic form ω, angular frequencies are denoted by Ω instead of the conventional lower case Greek letter.

Name	Symbol	Value	Units
Damping coefficient	γ	b/m	1/s
Undamped frequency	Ω_o	$\sqrt{k/m}$	1/s
Damped frequency	Ω_d	$\sqrt{\Omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$	1/s
Damping ratio	ζ	$\frac{b}{2\sqrt{mk}}$	–

Thermodynamics of the damped oscillator

When applying the thermodynamic framework discussed in the previous section to mechanical systems, we make the distinction between macroscopic and microscopic states. That is, energy can be stored in the movement of small particles, which does not manifest itself as an *observable* movement of the overall system, but rather as *temperature*. The associated kinetic energy is called internal energy U. On the other hand, the movement of the mass in the oscillator system *is* observable, and we consider its kinetic energy to be of a different kind. In addition, there is the energy which arises as a consequence of external potentials. Both the macroscopic kinetic energy and potential energy are subsumed in the *mechanical energy* E. This distinction between mechanical and internal energy is rather artificial, but it fits better in the notational and conceptual conventions of both mechanics and thermodynamics. The total energy in the system is then the sum of its total mechanical energy and total internal energy.

In the damped oscillator system, internal energy can be stored either in the form of macroscopic kinetic energy in the mass, potential energy in the spring and as microscopic kinetic energy of the 'heat bath'. The notion of a 'heat bath' is rather loosely defined: it can encompass the fluid in the damper, a body of surrounding air (or both), or anything to which we may attach the conceptual picture of a reservoir that absorbs the dissipated energy in the form of heat. If the heat bath is heterogeneous (for example, containing damper fluid and the surrounding air), it will not have a single temperature. We will not be concerned with all these possibilities and consider a single heat bath with single temperature: generalizations to more complex thermodynamic systems are immediate.

The overall system (i.e. mass, spring and heat bath) is assumed to be completely isolated: there is no exchange of energy nor matter with the environment. We therefore have, according to the First Law:

$$\mathrm{d}(U + E) = 0. \tag{3-7}$$

The mechanical energy of the mass-spring-damper system is a function of the position q of the mass and the momentum p of the mass, that is

$$E := E(p, q) = \frac{p^2}{2m} + \frac{1}{2}kq^2. \tag{3-8}$$

Furthermore, we assume that the heat bath is the only part of the system that can store entropy, and that it has no other degrees of freedom:

$$U := U(S). \tag{3-9}$$

Let uw now decompose the system analogous to E and U into two subsystems, one containing the mass and the spring and one the heat bath, as illustrated in Figure 3-2. Through the dissipative action of the damper, energy flows from the mechanical subsystem to the heat bath. We can apply the First Law to the subsystems separately, too: the first subsystem performs work on the damper, which manifests itself as heat added to the heat bath. We therefore have

$$\begin{aligned} \mathrm{d}E &:= \frac{\partial E}{\partial p}\,\mathrm{d}p + \frac{\partial E}{\partial q}\,\mathrm{d}q = -\beta, \\ \mathrm{d}U &:= \frac{\partial U}{\partial S}\,\mathrm{d}S = \eta, \end{aligned} \tag{3-10}$$

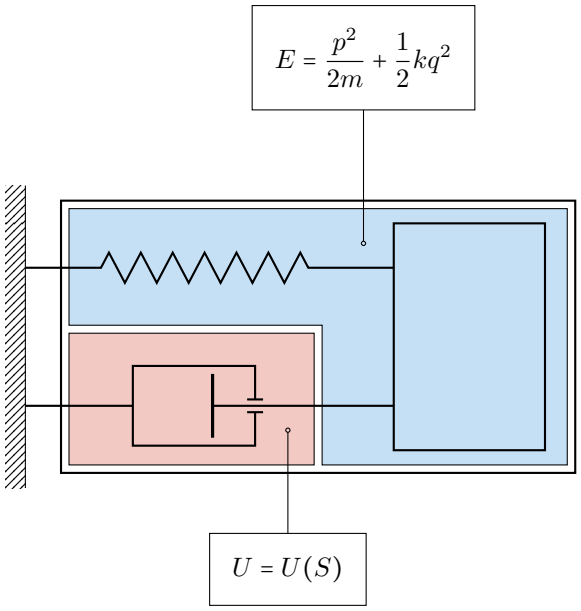


Figure 3-2: System boundaries of the damper-oscillator system. The mechanical subsystem stores mechanical energy E in the form of kinetic and potential energy, while the heat bath stores internal energy in the form of heat. They interface through the action of the damper.

where β is the (differential) work done *by* the mechanical subsystem on the damper and η is the (differential) heat added *to* the second subsystem as a result of this. From the preceding discussion we have that, because the total system is isolated, that $\beta = \eta$; i.e. all the work done by the damper enters the fluid as heat. For a linearly damped system, the work form is by definition equal to

$$\beta := \gamma p \, \mathrm{d}q \,, \tag{3-11}$$

with γ being the damping coefficient as defined in Table 3-1.

The above equations specify a contact structure on the three-dimensional phase space M constituted by the internal energy U, the momentum p and the position p, or equivalently, on the other space $\tilde{M}$ with the mechanical energy E, p and q:

$$\mathrm{d}U - \gamma p \, \mathrm{d}q \in \mathrm{T}^*M, \qquad \mathrm{d}E + \gamma p \, \mathrm{d}q \in \mathrm{T}^*\tilde{M}.$$

Formally, we have that the position q is a coordinate of the configuration space Q, to which we attach the cotangent bundle whose fibers contain the momentum: $(q, p) \in \mathrm{T}^*Q$. The latter space is then extended with a single dimension for either U or E. Both spaces M and $\tilde{M}$ are isomorphic to the three-dimensional real coordinate space $\mathbb{R}^3$, but are equipped with a different contact structure. Because the topology of these spaces is trivial, we assume the contact forms to be defined globally: they are *exact* or *strictly* contact manifolds.

In the following section, we define the contact Hamiltonian system on M (i.e. in terms of the internal energy), and denote the associated contact form by α, i.e.

$$\alpha := \mathrm{d}U - \gamma p \, \mathrm{d}q \tag{3-12}$$

In the following sections, it is important to keep in mind that contact forms are by definition nonunique: α is only determined up to multiplication by a function $f \in C^\infty(M)$ without zeros — that is to say, any of those forms gives rise to the same contact *structure*.

Contact Hamiltonian system for the damped oscillator

Given the contact form in Equation (3-12), we are now to find the contact Hamiltonian function to complete the picture of the contact Hamiltonian system. A contact Hamiltonian system is a triple (M, α, H), where M is a manifold, $\alpha \in \mathrm{T}^*M$ is a contact form on the manifold and $H \in C^\infty(M)$ is the Hamiltonian function that generates the dynamics. The contact structure α provides an isomorphism Ψ_α between the functions on the manifold and the infinitesimal strict contactomorphism. A strict infinitesimal contactomorphism is a vector field that preserves the strictly contact structure. That is to say, it preserves the contact structure that is *globally* determined by a contact 1-form. This is not in general the case, so strict contactomorphisms do not constitute the full class of contactomorphisms [15, 17].

The isomorphism Φ_α is to be compared to the traditional Hamiltonian formalism for conservative systems, where it is the symplectic 2-form ω that provides the isomorphism between the Hamiltonian function and the associated symplectic vector field. For *any* given function H (which does not necessarily has to be 'the' Hamiltonian), the associated *Hamiltonian vector field* is denoted by X_H:

$$\Psi_\alpha : \mathfrak{X}_c(M) \to C^\infty(M) : X_H \mapsto H. \tag{3-13}$$

The precise nature of Ψ_α is discussed in detail in Appendix B-3. The 'forward mapping' is defined as

$$\Psi_\alpha(X_H) \coloneqq X_H \lrcorner \alpha.$$

To go the other way is not so simple, since Ψ_α^{-1} must map the general class of smooth functions to a special subclass of inifinitesimal exact contactomorphims $\mathfrak{X}_c(M)$. For X_H to be an infinitesimal exact contactomorphism means that it must satisfy the following condition:

$$\pounds_{X_H}\alpha = s\alpha,$$

with s some function (zeros allowed) on M.

The trick to find $\Psi_\alpha^{-1}(X_H)$ is to separate the Hamiltonian vector field into a *horizontal* vector field X_H^{hor} and a vertical vector field X_H^{ver} according to the following splitting of the tangent bundle to M [15]

$$\mathrm{T}M = \ker\alpha \oplus \ker \mathrm{d}\alpha\,,$$

where the vertical field is in the kernel of $\mathrm{d}\alpha$, the horizontal vector field in the kernel of α, and $\oplus$ denotes the Whitney sum. The Hamiltonian vector field is then

$$X_H = X_H^{\mathrm{ver}} + X_H^{\mathrm{hor}},$$

with $X_H^{\mathrm{ver}} \in \ker \mathrm{d}\alpha$ and $X_H^{\mathrm{hor}} \in \ker\alpha$.

The vertical part of the vector field is easy to find based on the definition of Ψ_α, and is equal to

$$X_H^{\mathrm{ver}} = -HR_\alpha, \tag{3-14}$$

where R_α refers to the Reeb vector field of the contact form. In components, the Reeb vector field is[3]

$$R_\alpha = \frac{\partial}{\partial U}.$$

[3]Clearly, the vertical part of the vector field is always colinear with the Reeb vector field. However, this does not preclude the horizontal vector field to have a component in the direction of the Reeb vector field as well; as long as it is part of the kernel of α. It is therefore not the case that the horizontal vector field is symplectic, which one might intuitively assume.

For reasons that will become later, it is the horizontal part that is our prime interest, rather than the vertical part. The horizontal vector field is obtained using the second (and first) condition, which is equivalent to (by virtue of Cartan's magic formula, see Appendix B-3)

$$X_H^{\text{hor}} \lrcorner \, \mathrm{d}\alpha = \mathrm{d}H - s\alpha, \tag{3-15}$$

with $s = R_\alpha \lrcorner \, \mathrm{d}H = \frac{\partial H}{\partial U}$. Hence, the function s reflects the dependence of the Hamiltonian function on the variable U. Let us now reason from the standpoint of a classical 'symplectic' Hamiltonian system. In that case, the Hamiltonian would be equal to the mechanical energy, and the Hamiltonian vector field (associated to E) is defined through the relation

$$X_E \lrcorner \, \omega = \mathrm{d}E\,, \tag{3-16}$$

with $\omega = \mathrm{d}q \wedge \mathrm{d}p$ being the symplectic 2-form. In addition, observe that

$$\mathrm{d}\alpha = \gamma \, \mathrm{d}q \wedge \mathrm{d}p = \gamma \, \omega.$$

As such, Equation (3-15) becomes

$$\gamma\big(X_H^{\text{hor}} \lrcorner \, \omega\big) = \underbrace{\mathrm{d}H - \frac{\partial H}{\partial U}\,\mathrm{d}U}_{\text{closed form}} + \underbrace{\frac{\partial H}{\partial U}\gamma p\,\mathrm{d}q}_{\text{dissipation}}\,.$$

The left hand side of the above expression and the symplectic case given by Equation (3-16) look very much alike. The interior product of X_H and ω can decidedly not produce a closed form (which the left two terms on the right hand side are), because then X_H would be a symplectic vector field. Hence, the rightmost term represents the 'deficit' from a purely symplectic vector field: the work form multiplied by $\frac{\partial H}{\partial U}$. The other part of the right hand side should then be analogous to the purely symplectic case (multiplied by γ); that is to say, it is equal to $\mathrm{d}E$. This means that

$$\mathrm{d}H = \gamma \, \mathrm{d}E + \frac{\partial H}{\partial U}\,\mathrm{d}U\,.$$

Furthermore, choose $\dfrac{\partial H}{\partial U} \coloneqq \gamma$, such that

$$\mathrm{d}H = \gamma\,(\mathrm{d}E + \mathrm{d}U).$$

As a result, the contact Hamiltonian is equal to

$$H = \gamma(E + U) = \gamma\left(\frac{p^2}{2m} + \frac{1}{2}kq^2 + U\right), \tag{3-17}$$

at least up to the addition of a closed form (i.e. a first integral or total differential). The Hamiltonian is therefore equal to the total energy of the system multiplied by the damping coefficient.

Based on the expression for the Hamiltonian, the horizontal component of the Hamiltonian vector field is then found to be

$$X_H^{\text{hor}} = \frac{p}{m}\frac{\partial}{\partial q} - (kq + \gamma p)\frac{\partial}{\partial p} + \gamma\frac{p^2}{m}\frac{\partial}{\partial U}.$$

This vector field gives the correct dynamics for q and p. The rate of change of U (by the horizontal vector field) is given by the $\gamma\frac{p^2}{m}$, which is precisely equal to the power dissipated by the damper. As such, it seems that the correct dynamics are entirely represented by the horizontal vector field. On the other hand, the vertical vector field, given by Equation (3-14), is proportional to the numerical value of the Hamiltonian, and it also contributes to the time-rate of change of U. The presence of the vertical vector field gives rise to an additional exponential growth (since X_H^{ver} is proportional to the Hamiltonian, which contains U itself). Hence, if we we want to impose that U indeed be the internal energy of the heat bath, the *vertical vector field must vanish.* This is only the case if the Hamiltonian is numerically equal to zero, i.e. $H = 0$. From a thermodynamic standpoint, energy is only determined up to an additive constant, so this assertion would be admissible from a conceptual standpoint. Additionally, a value of 0 for the total energy is a common convention in literature, see for example Fermi [41].

The assumption that the contact Hamiltonian should be equal to zero is rather striking, and the preceding arguments do not provide a sound mathematical basis for it. Indeed, we could (and should) be quite leery of cancelling terms using zero factors, for it often leads to unanticipated consequences or even downright contradictions. But it is clear, from the discussion above that the fact that $H = 0$ is crucial for its interpretation as (the scaled) total energy of the system; this point is missed entirely by many applied texts on this subject, including Bravetti et al. [37], but also Valcázar and de León [42]. Leaving the vertical vector field in leads to extra 'parasitic' dynamics that are unphysical and delude us from the actual significance of the variable U.

Now to the mathematics. Recall that by definition $H = X_H \lrcorner \alpha$. In the previous section we defined Legendre submanifolds as manifolds on which the contact form pulls back to zero: in other words, tangent vectors to a Legendre manifold produce zero when contracted with the contact form. So, H measures in essence how 'non-Legendrian' an integral manifold of X_H is. We have stipulated earlier that Legendre submanifolds are the only ones with physical meaning, which is why this condition is essential. For the same reason, we can show that

$$\pounds_{X_H^{\text{hor}}} H = 0 \quad \Rightarrow \quad \frac{\mathrm{d}H}{\mathrm{d}t} = \pounds_{X_H^{\text{ver}}} H = -H\frac{\partial H}{\partial U}.$$

Hence, if the Hamiltonian does not vanish, it changes exponentially over time (for its change is proportional to its own value). If $\gamma > 0$, the Hamiltonian decays exponentially from its initial value:

$$H(t) = H_0\,\mathrm{e}^{-\gamma t}.$$

As a result, any nonphysical trajectories will approach a Legendre submanifold as time proceeds; the associated vector fields also become ever more tangent to the Legendre submanifold. Based on the expression for H, an expression for U may also be derived:

$$U(t) = \frac{1}{\gamma}\left(H_0\,\mathrm{e}^{-\gamma t} - E\right).$$

Hence, if $H = 0$, the internal energy is equal to the negative of the mechanical energy in the system. These findings are illustrated by Figure 3-3: the left plot shows perturbations of the 'ideal' physical trajectory. The right plot shows the trajectories for an initial value of $U = 0$ (also with perturbations); as a result of the above equation, U 'wobbles' around its zero point; but it is clearly not a physical trajectory.

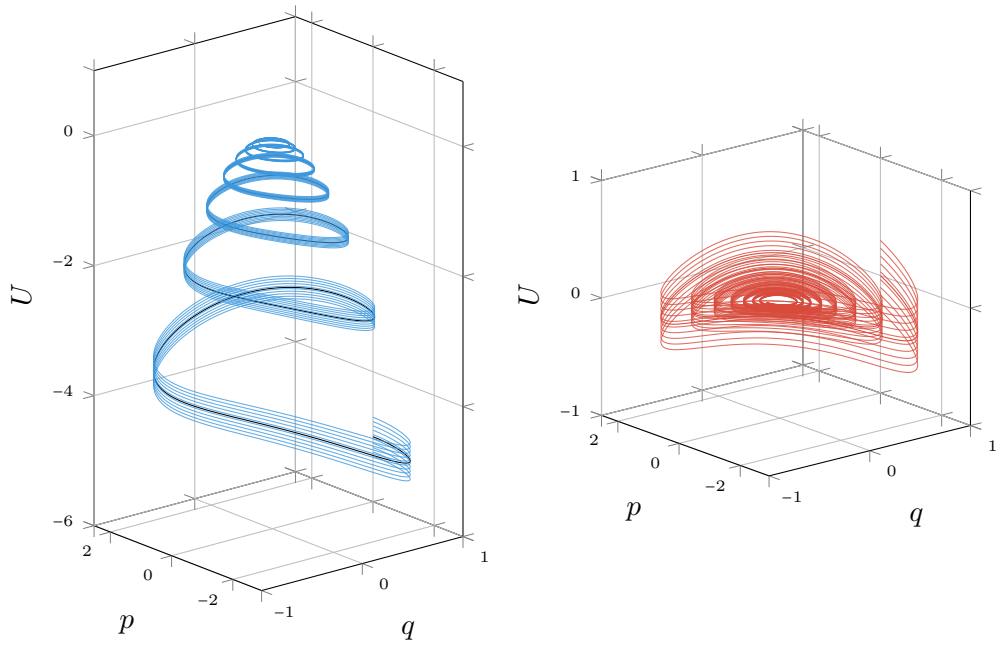


Figure 3-3: Integral curves of X_H for $b = 0.3\,\mathrm{kg/s}$, $m = 1\,\mathrm{kg}$ and $k = 10\,\mathrm{kg/s^2}$. The left plot shows the physical trajectory ($H = 0$) in black, together with some neighboring non-physical trajectories that approach the black trajectory with increasing time. The trajectories on the right are all unphysical, but show the case where we would choose a zero initial value for U (also with some perturbations). The 'wobble' is caused by the exponentially decaying value of the Hamiltonian being counterbalanced with the nonuniform decrease of the mechanical energy in the system. In this case, U clearly does *not* represent the internal energy of the heat bath, or any other physical variable.

As an additional argument, recall that the contact form α is not unique: multiplication with any nonzero function gives rise to the same contact *structure*. We can consider this to be a gauge transformation of the system (cf. Balian and Valentin [31]). However, the Hamiltonian is not intrinsically invariant under these transformations; if $\tilde{alpha} = f\alpha$ (f being a function without zeros), then the mapping Ψ'_α and the corresponding Hamiltonians are also different: [15, p. 321]

$$\Psi_{\tilde{\alpha}}^{-1}(H) = \Psi_\alpha^{-1}\left(\frac{1}{f}H\right). \tag{3-18}$$

The vertical component *is* directly dependent on the numerical value of the Hamiltonian. As a result, the *only* way to maintain invariance under the gauge transformation (which we assert to be crucial for it to be of physical siginifance) is to set $H = 0$.

The Hamiltonian and contact structure used here are not precisely the same as the one proposed by Bravetti et al. [37], for they differ by multiplication of γ. It is clear from the derivations in this section that, although the thermodynamic picture is clear, the derivations in the contact framework are implicit and inintuitive. An additional disadvantage presents itself in the fact that α is not specified in Darboux coordinates, which makes the derivations even more clumsy and precludes us from using the standard coordinate expressions. In the next section, a more elegant approach is taken using the symplectification of the contact manifold.

3-3 Liouville structure from time-dependence

As mentioned in Section 3-1, the Caldirola-Kanai method uses an explicit time-dependence in the Lagrangian and the Hamiltonian function to accomodate the effect of dissipation. In this section, we will explain how the Caldirola-Kanai method is really a 'concealed' version of a *homogeneous* Hamiltonian function. This is a special type of Hamiltonian function that arises after a contact Hamiltonian system is lifted to a symplectic space, a procedure colloquially called *symplectization* of the contact structure. The result is a symplectic structure with an extra homogeneity structure, called a *(symplectic) Liouville structure*[4].

First, the Caldirola-Kanai method is briefly explained in Section 3-3-1. Then, the theory of symplectization and Liouville structures is presented in Section 3-3-2, after which it is used to construct homogeneous Hamiltonian systems in Section 3-3-3.

3-3-1 The Caldirola-Kanai method

The Caldirola-Kanai Lagrangian is

$$L_{\mathrm{CK}}(q,\dot{q},t) = \mathrm{e}^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq_1^2\right), \tag{3-19}$$

with the corresponding Hamiltonian function:

$$H_{\mathrm{CK}}(q,\rho,t) = \frac{\rho^2}{2m}\mathrm{e}^{-\gamma t} + \frac{1}{2}kq^2\mathrm{e}^{\gamma t}. \tag{3-20}$$

In latter equation, ρ refers to a special 'canonical momentum', that is

$$\rho := \frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}, \tag{3-21}$$

which is related to the 'true' kinematic momentum by the relation $\rho = p\mathrm{e}^{\gamma t} = m\dot{q}\mathrm{e}^{\gamma t}$. As such, it is also clear that the Caldirola-Kanai Lagrangian and Hamiltonian functions are related by the Legendre transformation *with respect to the canonical momentum*:

$$H_{\mathrm{CK}} = \rho\dot{q} - L_{\mathrm{CK}}.$$

From either Equation (3-19) or Equation (3-20), the equations of motion are readily derived (for the Hamiltonian case with respect to ρ after which the transformation to p can be effected). Indeed, after taking the appropriate derivatives, one obtains:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}\right) - \frac{\partial L_{\mathrm{CK}}}{\partial q} = 0$$

$$\Rightarrow \mathrm{e}^{\gamma t}(m\ddot{q} + m\gamma\dot{q} + kq) = 0$$

[4]The name 'Liouville structure' stems from the fact that the associated structure is based on the Liouville 1-form, which makes its appearance under a myriad of names, such as the canonical 1-form, tautological 1-form, Poincaré 1-form or the symplectic potential.

for the Lagrangian case. Likewise, Hamilton's equations yield: [43]

$$\dot{q} = \frac{\partial H_{\mathrm{CK}}}{\partial \rho} = \frac{\rho}{m} \mathrm{e}^{-\gamma t} = \frac{p}{m},$$

$$\dot{\rho} = -\frac{\partial H_{\mathrm{CK}}}{\partial q} = -kq\mathrm{e}^{\gamma t}.$$

The relation between the time derivatives of the momenta $\dot{p}$ and $\dot{\rho}$ is slightly more involved since one must invoke the product rule as a result of their time-dependent relation:

$$\dot{\rho} = \mathrm{e}^{\gamma t}(\dot{p} + \gamma p). \tag{3-22}$$

Substition yields the correct equation for p, though the equation is again multiplied by $\mathrm{e}^{\gamma t}$. Because the latter is sufficiently well-behaved (that is, it has no zeros), it can be removed without any problems.

Geometric approach

The Hamiltonian Equation (3-20) is explicitly time-dependent. This will give rise to a time-dependent vector field governing the solution curves.[5] The construction of the vector field associated with a time-dependent Hamiltonian follows the same construction rules as a normal Hamiltonian (using the isomorphism given by ω), but 'frozen' at each instant of t. Even more bluntly speaking, one simply ignores the t-coordinate during the derivation, only to acknowledge the dependence at the very end. This leads to the following vector field, 'suspended' on the $\mathbb{R} \times Q$ space:

$$\tilde{X}_{H_{\mathrm{CK}}} = -\mathrm{e}^{\gamma t} kq \frac{\partial}{\partial \rho} + \mathrm{e}^{-\gamma t} \frac{\rho}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

The suspension is important to make the final coordinate transformation from ρ to p work properly. Indeed, effecting the transformation $(q, \rho, t) \mapsto (q, \mathrm{e}^{-\gamma t}\rho, t)$, one obtains

$$\tilde{X}_{H_{\mathrm{CK}}} = (-kq - \gamma p) \frac{\partial}{\partial p} + \frac{p}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

It is worthwile to ponder on some apparent peculiarities in the Caldirola-Kanai method, for they will be explained elegantly by the contact-Hamiltonian formalism. Firstly, the role of the two different momenta is not very clear from the get-go, apart from being a consequence of the way the Caldirola-Kanai Lagrangian is formulated. This has also been the reason for considerable confusion in the academic community (see Schuch [44]). Furthermore, there is the special role of the time coordinate, which is merely a parameter in the Hamiltonian function; for it does not partake in the dynamics of the system. Finally, there is the time dependence, which does indeed produce the correct equation of motion but seems to be out

[5] A *time-dependent vector field* on a manifold N is a mapping $X : N \times \mathbb{R} \to \mathrm{T}N$ such that for each $t \in \mathbb{R}$, the restriction X_t of X to $N \times \{t\}$ is a vector field on N. [15] An additional construction of importance, called the *suspension* of the vector field, is a mapping

$$\tilde{X} : \mathbb{R} \times N \to \mathrm{T}(\mathbb{R} \times N) \quad (t, m) \mapsto ((t, 1), (m, X(t, m))),$$

that is to say, the suspension lifts the vector field to the extended space that also includes t and assigns the time coordinate with a trivial velocity of 1. [1]

of place, for an explicit time-dependence is preferably reserved for *exogenous* effects. The dissipation is inherently part of the system (not an 'input'), which is why we would like to see it treated on the same footing as the mass and the spring.

3-3-2 Symplectization and Liouville structures

In Section 3-2, a contact Hamiltonian system is constructed for the damped harmonic oscillator based on the principles of thermodynamics. However, performing calculations in contact geometry directly is rather cumbersome and uninsightful: to quote Vladimir Arnol'd, 'one is advised to calculate symplectically but to think rather in contact geometry terms'. [17, 32] This is why we will raise the dimension of the three-dimensional contact manifold M by one more to arrive at an even-dimensional manifold, which has a canonical symplectic (and Liouville) structure.

The contact 1-form α considered in Section 3-2 is not expressed in Darboux coordinates. Although it does make sense to use it that way from a physical standpoint (for it directly represents the work done by the damper), using Darboux coordinates simplifies calculations and allows us to use standard coordinate expressions. Because contact structures are only unique up to multiplication by a constant, define

$$\alpha_\gamma := \frac{\alpha}{\gamma},$$

and the substitution

$$q_0 := \frac{U}{\gamma}.$$

Hence, the new contact 1-form is

$$\alpha_\gamma = \mathrm{d}q_0 - p\,\mathrm{d}q\,, \tag{3-23}$$

which means that q, p, q_0 are Darboux coordinates. The latter coordinate is, just a like U in the previous chapter, a special coordinate to keep track of the dissipated energy.

For the construction of the symplectization, we need a space that is slightly different from the manifold M of Section 3-2. Whereas M, is

$Q_\mathrm{e} = Q \times \mathbb{R} \cong \mathbb{R}^2$ for which q and q_0 are coordinates. This is a bundle with the extended configuration space Q_e as a base space, and its fibers consist of all the *contact elements* (i.e. 1-dimensional linear subspaces, or lines through the origin) of the tangent space at a given point. The fibers therefore have a projective structure;

Hence, denoted by $\mathbb{P}(\mathrm{T}^*Q_e)$. The contact form on $\mathbb{P}(\mathrm{T}^*Q_e)$ is given by

which accentuates the special role of the q_0 in the system dynamics. Contact forms are, by their very nature, ambiguous: they represent a distribution of hyperplanes, which coincides with the kernel of the contact form. Multiplication with a nonzero factor yields a different contact form with the same kernel, that is to say, they represent the same contact structure. This is the reason behind the 'projective' nature of contact mechanics[6]. Hence, one may just as well multiply the 1-form with a nonzero factor λ:

$$\lambda(\mathrm{d}q_0 - p\,\mathrm{d}q) \quad \lambda \in \mathbb{R}_0.$$

[6] As explained in Appendix B, the manifold of contact elements is bundle-isomorphic to the projectivization of the cotangent bundle.

The factor λ can be considered to be an extra degree of freedom (leaving the contact structure unaffected), which provides a 'lift' from the odd-dimensional manifold to an even-dimensional one, which is called the symplectification of the contact manifold. [18]

To restate the above in canonical coordinates, choose[7]

$$\rho_0 = \lambda \quad \text{and} \quad \rho = -\lambda p \tag{3-24}$$

such that

$$\vartheta = \rho_0 \, \mathrm{d}q_0 + \rho \, \mathrm{d}q \,, \tag{3-25}$$

which is the Liouville form on T^*M. [15, p. 308] The Liouville form defines a symplectic structure given by[8]

$$\omega = -\,\mathrm{d}\vartheta = \mathrm{d}q_0 \wedge \mathrm{d}\rho_0 + \mathrm{d}q \wedge \mathrm{d}\rho \,.$$

Principal bundles Let us now formalize the Liouville structure in the language of principal bundles. The projectivized cotangent bundle $\mathbb{P}(\mathrm{T}^*M)$ has as its fiber the space of lines passing through the origin. These lines are also the orbits of the multiplicative group $\mathbb{R}_\times$ acting through dilations on the fiber of a bundle with two-dimensional fibers: the cotangent bundle of M without zero section, denoted by T^*_0M. Using the canonical coordinates defined previously, coordinates for T^*_0M are (q_0, q, ρ_0, ρ), where ρ and ρ_0 cannot vanish at the same time (the zero section).[9]

Define the $\mathbb{R}_\times$-*action* $\blacktriangleleft$ on T^*_0M as:

$$\blacktriangleleft : \mathrm{T}^*_0M \times \mathbb{R}_\times \to \mathrm{T}^*_0M : \quad (q_0, q, \rho_0, \rho) \blacktriangleleft \lambda = (q_0, q, \lambda\rho_0, \lambda\rho) \qquad \lambda \in \mathbb{R}_\times,$$

which are referred to as dilations of the fiber.

As illustrated in Figure 3-4, the *orbit space* of T^*_0M with respect to the group action $\blacktriangleleft$ is the space of all points in T^*_0M with all points on the same line through the origin (in the fiber) identified. This space is precisely equal to the projectivization of the cotangent bundle $\mathbb{P}(\mathrm{T}^*M)$. Hence, consider the *principal* $\mathbb{R}_\times$-bundle $\mathrm{T}^*_0M \xrightarrow{\sigma} \mathbb{P}(\mathrm{T}^*M)$

$$\begin{array}{c} \mathrm{T}^*_0M \\ \blacktriangleleft \mathbb{R}_\times \big\uparrow \\ \mathrm{T}^*_0M \\ \big\downarrow \sigma \\ \mathbb{P}(\mathrm{T}^*M) \cong \mathrm{C}M \end{array}$$

[7] The minus sign is there to obtain the convential form of the Liouville form in symplectic geometry.

[8] The nondegeneracy condition on the contact structure guarantees that this structure is indeed symplectic.

[9] An instructive example of principal bundles in system theory is the set of all controllable and observable LTI systems, specified by the matrices A, B, C and D. This is the total space. The base space is the space of all transfer functions of the appropriate dimensions. The correspondence of matrix systems with transfer functions is not injective: the matrix systems are only 'unique' up to a similarity transform, which is a $\mathrm{GL}(n, \mathbb{R})$-action on the manifold of matrix systems. Hence, we have a principal bundle with (i) as total space the observable and controllable systems, (ii) as base space of the transfer functions (of appropriate order) and (iii) a $\mathrm{GL}(n, \mathbb{R})$-action in the form of a similarity transform. [45]

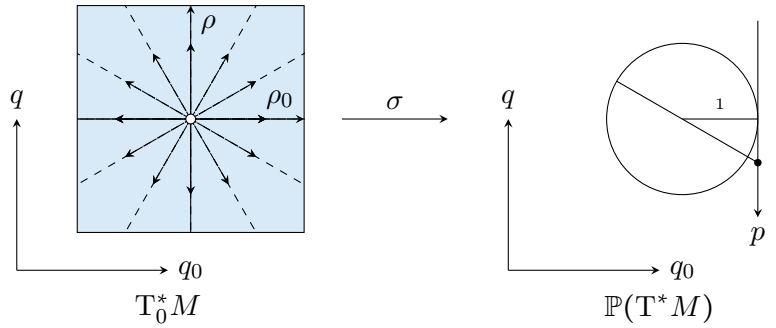


Figure 3-4: Illustration of the principal $\mathbb{R}_\times$-bundle $\mathrm{T}^*_0 M \xrightarrow{\pi} \mathbb{P}(\mathrm{T}^* M)$. The total space $\mathrm{T}^*_0 M$ is the cotangent bundle to M with zero section removed, which is shown on the left. The action by the multiplicative group $\mathbb{R}_\times$ is illustrated by the arrows, for it acts as a scaling (dilation) on all the cotangent variables. The origin is not part of the fiber, for it is part of the zero section. The bundle projection π projects all points that are on the same orbit (straight lines through the origin) to a single point on the base manifold: the projectivized cotangent bundle $\mathbb{P}(\mathrm{T}^* M)$. The former space has a symplectic structure while the latter space has a contact structure. Observe from Equation (3-24) that $p = \rho/\rho_0$, i.e. such that p is a coordinate for the projectivization by stereographic projection, as shown on the right.

The removal of the zero section is required for the group action to be free. The principal bundle $\mathrm{T}^*_0 M \xrightarrow{\pi} \mathbb{P}(\mathrm{T}^* M)$ admits a *fibered symplectic Liouville structure*, given by the Liouville form [15]

$$\vartheta = \rho_0 \, \mathrm{d}q_0 + \rho \, \mathrm{d}q \,,$$

and the associated two-form $\omega = -\,\mathrm{d}\vartheta$. The distinctive feature of these forms that makes this a Liouville structure is that they both commute with the group action $\blacktriangleleft$: [15]

$$(\blacktriangleleft \lambda)^* \vartheta = \lambda \vartheta \qquad \lambda \in \mathbb{R}_* ,$$

which makes them homogeneous forms of degree 1.

The projection map σ of the principal bundle is locally defined as

$$\sigma : \mathrm{T}^*_0 M \to \mathbb{P}(\mathrm{T}^* M) : (q_0, q, \rho_0, \rho) \mapsto (q_0, q, -\rho/\rho_0), \tag{3-26}$$

with $p := -\rho/\rho_0$ a coordinate for the projectivized fiber. This coordinate does not cover the entire fiber: the points for which $\rho_0 = 0$ is missing (in Figure 3-4, this point is the only point on the circle that cannot be projected on the p-axis). However, we will make the deliberate assumption that in our application, ρ_0 is never equal to zero.

Finally, the *Liouville vector field* Z associated with the Liouville structure is the vector field that represents the dilation of the fiber in the symplectization. It is defined as

$$Z = \omega^\sharp(\vartheta) = \rho_0 \frac{\partial}{\partial \rho_0} + \rho \frac{\partial}{\partial \rho}. \tag{3-27}$$

Vector field (components) colinear with the Liouville vector fields are called *vertical*; they represent dissipative action in the system. After the vertical components are removed, they remaining vector field is called *horizontal*.

[Liouville automorphisms, commute with the Liouville vector field -> very important in the chapter about split-quaternions]

To summarize, we lifted the original system with symplectic structure $\mathrm{d}q \wedge \mathrm{d}p$ to a contact manifold through the addition of a gauge variable q_0. We then symplectified the contact manifold to a four-dimensional system, with 'positions' (q_0, q) and 'momenta' (ρ_0, ρ).

3-3-3 Homogeneous Hamiltonian systems

The theoretical construction of the past section serves an important purpose, because it is the symplectified space which is the proper setting for the Caldirola-Kanai Hamiltonian discussed in Section 3-3-1. Along with the symplectification of the contact structure described in the past section, we can do the same with a contact Hamiltonian system.

There is a one-to-one correspondence between contact Hamiltonians on $\mathbb{P}(\mathrm{T}^*M)$ and a special class of Hamiltonians on the symplectified space $\mathrm{T}^*_0 M$. These are the Hamiltonians which are *homogeneous* in the cotangent variables with degree 1:[10]

$$\mathcal{H}(q_0, q, \lambda\rho_0, \lambda\rho) = \lambda\,\mathcal{H}(q_0, q, \rho_0, \rho) \quad \text{or} \quad \pounds_Z \mathcal{H} = \mathcal{H}, \tag{3-28}$$

with $\lambda \in \mathbb{R}_0$, $H \in C^\infty(\mathrm{T}^*_0 M)$ and Z defined according to Equation (3-27). Given that H is indeed homogeneous of degree 1, this correspondence is in canonical coordinates:

$$\mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0\, H\!\left(q_0, q, -\frac{\rho}{\rho_0}\right) \tag{3-29}$$

where $\mathcal{H} \in C^\infty(\mathrm{T}^*_0 M)$, $H \in C^\infty(\mathbb{P}(\mathrm{T}^*M))$ and $p = -\rho/\rho_0$ is a coordinate for the projectivized fiber. Likewise, there is also a direct correspondence between the vector fields generated by these Hamiltonians, and therefore the system dynamics. This is the reason why we go through the trouble of symplectification in the first place, it offers significant computational advantages. It is possible to derive the contact equations directly (as Bravetti et al. [37] does), but it does not offer the same amount of insight as its symplectified counterpart. [18, 32]

Now, recall the Caldirola-Kanai Hamiltonian in Equation (3-20). Instead of assuming a direct time-dependence, we will think of the time-dependence as the gauge momentum, i.e. ρ_0 " = " $-\mathrm{e}^{\gamma t}$. However, we will now consider ρ_0 to be a coordinate in its own right, instead of directly using the expression above — the equality sign must therefore not be taken too literally. The Caldirola-Kanai Hamiltonian is then written as (cf. Equation (3-20)):

$$-\rho_0\left[\frac{1}{2m}\left(-\frac{\rho}{\rho_0}\right)^2 + \frac{1}{2}kq^2\right].$$

[10] This is a consequence of the Euler theorem for homogeneous functions. If $\mathcal{H} = \mathcal{H}(\boldsymbol{q}, \boldsymbol{\rho})$ is homogeneous of degree r in $\boldsymbol{\rho}$, then

$$\sum_{i=1}^{n} \rho_i \frac{\partial \mathcal{H}}{\partial \rho_i} = r\mathcal{H}.$$

Therefore, for homogeneity of degree 1, we have:

$$\pounds_Z \mathcal{H} = Z(\mathcal{H}) = \sum_{i=1}^{n} \rho_i \frac{\partial \mathcal{H}}{\partial \rho_i} = \mathcal{H} \quad \text{with} \quad Z := \sum \rho_i \frac{\partial}{\partial \rho_i}.$$

The correspondence between the

The motivation to make this particular choice is twofold: first, observe that $\mathcal{H}$ is homogeneous in the cotangent variables ρ_0, ρ, and second, that their fraction yields the *real* momentum: $p = -\rho/\rho_0$. However, we must acknowledge a potential dependence on q_0, since we want to convert the explicitly time-dependent Hamiltonian into a contact Hamiltonian. We therefore add the arbitrary function $f = f(q_0)$, whose value is to be determined later, and also multiply by ρ_0 to maintain homogeneity. The homogeneous Hamiltonian $\mathcal{H}$ is then:

$$\mathcal{H} : \mathrm{T}^*_0 M \to \mathbb{R} : \quad \mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0 \left[\frac{1}{2m} \left(-\frac{\rho}{\rho_0} \right)^2 + \frac{1}{2} k q^2 + f(q_0) \right]. \tag{3-30}$$

Using the correspondence given by Equation (3-29), the homogeneous Hamiltonian may be 'projected' to the contact Hamiltonian H:

$$H : \mathbb{P}(\mathrm{T}^* M) \to \mathbb{R} : \quad H(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2} k q^2 + f(q_0). \tag{3-31}$$

Numerically, this contact Hamiltonian is the same as the Hamiltonian for an undamped mass-spring system but it is defined on the contact manifold that also takes into account the gauge variable q_0.

Equations of motion Now to derive the equations of motion. As mentioned, this is easiest in the symplectified space because Hamilton's equations can be readily applied (the reader can consult Appendix B for the direct derivation). Because we are using canonical coordinates, the Hamiltonian vector field

$$X_{\mathcal{H}} = \omega^\sharp(\mathrm{d}\mathcal{H}) \tag{3-32}$$

corresponds to Hamilton's equations in the familiar form:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q}, \quad \frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_0}, \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \rho}, \quad \frac{\mathrm{d}q_0}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \rho_0}. \tag{3-33}$$

Observe that this motivates why one has to take the partial with respect to the 'other' momentum in the Caldirola-Kanai momentum: we are dealing with a specific instance of a more general class of homogeneous coordinates of the cotangent variables. Of course, the variable of interest is the actual momentum p, not the scaled version ρ. The time-derivative of p can be written in terms of ρ and ρ_0, completely analogous to Equation (3-22):

$$p = -\rho/\rho_0 \quad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho}{\rho_0^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{p}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}t}. \tag{3-34}$$

Given Equation (3-30), the partial derivatives of $\mathcal{H}$ and H are related through the by relations: [18]

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial q} &= -\rho_0 \frac{\partial H}{\partial q}, \\
\frac{\partial \mathcal{H}}{\partial q_0} &= -\rho_0 \frac{\partial H}{\partial q_0}, \\
\frac{\partial \mathcal{H}}{\partial \rho} &= -\rho_0 \frac{\partial H}{\partial p}\frac{\partial p}{\partial \rho} = \frac{\partial H}{\partial p}, \\
\frac{\partial \mathcal{H}}{\partial \rho_0} &= -H - \rho_0 \frac{\partial H}{\partial p}\frac{\partial p}{\partial \rho_0} = -H - \frac{\partial H}{\partial p}\frac{\rho}{\rho_0} = \frac{\partial H}{\partial p} p - H.
\end{aligned} \tag{3-35}$$

Hence, the *contact* equations of motion can be found by combining of Equation (3-22) and Equation (3-35):

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{\partial H}{\partial p}\\ \frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{1}{\rho_0}\frac{\partial \mathcal{H}}{\partial q} + \frac{p}{\rho_0}\frac{\partial \mathcal{H}}{\partial q_0} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial q_0}\\ \frac{\mathrm{d}q_0}{\mathrm{d}t} &= \frac{\partial H}{\partial p}p - H.\end{aligned} \tag{3-36}$$

Some observations are important to note:

- The evolution of the position q remains the same as for the 'normal' (undamped) case.
- The evolution of the momentum operator picks up a term that is depends on presence of the gauge variable in the contact Hamiltonian.
- The evolution of the gauge variable is equal to the Legendre transformation of the contact Hamiltonian with respect to p.

We are now ready to determine the nature of the as of yet unknown function f to obtain the correct equations of motion. By comparing Equation (3-34) and Equation (3-36), the following relation must hold:

$$\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = \frac{\partial H}{\partial q_0} = \frac{\mathrm{d}f}{\mathrm{d}q_0}.$$

Furthermore, since we initial 'substituted' $\mathrm{e}^{\gamma t}$ in favor of ρ_0, the left hand side of the equation should be equal to γ for it to be consistent with the Caldirola-Kanai Hamiltonian. Hence, we know that $\frac{\mathrm{d}f}{\mathrm{d}q_0} = \gamma$, or $f(q_0) = \gamma q_0$ up to a constant, which we choose to be zero. Although this may seem like an odd construction, the only thing we did is made the contact equation of motions equivalent with the time-dependent equations of motion. Now, the fact that $\rho_0 = \mathrm{e}^{\gamma t}$, ceases to be an a priori assumption, and is derivable through Hamilton's equations:

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_0} = \gamma\rho_0 \quad \Rightarrow \quad \rho_0 = \mathrm{e}^{\gamma t} + C$$

As such, we have for the contact Hamiltonian:

$$H : \mathbb{P}(\mathrm{T}^*M) \to \mathbb{R} : \quad H(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 + \gamma q_0, \tag{3-37}$$

and this is precisely Bravetti's result. For the homogenous Hamiltonian on the symplectified space, we have:

$$\mathcal{H} : \mathrm{T}_0^*M \to \mathbb{R} : \quad \mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0\left[\frac{1}{2m}\left(-\frac{\rho}{\rho_0}\right)^2 + \frac{1}{2}kq^2 + \gamma q_0\right]. \tag{3-38}$$

The Hamiltonian vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathrm{T}_0^*M)$ is then, using Equation (3-38) and Equation (3-32):

$$X_{\mathcal{H}} = -\frac{1}{m}\frac{\rho}{\rho_0}\frac{\partial}{\partial q} + \left[\frac{1}{2m}\left(\frac{\rho}{\rho_0}\right)^2 - \frac{1}{2}kq^2 - \gamma q_0\right]\frac{\partial}{\partial q_0} + \rho_0 kq\frac{\partial}{\partial \rho} + \gamma\rho_0\frac{\partial}{\partial \rho_0}.$$

The contact Hamiltonian vector field $X_H \in \mathfrak{X}(\mathbb{P}(\mathrm{T}^*M))$ is obtained either by using the contact Hamilton equations given by Equation (3-36), or by using the pushforward of the projection map σ:

$$X_H = \sigma_* X_{\mathcal{H}} = \frac{p}{m}\frac{\partial}{\partial q} + \left(\frac{p^2}{2m} - \frac{1}{2}kq^2 - \gamma q_0\right)\frac{\partial}{\partial q_0} - (kq + \gamma p)\frac{\partial}{\partial p}$$

This is essentially the equivalent of the time-dependent transformation performed in Section 3-3-1. Clearly, these yield the correct equations of motion for the damped harmonic oscillator.

Mechanical energy One of the computational advantages of the symplectified space is the fact that 'regular' Poisson brackets can be used in contrast to their slightly unwieldy contact counterparts.[11] The mechanical energy, denoted by E, is equal to

$$E : \mathbb{P}(\mathrm{T}^*M) \to \mathbb{R} : \quad E(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2,$$

With some abuse of notation, we will denote both functions on the contact space and their lifted version to the symplectified space by the same symbol. That is to say, the function E both refers to $E \in C^\infty(\mathbb{P}(\mathrm{T}^*M))$ and $(E \circ \sigma) \in C^\infty(\mathrm{T}^*_0)$.

The change of mechanical energy in the system is then readily determined using the Poisson brackets:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \{E, \mathcal{H}\} = \pounds_{X_{\mathcal{H}}} E = -\frac{\gamma}{2m}\left(\frac{\rho}{\rho_0}\right)^2 = -\frac{\gamma}{2m}p^2,$$

which is precisely the dissipative power in the damping element.

TODO

Check minus signs of the change in mechanical energy, should be opposite to heat generated.

TODO

Finish Poincaré lemma discussion.

The importance of the zero section

It may be tempting to disregard the removal of the zero section from the cotangent bundle as a mathematical technicality. It has, however, deep implications for the nature of the Hamiltonian systems that can be defined on it, encoded in the so-called de Rham cohomology groups.

Suppose that Y is a symplectic vector field with ω the symplectic form. The 1-form

$$\xi = Y \lrcorner \omega$$

[11] The Poisson bracket is defined as

$$\{f, g\} = \omega(X_f, X_g) = \pounds_{X_g} f,$$

where X_f and X_g are the Hamiltonian vector fields of f and g. [15]

is necessarily closed, because

$$\mathrm{d}Y \lrcorner \omega = \underbrace{\cancel{\pounds_Y \omega}}_{Y \text{ symp.}} - \underbrace{\cancel{Y \lrcorner \mathrm{d}\omega}}_{\omega \text{ closed}} = 0.$$

The Poincaré lemma (a specific instance of the de Rham cohomology) says that on a *contractible domain*, all closed forms are necessarily also exact (the converse is true on any manifold, for $\mathrm{d}^2 = 0$). This would mean that, if ξ were to be defined on a contractible manifold, it would automatically be an exact form (this is the same as saying that on these types of manifolds, all symplectic vector fields are Hamiltonian). In other words, there must be a function $\mathcal{H}$ such that $\xi = \mathrm{d}\mathcal{H}$.

Integration around curve to show that ξ in our case is not exact. Hard, because over two charts.

Also show that the region of interest is not simply connected in order to use de Rham in the first place. $\mathbb{R}^4/\{(q_0, q, 0, 0)\}$ not simply connected

Another advantage of using the purely symplectic formalism on the lifted space is the fact that the homogeneous Hamiltonian is invariant under the flow it generates, since the explicit time-dependence has been removed:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \{\mathcal{H}, \mathcal{H}\} = \pounds_{X_\mathcal{H}} \mathcal{H} = 0.$$

We may therefore associate the homogeneous Hamiltonian with a constant. By inspection of Equation (3-29),

$$\mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0 \underbrace{\left(C\rho_0^{-1}\right)}_{H} \qquad C \in \mathbb{R}.$$

Recall that we made the assumption earlier that ρ_0 is a function *without zeros*.

The constant C is a degree of freedom in the system that we are free to choose, for the equations of motion will be consistent with any chosen value. This called a *gauge* of the system, and its choice will influence the value of the gauge variable directly. The contact Hamiltonian is not a constant of motion however (at least, for dissipative systems). With some abuse of notation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = C\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -C\frac{\partial \mathcal{H}}{\partial q_0}.$$

Until now, there were no assumptions regarding the value of the damping constant γ: indeed, the 'normal' equations of motion are readily derived when γ is set to zero. We will now add the assumption that there is at least some dissipation present in the system ($\gamma \neq 0$) to assign further intrepretation to the gauge variable. The evolution of q_0 is directly related to the evolution of H by Equation (3-37)

$$q_0 = \frac{1}{\gamma}\left(\rho_0 C - \underbrace{\frac{p^2}{2m} - \frac{1}{2}kq^2}_{E}\right)$$

Because we are free to choose the value of C, let us now make a choice of particular interest; namely $C = 0$. In that case, both $\mathcal{H}$ and H vanish *weakly*;[12] this choice rids us from the additional freedom in C that would also show up in the equation of motion for q_0. Instead, $q_0 = -E/\gamma$; which can be interpreted as the heat dissipated by the system (one can add a suitable initial condition for $q_0(0) = E(0)$ to make this also numerically correct). This 'heat' function will be called Q; we therefore have

$$q_0 = Q/\gamma.$$

The vanishing of the Hamiltonians reflects the energy balance that is maintained throughout the evolution of the system:

$$\begin{aligned} H &= E + Q \\ &= \text{MECHANICAL ENERGY} + \text{DISSIPATED HEAT}. \end{aligned}$$

The fact that H vanishes makes it a constant of motion on par with the symplectified Hamiltonian $\mathcal{H}$. This careful choice of for the gauge variable removes a lot of the ambiguity that is naturally present in contact systems; this rather subtle point is a bit overlooked by past research. [37] Furthermore, the evolution of the dissipated heat is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \gamma \frac{\mathrm{d}q_0}{\mathrm{d}t} = \gamma \left[\frac{1}{2m} \left(\frac{\rho}{\rho_0} \right)^2 - \frac{1}{2} k q^2 - \gamma q_0 \right] = \gamma \frac{1}{2m} \left(\frac{\rho}{\rho_0} \right)^2,$$

which is the expected result.

This very particular choice for the gauge variable may seem a little arbitrary. However, *in general*, the time-rate of change

$$[\![f, g]\!]$$

TODO

Canonical transformation

TODO

Action-angle coordinates

TODO

Generalization using Jacobi problems

3-4 Contact Lagrangian systems

In the classic, symplectic case, the Legendre transformation is used to pass from the Hamiltonian to the Lagrangian formalism and vice versa. This is because the Legendre transform

[12] The weak equality, as opposed to the strong equality, is not maintained under variations. Hence, although the numerical value of the function is zero, its partial derivatives do not necessarily vanish. The reader is referred to Dirac [46] for a more elaborate discussion.

facilitates a mapping between the tangent and cotangent bundle. If the Lagrangian (or Hamiltonian) is (hyper)regular (i.e. the mass matrix is invertible), this mapping is a diffeomorphism. [47]

One would be tempted to use the normal Legendre transformation on the symplectified Hamiltonian $\mathcal{H}$. This approach will meet some problems though:

- A homogeneous function is not regular in the homogeneous variables — naturally, a degree of freedom still resides in the action of the multiplicative group. Therefore, the mapping from the cotangent to the tangent bundle is not a diffeomorphism. Said otherwise, there is not a one-to-one correspondence between the homogeneous momenta and the associated velocities in the Lagrangian description.
- As a consequence of Euler's theorem for homogeneous functions, the Legendre transformation for a homogeneous function is necessarily equal to zero. For any homogeneous function H (of degree 1), Euler's theorem states that

$$\sum_{i=1}^{n} \rho_i \frac{\partial \mathcal{H}}{\partial \rho_i} = \mathcal{H},$$

  i.e. the function is equal to its associated 'action', and therefore the expression for the Legendre transformation vanishes. [46, 48]

There is a better path to take. In essence the Legendre transform is (and was originally meant to be) a *contact transformation*.

3-5 Generalization to other mechanical systems

Chapter 4

Split-Quaternions as Dynamical Systems

In this chapter, the geometric connection is made between the algebra of split-quaternions and the qualitative behavior of two-dimensional linear dynamical systems.

4-1 Split-quaternion algebra

4-1-1 Basic properties

Like conventional quaternions, the split-quaternions form a number system that consists of linear combinations of four basis elements, which will be denoted by 1, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.[1] The algebra of split-quaternions is associative but not commutative — formally speaking, we are dealing with an algebraic structure called a *noncommutative ring*. The multiplication table for the split-quaternion algebra is shown in Table 4-1. The set of split-quaternions is denoted by $\hat{\mathbb{H}}$, since $\mathbb{H}$ is reserved for conventional quaternions.[2]

Table 4-1: Multiplication table for the split-quaternion algebra.

	1	$\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{k}}$
1	1	$\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{k}}$
$\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\imath}}$	-1	$\hat{\boldsymbol{k}}$	-$\hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{\jmath}}$	-$\hat{\boldsymbol{k}}$	1	-$\hat{\boldsymbol{\imath}}$
$\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{\imath}}$	1

[1] Even though they behave similarly, the imaginary unit i is not to be confused with the split-quaternion basis element $\hat{\boldsymbol{\imath}}$, because they belong to different number systems.

[2] The '$\mathbb{H}$' is in honour of sir William Rowan Hamilton, who also developed the Hamiltonian formalism: the fruits of his work truly form the central theme in this thesis. [49]

The important distinction from conventional quaternions resides in the diagonal elements of Table 4-1. For quaternions, all the nonreal basis elements square to -1, this is not the case for the split-quaternions (only $\hat{\boldsymbol{\imath}}$ does). This is precisely the reason why split-quaternions are 'split', for this difference in sign makes their norm (to be defined later) into an indefinite form. That is to say, whereas quaternions have a 'metric signature' (in a very imprecise sense of the word metric) of $(+,+,+,+)$, for the split-quaternions we have $(+,+,-,-)$. The distinctive 'metric' signature makes the algebra of split-quaternions different from its conventional quaternion counterpart.

Dihedral group The basis elements of the split-quaternions $\left\{1, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\right\}$ form a finite group under multiplication, namely the *dihedral group* D_4, which represents all the symmetries of a square: the identity, a 90-degree rotation and two reflections, as illustrated in Figure 4-1. [50]

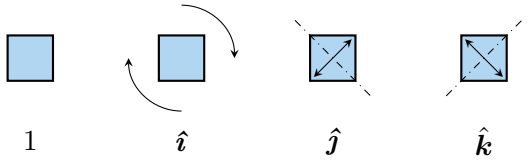


Figure 4-1: The dihedral group D_4 is the symmetry group of a square. This group is isomorphic to the group formed by $1, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ under multiplication.

The structure of the dihedral group can be visualized by its *cycle graph* in Figure 4-2. Many important properties of the split-quaternion algebra and the applications in this chapter can be traced back to the shape of this cycle graph. One example is the split nature of the quaternions: the $\hat{\boldsymbol{\imath}}$-element generates an order-4 cycle, while $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ generate order-2 cycles (in contrast, the cycle graph for conventional quaternions is entirely symmetric for all these elements). [50]

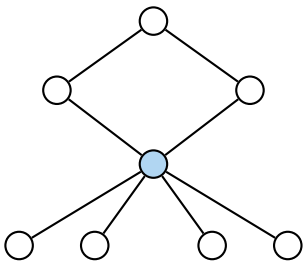


Figure 4-2: Cycle graph of the dihedral group. There are five cycles: one of order four which represents the rotations (or the element $\hat{\boldsymbol{\imath}}$), and four order-2 cycles, which are all the possible reflections. The colored element represents the identity.

Split-quaternion norm Complex numbers have a real and imaginary part. Likewise, (split)-quaternions have a similar notion: a *scalar* (or real) and *vector* (or imaginary) components. For an arbitrary split-quaternion $a \in \hat{\mathbb{H}}$, [51]

$$a = a_0 + a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$$

the real part is $\text{sca}(h) = a_0$ and the vector part is $\text{vec}(a) = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$. For convenience, the vector part will be referred to in traditional bold vector notation:

$$\boldsymbol{a} \equiv \text{vec}(a) \equiv a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}.$$

Furthermore, for every split-quaternion there is a unique *conjugate*

$$a^* = \text{sca}(a) - \text{vec}(a) = a_0 - a_1\hat{\boldsymbol{\imath}} - a_2\hat{\boldsymbol{\jmath}} - a_3\hat{\boldsymbol{k}},$$

which we require to define the *squared split-quaternion norm*:

$$\mathcal{N} : \hat{\mathbb{H}} \to \mathbb{R} : \quad \mathcal{N}(a) \equiv aa^* = a_0^2 + a_1^2 - a_2^2 - a_3^2. \tag{4-1}$$

As mentioned, this norm is not positive definite, in stark contrast to quaternions (or complex numbers, for that matter). Split-quaternions can be categorized into three regimes based on their norm being negative, zero or positive. In the tradition of special relativity, these regimes are named[3] [52, 53]

- *timelike* if $\mathcal{N}(a) > 0$,
- *lightlike* if $\mathcal{N}(a) = 0$,
- *spacelike* if $\mathcal{N}(a) < 0$.

The *split-quaternion norm* is then defined as

$$\|a\| \equiv \sqrt{|\mathcal{N}(a)|}.$$

Vector norm Apart from the split-quaternion norm, we may also define a (square) norm that only considers the *vector part* of the split-quaternion. This squared norm is defined in accordance with the overall squared split-quaternion norm given by Equation (4-1):

$$\mathcal{N}(\boldsymbol{a}) = a_1^2 - a_2^2 - a_3^2.$$

The notation used here is not abusive: $\boldsymbol{a}$ simply refers to the split-quaternion with the same vector part as a but with zero scalar part. We can therefore use the same function with no ambiguity. Likewise, the vector norm is $\|\boldsymbol{a}\| = \sqrt{|\mathcal{N}(\boldsymbol{a})|}$.

The quadratic form of the squared vector norm is not positive-definite either; in the the same vein as before, we can therefore classify split-quaternions by the 'sign' of their vector part again. We refer to (vectors in) these regimes as *timelike (vectors)*, *spacelike (vectors)* and *lightlike (vectors)* in the same fashion.

In contrast to the larger space of split-quaternions, the space of vectors *does* have a traditional Lorentz (i.e. 'special relativity') structure, but in three dimensions instead of four. This is because the signature of the squared vector norm only has one minus sign instead of two. The

[3] Spacetime is also four-dimensional, but the signature of the Minkowski metric is different from the split-quaternion signature: it is either $(-,+,+,+)$ or equivalently $(+,-,-,-)$ depending on the sign convention we choose to follow. However, the terminology (i.e. spacelike, timelike, lightlike) still applies.

above expression is equivalent the Lorentz norm applied to a vector in $\mathbb{R}^3$; we will denote $\mathbb{R}^3$ equipped with the Lorentz norm by $\mathbb{R}^{1,2}$, and call it the Lorentzian three-space. [51]

Observe that $\mathcal{N}(a) < 0 \Rightarrow \mathcal{N}(\boldsymbol{a}) < 0$; that is to say, *a spacelike split-quaternion always has a spacelike vector part.* The converse is not necessarily true. Along the same line, a lightlike split-quaternion can only have a lightlike or spacelike vector part. All possible regime combinations for split-quaternions and their vector parts are listed in Table 4-2. This classification is important because, as discussed in Section 4-2, this classification is precisely equivalent to the qualitative classification of dynamical systems. It will play a central role throughout this chapter.

Table 4-2: All the possible combinations of the regime of a split-quaternion and its vector part. Spacelike split-quaternions can only have a spacelike vector, while lightlike split-quaternions can only have lightlike or spacelike vector parts.

		$\mathcal{N}(\boldsymbol{a})$		
		spacelike	*lightlike*	*timelike*
$\mathcal{N}(a)$	*spacelike*	①	—	—
	lightlike	②	③	—
	timelike	④	⑤	⑥

4-1-2 Relation with two-dimensional matrix algebra

The usefulness of split-quaternions (for our purpose) originates from the fact that the algebra of split-quaternions is isomorphic to the algebra of real two-dimensional matrices. This underlies this entire chapter, for it allows us to find an alternative for the traditional matrix description of linear dynamical systems.

Formally, an algebra is a vector space combined with a vector space V over a field $\mathbb{F}$, combined with an addition operation, scalar multiplication, and an $\mathbb{F}$-bilinear product operation $V \times V \to V$. [54]

- The split-quaternion algebra is an algebra over the field real numbers ($\mathbb{F} = \mathbb{R}$), where the multiplication is governed by the split-quaternion multiplication rules (see Table 4-1).
- The set of 2×2-matrices also constitutes an $\mathbb{R}$-vector space; matrix multiplication makes it into an algebra.

An algebra isomorphism is an isomorphism between vector spaces that also commutes with the respective product operations in both vector spaces. If $(V, \bullet)$ and $(W, \diamond)$ are vector spaces equipped with their product operations, then $\phi : V \to W$ is an algebra isomorphism if (i) ϕ is a vector space isomorphism between V and W, and (ii) [55]

$$\phi(v_1 \bullet v_2) = \phi(v_1) \diamond \phi(v_2) \qquad v_1, v_2 \in V.$$

In the case of the split-quaternions and two-dimensional matrices, it is sufficient to map the basis elements of the split-quaternions to four linearly independent 'basis' matrices, and show

that the resulting matrices observe the same multiplication rules as defined in Table 4-1. Indeed, define the mapping ϕ by

$$\begin{aligned} \phi : \hat{\mathbb{H}} \to \mathbb{R}^{2\times 2} : \quad 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{\boldsymbol{\imath}} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \hat{\boldsymbol{\jmath}} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\boldsymbol{k}} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4-2}$$

It is easily verified that (i) these matrices span $\mathbb{R}^{2\times 2}$ and (ii) that the multiplication rules for split-quaternions are in accordance when translated to the respective matrices under matrix multiplication. Due to the bilinearity of the product, any linear combination of the basis elements will therefore satisfy the rules as well. Hence, we have established an algebra isomorphism between the split-quaternions and the 2×2-matrices. Based on this mapping for the basis vectors, a general quaternion maps to

$$\phi(a_0 + a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}) \quad = \quad \begin{pmatrix} a_0 + a_3 & a_1 + a_2 \\ a_2 - a_1 & a_0 - a_3 \end{pmatrix}. \tag{4-3}$$

Likewise, the inverse mapping on an arbitrary matrix yields

$$\phi^{-1}\begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} \quad = \quad \frac{b_0 + b_3}{2} + \left(\frac{b_1 - b_2}{2}\right)\hat{\boldsymbol{\imath}} + \left(\frac{b_1 + b_2}{2}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{b_0 - b_3}{2}\right)\hat{\boldsymbol{k}}. \tag{4-4}$$

One of the powerful features of the mapping ϕ is that it maps natural properties of the split-quaternion to natural properties of the associated matrix. Hence, given that $A = \phi(a)$ with $a \in \hat{\mathbb{H}}$ and $A \in \mathbb{R}^{2\times 2}$, we have the following properties:

- The *conjugate* of the split-quaternion maps to the *adjugate* of the matrix:[4]

$$\phi(a^*) = \mathrm{adj}(A).$$

- The *trace* of the matrix coincides with the *real or scalar part* of the split-quaternion:

$$\mathrm{sca}(a) = a_0 = \frac{\mathrm{tr}(A)}{2}.$$

- The *determinant* of the matrix is equal to the *squared norm* of the split-quaternion:

$$\mathcal{N}(a) = \det(A).$$

- The equivalence of the determinant and the split-quaternion norm hints at the fact that the multiplicative inverse of a split-quaternion does not always exist: only when its norm is nonzero. In that case, it is clear that

$$\phi\left(a^{-1}\right) = A^{-1} \qquad \mathcal{N}(a) \neq 0.$$

  The determinant property also shows us what the regime will be of the product of two split-quaternions; this is shown in Table 4-3.

[4]The adjugate of a matrix is the transpose of its cofactor matrix. [56]

Table 4-3: Regime transition under the action of split-quaternion multiplication. The timelike split-quaternions form a group under multiplication, the timelike and spacelike split-quaternions do not: timelike split-quaternions do not have an inverse and the spacelike split-quaternions are not closed.

×	*space*	*light*	*time*
space	time	light	space
light	light	light	light
time	space	light	time

- The eigenvalues of a 2×2-matrix can be expressed in terms of its trace and its determinant:

$$\lambda_A = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}.$$

The argument of the square root is equal to the *negative of the squared vector norm* of a. We therefore have:

$$\lambda_A = \frac{2a_0 \pm \sqrt{4a_0^2 - 4\mathcal{N}(a)}}{2} = \begin{cases} a_0 \pm \mathrm{i}\|\boldsymbol{a}\| & \boldsymbol{a} \text{ timelike}, \\ a_0 \pm 0 & \boldsymbol{a} \text{ lightlike}, \\ a_0 \pm \|\boldsymbol{a}\| & \boldsymbol{a} \text{ spacelike}. \end{cases} \tag{4-5}$$

Hence, the 'real' (scalar) and the magnitude of the 'imaginary' (vector) parts of the quaternion coincide with the real and imaginary part of the eigenvalues of the matrix.

The algebra of 2×2-matrices (or equivalently, of the split-quaternions) also consitute the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ of the two-dimensional general linear group $\mathrm{GL}(2,\mathbb{R})$. Furthermore, the traceless matrices, or equivalently, the split-quaternions with zero real part form the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ of the special linear group $\mathrm{SL}(2,\mathbb{R})$. These are the volume-preserving automorphisms on $\mathbb{R}^2$. Because in $\mathbb{R}^2$, volume and area coincide, the special linear group and the symplectic group $\mathrm{Sp}(1)$ are equivalent. For higher dimensions, this is not the case: area preservation is generally a stronger condition than volume preservation. The Lie algebra elements of the symplectic group are called Hamiltonian matrices; therefore, split-quaternions without real part are referred to as *Hamiltonian*.

4-2 Split-quaternion representation of dynamical systems

4-2-1 The algebra of vector fields

The isomorphism between the split-quaternions and the algebra of two-dimensional square matrices exposed in the preceding section can be used to develop an alternative representation of linear dynamical systems. Indeed, an autonomous dynamical system is defined by a *vector field* on the state space. If this vector field is a linear mapping from the state space into the tangent space, it can be represented by a matrix. These vector fields form a vector space on their own, spanned (for example) by the four basis elements shown in Equation (4-2). Each

of the basis elements 1, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ corresponds to a specific 'basis' vector field, denoted by X_1, $X_{\hat{\boldsymbol{\imath}}}$, $X_{\hat{\boldsymbol{\jmath}}}$ and $X_{\hat{\boldsymbol{k}}}$ respectively. The basis vector fields are shown in Figure 4-3.

The vector field element X_1, corresponding to the identity element is an infinitesimal dilation, while $X_{\hat{\boldsymbol{\imath}}}$ represents an infinitesimal clockwise rotation, and $X_{\hat{\boldsymbol{\jmath}}}$ and $X_{\hat{\boldsymbol{\jmath}}}$ are infinitesimal 'squeeze mappings', hyperbolic rotations or *Lorentz transformations* along two differents sets of principal axes. The binary operation of matrix multiplication translates to the composition of the vector fields.

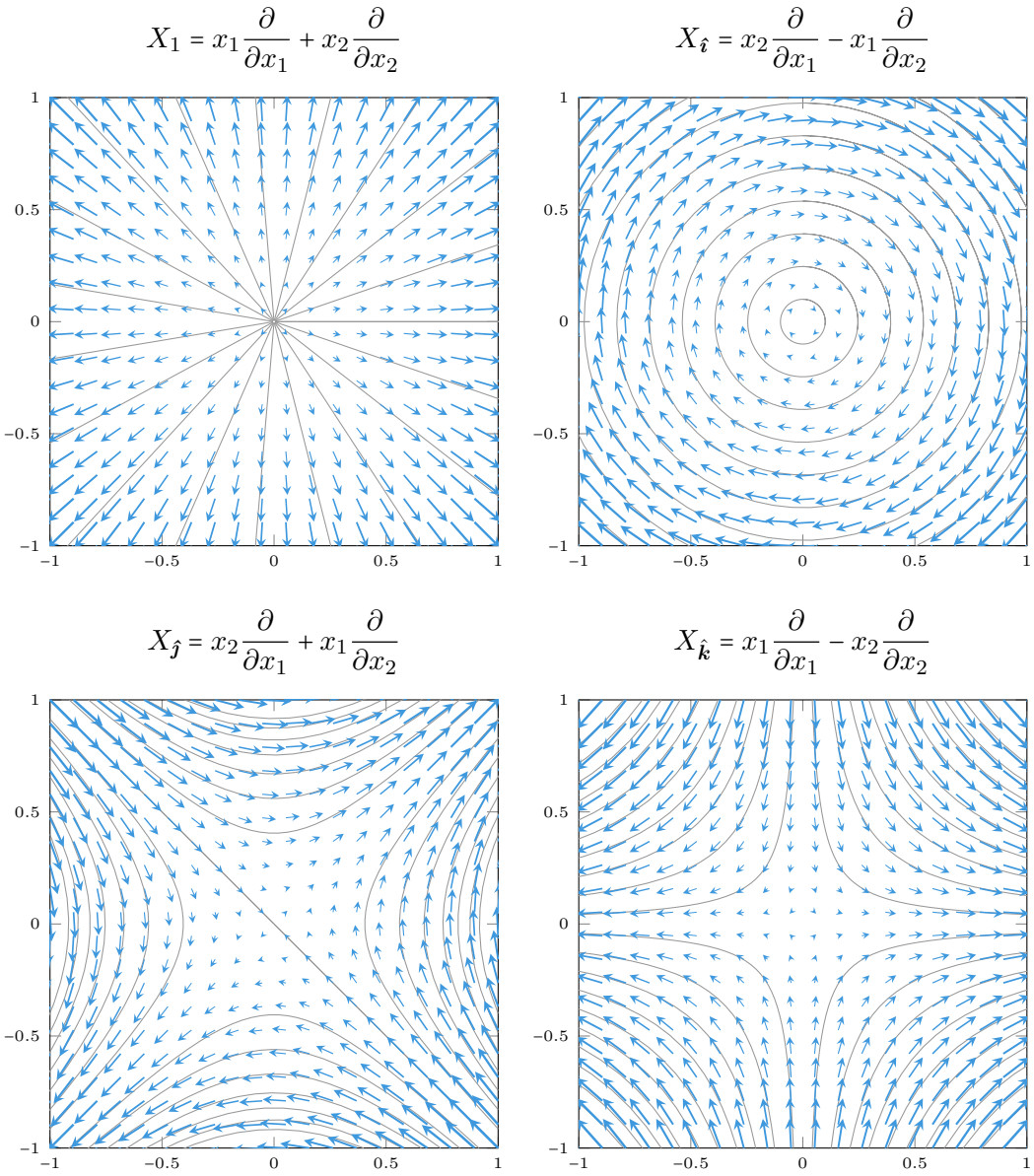


Figure 4-3: Basis vector fields corresponding to the basis elements of the split-quaternions.

Apart from the multiplication operation for split-quaternions and matrices, we can also define the *commutator* of a binary operation, which measures exactly by how much two elements fail to commute. For split-quaternions, matrices and vector fields the commutator is defined

as (in that order):

$$\begin{aligned}[a,\, b] &= ab - ba, \quad a, b \in \hat{\mathbb{H}},\\ [A,\, B] &= AB - BA, \quad A, B \in \mathbb{R}^{2\times 2},\\ [X,\, Y] &= \pounds_X Y, \quad X, Y \in \mathfrak{X}(M),\end{aligned} \tag{4-6}$$

for some smooth manifold M of the appropriate dimension. For vector fields, the commutator is also referred to as the *Lie bracket* (it therefore defines the *Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$). The commutation relations (or *structure constants*) for the basis vector fields are [54]

$$[X_1,\, X_{\hat{\imath}}] = \left[X_1,\, X_{\hat{\jmath}}\right] = \left[X_1,\, X_{\hat{k}}\right] = 0, \tag{4-7}$$

$$\left[X_{\hat{\imath}},\, X_{\hat{\jmath}}\right] = 2X_{\hat{k}}, \qquad \left[X_{\hat{\imath}},\, X_{\hat{k}}\right] = -2X_{\hat{\jmath}}, \qquad \left[X_{\hat{\jmath}},\, X_{\hat{k}}\right] = -2X_{\hat{\imath}}. \tag{4-8}$$

Of course, these commutation relations are exactly the same for the corresponding split-quaternion or matrix basis elements. Scalar multiples of the identity element commute with every other element of the algebra; they are in the *center* of the algebra. Importantly, the associate vector field also commutes with all the other vector fields.

4-2-2 Classification of dynamical systems

The classification of two-dimensional linear dynamical systems is important, for they also locally represent the fixed points of general nonlinear systems. Traditionally, this decomposition is done according to the eigenvalues of the state transition matrix matrix A, or equivalently, through a Poincaré diagram as shown in Figure 4-4. Because the split-quaternion norms are directly related to the real and imaginary part of the eigenvalues of the associated matrix, this classification is more naturally done in the realm of split-quaternions, based on their squared (vector) norm, on par with the regimes defined in Table 4-2.

Spacelike split-quaternion norm

① For spacelike split-quaternions, there is only one possibility: a negative split-quaternion norm corresponds to a negative determinant, which means that the fixed point is a *saddle*. We can distinguish one particular case: if the scalar part of the split-quaternion is zero ($a_0 = 0$), the saddle is 'balanced', and generates a proper *squeeze mapping*, which is a symplectomorphism of the phase space. The split-quaternion is therefore Hamiltonian. An example of the latter is the linearization of the unstable fixed point of a rotational pendulum.

Lightlike split-quaternion norm

② *Spacelike vector norm*: in this case, there is not just a fixed point but a fixed line in the phase space. This fixed line is stable or unstable depending on the sign of the scalar part of the quaternion.

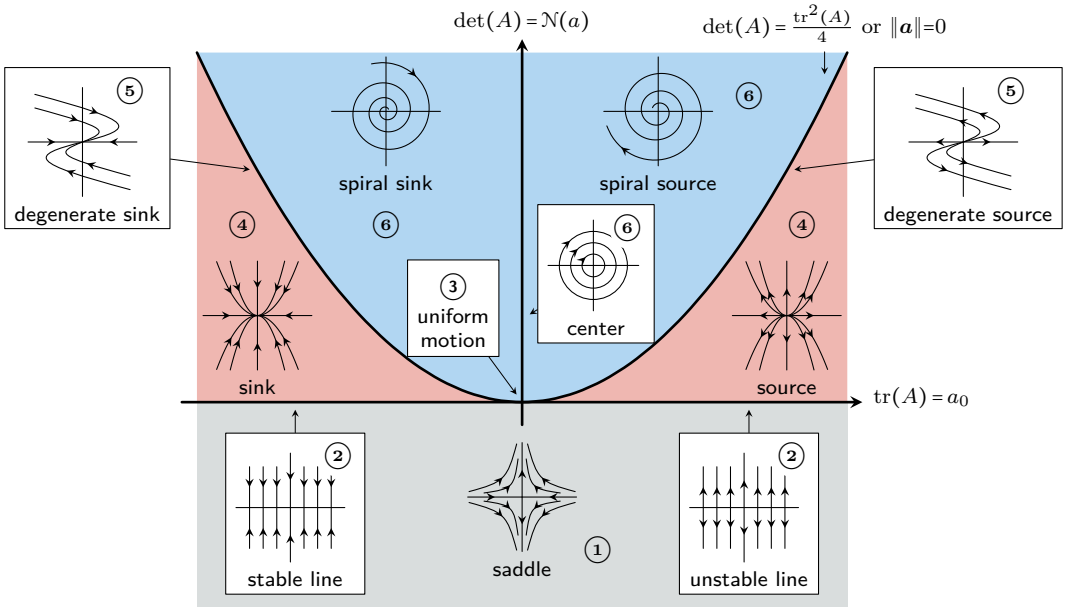


Figure 4-4: The classic Poincaré diagram, based on the conventional classification of fixed points based on the trace and determinant of the state transition matrix A. The corresponding split-quaternion regimes defined in Table 4-2 are displayed as well. The determinant axis coincides with the squared norm of the split quaternion being 0, while all the points on the parabolic line correspond to split-quaternions with zero *vector* norm. A further distinction is made with the scalar part of the split-quaternion, which, for each of the regimes, determines (asymptotic) (in)stability.

③ *Lightlike vector norm*: this case is degenerate of the second degree; it coincides with the origin in the Poincaré diagram. The associated vector field is purely translational. An example is an object in uniform motion.

Timelike split-quaternion norm

④ *Spacelike vector norm*: this case gives rise to eigenvalues that are purely real; the fixed point is called a *node*. Depending on the sign of the scalar part, the fixed point can be an unstable node or *source* ($a_0 > 0$) or a stable node or *sink* ($a_0 < 0$). An example of such a system is the overdamped harmonic oscillator.

⑤ *Lightlike vector norm*: the eigenvalues of the associated matrix are real and equal; this type of fixed point is named a *degenerate node*. More specifically, in the unstable case ($a_0 > 0$) it is called a *degenerate source*, while in the stable case it is referred to as a *degenerate sink*. An example is a critically damped harmonic oscillator.

We can also relate the vector norm to the Jordan form of the associated matrix. Recall that the Jordan form is 'special' if a matrix is not completely diagonalizable: in the case of two identical eigenvalues, their geometric multiplicity is equal to one instead of two. In terms of the corresponding matrix, the vector norm is equal to:

$$\mathcal{N}(\boldsymbol{a}) = \det\left(A - \frac{\mathrm{tr}(A)}{2}\right)$$

when two eigenvalues are identical, $\mathrm{tr}(A) = 2\lambda$, the above expression vanishes and the split-quaternion has a lightlike vector. If the matrix is nevertheless diagonalizable, it must be a scalar multiple of the identity. The corresponding split-quaternion is then purely real — it has no vector part (of course, the vector is then lightlike in a trivial sense). The associate fixed point is then a *proper node*. Conversely, if the vector norm is zero but the vector part not, the matrix is not diagonalizable, and the fixed point is an *improper node*.

⑥ *Timelike vector norm*: this really is the only general case where the eigenvalues of A are complex. If $a_0 = 0$, the eigenvalues are imaginary and the fixed point is a *center*. Likewise, for $a_0 > 0$ it is an *unstable spiral node* and for $a_0 < 0$ a *stable spiral node*. An example is an underdamped (or even undamped) harmonic oscillator.

It is clear from the present discussion that the split-quaternions offer a very natural representation of linear dynamical sytems, and their natural properties translate directly to distinctions in the qualitative behaviour of these systems.

4-2-3 The exponential function of split-quaternions

Just like the concept of the exponential function was originally generalized for square matrices, we can do the same for split-quaternions in an analogous manner. As such, the *split-quaternion exponential function* is defined as

$$\exp(a) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} a^k \qquad a \in \hat{\mathbb{H}}.$$

Because this definition is identical to the one for matrices, we may, as a result of the isomorphism defined in Section 4-1-2, also expect the exact same result; i.e.

$$\exp(a) = \phi^{-1}(\exp(\phi(a)))$$

where the second exponential function refers to the matrix exponential instead.

To evaluate the exponential function of a split-quaternion, let us first use the following property of the matrix exponential [57]

$$AB = BA \quad \Rightarrow \quad \exp(A + B),$$

i.e. we can only 'split' the exponential of a sum if the two elements *commute*. We can regard an arbitrary split-quaternion

$$a = a_0 + a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$$

as the sum of a_0 and $a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$. The real part is distinguished from the other three parts in the sense that it commutes with every other element (cf. Section 4-2-1). We may therefore use the former property and apply it to the split-quaternion exponential as well:

$$\exp(a) = \mathrm{e}^{a_0} \exp\big(a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}\big).$$

We therefore only have to be considered with the evaluation exponential of $\boldsymbol{a}$. To do so, observe that we can consider the vector part of a split-quaternion to be a split-quaternion in

its own right, but with zero real part. This means that $\boldsymbol{a}^* = -\boldsymbol{a}$, and the squared vector norm is simply the negative of the square of the vector part:

$$\mathcal{N}(\boldsymbol{a}) = \boldsymbol{a}\boldsymbol{a}^* = -\boldsymbol{a}^2.$$

Let us now introduce the concept of *unit split-quaternion vectors*, which are vector split-quaternions with a vector norm of ±1. The unit vector may be obained by normalization of the vector part:

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{\sqrt{|\mathcal{N}(\boldsymbol{a})|}} \qquad \mathcal{N}(\boldsymbol{a}) \neq 0,$$

which squares to

$$\hat{\boldsymbol{a}}^2 = -\mathcal{N}(\hat{\boldsymbol{a}}) = -\frac{\mathcal{N}(\boldsymbol{a})}{|\mathcal{N}(\boldsymbol{a})|} = -\operatorname{sgn}(\mathcal{N}(\boldsymbol{a})).$$

Normalizing lightlike vectors is not possible, because they have all the same length of zero: there is no point in making the distinction between vector and unit vector. Based on the regime of the vector part, three possiblities arise: [58, 59]

- If $\boldsymbol{a}$ is timelike, then $\hat{\boldsymbol{a}}^2 = -1$. We can therefore say that the unit vector 'behaves' like the imaginary unit i ($\mathrm{i}^2 = -1$). In general, we can identify the split-quaternion (with timelike vector part) $a_0 + \|a\|\hat{\boldsymbol{a}}$ with the *complex number* $a_0 + \|\boldsymbol{a}\|\mathrm{i}$.
- If $\boldsymbol{a}$ is lightlike, then $\boldsymbol{a}^2 = 0$, and the notion of the unit vector is not well-defined. Because the vector is nilpotent with degree 2, it is analogous to the nilpotent unit ε (for which we have that $\varepsilon^2 = 0$). Split-quaternions with timelike vector part can be identified with the *dual number* $a_0 + \varepsilon$.
- Finally, if $\boldsymbol{a}$ is spacelike, then $\hat{\boldsymbol{a}}^2 = 1$. The unit vector behaves like the idempotent unit j, with defining property $\mathrm{j}^2 = 1$ ($\mathrm{j} \notin \mathbb{R}$!).[5] Likewise, a split-quaternion with spacelike vector part is analogous to the *split-complex number* (or hyperbolic number) $a_0 + \|\boldsymbol{a}\|\mathrm{j}$.

This connections with the generalized complex numbers[6] sheds some additional light on the behavior of the eigenvalues of the associated matrix A by means of the root locus plot (see Equation (4-5)). A common type of root locus branch consists of a complex pole pair approaching the real axis when the gain is increased. When they finally collide on the real axis, they each go their opposite ways on the real axis, essentially breaking the symmetry with respect to the real axis. The split-quaternions and hypercomplex numbers paint a slightly more elegant picture, which is shown in Figure 4-5:

- As shown above, when the pole pair is complex, the associated split-quaternion vector is timelike. The eigenvalues are naturally conjugate with respect to the real axis, i.e.

$$\lambda_A = a_0 \pm \|a\|\mathrm{i}.$$

[5] Again, we must take care not to confuse the hyperbolic unit with the split-quaternion basis element $\hat{\boldsymbol{\jmath}}$. They behave the same, and are related in the sense that they give rise to 'split' behavior, but are part of a very different number systems.

[6] For more details on generalized complex numbers, the reader is referred to Harkin and Harkin [59].

- When the pole pair collides on the real axis (often called the *branch point*), the imaginary part of the eigenvalue is zero, and the vector is timelike. Observe that we can make the case that, because the branches continue afterwards in a separate manner, they cannot be *exactly* the same. Indeed, the eigenvalues are

$$\lambda_A = a_0 \pm \varepsilon.$$

  The nilpotent unit ε is often interpreted as a differential, or an infinitesimally small quantity.[7] We argue that in this case, the pole pair is still conjugate, but the poles differ only by an infinitesimal amount.

- When the gain is increased further, the poles are real and the symmetry with respect to the real axis is broken. However, we can infer from the preceding discussion that the imaginary part is now hyperbolic instead, i.e.

$$\lambda_A = a_0 \pm \|a\|\mathrm{j}.$$

  Of course, it is possible to project these points on the real axis, but this obscures the natural symmetry of the root locus branch. In Figure 4-5, we therefore put the hyperbolic part on a third axis.

Let us now return to the exponential function. We can manipulate the definition of $\exp(a)$ as follows:

$$\begin{aligned}\exp(a) &= \mathrm{e}^{a_0}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{a}^k\right)\\ &= \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty}\frac{\left(\boldsymbol{a}^2\right)^k}{(2k)!}+\sum_{k=0}^{\infty}\frac{\boldsymbol{a}\left(\boldsymbol{a}^2\right)^k}{(2k+1)!}\right].\end{aligned}$$

Furthermore, if $\boldsymbol{a}$ is not lightlike, we have:

$$\exp(a) = \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k}\left(\hat{\boldsymbol{a}}^2\right)^k}{(2k)!}+\hat{\boldsymbol{a}}\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k+1}\left(\hat{\boldsymbol{a}}^2\right)^k}{(2k+1)!}\right].$$

Once again, there are three possibilities, depending on the regime of $\boldsymbol{a}$:

- If $\boldsymbol{a}$ is *timelike*, then the above expression reverts to

$$\begin{aligned}\exp(a) &= \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k}(-1)^k}{(2k)!}+\hat{\boldsymbol{a}}\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k+1}(-1)^k}{(2k+1)!}\right],\\ &= \mathrm{e}^{a_0}\Big[\cos(\|a\|)+\hat{\boldsymbol{a}}\sin(\|a\|)\Big].\end{aligned} \tag{4-9}$$

  This is roughly equivalent to the Euler identity for complex numbers: this is not at all surprising, since we found before that $\hat{\boldsymbol{a}}$ can be associated with the imaginary unit if $\boldsymbol{a}$ is timelike.

[7] A common application of dual numbers is automatic differentiation: because higher powers vanish, they can be used to generate first-order polynomial approximations. The unit 'circle' for dual numbers consists of two vertical lines crossing the horizontal axis at ±1. These lines can again be interpreted as linear approximations of the actual unit circle (or unit hyperbola) associated to (split-)complex numbers. The plane spanned by the j-axis and the real axis is the split-complex plane. The 'projection' to the real axis is in this plane a reflection with respect to the light cone (first diagonal).

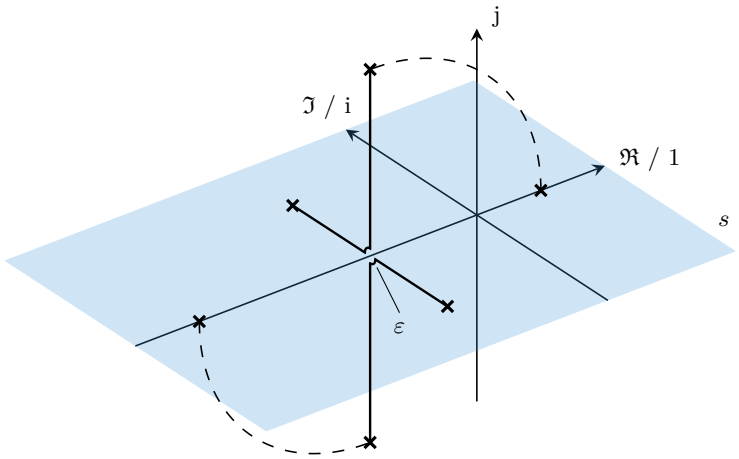


Figure 4-5: Generalized version of a root locus plot in terms of hypercomplex numbers. The traditional root locus is set in the complex s-plane (shown in blue), but we added a third axis for the hyperbolic part of the eigenvalue. When the gain is increased, the initially complex pole pair ventures towards the real axis. If the pole pair is critically damped, both poles are separated from the real axis by an infinitesimal distance of ε. Increasing the gain even more pushes the pole pair into the hyperbolic regime (the associated split-quaternion vector is now spacelike). Observe that in this picture, the symmetry with respect to the real axis is preserved. In the traditional root locus, these points are projected onto the real axis, indicated by the dashed lines.

- Secondly, if $\boldsymbol{a}$ is *lightlike*, we can simply use the definition of the expontial in its original form: [8]

$$\begin{aligned} \exp(a) &= \mathrm{e}^{a_0} \sum_{k=0}^{\infty} \frac{\boldsymbol{a}^k}{k!}, \\ &= \mathrm{e}^{a_0}\left[1 + \boldsymbol{a} + \sum_{k=2}^{\infty} \frac{\boldsymbol{a}^{k-2}\boldsymbol{a}^2}{k!}\right], \\ &= \mathrm{e}^{a_0}(1 + \boldsymbol{a}). \end{aligned} \tag{4-10}$$

- Finally, if $\boldsymbol{a}$ is *spacelike*, we have

$$\begin{aligned} \exp(a) &= \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty} \frac{\|\boldsymbol{a}\|^{2k}}{(2k)!} + \hat{\boldsymbol{a}} \sum_{k=0}^{\infty} \frac{\|\boldsymbol{a}\|^{2k+1}}{(2k+1)!}\right], \\ &= \mathrm{e}^{a_0}\big[\cosh(\|a\|) + \hat{\boldsymbol{a}}\sinh(\|a\|)\big]. \end{aligned} \tag{4-11}$$

The relevance of the exponential map lies of course in the fact that the solution of the linear differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = A\boldsymbol{x}$$

[8] We can also use the 'split' expression, defining that $0^0 \equiv 1$, a common convention in power series and algebra. Observe that the trigonometric functions associated to the dual numbers (i.e. the lightlike vectors) are then equal to the small-angle approximation for sin and cos. [59, 60]

is equal to [61]

$$\boldsymbol{x}(t) = \exp(At)\boldsymbol{x}_0,$$

where the one-parameter group of transformations generated by $\exp(At)$ is referred to as the *flow* of the vector field $A\boldsymbol{x}$. Hence, for two-dimensional systems, the matrix A can be represented by a split-quaternion, and we have just derived easy and insightful ways to evaluate its exponential:

$$\exp(at) = \begin{cases} \mathrm{e}^{a_0 t}\big[\cos(\|\boldsymbol{a}\|t) + \hat{\boldsymbol{a}}\sin(\|\boldsymbol{a}\|t)\big] & \boldsymbol{a} \text{ timelike}, \\ \mathrm{e}^{a_0 t}(1 + \|\boldsymbol{a}\|t) & \boldsymbol{a} \text{ lightlike}, \\ \mathrm{e}^{a_0 t}\big[\cosh(\|\boldsymbol{a}\|t) + \hat{\boldsymbol{a}}\sinh(\|\boldsymbol{a}\|t)\big] & \boldsymbol{a} \text{ spacelike}. \end{cases}$$

Evaluating a matrix exponential by hand usually involves diagonalizing (strictly speaking, finding the Jordan form). The convenience of using split-quaternions instead resides in the fact that they resolve the ambiguity that is naturally present in the eigenvectors of the matrix A; especially when they are complex. In the next section, the relation between the eigenvectors and the unit vector $\hat{\boldsymbol{a}}$ are discussed in greater detail. [62]

4-3 Application to mechanical systems

We will now proceed by using a mechanical 'prototype' example for our mechanical system: the harmonic oscillator with *two* dampers: one in series and one in parallel. The two dampers are interesting because they completely fill the state transition matrix: as such, this system can represent all the possible cases discussed in the previous section. Furthermore, the two dampers have a very distinctive interpretation within the field of economic engineering.

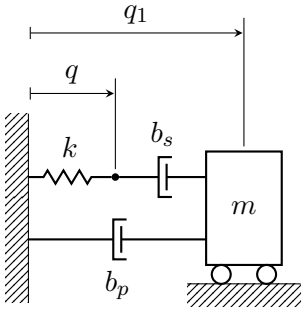


Figure 4-6: Schematic of the harmonic oscillator with two dampers: one in series and one in parallel.

4-3-1 Equations of motion

The harmonic oscillator with two dampers is shown in Figure 4-6. The equations of motion can be readily derived:

$$\begin{aligned} m\ddot{q}_1 &= -kq - b_p\dot{q}_1 \\ kq &= b_s(\dot{q}_1 - \dot{q}) \end{aligned} \tag{4-12}$$

Due to the presence of the serial damper, the situation is somewhat curious, since there are two positions in the system; one measuring the spring deflection q and the position of the mass q_1. However, the node connecting the serial damper and the spring has no mass, and therefore no second-order dynamics: as such, the overall order of the system is 2. In accordance with the economic analogy, we will say that position is stored in the spring, but momentum is stored in a mass. Hence, let $p = m\dot{q}_1$ — but $\dot{q} \neq p/m$ in general. The equations of motion than have the matrix form:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -\frac{k}{b_s} & \frac{1}{m} \\ -k & -\frac{b_p}{m} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

or, using the parameters defined in Table 4-4:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} -\gamma_s & \frac{1}{m} \\ -m\Omega_n^2 & -\gamma_p \end{pmatrix}}_{A} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{4-13}$$

The nontrivial relation between momentum and the $\dot{q}$ also precludes us from casting this system directly to the Lagrangian formalism, because the vector field is not second-order. A second-order vector field implies that the 'velocities' really are the time derivatives of the 'positions', (cf. Appendix A for a more formal discussion).

Table 4-4: Substition parameters for the harmonic oscillator with serial and parallel damping, shown in Figure 4-6.

Name	Symbol	Value	Units
Serial damping coefficient	γ_s	k/b_s	1/s
Parallel damping coefficient	γ_p	b_p/m	1/s
Natural frequency	Ω_n	$\sqrt{k/m}$	1/s
Damped frequency	Ω_d		1/s

The split-quaternion associated with the A-matrix of the doubly damped system can easily be found using the mapping defined by Equation (4-4). We must, however, be careful when dealing with physical systems, because the entries of the A-matrix are not dimensionless. In a vector space, we associate the units with the basis vectors, not with the components. For example, in a two-dimensional vector space spanned by a axis for apples and and an axis for pears, and we wish to represent that someone possesses three apples and four pairs, the *components* of that vector are $(3, 4)$, and the *unit vectors* are $(1\text{ apple}, 1\text{ pear})$. Along the same line, we must define the units in the A-matrix in the split-quaternion basis elements 1, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$. To do so, we define the reference quantities and m_0, t_0. The basis elements are then mapped in terms of these reference quantities:

$$\phi(1) = \begin{pmatrix} \frac{1}{t_0} & 0 \\ 0 & \frac{1}{t_0} \end{pmatrix}, \quad \phi(\hat{\boldsymbol{\imath}}) = \begin{pmatrix} 0 & \frac{1}{m_0} \\ -\frac{m_0}{t_0^2} & 0 \end{pmatrix}, \quad \phi(\hat{\boldsymbol{\imath}}) = \begin{pmatrix} 0 & \frac{1}{m_0} \\ \frac{m_0}{t_0^2} & 0 \end{pmatrix}, \quad \phi(\hat{\boldsymbol{k}}) = \begin{pmatrix} \frac{1}{t_0} & 0 \\ 0 & -\frac{1}{t_0} \end{pmatrix},$$

where, in case we would use SI units, $m_0 = 1\,\text{kg}$ and $t_0 = 1\,\text{s}$. As a result, the split-quaternion associated with the A-matrix given in Equation (4-13) becomes

$$a = -\frac{1}{2}(t_0\gamma_s + t_0\gamma_p) + \frac{1}{2}\left(\frac{m_0}{m} + \frac{m\Omega_n^2 t_0^2}{m_0}\right)\hat{\boldsymbol{\imath}} + \frac{1}{2}\left(\frac{m_0}{m} + \frac{m\Omega_n^2 t_0^2}{m_0}\right)\hat{\boldsymbol{\jmath}} + \frac{1}{2}(t_0\gamma_p - t_0\gamma_s)\hat{\boldsymbol{k}}. \tag{4-14}$$

Clearly, all the components of the split-quaternion are dimensionless. This really is not too wild of an idea: after all, we are translating the matrix itself, and *not* the two-dimensional vector space that it acts on. The units are inherited from the vector space, so we should only add them when returning from the split-quaterions back to the realm of the matrices.

The preceding argument only explains why we can work around this issue without performing illegal operations, but it does not give a satisfactory answer as to why we would be interested to add numbers that are seemingly incompatible. Indeed, observe that γ_s and γ_p have the same units, whereas $\frac{1}{m}$ and Ω_n^2 do not. So, in which sense can the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$-components be of any signficance? To answer this question, we first note that 'rescaling of units' is a linear operation on the vector space given by a diagonal matrix (with nonzero diagonal entries):

$$N = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} \qquad \nu_1, \nu_2 \in \mathbb{R}^*,$$

which form the group isomorphic to $(\mathbb{R}^*)^2$. This transformation of the vector space manifests itself on the A-matrix as: $A' = N^{-1}AN$. It is easy to see that the basis matrices (or vector fields) for '1' and $\hat{\boldsymbol{k}}$ are invariant under this transformation, while the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$-matrices are not (that is, without making use of the reference quantities). A geometric explanation is that the eigenvectors of the identity matrix and the $\hat{\boldsymbol{k}}$-matrix point along the axes; and are therefore invariant under rescaling of these axes. As a result of this fact, the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components will not transform properly under a unit transformation. It is common practice in physics to rescale the state space of the undamped harmonic oscillator as follows [5, 9]

$$p \mapsto \frac{p}{m} \qquad q \mapsto m\Omega q,$$

such that the Hamiltonian reverts to a particularly convenient form. We can see that this is precisely the transformation that kills the $\hat{\boldsymbol{\jmath}}$-component of the split-quaternion. This would essentially resolve this 'unit problem', because it only arises when we attempt to make the *distinction* between the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$-component.

In contrast to common practice in physics, we are interested in the full range of geometrical properties that the trajectories in the phase plane can exhibit, including those that are not invariant under the action of the structure group $(\mathbb{R}^*)^2$ that contains the changes of units. Furthermore, many invariants, such as the split-quaternion (vector) norm, scalar part, etc. that we use to draw conclusions about the nature of the system *do* commute with this group action, and are therefore remain valid. It is even possible to effect unit transformations within the split-quaternion transformations by translating the matrix N to the appropriate split-quaternion using the isomorphism. We can indeed observe that the action of $n^{-1}an$ (where $n = \phi^{-1}(n)$) produces a split-quaternion with zero $\hat{\boldsymbol{\jmath}}$-component.

As a final argument, we can say that the 'rescaling of the axes', while common in physics and mathematically allowed, is of little use for engineers, since they tend to stick to SI units in the first place. The 'scale of the axes' is therefore a physical reality. This is why we choose not to discard the $\hat{\boldsymbol{\jmath}}$-component through a rescaling.

To conclude, it is not so much the case that unit transformations are not allowed in the split-quaternion space, but the question as to what the units of the $\hat{\boldsymbol{\jmath}}$-components are is moot. Unfortunately, the notation in Equation (4-14) is rather obfuscating. Hence, we take the freedom to choose $m_0 = 1(\mathrm{kg})$ and $t_0 = 1(\mathrm{s})$, and write the split-quaternion as follows:

$$a = -\tfrac{1}{2}(\gamma_s + \gamma_p) + \tfrac{1}{2}\left(\tfrac{1}{m} + m\Omega_n^2\right)\hat{\boldsymbol{\imath}} + \tfrac{1}{2}\left(\tfrac{1}{m} + m\Omega_n^2\right)\hat{\boldsymbol{\jmath}} + \tfrac{1}{2}(\gamma_p - \gamma_s)\hat{\boldsymbol{k}}. \tag{4-15}$$

This of course requires the implicit understanding that all the components are dimensionless, and that we are not just adding apples and pears.

4-3-2 Eigenvalues

Underdamped, overdamped critically damped.

4-3-3 Eigenvector geometry in SO(3, R)

We have learned in Section 4-1-2 that the eigenvalues of a 2×2-matrix can be expressed in terms of the scalar part and vector norm of the associated split-quaternion. Once we have removed the scalar part and normalized the vector part with its norm (if admissible), we are left with the unit vector part of the split-quaternion. Hence, this two-dimensional object contains the remaining information present in the matrix: the eigenvectors. The great advantage of split-quaternions is that the unit vector is entirely unambiguous, which is not the case for eigenvectors, for they represent only a direction, while the length of the eigenvector is arbitrary. The unit vector part of the split-quaternion encodes this information in a *single two-dimensional object*.

As mentioned in Section 4-1-1, the split-quaternion vectors live in the Lorentzian three-space $\mathbb{R}^{1,2}$. The *unit vectors* in this space live in a particular subspace, the Lorentzian equivalent of the unit sphere in this space. Due to the indefiniteness of the Lorentz norm, this subspace is disconnected: it consists out of three connected parts (shown in Figure 4-7):

- The *one-sheet hyperboloid*, which contains all the *spacelike* unit vectors (overdamped systems).
- The *light cone*, which contains all the *lightlike* vectors (the notion of a unit vector is not well defined in this region) (critically damped systems).
- The *two-sheet hyperboloid*; which contains all the *timelike* unit vectors (underdamped systems).

Given the regime of the mechanical systems, the eigenvectors of the state transition matrix determine the particular shape of the trajectories in the phase plane. We distinguish three types of shapes:

① Saddle point: two directions

② Stable line: two direction (?)

③ Pure translation: translation direction?

④ Node: two directions

⑥ A center/spiral, which has elliptic trajectories. These also include the spiral nodes, since the scalar part (contraction) is not part of the vector. We can characterize the shape of the elliptic trajectories by the *eccentricity* and their *tilt* (i.e. the rotation of the major axes of the ellipse with respect to the phase plane axes).

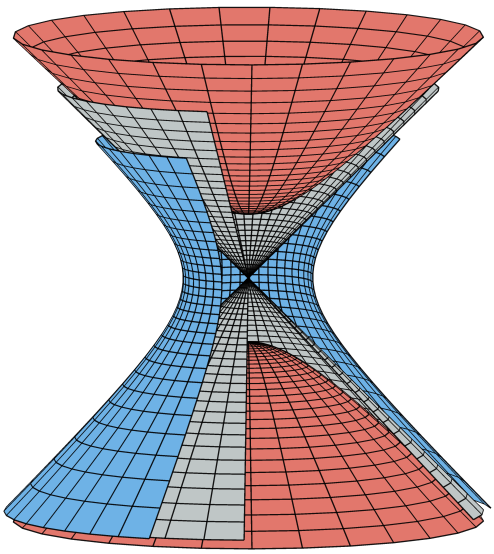


Figure 4-7: The disconnected 'unit sphere' in the Lorentzian 3-space. The blue surface is the one-sheet hyperboloid, containing all the spacelike unit vectors; the gray sheet is the light cone, that contains all the lightlike 'null' vectors with zero norm. Finally, the red surface is the two-sheet hyperboloid, which is the space of all timelike unit vectors.

Compute real eigenvectors by substituting the action of multiplying with a vector *in* the split-quaternion and solving appropriately.

Underdamped systems

TODO

Master figure with multiple trajectories and their respective locations on the poincaré disk, with poincaré disk as map

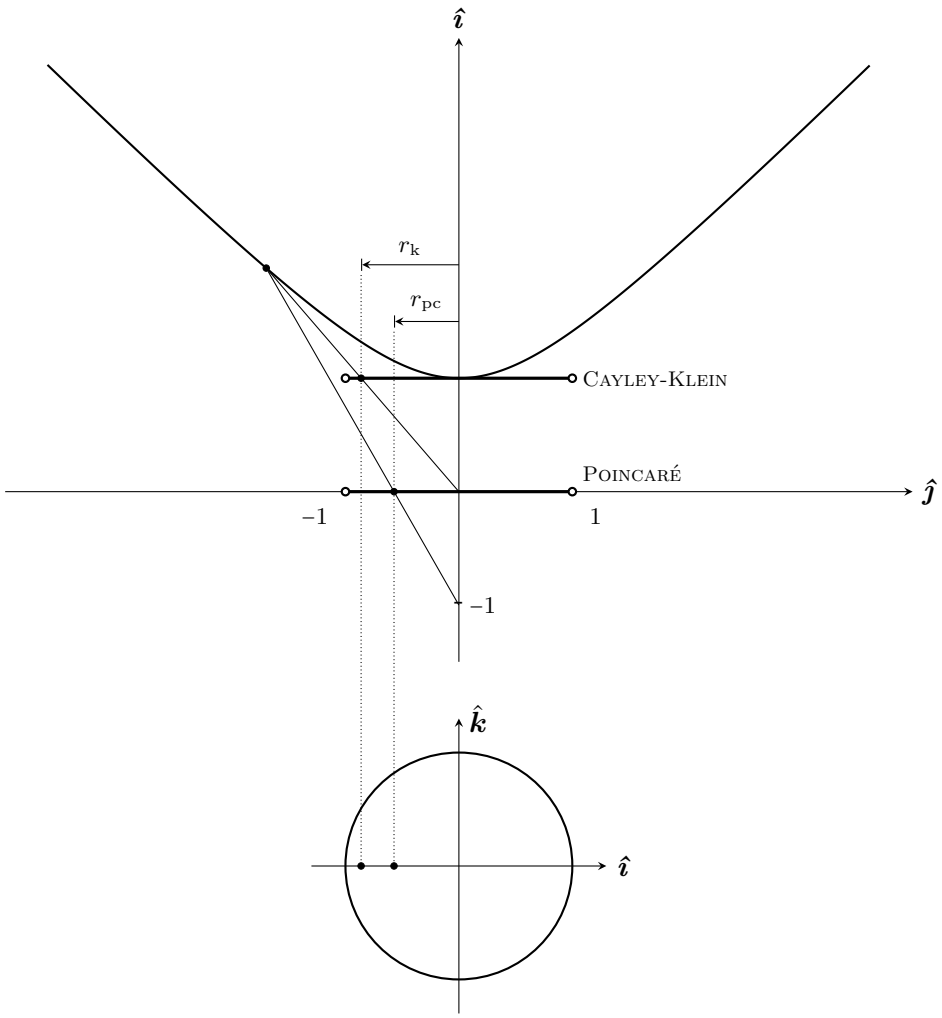


Figure 4-8: Illustration of the projection on the Poincaré disk and the Cayley-Klein disk.

4-4 Notes

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{i}} - u_2\hat{\boldsymbol{j}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned} \boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \hat{\boldsymbol{n}} \\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}), \end{aligned} \tag{4-16}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, we obtain:

$$\begin{aligned} \hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\ &= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}. \end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\langle \boldsymbol{c}, \boldsymbol{a} \rangle - \boldsymbol{c}\langle \boldsymbol{a}, \boldsymbol{b} \rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b}\langle \boldsymbol{c}, \boldsymbol{a} \rangle_{\mathrm{L}} - \boldsymbol{c}\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$\begin{aligned}
&\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} - \hat{\boldsymbol{n}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle_{\mathrm{L}}) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \boldsymbol{e}_2 \mathrm{i}.
\end{aligned}$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ■

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

4-4-1 Relation with complex Hamiltonians

Chapter 5

Conclusion

Appendix A

Symplectic geometry in Analytical Mechanics

A-1 Symplectic geometry

A *symplectic manifold* (M, ω) is a manifold M equipped with a closed, nondegenerate (and therefore symplectic) 2-form ω.

- Darboux theorem
- Lagrangian submanifolds
- Canonical symplectic structure on the cotangent bundle

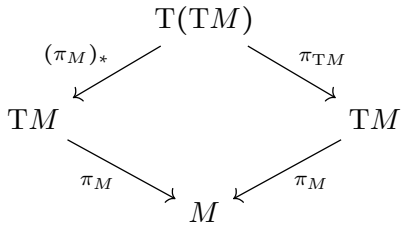
A-2 Hamiltonian mechanics

- Hamiltonian vector field
- Poisson bracket
- Integral invariant (?)

A-3 Lagrangian mechanics

Just like the cotangent bundle, the tangent bundle admits a canonical structure, which is called the *vertical endomorphism*. Its construction is slightly more convoluted than the canonical symplectic structure of the cotangent bundle, but nevertheless essential for a proper geometric interpretation of Lagrangian mechanics.

The vertical endomorphism The *double tangent bundle* is the tangent bundle to $\mathrm{T}M$, denoted by $\mathrm{T}(\mathrm{T}M)$. This space has not one but two canonical vector bundle structures, defined by projection maps from $\mathrm{T}(\mathrm{T}M) \to \mathrm{T}M$. First, there is the trivial projection $\pi_{\mathrm{T}M}$ thats 'forgets' about the tangent elements to $\mathrm{T}M$. Secondly, there is $(\pi_M)_*$ the pushforward (tangent map) of the projection map $\pi_M : \mathrm{T}M \to M$. [1]



Vectors on the tangent bundle $\mathrm{T}M$ (they live in $\mathrm{T}(\mathrm{T}M)$) are called vertical if they vanish under the action of $(\pi_M)_*$. These vectors point entirely in the 'direction' of the fiber: in the Lagrangian formalism, they reflect a pure change in velocity, and no change in the generalized position. The *vertical lift* Ψ maps a vector on M to a vertical vector on $\mathrm{T}M$. [47]

$$\begin{aligned} &\Psi_{\boldsymbol{v}} : \mathrm{T}_q M \to \mathrm{T}_{\boldsymbol{v}}(\mathrm{T}_q M) : \\ &\qquad \Psi_{\boldsymbol{v}}(\boldsymbol{w})\, f = \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{v} + t\boldsymbol{w})\Big|_{t=0} \qquad q \in M,\ \ \boldsymbol{v}, \boldsymbol{w} \in \mathrm{T}_q M,\ \ f \in C^\infty(\mathrm{T}M). \end{aligned} \tag{A-1}$$

In components, the effect of the vertical lift is as follows:

$$\Psi_{\boldsymbol{v}} : \quad \boldsymbol{w} = w_i \frac{\partial}{\partial q_i}\Big|_q \quad \mapsto \quad \Psi_{\boldsymbol{v}}(\boldsymbol{w}) = w_i \frac{\partial}{\partial v_i}\Big|_{(q,\boldsymbol{v})}.$$

The vertical lift can also lift entire sections of $\mathrm{T}M$ by simply applying the vertical lift pointwise.

Using the concept of the vertical lift, we can define the *vertical isomorphism* S from the double tangent bundle to itself, first by projecting with $(\pi_M)_*$ and then lifting again:

$$S : \mathrm{T}(\mathrm{T}M) \to \mathrm{T}(\mathrm{T}M) : \quad S(q, \boldsymbol{v})\, u = (\Psi_{\boldsymbol{v}} \circ (\pi_M)_*)\, u \qquad u \in \mathrm{T}_{(q,\boldsymbol{v})}\mathrm{T}M.$$

The action of S can also be stated in the form of the following diagram:

$$\begin{array}{ccc} \mathrm{T}(\mathrm{T}M) & \xrightarrow{S} & \mathrm{T}(\mathrm{T}M) \\ {\scriptstyle (\pi_M)_*}\downarrow & & \uparrow{\scriptstyle \Psi} \\ \mathrm{T}M & \xrightarrow[\mathrm{id}_{\mathrm{T}M}]{} & \mathrm{T}M \end{array}.$$

The action of the vertical endomorphism on the chart-induced basis is:

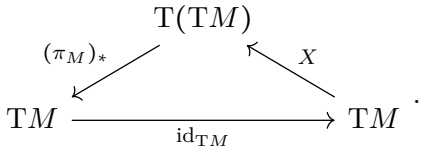
$$S : \quad \frac{\partial}{\partial q_i}\Big|_{(q,\boldsymbol{v})} \mapsto \frac{\partial}{\partial v_i}\Big|_{(q,\boldsymbol{v})} \qquad \frac{\partial}{\partial v_i}\Big|_{(q,\boldsymbol{v})} \mapsto 0.$$

The vertical isomorphism is therefore a tensor of valence (1, 1) — it takes a vector and produces another. Locally, S can be expressed as:

$$S = \frac{\partial}{\partial v_i} \otimes \mathrm{d}q_i \,.$$

with $\otimes$ being the tensor product. [47]

The Lagrangian formalism only applies to second-order vector fields. A second-order vector field is a vector field X such that $(\pi_M \circ X) = \mathrm{id}_{\mathrm{T}M}$; i.e. the following diagram commutes: [1]



The identity on $\mathrm{T}M$ is $\mathrm{id}_{\mathrm{T}M} : (q, \boldsymbol{v}) \mapsto (q, \boldsymbol{v})$. Therefore, for a vector field X to be second order, we should have that the component in $\frac{\partial}{\partial q_i}$ that is picked out by $(\pi_M)_*$ should be equal to v_i; for example

$$X = \sum_{i=1}^{n} \left[v_i \frac{\partial}{\partial q_i} + F_i \frac{\partial}{\partial v_i} \right].$$

The corresponding differential equations are

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = v_i \qquad \frac{\mathrm{d}v_i}{\mathrm{d}t} = F_i,$$

which means that the second-order vector field coincides with the notion of a 'second-order differential equation' in q_i.

The Euler-Lagrange equations With the infrastructure set up in the preceding paragraph, we can now define the precise geometric setting of Lagrangian mechanics. Given a Lagrangian function $L \in C^\infty(\mathrm{T}M)$, define the *Lagrange 1-form*[1]

$$\vartheta_L \equiv \mathrm{d}L \circ S = \sum_{j=1}^{n} \frac{\partial L}{\partial v^j} \mathrm{d}q^j \,. \tag{A-2}$$

Observe that the Lagrange 1-form is also equal to the pullback of the Liouville form under the Legendre transformation: $\vartheta_L = (\mathbb{F}L)^* \vartheta$. [1] Secondly, we define the *Lagrange 2-form* as: [1, 47]

$$\omega_L \equiv -\mathrm{d}\vartheta_L = \frac{\partial^2 L}{\partial v^i \partial v^j} \mathrm{d}q^j \wedge \mathrm{d}v^i + \frac{\partial^2 L}{\partial q^i \partial v^j} \mathrm{d}q^j \wedge \mathrm{d}q^i \,. \tag{A-3}$$

Because the exterior derivative and the pullback commute, the Lagrange 2-form is equal to the pullback of the symplectic 2-form under the Legendre transform. If the rank of the Hessian $\frac{\partial^2 L}{\partial v^i \partial v^j}$ is full (and constant), then ω_L is nondegenerate and therefore defines a symplectic structure on $\mathrm{T}M$. However, observe that whether ω_L is symplectic or not depends on the

[1] Cariñena [47] calls ϑ the Euler-Poincaré 1-form.

nature of the Lagrangian, while the symplectic structure in the Hamiltonian setting is canonically derived from the cotangent bundle itself — there is no need for the Hamiltonian to be regular.

The final ingredient for the Euler-Lagrange equations is the *energy function*

$$E \equiv Z(L) - L,$$

where $Z = \sum v^i \frac{\partial}{\partial v^i}$ is the Liouville vector field on $\mathrm{T}M$.

The *Lagrangian vector field* X_L is then the unique vector field that satisfies the equation: [63]

$$X_L \lrcorner \omega_L = \mathrm{d}E\,, \tag{A-4}$$

In components, the right hand side of this equation is:

$$\begin{aligned} \mathrm{d}E &= \sum_{i,j}\left(\frac{\partial^2 L}{\partial v_j \partial q_i} v_j\,\mathrm{d}q_i + \frac{\partial^2 L}{\partial v_j \partial v_i} v_j\,\mathrm{d}v_i + \frac{\partial L}{\partial v_j}\,\mathrm{d}v_j\right) - \mathrm{d}L\,, \\ \mathrm{d}E &= \sum_{i,j}\left(\frac{\partial^2 L}{\partial v_j \partial q_i} v_j\,\mathrm{d}q_i + \frac{\partial^2 L}{\partial v_j \partial v_i} v_j\,\mathrm{d}v_i - \frac{\partial L}{\partial q_j}\,\mathrm{d}q_j\right). \end{aligned} \tag{A-5}$$

Let $X_L = \sum_i \left(A_i \frac{\partial}{\partial q_i} + B_i \frac{\partial}{\partial v_i}\right)$; the left hand side can then be written as follows:

$$X_L \lrcorner \omega_L = -\sum_{i,j} A_i \frac{\partial^2 L}{\partial q_i \partial v_j}\,\mathrm{d}q_j + \sum_{i,j} A_j \frac{\partial^2 L}{\partial q_i \partial v_j}\,\mathrm{d}q_i - \sum_{i,j} B_i \frac{\partial^2 L}{\partial v_i \partial v_j}\,\mathrm{d}q_j + \sum_{i,j} A_j \frac{\partial^2 L}{\partial v_i \partial v_j}\,\mathrm{d}v_i\,. \tag{A-6}$$

Comparing this expression with Equation (A-5), it is immediately clear that

$$A_j \frac{\partial^2 L}{\partial v_i \partial v_j} = v_j \frac{\partial^2 L}{\partial v_i \partial v_j}.$$

We therefore have that $A_j = v_j$, but *only* if the Hessian of L with respect to the velocities is nonsingular. If this is indeed the case (i.e. L is regular), and the condition implies that the vector field X_L is second-order. We can use this knowledge to obtain a second condition (since the terms in $\mathrm{d}q_i$ cancel):

$$\sum_i B_i \frac{\partial^2 L}{\partial v_i \partial v_j} = \frac{\partial L}{\partial q_j} - \sum_i v_i \frac{\partial^2 L}{\partial q_i \partial v_j}.$$

The Hessian of L in the velocities $M_{ij} = \frac{\partial^2 L}{\partial v_i \partial v_j}$ is also called the mass matrix of the system. We have already assumed that this matrix is invertible (i.e. L is regular). As such, we have that

$$\sum_i \frac{\partial^2 L}{\partial v_i \partial v_j} \frac{\mathrm{d}^2 q_j}{\mathrm{d}t^2} + \sum_i \frac{\partial^2 L}{\partial q_i \partial v_j} \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial L}{\partial q_j},$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

which is the traditional form of the Euler-Lagrange equations.

Provided that X_L is a second-order vector field, the equation Equation (A-4) is equivalent to the following statement:

$$\pounds_{X_L}\vartheta_L = \mathrm{d}L\,. \tag{A-7}$$

The equivalence is easily shown using the Cartan formula:

$$\pounds_{X_L}\vartheta_L = \mathrm{d}L$$

$$\mathrm{d}(X_L \lrcorner\, \vartheta_L) + X_L \lrcorner\, \mathrm{d}\vartheta_L = \mathrm{d}L$$

$$\mathrm{d}(X_L \lrcorner\, \vartheta_L) - X_L \lrcorner\, \omega_L = \mathrm{d}L$$

The fact that X_L is second-order implies that $X_L \lrcorner\, \vartheta_L = Z(L)$. Therefore

$$\mathrm{d}(Z(L)) - X_L \lrcorner\, \omega_L = \mathrm{d}L$$

$$X_L \lrcorner\, \omega_L = \mathrm{d}Z(L) - L$$

$$X_L \lrcorner\, \omega_L = \mathrm{d}E\,.$$

Lagrangians are not unique: from Equation (A-4) we can deduce that the addition of a closed 1-form (as a map from $\mathrm{T}M \to \mathbb{R}$) to the Lagrangian will not alter the Euler-Lagrange equations. The closed 1-forms on M therefore constitute the *gauge group* of Lagrangian mechanics. An equivalent statement is that the Euler-Lagrange equations remain invariant if a total derivative is added to the Lagrangian function. [1]

Appendix B

Contact geometry

This appendix provides a short introduction to the basic concepts of contact geometry that are relevant in this thesis, particularly Chapter 3.

B-1 Contact structures

A *contact element* on a manifold M is a point $m \in M$ combined with a tangent hyperplane $\xi_m \subset \mathrm{T}_m M$ (a subspace of the tangent space with codimension 1). The term 'contact' refers to the intuitive notion that if two submanifolds 'touch', they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [2] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

A *contact manifold* is a manifold M (of dimension $2n+1$) with a *contact structure*, which is a smooth field (or distribution) of contact elements on M. Locally, any contact element determines a 1-form α (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{B-1}$$

This α is called the (local) *contact form*, and it acts like a 'normal (co-)vector' to the hyperplane. For the field hyperplanes to be a constact structure, it must satisfy a nondegeneracy condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [1, 2, 18]

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of M such that ξ is everywhere tangent to it, or (ii) that $\mathrm{d}\alpha|_\xi$ is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient TM/ξ is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [14].

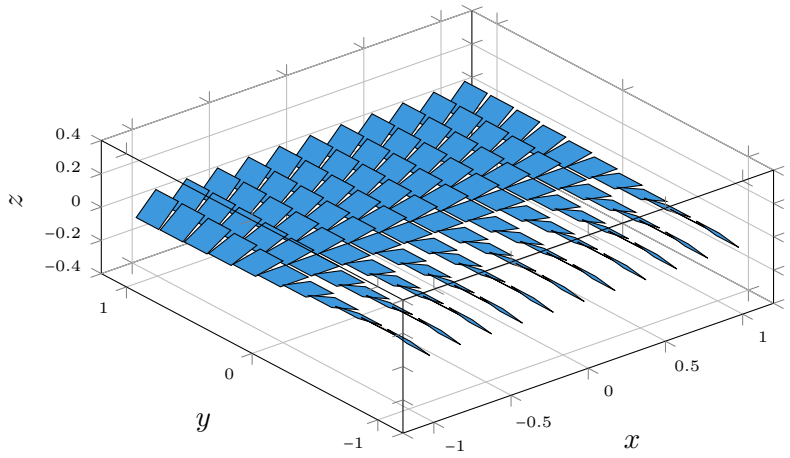


Figure B-1: The standard contact structure on $\mathbb{R}^3$, given by the contact form $\mathrm{d}z - y\,\mathrm{d}x$; the hyperplanes tilt more in the increasing y-direction.

The *Darboux theorem* for contact manifolds states that it is always possible to find coordinates z, x_i, y_i such that locally, the contact form is equal to

$$\mathrm{d}z - \sum y_i\,\mathrm{d}x_i\,,$$

which is also called the standard or natural contact structure. The standard contact structure on $\mathbb{R}^3$ is illustrated in Figure B-1. Finally, it is clear that the contact form singles out a 'special direction' in the tangent space at every point of the manifold. This direction is given by the unique *Reeb vector field*,

$$R_\alpha \in \mathfrak{X}(M): \quad R_\alpha \lrcorner\, \mathrm{d}\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner\, \alpha = 1. \tag{B-2}$$

The special direction identified by the Reeb vector field is referred to as the *vertical* direction. Likewise, vector field components in the direction of the Reeb vector field are vertical. A vector field with no vertical component is called *horizontal*.

B-2 The manifold of contact elements

A contact manifold is a manifold with a contact structure. One can, however, associate a *canonical* $(2n-1)$-dimensional contact manifold to *any* n-dimensional manifold Q, just like one can always find a canonical symplectic structure on T^*Q. Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold Q. As it turns out, this 'manifold of contact elements' has a natural contact structure.

The *manifold of contact elements* of an n-dimensional manifold is [2]

$$\mathrm{C}Q = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } \mathrm{T}_qQ\}.$$

This manifold $\mathrm{C}Q$ has dimension $2n-1$. It is clear that C has a natural bundle structure, i.e. $C \xrightarrow{\pi} Q$ where the bundle projection 'forgets' the contact element, that is

$$\pi : \mathrm{C}Q \to Q : (q, \xi_q) \mapsto q.$$

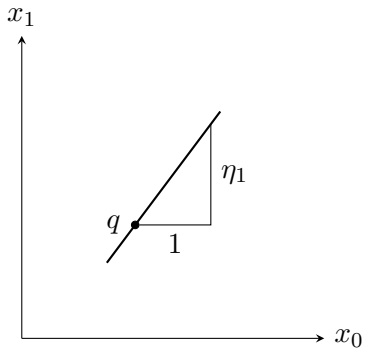


Figure B-2: A point in the manifold of contact elements on $Q = \mathbb{R}^2$. A coordinate system for $\mathrm{C}Q$ consists of (x_0, x_1) to indicate a point on Q, and projective coordinates $[\eta_0 : \eta_1]$, which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$, and the remaining coordinate η_1 covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the 'hyperplane' and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$.

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to Q, denoted by $\mathbb{P}(\mathrm{T}^*Q)$. This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in \mathrm{T}^*_q Q \setminus \{\mathbf{0}\}: \quad (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in Equation (B-1), in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure. $\mathbb{P}(\mathrm{T}^*Q)$ is then the quotient set of T^*Q (without zero section) with respect to the equivalence relation $\sim$. Visually, the projectivization of an n-dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension $n - 1$. It can be shown that this space is bundle-isomorphic to the manifold $\mathrm{C}Q$. [2]

As shown in Figure B-2, coordinates of the equivalence class of 1-forms are 'projective coordinates', $[\eta_0 : \eta_1 : \ldots : \eta_{n-1}]$, where η_i are coordinates for $\mathrm{T}^*_q Q$. The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes η_0 to be nonzero, the tuple $(1, \eta_1, \ldots, \eta_n)$ provides coordinates that cover most of $\mathbb{P}(\mathrm{T}^*Q)$.

Now, it remains to be explained why the 'manifold of contact elements' is itself a contact manifold. Indeed, there is a canonical field of hyperplanes *on* $\mathrm{C}Q$, which lifts the hyperplane tangent to Q to a hyperplane tangent to $\mathrm{C}Q$ (this is akin to the 'tautological' trick played in the symplectic structure of the cotangent bundle). The contact structure distinguishes the curves in $\mathrm{C}Q$ that are lifted versions from curves in Q. This is illustrated in Figure B-3. [21] Said otherwise, a tangent vector on $\mathrm{C}Q$ lies in the hyperplane defined by the contact structure if its projection down on Q lies in the hyperplane on Q defined by the given point on the $\mathrm{C}Q$.

This contact structure is associated with the 1-form:

$$\alpha = \mathrm{d}x_0 + \sum_{i=1}^{n-1} \eta_i \, \mathrm{d}x_i \,,$$

given that the η_0 is the 'special' coordinate wich is chosen to be 1.

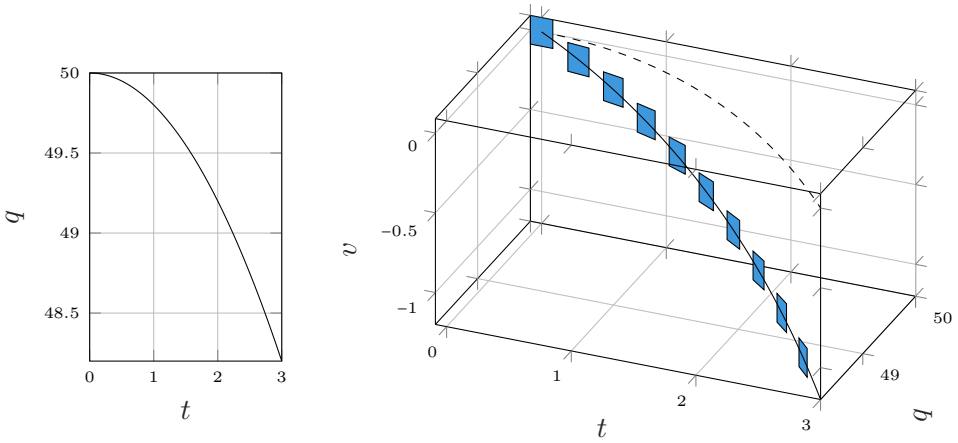


Figure B-3: Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let $(t, q) \in Q$, and let v be a coordinate for the contact (line) element. The standard contact form is then $\mathrm{d}q - v\,\mathrm{d}t$. On the left, the curve corresponding to a falling object is shown in Q. When this curve is 'lifted' to $\mathrm{C}Q$, the contact structure imposes that it be locally tangent to the contact structure, or that $v = \frac{\mathrm{d}q}{\mathrm{d}t}$. If the vertical direction is projected down onto the $(q - t)$-plane ($\mathrm{C}(Q) \to Q$), the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making v the actual velocity of the curve.

B-3 Contact Hamiltonian systems

TODO

Introduction

B-3-1 Contact Hamiltonian vector fields

Just like in the symplectic case, the contact Hamiltonian formalism defines an automorphism between a function on the contact manifold, $H \in C^\infty(M)$, and an associated 'Hamiltonian' vector field $X_H \in \mathfrak{X}(M)$. While the isomorphism is rather straightforward for symplectic manifolds, the contact counterpart is not so perspicuous: this is the prime reason behind the computational advantage of symplectification, as opposed to performing the calculations directly on the contact manifold.

Coordinate-free derivation Given a contact manifold (M, ξ) with contact form α (i.e. $\xi \in \ker \alpha$), the tangent bundle M can be decomposed into two subbundles: [2, 15]

$$\mathrm{T}M = \ker \alpha \oplus \ker \mathrm{d}\alpha \,,$$

where $\oplus$ denotes the Whitney sum. The first subbundle is referred to as the *horizontal* bundle, the second as the *vertical* bundle. The vertical subbundle is of rank 1 and its fiber is spanned by the Reeb vector field (cf. Equation (B-2)). As mentioned to in Appendix B-1, *any* vector field $X \in \mathfrak{X}(M)$ may be decomposed accordingly. This decomposition is unique and given by

$$X = \underbrace{(X \lrcorner \alpha)R_\alpha}_{X^{\text{ver}}} + \underbrace{[X - (X \lrcorner \alpha)R_\alpha]}_{X^{\text{hor}}}. \tag{B-3}$$

Observe that indeed $X^{\text{ver}} \in \ker \mathrm{d}\alpha$ and $X^{\text{hor}} \in \ker \alpha$. [2, 15, 64]

We now wish to find the relation between the contact Hamiltonian $H \in C^\infty(M)$ and the associated Hamiltonian vector field $X_H \in \mathfrak{X}(M)$. This one-to-one relation is uniquely determined by two conditions. Firstly, we impose that[1]

$$H \equiv -X_H \lrcorner \alpha.$$

This condition already provides us with the vertical component of the Hamiltonian vector field, namely

$$X_H^{\text{ver}} = -HR_\alpha.$$

Secondly, the automorphism generated by the Hamiltonian vector field must be a *contact automorphism*: it must preserve the contact structure. This condition is encoded in terms of the Lie derivative:[2]

$$X_H \text{ is an infinitesimal contact automorphism} \Leftrightarrow \pounds_{X_H}\alpha = s\alpha,$$

where $s \in C^\infty(M)$. The function s is there because $s\alpha$ and α give rise to the same hyperplane distribution. Using Cartan's 'magic' formula, the Lie derivative can be expanded as follows:

$$\pounds_{X_H}\alpha = s\alpha$$

$$\mathrm{d}(X_H \lrcorner \alpha) + X_H \lrcorner \mathrm{d}\alpha = s\alpha$$

$$-\mathrm{d}H + X_H \lrcorner \mathrm{d}\alpha = s\alpha$$

Contracting both sides with the Reeb vector field yields:

$$R_\alpha \lrcorner (-\mathrm{d}H + X_H \lrcorner \mathrm{d}\alpha) = R_\alpha \lrcorner (s\alpha)$$

$$-R_\alpha \lrcorner \mathrm{d}H + R_\alpha \lrcorner X_H \lrcorner \mathrm{d}\alpha = s\, R_\alpha \lrcorner \alpha$$

$$-R_\alpha \lrcorner \mathrm{d}H - X_H \lrcorner R_\alpha \lrcorner \mathrm{d}\alpha = s.$$

Hence, we have $s = -R_\alpha \lrcorner \mathrm{d}H$. Because the vertical component of X_H is spanned by the Reeb vector field, its contraction with $\mathrm{d}\alpha$ vanishes. As a result, we can rewrite the previous expression in terms of the *horizontal* component of X_H:

$$X_H \lrcorner \mathrm{d}\alpha = X_H^{\text{hor}} \lrcorner \mathrm{d}\alpha = [\mathrm{d}H - (R_\alpha \lrcorner \mathrm{d}H)\alpha], \tag{B-4}$$

[1] This is the sign convention observed by Bravetti et al. [37] en van der Schaft [65], as opposed to Libermann and Marle [15].

[2] Terminology differs somewhat in literature on this point: some authors, such as de León and Lainz [64] only refer to contactomorphisms as the special case where $g = 0$; while the more general case is called *conformal* contactomorphisms.

We must therefore recover X_H^{hor} from the above expression. Define the mapping

$$\alpha^\flat : \mathrm{T}M \to \mathrm{T}^*M : X \mapsto X \lrcorner \, \mathrm{d}\alpha \,,$$

when restricted to the space of horizontal vector fields, this mapping is an isomorphism onto the 'semi-basic' forms[3]. Define the inverse mapping of $\alpha^\flat$ by $\alpha^\sharp$, such that

$$X_H^{\text{hor}} = \alpha^\sharp(\mathrm{d}H - (R_\alpha \lrcorner \, \mathrm{d}H)\,\alpha).$$

As such, the Hamiltonian vector field associated to the contact Hamiltonian H is

$$X_H = HR_\alpha + \alpha^\sharp(-\mathrm{d}H + (R_\alpha \lrcorner \, \mathrm{d}H)\,\alpha). \tag{B-5}$$

Coordinate expression Given the contact manifold (M, ξ) with contact form

$$\mathrm{d}q_0 - \sum_{i=1}^{n} p_i \, \mathrm{d}q_i \,,$$

and define the contact Hamiltonian $H = H(q_0, q_1, \ldots, q_n, p_1, \ldots, p_n)$. The vertical component of the Hamiltonian vector field is straightforward (cf. Equation (B-2)):

$$X_H^{\text{ver}} = -H \frac{\partial}{\partial q_0}.$$

For the horizontal component, first evaluate the right hand side of Equation (B-4) in coordinates:

$$X_H^{\text{hor}} \lrcorner \, \mathrm{d}\alpha = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0} \right) \mathrm{d}q_i + \frac{\partial H}{\partial p_i} \, \mathrm{d}p_i \,.$$

In terms of the basis vectors, the mapping $\alpha^\flat$ is

$$\frac{\partial}{\partial q_i} \mapsto \mathrm{d}p_i \qquad \frac{\partial}{\partial p_i} \mapsto -\mathrm{d}q_i \qquad \frac{\partial}{\partial q_0} \mapsto 0 \qquad i = 1, \ldots, n.$$

The inverse transformation is slightly ambiguous at first sight, since any $\frac{\partial}{\partial q_0}$ cannot be recovered directly from the 'forward' mapping. However, we know that $\alpha^\sharp$ must produce a horizontal vector field. Therefore, first perform the inverse mapping in the q_i, p_i-components to obtain

$$-\sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}.$$

Contracting this expression with α produces $-\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i}$. Hence, we can use this knowledge to find the actual horizontal component:

$$X_H^{\text{hor}} = \sum_{i=1}^{n} p \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_0} - \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}.$$

[3] Semi-basic forms are forms that vanish when contracted with a vertical vector field. [15]

As such, the coordinate expression of Equation (B-5) is

$$X_H = \left(\sum_{i=1}^{n} p\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial q_0} - \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} + p_i\frac{\partial H}{\partial q_0}\right)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \tag{B-6}$$

Furthermore, we have

$$\pounds_{X_H}\alpha = -\frac{\partial H}{\partial q_0}\alpha,$$

and

$$\pounds_{X_H}H = -H\frac{\partial H}{\partial q_0}.$$

B-3-2 Jacobi brackets

Just like the Poisson brackets define a Poisson algebra of the smooth functions on a symplectic manifold, there is a bracket operation on contact manifolds that serves (about) the same purpose. These brackets do not define a Poisson structure, but rather a *Jacobi structure*, which is a more general notion that includes the Poisson structure as a particular instance. In this treatment we will only focus on the associated *Jacobi bracket* for contact Hamiltonian systems. For more details regarding Jacobi manifolds, the reader is referred to [15, chap. V] and [64].

For two smooth functions $f, g \in C^\infty(M)$, and M a contact manifold with contact form α, the *Jacobi bracket* is defined as

$$[\![\,,\,]\!] : C^\infty(M)\times C^\infty(M) \to C^\infty(M) : [\![f,\,g]\!] = -\big[X_f,\,X_g\big] \lrcorner\, \alpha, \tag{B-7}$$

where $X_f, X_g \in \mathfrak{X}(M)$ are the contact Hamiltonian vector fields of f and g respectively, and $[\cdot\,,\cdot]$ is the Lie bracket (i.e. the commutator of vector fields). Equivalent expressions for the Jacobi bracket are: [15]

$$\begin{aligned}[\![f,\,g]\!] &= -X_f \lrcorner\, \mathrm{d}g + g(R_\alpha \lrcorner\, \mathrm{d}f)\\ &= X_g \lrcorner\, \mathrm{d}f - f(R_\alpha \lrcorner\, \mathrm{d}g)\\ &= -\mathrm{d}\alpha\,(X_f, X_g) - f(R_\alpha \lrcorner\, \mathrm{d}g) + g(R_\alpha \lrcorner\, \mathrm{d}f).\end{aligned} \tag{B-8}$$

From these expressions, it is also clear that the Jacobi bracket is antisymmetric, i.e. $[\![f,\,g]\!] = -[\![g,\,f]\!]$ and $[\![f,\,f]\!] = 0$. As a time evolution operator (with respect to the Hamiltonian H), we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [\![f,\,H]\!] + f(R_\alpha \lrcorner\, \mathrm{d}H) = [\![f,\,H]\!] - fs.$$

Using the same coordinates as in Appendix B-3-1, the Jacobi bracket is equal to:

$$[\![f,\,g]\!] = \left(\sum_{i=1}^{n} p_i\frac{\partial g}{\partial p_i} - g\right)\frac{\partial f}{\partial q_0} - \left(\sum_{i=1}^{n} p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial g}{\partial q_0} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right).$$

TODO

Check signs of Jacobi bracket, sign convention is again different from Libermann and Marle + mistake?

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Glossary

Economic symbols

$\dot{q}$	Quantity demanded; quantity supplied; flow of goods
p	Price
q	Quantity; amount of goods

Physical symbols

β	Work 1-form
γ	Damping coefficient
η	Heat 1-form
E	(Mechanical) energy
n_{s}	Amount of substance
P	Pressure
p	Momentum
q	Position
R_{g}	Universal gas constant
S	Entropy
T	Temperature
U	Internal energy
V	Volume

Mathematical symbols

α	General contact 1-form
ω	Symplectic 2-form

ϑ	Liouville 1-form
M	Phase space; general manifold
Q	Configuration space
Z	Liouville vector field
$\mathbb{R}^n$	Real coordinate space of dimension n
$\wedge$	Wedge (or exterior) product
$\lrcorner$	Interior product
d	Exterior derivative
$\pounds_X$	Lie derivative with respect to the vector field X
$\oplus$	Whitney sum; direct sum
$\times$	Cartesian product; cross product (depending on context)
$\otimes$	Tensor product
$E \xrightarrow{\pi} B$	Bundle with total space E, projection map π and base space B
$\mathrm{T}_x M$	Tangent space to the manifold M at the point x
$\mathrm{T}_x^* M$	Cotangent space of the manifold M at the point x
$\mathrm{T}M$	Tangent bundle of the manifold M
$\mathrm{T}^* M$	Cotangent bundle of the manifold M
$\mathfrak{X}(M)$	Set of vector fields (smooth sections of $\mathrm{T}M$) on the manifold M
$C^\infty(M)$	Set of smooth functions on the manifold M
$\Lambda^n(M)$	Set of differential n-forms on the manifold M