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E. B. Legrand

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The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

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by

E. B. Legrand

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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis…

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology
July 18, 2022

E. B. Legrand

Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*

Chapter 1

Introduction

Idea: introduction using Arnol'ds thermodynamics quote.

Chapter 2

Symplectic and Contact Geometry in Economic Engineering

Chapter 3

Geometric Structures in Dissipative Mechanics

[intro here]

3-1 Symplectic mechanical systems

In the traditional sense, Hamiltonian mechanics take place on *symplectic manifolds*. A symplectic manifold (M, ω) is a smooth manifold M equipped with a *closed*, *nondegenerate* 2-form ω. Because ω must be nondegenerate, symplectic manifolds are necessarily even-dimensional. The celebrated Darboux theorem states that locally, all symplectic manifolds of the same dimension (say $2n$) are all symplectomorphic to each other. As a result, we define the prototypical symplectic 2-form that serves as a representative for *all* symplectic structures of that dimension as

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i \,, \tag{3-1}$$

where p_i and q_i are coordinates for the manifold M. A coordinate chart in which the symplectic 2-form has the above form is called a *Darboux charts*, and the associated coordinates *Darboux coordinates* [1, 2].

In mechanics, the *configuration manifold Q*: is the manifold specified by all the possible generalized positions q_i (or configurations) of the mechanical system. The *generalized momenta* associated with each of the generalized positions live in the collection of cotangent spaces to the configuration manifold. This is because from the Lagrangian viewpoint the generalized momenta are given by[1]

$$p_i = \frac{\partial L}{\partial q_i},$$

and the vector of p_i's is a cotangent vector to Q. Hence, the *cotangent bundle* of the configuration manifold contains all the possible position and momentum pairs.

The structure that associates each position with its corresponding momentum is given by the *Liouville 1-form* [2] ϑ on T^*Q. The Liouville form is defined at every point $(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathrm{T}^*M$ as

$$\vartheta = \sum_{i=1} p_i \,\mathrm{d}q_i \,. \tag{3-2}$$

Hence, the Liouville form tells us which momentum coordinate corresponds to a given position coordinate and vice versa. This turns out to be an essential piece of the geometric structure that underpins classical mechanics.

Every cotangent bundle is canonically endowed with a Liouville form. The exterior derivative of the Liouville form produces a *symplectic* 2-form. By convention, we define this symplectic form as follows:[3]

$$\omega = -\,\mathrm{d}\vartheta = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i \,.$$

Hence, the space of generalized positions and momenta (i.e. the cotangent bundle the configuration manifold Q) is canonically symplectic. The symplectic structure pairs the corresponding position and momentum coordinates in a skew-symmetric fashion.

The idea of Hamiltonian mechanics is that the equations of motion are generated by *Hamilton's equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

given the Hamiltonian function H, which is equal to the mechanical energy in the system. Observe that the above equation assumes that the pairing between the positions and momenta is known a priori.

In the language of differential geometry, Hamilton's equations are specified by a symplectic structure on T^*Q and an appropriate Hamiltonian function on that manifold: the pairing between the positions and momenta is therefore built in. A generic Hamiltonian system is a triple (M, ω, H), where (M, ω) is a symplectic manifold. In mechanics, we have that $M = \mathrm{T}^*Q$. To produce the equations of motion, the symplectic structure provides a mapping between the smooth functions on the manifold and the Hamiltonian vector fields on the manifold[4]. First, define the mapping

$$\omega^\flat : \mathrm{T}M \to \mathrm{T}^*M : X \mapsto X \lrcorner\, \omega.$$

[1] In coordinate-free language, the Liouville 1-form is defined pointwise through its action on a tangent vector $\boldsymbol{\xi}$ to T^*Q as follows

$$\vartheta(\boldsymbol{\xi}) = (x \circ \pi_*)(\boldsymbol{\xi}).$$

Here π_* is the pushforward of the bundle projection map $\pi : \mathrm{T}^*Q \to Q : (\boldsymbol{q}, \boldsymbol{p}) mapsto \boldsymbol{q}$. The point $x \in \mathrm{T}^*Q$ is interpreted as a map on the tangent space to $\pi(x)$ on the base manifold.

[2] The Liouville 1-form makes its appearance in literature under a myriad of names, such as the canonical 1-form, tautological 1-form, Poincaré 1-form or the symplectic potential.

[3] In this text, the 'q-first' sign convention used by Abraham and Marsden [3] and Cannas da Silva [1] is observed and maintained in the following sections concerning contact and Jacobi manifolds.

[4] These are a special class of vector fields on M that arise as a result of this mapping.

Because ω is nondegenerate by definition, the mapping $\omega^\flat$ is an isomorphism. Thus, the inverse mapping is well-defined, and is denoted by $\omega^\sharp$.

This isomorphism specified by ω allows us to find the corresponding Hamiltonian vector field X_H to a Hamiltonian function, given by:

$$X_H = \omega^\sharp(\mathrm{d}H).$$

In (Darboux) coordinates, the action of $\omega^\sharp$ on the basis 1-forms is

$$\mathrm{d}p_i \mapsto \frac{\partial}{\partial q_i} \qquad \mathrm{d}q_i \mapsto -\frac{\partial}{\partial p_i}.$$

The minus sign arises as a consequence of the anticommutativity of the wedge product in appearing in ω.

A classical example of this formalism is the harmonic oscillator (undamped) shown in Figure 3-1, with Hamiltonian function

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2,$$

where m is the mass and k the spring constant. The Hamiltonian vector field is then

$$X_H = \frac{p}{m}\frac{\partial}{\partial q} - kq\frac{\partial}{\partial p},$$

or stated as a system of differential equations

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -kq.$$

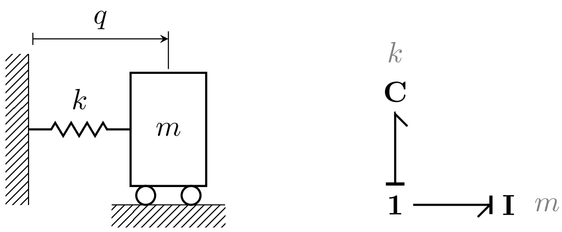

Figure 3-1: Blabla

In the context of bond graphs, the symplectic form represent the dual nature of a bond. That is to say, a bond represents an exchange of both an effort and a flow, and they are inherently tied to each other. The flow is a change in position or generalized velocity, and the effort is a change in momentum. The effort and flow associated to a bond are conjugate: the symplectic form provides precisely the structure that is visually present in a bond graphs (e.g. Figure 3-1).

The symplectic form endows the manifold M also with a *Poisson structure*, i.e. a Lie algebra structure on the vector space space of functions on M. The commutator of this algebra structure is the *Poisson bracket*,

$$
\begin{aligned}
\{\,,\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) : \{f, g\} &= \omega\big(\omega^\sharp \,\mathrm{d}f \,, \omega^\sharp \,\mathrm{d}g\big) \\
&= \omega\big(X_f, X_g\big) \\
&= \pounds_{X_f} g.
\end{aligned}
\tag{3-3}
$$

Poisson brackets are anticommutative, bilinear and satisfy the Jacbobi identity. Additionally, they also satisfy the *Leibniz rule*,

$$
\{f, gh\} = \{f, g\}h + \{f, h\}g.
$$

Due to the Poisson structure, symplectic manifolds are also Poisson manifolds. In Sections 3-2 and 3-3 the notion of Poisson manifolds will be generalized to *Jacobi manifolds* to cover more general mechanical systems. In contrast to the Poisson structure, does the Jacobi structure not have the Leibniz property. The usefulness of Poisson brackets is due to the fact that they provide a convenient way to calculate the time-rate of change of an observable f:

$$
\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t}.
$$

If the Hamiltonian does not depend on time, it is conserved under its own Hamiltonian vector field. This is easily seen from the anticommutativity of Poisson brackets ($\{H, H\}$). Therefore, Hamiltonian systems conserve energy; they do not allow for dissipative (friction) forces in a straightforward manner, unless included in the form of an explicit time depence. This is clearly a consequence of the symplectic structure. In the next section we will extend the Hamiltonian formalism to a related type of geometry called *contact geometry* to incorporate dissipation in the Hamiltonian system.

3-2 Contact mechanical systems

3-3 Jacobi structure for general systems

Chapter 4

Split-Quaternions as Dynamical Systems

In this chapter, the geometric connection is made between the algebra of split-quaternions and the qualitative behavior of two-dimensional linear dynamical systems.

4-1 Split-quaternion algebra

4-1-1 Basic properties

Like conventional quaternions, the split-quaternions form a number system that consists of linear combinations of four basis elements, which will be denoted by 1, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.[1] The algebra of split-quaternions is associative but not commutative — formally speaking, we are dealing with an algebraic structure called a *noncommutative ring*. The multiplication table for the split-quaternion algebra is shown in Table 4-1. The set of split-quaternions is denoted by $\hat{\mathbb{H}}$, since $\mathbb{H}$ is reserved for conventional quaternions.[2]

Table 4-1: Multiplication table for the split-quaternion algebra.

	1	$\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{k}}$
1	1	$\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{k}}$
$\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\imath}}$	-1	$\hat{\boldsymbol{k}}$	-$\hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{\jmath}}$	-$\hat{\boldsymbol{k}}$	1	-$\hat{\boldsymbol{\imath}}$
$\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{\imath}}$	1

[1] Even though they behave similarly, the imaginary unit i is not to be confused with the split-quaternion basis element $\hat{\boldsymbol{\imath}}$, because they belong to different number systems.
[2] The '$\mathbb{H}$' is in honour of sir William Rowan Hamilton, who also developed the Hamiltonian formalism: the fruits of his work truly form the central theme in this thesis. [4]

The important distinction from conventional quaternions resides in the diagonal elements of Table 4-1. For quaternions, all the nonreal basis elements square to -1, this is not the case for the split-quaternions (only $\hat{\boldsymbol{\imath}}$ does). This is precisely the reason why split-quaternions are 'split', for this difference in sign makes their norm (to be defined later) into an indefinite form. That is to say, whereas quaternions have a 'metric signature' (in a very imprecise sense of the word metric) of $(+,+,+,+)$, for the split-quaternions we have $(+,+,-,-)$. The distinctive 'metric' signature makes the algebra of split-quaternions different from its conventional quaternion counterpart.

Dihedral group The basis elements of the split-quaternions $\left\{1, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\right\}$ form a finite group under multiplication, namely the *dihedral group* D_4, which represents all the symmetries of a square: the identity, a 90-degree rotation and two reflections, as illustrated in Figure 4-1. [5]

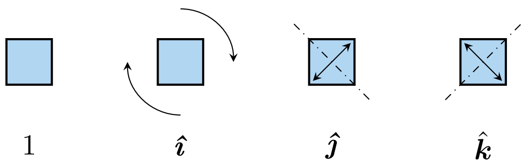

Figure 4-1: The dihedral group D_4 is the symmetry group of a square. This group is isomorphic to the group formed by $1, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ under multiplication.

The structure of the dihedral group can be visualized by its *cycle graph* in Figure 4-2. Many important properties of the split-quaternion algebra and the applications in this chapter can be traced back to the shape of this cycle graph. One example is the split nature of the quaternions: the $\hat{\boldsymbol{\imath}}$-element generates an order-4 cycle, while $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ generate order-2 cycles (in contrast, the cycle graph for conventional quaternions is entirely symmetric for all these elements). [5]

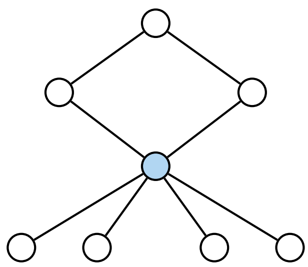

Figure 4-2: Cycle graph of the dihedral group. There are five cycles: one of order four which represents the rotations (or the element $\hat{\boldsymbol{\imath}}$), and four order-2 cycles, which are all the possible reflections. The colored element represents the identity.

Split-quaternion norm Complex numbers have a real and imaginary part. Likewise, (split)-quaternions have a similar notion: a *scalar* (or real) and *vector* (or imaginary) components. For an arbitrary split-quaternion $a \in \hat{\mathbb{H}}$, [6]

$$a = a_0 + a_1 \hat{\boldsymbol{\imath}} + a_2 \hat{\boldsymbol{\jmath}} + a_3 \hat{\boldsymbol{k}}$$

the real part is $\mathrm{sca}(h) = a_0$ and the vector part is $\mathrm{vec}(a) = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$. For convenience, the vector part will be referred to in traditional bold vector notation:

$$\boldsymbol{a} \equiv \mathrm{vec}(a) \equiv a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}.$$

Furthermore, for every split-quaternion there is a unique *conjugate*

$$a^* = \mathrm{sca}(a) - \mathrm{vec}(a) = a_0 - a_1\hat{\boldsymbol{\imath}} - a_2\hat{\boldsymbol{\jmath}} - a_3\hat{\boldsymbol{k}},$$

which we require to define the *squared split-quaternion norm*:

$$\mathcal{N} : \hat{\mathbb{H}} \to \mathbb{R} : \ \mathcal{N}(a) \equiv aa^* = a_0^2 + a_1^2 - a_2^2 - a_3^2. \tag{4-1}$$

As mentioned, this norm is not positive definite, in stark contrast to quaternions (or complex numbers, for that matter). Split-quaternions can be categorized into three regimes based on their norm being negative, zero or positive. In the tradition of special relativity, these regimes are named[3] [7, 8]

- *timelike* if $\mathcal{N}(a) > 0$,
- *lightlike* if $\mathcal{N}(a) = 0$,
- *spacelike* if $\mathcal{N}(a) < 0$.

The *split-quaternion norm* is then defined as

$$\|a\| \equiv \sqrt{|\mathcal{N}(a)|}.$$

Vector norm Apart from the split-quaternion norm, we may also define a (square) norm that only considers the *vector part* of the split-quaternion. This squared norm is defined in accordance with the overall squared split-quaternion norm given by Equation (4-1):

$$\mathcal{N}(\boldsymbol{a}) = a_1^2 - a_2^2 - a_3^2.$$

The notation used here is not abusive: $\boldsymbol{a}$ simply refers to the split-quaternion with the same vector part as a but with zero scalar part. We can therefore use the same function with no ambiguity. Likewise, the vector norm is $\|\boldsymbol{a}\| = \sqrt{|\mathcal{N}(\boldsymbol{a})|}$.

The quadratic form of the squared vector norm is not positive-definite either; in the the same vein as before, we can therefore classify split-quaternions by the 'sign' of their vector part again. We refer to (vectors in) these regimes as *timelike (vectors)*, *spacelike (vectors)* and *lightlike (vectors)* in the same fashion.

In contrast to the larger space of split-quaternions, the space of vectors *does* have a traditional Lorentz (i.e. 'special relativity') structure, but in three dimensions instead of four. This is because the signature of the squared vector norm only has one minus sign instead of two. The

[3]Spacetime is also four-dimensional, but the signature of the Minkowski metric is different from the split-quaternion signature: it is either $(-,+,+,+)$ or equivalently $(+,-,-,-)$ depending on the sign convention we choose to follow. However, the terminology (i.e. spacelike, timelike, lightlike) still applies.

above expression is equivalent the Lorentz norm applied to a vector in $\mathbb{R}^3$; we will denote $\mathbb{R}^3$ equipped with the Lorentz norm by $\mathbb{R}^{1,2}$, and call it the Lorentzian three-space. [6]

Observe that $\mathcal{N}(a) < 0 \Rightarrow \mathcal{N}(\boldsymbol{a}) < 0$; that is to say, *a spacelike split-quaternion always has a spacelike vector part.* The converse is not necessarily true. Along the same line, a lightlike split-quaternion can only have a lightlike or spacelike vector part. All possible regime combinations for split-quaternions and their vector parts are listed in Table 4-2. This classification is important because, as discussed in Section 4-2, this classification is precisely equivalent to the qualitative classification of dynamical systems. It will play a central role throughout this chapter.

Table 4-2: All the possible combinations of the regime of a split-quaternion and its vector part. Spacelike split-quaternions can only have a spacelike vector, while lightlike split-quaternions can only have lightlike or spacelike vector parts.

		$\mathcal{N}(\boldsymbol{a})$		
		spacelike	*lightlike*	*timelike*
$\mathcal{N}(a)$	*spacelike*	①	—	—
	lightlike	②	③	—
	timelike	④	⑤	⑥

4-1-2 Relation with two-dimensional matrix algebra

The usefulness of split-quaternions (for our purpose) originates from the fact that the algebra of split-quaternions is isomorphic to the algebra of real two-dimensional matrices. This underlies this entire chapter, for it allows us to find an alternative for the traditional matrix description of linear dynamical systems.

Formally, an algebra is a vector space combined with a vector space V over a field $\mathbb{F}$, combined with an addition operation, scalar multiplication, and an $\mathbb{F}$-bilinear product operation $V \times V \to V$. [9]

- The split-quaternion algebra is an algebra over the field real numbers ($\mathbb{F} = \mathbb{R}$), where the multiplication is governed by the split-quaternion multiplication rules (see Table 4-1).
- The set of 2×2-matrices also constitutes an $\mathbb{R}$-vector space; matrix multiplication makes it into an algebra.

An algebra isomorphism is an isomorphism between vector spaces that also commutes with the respective product operations in both vector spaces. If $(V, \bullet)$ and $(W, \diamond)$ are vector spaces equipped with their product operations, then $\phi : V \to W$ is an algebra isomorphism if (i) ϕ is a vector space isomorphism between V and W, and (ii) [10]

$$\phi(v_1 \bullet v_2) = \phi(v_1) \diamond \phi(v_2) \qquad v_1, v_2 \in V.$$

In the case of the split-quaternions and two-dimensional matrices, it is sufficient to map the basis elements of the split-quaternions to four linearly independent 'basis' matrices, and show that the resulting matrices observe the same multiplication rules as defined in Table 4-1. Indeed, define the mapping ϕ by

$$\begin{aligned} \phi : \hat{\mathbb{H}} \to \mathbb{R}^{2\times2} : \quad 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \hat{\boldsymbol{\imath}} &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \hat{\boldsymbol{\jmath}} &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \hat{\boldsymbol{k}} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4-2}$$

It is easily verified that (i) these matrices span $\mathbb{R}^{2\times2}$ and (ii) that the multiplication rules for split-quaternions are in accordance when translated to the respective matrices under matrix multiplication. Due to the bilinearity of the product, any linear combination of the basis elements will therefore satisfy the rules as well. Hence, we have established an algebra isomorphism between the split-quaternions and the 2×2-matrices. Based on this mapping for the basis vectors, a general quaternion maps to

$$\phi(a_0 + a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}) = \begin{pmatrix} a_0 + a_3 & a_1 + a_2 \\ a_2 - a_1 & a_0 - a_3 \end{pmatrix}. \tag{4-3}$$

Likewise, the inverse mapping on an arbitrary matrix yields

$$\phi^{-1}\begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} = \frac{b_0 + b_3}{2} + \left(\frac{b_1 - b_2}{2}\right)\hat{\boldsymbol{\imath}} + \left(\frac{b_1 + b_2}{2}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{b_0 - b_3}{2}\right)\hat{\boldsymbol{k}}. \tag{4-4}$$

One of the powerful features of the mapping ϕ is that it maps natural properties of the split-quaternion to natural properties of the associated matrix. Hence, given that $A = \phi(a)$ with $a \in \hat{\mathbb{H}}$ and $A \in \mathbb{R}^{2\times2}$, we have the following properties:

- The *conjugate* of the split-quaternion maps to the *adjugate* of the matrix:[4]

$$\phi(a^*) = \operatorname{adj}(A).$$

- The *trace* of the matrix coincides with the *real or scalar part* of the split-quaternion:

$$\operatorname{sca}(a) = a_0 = \frac{\operatorname{tr}(A)}{2}.$$

- The *determinant* of the matrix is equal to the *squared norm* of the split-quaternion:

$$\mathcal{N}(a) = \det(A).$$

- The equivalence of the determinant and the split-quaternion norm hints at the fact that the multiplicative inverse of a split-quaternion does not always exist: only when its norm is nonzero. In that case, it is clear that

$$\phi\left(a^{-1}\right) = A^{-1} \qquad \mathcal{N}(a) \neq 0.$$

  The determinant property also shows us what the regime will be of the product of two split-quaternions; this is shown in Table 4-3.

[4]The adjugate of a matrix is the transpose of its cofactor matrix. [11]

Table 4-3: Regime transition under the action of split-quaternion multiplication. The timelike split-quaternions form a group under multiplication, the timelike and spacelike split-quaternions do not: timelike split-quaternions do not have an inverse and the spacelike split-quaternions are not closed.

$\times$	*space*	*light*	*time*
space	time	light	space
light	light	light	light
time	space	light	time

- The eigenvalues of a 2×2-matrix can be expressed in terms of its trace and its determinant:

$$\lambda_A = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det(A)}}{2}.$$

  The argument of the square root is equal to the *negative of the squared vector norm* of a. We therefore have:

$$\lambda_A = \frac{2a_0 \pm \sqrt{4a_0^2 - 4\mathcal{N}(a)}}{2} = \begin{cases} a_0 \pm \mathrm{i}\|\boldsymbol{a}\| & \boldsymbol{a} \text{ timelike,} \\ a_0 \pm 0 & \boldsymbol{a} \text{ lightlike,} \\ a_0 \pm \|\boldsymbol{a}\| & \boldsymbol{a} \text{ spacelike.} \end{cases} \tag{4-5}$$

  Hence, the 'real' (scalar) and the magnitude of the 'imaginary' (vector) parts of the quaternion coincide with the real and imaginary part of the eigenvalues of the matrix.

The algebra of 2×2-matrices (or equivalently, of the split-quaternions) also consitute the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ of the two-dimensional general linear group $\mathrm{GL}(2, \mathbb{R})$. Furthermore, the traceless matrices, or equivalently, the split-quaternions with zero real part form the subalgebra $\mathfrak{sl}(2, \mathbb{R})$ of the special linear group $\mathrm{SL}(2, \mathbb{R})$. These are the volume-preserving automorphisms on $\mathbb{R}^2$. Because in $\mathbb{R}^2$, volume and area coincide, the special linear group and the symplectic group $\mathrm{Sp}(1)$ are equivalent. For higher dimensions, this is not the case: area preservation is generally a stronger condition than volume preservation. The Lie algebra elements of the symplectic group are called Hamiltonian matrices; therefore, split-quaternions without real part are referred to as *Hamiltonian.*

4-2 Split-quaternion representation of dynamical systems

4-2-1 The algebra of vector fields

The isomorphism between the split-quaternions and the algebra of two-dimensional square matrices exposed in the preceding section can be used to develop an alternative representation of linear dynamical systems. Indeed, an autonomous dynamical system is defined by a *vector field* on the state space. If this vector field is a linear mapping from the state space into the tangent space, it can be represented by a matrix. These vector fields form a vector space on their own, spanned (for example) by the four basis elements shown in Equation (4-2). Each of the basis elements 1, $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ corresponds to a specific 'basis' vector field, denoted by X_1, $X_{\hat{\imath}}$, $X_{\hat{\jmath}}$ and $X_{\hat{k}}$ respectively. The basis vector fields are shown in Figure 4-3.

The vector field element X_1, corresponding to the identity element is an infinitesimal dilation, while $X_{\hat{\imath}}$ represents an infinitesimal clockwise rotation, and $X_{\hat{\jmath}}$ and $X_{\hat{\jmath}}$ are infinitesimal 'squeeze mappings', hyperbolic rotations or *Lorentz transformations* along two differents sets of principal axes. The binary operation of matrix multiplication translates to the composition of the vector fields.

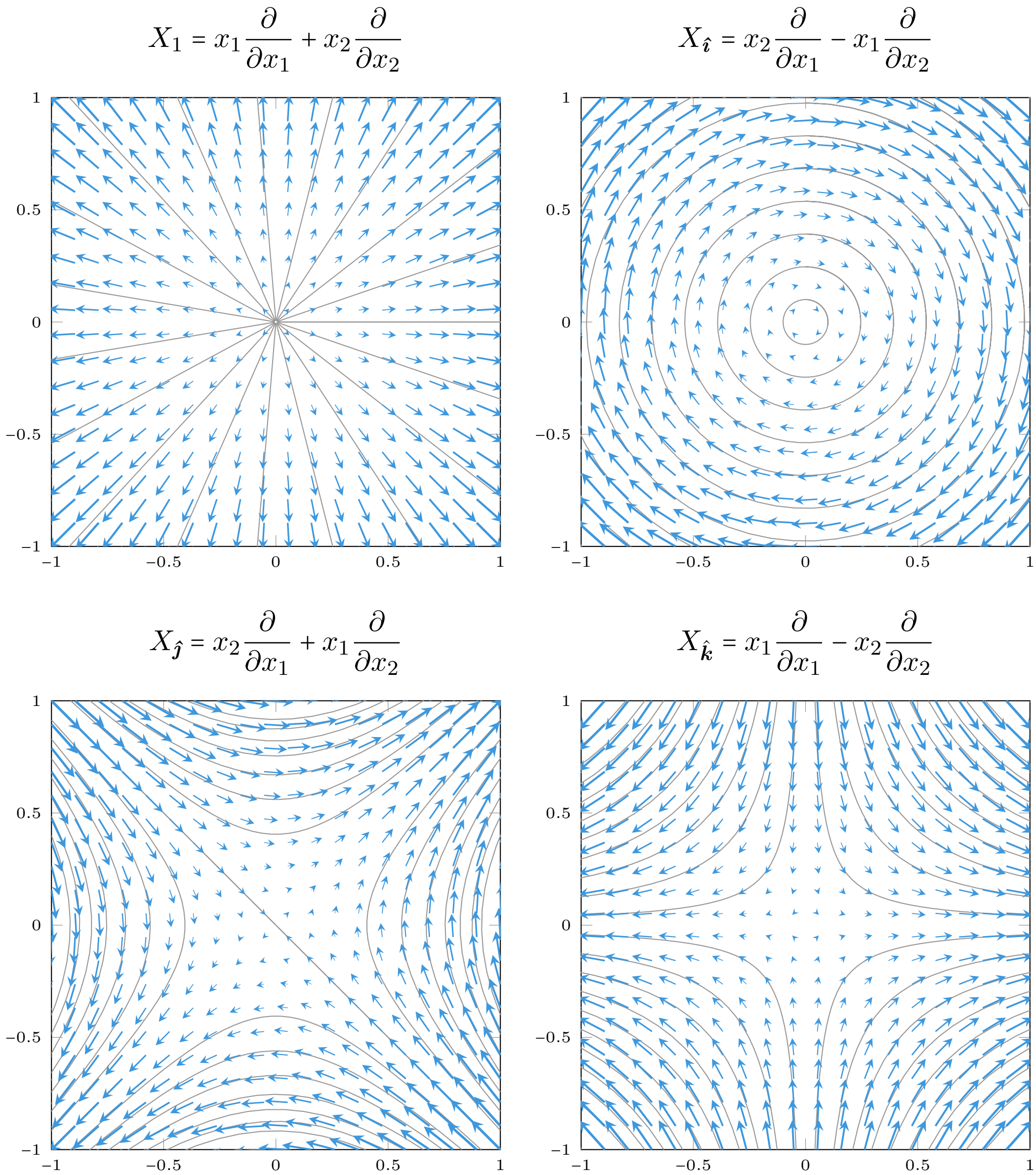

Figure 4-3: Basis vector fields corresponding to the basis elements of the split-quaternions.

Apart from the multiplication operation for split-quaternions and matrices, we can also define the *commutator* of a binary operation, which measures exactly by how much two elements fail to commute. For split-quaternions, matrices and vector fields the commutator is defined as (in that order):

$$
\begin{aligned}
[a, b] &= ab - ba, \quad a, b \in \hat{\mathbb{H}}, \\
[A, B] &= AB - BA, \quad A, B \in \mathbb{R}^{2\times 2}, \\
[X, Y] &= \pounds_X Y, \quad X, Y \in \mathfrak{X}(M),
\end{aligned}
\tag{4-6}
$$

for some smooth manifold M of the appropriate dimension. For vector fields, the commutator is also referred to as the *Lie bracket* (it therefore defines the *Lie algebra* $\mathfrak{sl}(2, \mathbb{R})$). The commutation relations (or *structure constants*) for the basis vector fields are [9]

$$
[X_1, X_{\hat{\imath}}] = [X_1, X_{\hat{\jmath}}] = [X_1, X_{\hat{k}}] = 0, \tag{4-7}
$$

$$
[X_{\hat{\imath}}, X_{\hat{\jmath}}] = 2X_{\hat{k}}, \qquad [X_{\hat{\imath}}, X_{\hat{k}}] = -2X_{\hat{\jmath}}, \qquad [X_{\hat{\jmath}}, X_{\hat{k}}] = -2X_{\hat{\imath}}. \tag{4-8}
$$

Of course, these commutation relations are exactly the same for the corresponding split-quaternion or matrix basis elements. Scalar multiples of the identity element commute with every other element of the algebra; they are in the *center* of the algebra. Importantly, the associate vector field also commutes with all the other vector fields.

4-2-2 Classification of dynamical systems

The classification of two-dimensional linear dynamical systems is important, for they also locally represent the fixed points of general nonlinear systems. Traditionally, this decomposition is done according to the eigenvalues of the state transition matrix matrix A, or equivalently, through a Poincaré diagram as shown in Figure 4-4. Because the split-quaternion norms are directly related to the real and imaginary part of the eigenvalues of the associated matrix, this classification is more naturally done in the realm of split-quaternions, based on their squared (vector) norm, on par with the regimes defined in Table 4-2.

Spacelike split-quaternion norm

① For spacelike split-quaternions, there is only one possibility: a negative split-quaternion norm corresponds to a negative determinant, which means that the fixed point is a *saddle*. We can distinguish one particular case: if the scalar part of the split-quaternion is zero ($a_0 = 0$), the saddle is 'balanced', and generates a proper *squeeze mapping*, which is a symplectomorphism of the phase space. The split-quaternion is therefore Hamiltonian. An example of the latter is the linearization of the unstable fixed point of a rotational pendulum.

Lightlike split-quaternion norm

② *Spacelike vector norm*: in this case, there is not just a fixed point but a fixed line in the phase space. This fixed line is stable or unstable depending on the sign of the scalar part of the quaternion.

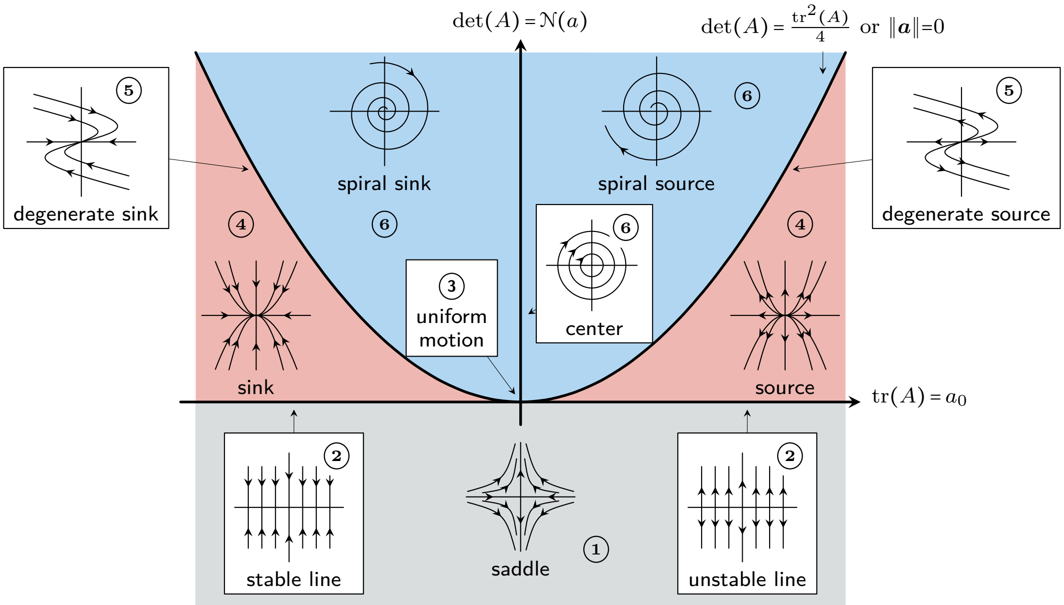

Figure 4-4: The classic Poincaré diagram, based on the conventional classification of fixed points based on the trace and determinant of the state transition matrix A. The corresponding split-quaternion regimes defined in Table 4-2 are displayed as well. The determinant axis coincides with the squared norm of the split quaternion being 0, while all the points on the parabolic line correspond to split-quaternions with zero *vector* norm. A further distinction is made with the scalar part of the split-quaternion, which, for each of the regimes, determines (asymptotic) (in)stability.

③ *Lightlike vector norm*: this case is degenerate of the second degree; it coincides with the origin in the Poincaré diagram. The associated vector field is purely translational. An example is an object in uniform motion.

Timelike split-quaternion norm

④ *Spacelike vector norm*: this case gives rise to eigenvalues that are purely real; the fixed point is called a *node*. Depending on the sign of the scalar part, the fixed point can be an unstable node or *source* ($a_0 > 0$) or a stable node or *sink* ($a_0 < 0$). An example of such a system is the overdamped harmonic oscillator.

⑤ *Lightlike vector norm*: the eigenvalues of the associated matrix are real and equal; this type of fixed point is named a *degenerate node*. More specifically, in the unstable case ($a_0 > 0$) it is called a *degenerate source*, while in the stable case it is referred to as a *degenerate sink*. An example is a critically damped harmonic oscillator.

We can also relate the vector norm to the Jordan form of the associated matrix. Recall that the Jordan form is 'special' if a matrix is not completely diagonalizable: in the case of two identical eigenvalues, their geometric multiplicity is equal to one instead of two. In terms of the corresponding matrix, the vector norm is equal to:

$$\mathcal{N}(\boldsymbol{a}) = \det\left(A - \frac{\operatorname{tr}(A)}{2}\right)$$

when two eigenvalues are identical, $\mathrm{tr}(A) = 2\lambda$, the above expression vanishes and the split-quaternion has a lightlike vector. If the matrix is nevertheless diagonalizable, it must be a scalar multiple of the identity. The corresponding split-quaternion is then purely real — it has no vector part (of course, the vector is then lightlike in a trivial sense). The associate fixed point is then a *proper node*. Conversely, if the vector norm is zero but the vector part not, the matrix is not diagonalizable, and the fixed point is an *improper node*.

⑥ *Timelike vector norm*: this really is the only general case where the eigenvalues of A are complex. If $a_0 = 0$, the eigenvalues are imaginary and the fixed point is a *center*. Likewise, for $a_0 > 0$ it is an *unstable spiral node* and for $a_0 < 0$ a *stable spiral node*. An example is an underdamped (or even undamped) harmonic oscillator.

It is clear from the present discussion that the split-quaternions offer a very natural representation of linear dynamical sytems, and their natural properties translate directly to distinctions in the qualitative behaviour of these systems.

4-2-3 The exponential function of split-quaternions

Just like the concept of the exponential function was originally generalized for square matrices, we can do the same for split-quaternions in an analogous manner. As such, the *split-quaternion exponential function* is defined as

$$\exp(a) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} a^k \qquad a \in \hat{\mathbb{H}}.$$

Because this definition is identical to the one for matrices, we may, as a result of the isomorphism defined in Section 4-1-2, also expect the exact same result; i.e.

$$\exp(a) = \phi^{-1}(\exp(\phi(a)))$$

where the second exponential function refers to the matrix exponential instead.

To evaluate the exponential function of a split-quaternion, let us first use the following property of the matrix exponential [12]

$$AB = BA \quad \Rightarrow \quad \exp(A + B),$$

i.e. we can only 'split' the exponential of a sum if the two elements *commute*. We can regard an arbitrary split-quaternion

$$a = a_0 + a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$$

as the sum of a_0 and $a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$. The real part is distinguished from the other three parts in the sense that it commutes with every other element (cf. Section 4-2-1). We may therefore use the former property and apply it to the split-quaternion exponential as well:

$$\exp(a) \;=\; \mathrm{e}^{a_0} \exp\!\left(a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}\right).$$

We therefore only have to be considered with the evaluation exponential of $\boldsymbol{a}$. To do so, observe that we can consider the vector part of a split-quaternion to be a split-quaternion in its own right, but with zero real part. This means that $\boldsymbol{a}^* = -\boldsymbol{a}$, and the squared vector norm is simply the negative of the square of the vector part:

$$\mathcal{N}(\boldsymbol{a}) = \boldsymbol{a}\boldsymbol{a}^* = -\boldsymbol{a}^2.$$

Let us now introduce the concept of *unit split-quaternion vectors*, which are vector split-quaternions with a vector norm of ±1. The unit vector may be obained by normalization of the vector part:

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{\sqrt{|\mathcal{N}(\boldsymbol{a})|}} \qquad \mathcal{N}(\boldsymbol{a}) \neq 0,$$

which squares to

$$\hat{\boldsymbol{a}}^2 = -\mathcal{N}(\hat{\boldsymbol{a}}) = -\frac{\mathcal{N}(\boldsymbol{a})}{|\mathcal{N}(\boldsymbol{a})|} = -\operatorname{sgn}(\mathcal{N}(\boldsymbol{a})).$$

Normalizing lightlike vectors is not possible, because they have all the same length of zero: there is no point in making the distinction between vector and unit vector. Based on the regime of the vector part, three possiblities arise: [13, 14]

- If $\boldsymbol{a}$ is timelike, then $\hat{\boldsymbol{a}}^2 = -1$. We can therefore say that the unit vector 'behaves' like the imaginary unit i ($\mathrm{i}^2 = -1$). In general, we can identify the split-quaternion (with timelike vector part) $a_0 + \|a\|\hat{\boldsymbol{a}}$ with the *complex number* $a_0 + \|\boldsymbol{a}\|\mathrm{i}$.

- If $\boldsymbol{a}$ is lightlike, then $\boldsymbol{a}^2 = 0$, and the notion of the unit vector is not well-defined. Because the vector is nilpotent with degree 2, it is analogous to the nilpotent unit ε (for which we have that $\varepsilon^2 = 0$). Split-quaternions with timelike vector part can be identified with the *dual number* $a_0 + \varepsilon$.

- Finally, if $\boldsymbol{a}$ is spacelike, then $\hat{\boldsymbol{a}}^2 = 1$. The unit vector behaves like the idempotent unit j, with defining property $\mathrm{j}^2 = 1$ ($\mathrm{j} \notin \mathbb{R}$!).[5] Likewise, a split-quaternion with spacelike vector part is analogous to the *split-complex number* (or hyperbolic number) $a_0 + \|\boldsymbol{a}\|\mathrm{j}$.

This connections with the generalized complex numbers[6] sheds some additional light on the behavior of the eigenvalues of the associated matrix A by means of the root locus plot (see Equation (4-5)). A common type of root locus branch consists of a complex pole pair approaching the real axis when the gain is increased. When they finally collide on the real axis, they each go their opposite ways on the real axis, essentially breaking the symmetry with respect to the real axis. The split-quaternions and hypercomplex numbers paint a slightly more elegant picture, which is shown in Figure 4-5:

- As shown above, when the pole pair is complex, the associated split-quaternion vector is timelike. The eigenvalues are naturally conjugate with respect to the real axis, i.e.

$$\lambda_A = a_0 \pm \|a\|\mathrm{i}.$$

[5] Again, we must take care not to confuse the hyperbolic unit with the split-quaternion basis element $\hat{\boldsymbol{\jmath}}$. They behave the same, and are related in the sense that they give rise to 'split' behavior, but are part of a very different number systems.

[6] For more details on generalized complex numbers, the reader is referred to Harkin and Harkin [14].

- When the pole pair collides on the real axis (often called the *branch point*), the imaginary part of the eigenvalue is zero, and the vector is timelike. Observe that we can make the case that, because the branches continue afterwards in a separate manner, they cannot be *exactly* the same. Indeed, the eigenvalues are

  $$\lambda_A = a_0 \pm \varepsilon.$$

  The nilpotent unit ε is often interpreted as a differential, or an infinitesimally small quantity.[7] We argue that in this case, the pole pair is still conjugate, but the poles differ only by an infinitesimal amount.

- When the gain is increased further, the poles are real and the symmetry with respect to the real axis is broken. However, we can infer from the preceding discussion that the imaginary part is now hyperbolic instead, i.e.

  $$\lambda_A = a_0 \pm \|a\|\mathrm{j}.$$

  Of course, it is possible to project these points on the real axis, but this obscures the natural symmetry of the root locus branch. In Figure 4-5, we therefore put the hyperbolic part on a third axis.

Let us now return to the exponential function. We can manipulate the definition of $\exp(a)$ as follows:

$$\begin{aligned}\exp(a) &= \mathrm{e}^{a_0}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{a}^k\right)\\ &= \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty}\frac{\left(\boldsymbol{a}^2\right)^k}{(2k)!}+\sum_{k=0}^{\infty}\frac{\boldsymbol{a}\left(\boldsymbol{a}^2\right)^k}{(2k+1)!}\right].\end{aligned}$$

Furthermore, if $\boldsymbol{a}$ is not lightlike, we have:

$$\exp(a) = \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k}\left(\hat{\boldsymbol{a}}^2\right)^k}{(2k)!}+\hat{\boldsymbol{a}}\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k+1}\left(\hat{\boldsymbol{a}}^2\right)^k}{(2k+1)!}\right].$$

Once again, there are three possibilities, depending on the regime of $\boldsymbol{a}$:

- If $\boldsymbol{a}$ is *timelike*, then the above expression reverts to

  $$\begin{aligned}\exp(a) &= \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k}(-1)^k}{(2k)!}+\hat{\boldsymbol{a}}\sum_{k=0}^{\infty}\frac{\|\boldsymbol{a}\|^{2k+1}(-1)^k}{(2k+1)!}\right],\\ &= \mathrm{e}^{a_0}\big[\cos(\|a\|)+\hat{\boldsymbol{a}}\sin(\|a\|)\big].\end{aligned} \tag{4-9}$$

  This is roughly equivalent to the Euler identity for complex numbers: this is not at all surprising, since we found before that $\hat{\boldsymbol{a}}$ can be associated with the imaginary unit if $\boldsymbol{a}$ is timelike.

[7] A common application of dual numbers is automatic differentiation: because higher powers vanish, they can be used to generate first-order polynomial approximations. The unit 'circle' for dual numbers consists of two vertical lines crossing the horizontal axis at ±1. These lines can again be interpreted as linear approximations of the actual unit circle (or unit hyperbola) associated to (split-)complex numbers. The plane spanned by the j-axis and the real axis is the split-complex plane. The 'projection' to the real axis is in this plane a reflection with respect to the light cone (first diagonal).

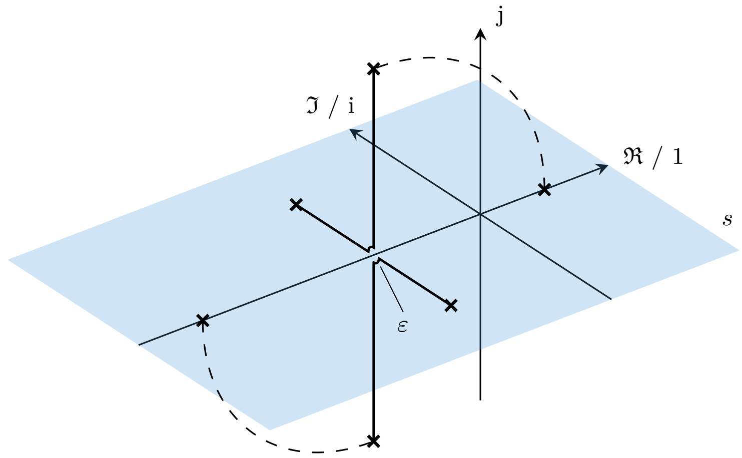

Figure 4-5: Generalized version of a root locus plot in terms of hypercomplex numbers. The traditional root locus is set in the complex s-plane (shown in blue), but we added a third axis for the hyperbolic part of the eigenvalue. When the gain is increased, the initially complex pole pair ventures towards the real axis. If the pole pair is critically damped, both poles are separated from the real axis by an infinitesimal distance of ε. Increasing the gain even more pushes the pole pair into the hyperbolic regime (the associated split-quaternion vector is now spacelike). Observe that in this picture, the symmetry with respect to the real axis is preserved. In the traditional root locus, these points are projected onto the real axis, indicated by the dashed lines.

- Secondly, if $\boldsymbol{a}$ is *lightlike*, we can simply use the definition of the expontial in its original form: [8]

$$\begin{aligned}\exp(a) &= \mathrm{e}^{a_0} \sum_{k=0}^{\infty} \frac{\boldsymbol{a}^k}{k!},\\ &= \mathrm{e}^{a_0}\left[1 + \boldsymbol{a} + \sum_{k=2}^{\infty} \frac{\boldsymbol{a}^{k-2}\boldsymbol{a}^2}{k!}\right],\\ &= \mathrm{e}^{a_0}(1 + \boldsymbol{a}).\end{aligned} \tag{4-10}$$

- Finally, if $\boldsymbol{a}$ is *spacelike*, we have

$$\begin{aligned}\exp(a) &= \mathrm{e}^{a_0}\left[\sum_{k=0}^{\infty} \frac{\|\boldsymbol{a}\|^{2k}}{(2k)!} + \hat{\boldsymbol{a}} \sum_{k=0}^{\infty} \frac{\|\boldsymbol{a}\|^{2k+1}}{(2k+1)!}\right],\\ &= \mathrm{e}^{a_0}\left[\cosh(\|a\|) + \hat{\boldsymbol{a}}\sinh(\|a\|)\right].\end{aligned} \tag{4-11}$$

The relevance of the exponential map lies of course in the fact that the solution of the linear differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = A\boldsymbol{x}$$

[8] We can also use the 'split' expression, defining that $0^0 \equiv 1$, a common convention in power series and algebra. Observe that the trigonometric functions associated to the dual numbers (i.e. the lightlike vectors) are then equal to the small-angle approximation for sin and cos. [14, 15]

is equal to [16]

$$\boldsymbol{x}(t) = \exp(At)\boldsymbol{x}_0,$$

where the one-parameter group of transformations generated by $\exp(At)$ is referred to as the *flow* of the vector field $A\boldsymbol{x}$. Hence, for two-dimensional systems, the matrix A can be represented by a split-quaternion, and we have just derived easy and insightful ways to evaluate its exponential:

$$\exp(at) = \begin{cases} \mathrm{e}^{a_0 t}\big[\cos(\|\boldsymbol{a}\|t) + \hat{\boldsymbol{a}}\sin(\|\boldsymbol{a}\|t)\big] & \boldsymbol{a} \text{ timelike}, \\ \mathrm{e}^{a_0 t}(1 + \|\boldsymbol{a}\|t) & \boldsymbol{a} \text{ lightlike}, \\ \mathrm{e}^{a_0 t}\big[\cosh(\|\boldsymbol{a}\|t) + \hat{\boldsymbol{a}}\sinh(\|\boldsymbol{a}\|t)\big] & \boldsymbol{a} \text{ spacelike}. \end{cases}$$

Evaluating a matrix exponential by hand usually involves diagonalizing (strictly speaking, finding the Jordan form). The convenience of using split-quaternions instead resides in the fact that they resolve the ambiguity that is naturally present in the eigenvectors of the matrix A; especially when they are complex. In the next section, the relation between the eigenvectors and the unit vector $\hat{\boldsymbol{a}}$ are discussed in greater detail. [17]

4-3 Application to mechanical systems

We will now proceed by using a mechanical 'prototype' example for our mechanical system: the harmonic oscillator with *two* dampers: one in series and one in parallel. The two dampers are interesting because they completely fill the state transition matrix: as such, this system can represent all the possible cases discussed in the previous section. Furthermore, the two dampers have a very distinctive interpretation within the field of economic engineering.

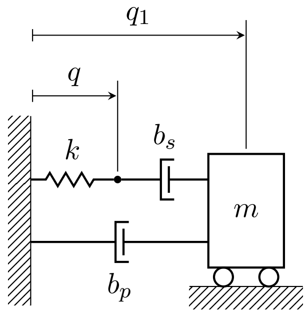

Figure 4-6: Schematic of the harmonic oscillator with two dampers: one in series and one in parallel.

4-3-1 Equations of motion

The harmonic oscillator with two dampers is shown in Figure 4-6. The equations of motion can be readily derived:

$$\begin{aligned} m\ddot{q}_1 &= -kq - b_p\dot{q}_1 \\ kq &= b_s(\dot{q}_1 - \dot{q}) \end{aligned} \tag{4-12}$$

Due to the presence of the serial damper, the situation is somewhat curious, since there are two positions in the system; one measuring the spring deflection q and the position of the mass q_1. However, the node connecting the serial damper and the spring has no mass, and therefore no second-order dynamics: as such, the overall order of the system is 2. In accordance with the economic analogy, we will say that position is stored in the spring, but momentum is stored in a mass. Hence, let $p = m\dot{q}_1$ — but $\dot{q} \neq p/m$ in general. The equations of motion than have the matrix form:

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}-\frac{k}{b_s} & \frac{1}{m}\\ -k & -\frac{b_p}{m}\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix},$$

or, using the parameters defined in Table 4-4:

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \underbrace{\begin{pmatrix}-\gamma_s & \frac{1}{m}\\ -m\Omega_n^2 & -\gamma_p\end{pmatrix}}_{A}\begin{pmatrix}q\\ p\end{pmatrix}. \tag{4-13}$$

The nontrivial relation between momentum and the $\dot{q}$ also precludes us from casting this system directly to the Lagrangian formalism, because the vector field is not second-order. A second-order vector field implies that the 'velocities' really are the time derivatives of the 'positions', (cf. Appendix A for a more formal discussion).

Table 4-4: Substition parameters for the harmonic oscillator with serial and parallel damping, shown in Figure 4-6.

Name	Symbol	Value	Units
Serial damping coefficient	γ_s	k/b_s	1/s
Parallel damping coefficient	γ_p	b_p/m	1/s
Natural frequency	Ω_n	$\sqrt{k/m}$	1/s
Damped frequency	Ω_d		1/s

The split-quaternion associated with the A-matrix of the doubly damped system can easily be found using the mapping defined by Equation (4-4). We must, however, be careful when dealing with physical systems, because the entries of the A-matrix are not dimensionless. In a vector space, we associate the units with the basis vectors, not with the components. For example, in a two-dimensional vector space spanned by a axis for apples and and an axis for pears, and we wish to represent that someone possesses three apples and four pairs, the *components* of that vector are $(3, 4)$, and the *unit vectors* are (1 apple, 1 pear). Along the same line, we must define the units in the A-matrix in the split-quaternion basis elements 1, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$. To do so, we define the reference quantities and m_0, t_0. The basis elements are then mapped in terms of these reference quantities:

$$\phi(1) = \begin{pmatrix}\frac{1}{t_0} & 0\\ 0 & \frac{1}{t_0}\end{pmatrix},\quad \phi(\hat{\boldsymbol{\imath}}) = \begin{pmatrix}0 & \frac{1}{m_0}\\ -\frac{m_0}{t_0^2} & 0\end{pmatrix},\quad \phi(\hat{\boldsymbol{\imath}}) = \begin{pmatrix}0 & \frac{1}{m_0}\\ \frac{m_0}{t_0^2} & 0\end{pmatrix},\quad \phi(\hat{\boldsymbol{k}}) = \begin{pmatrix}\frac{1}{t_0} & 0\\ 0 & -\frac{1}{t_0}\end{pmatrix},$$

where, in case we would use SI units, $m_0 = 1\,\text{kg}$ and $t_0 = 1\,\text{s}$. As a result, the split-quaternion associated with the A-matrix given in Equation (4-13) becomes

$$a \;=\; -\frac{1}{2}(t_0\gamma_s + t_0\gamma_p) \;+\; \frac{1}{2}\left(\frac{m_0}{m} + \frac{m\Omega_n^2 t_0^2}{m_0}\right)\hat{\boldsymbol{\imath}} \;+\; \frac{1}{2}\left(\frac{m_0}{m} + \frac{m\Omega_n^2 t_0^2}{m_0}\right)\hat{\boldsymbol{\jmath}} \;+\; \frac{1}{2}(t_0\gamma_p - t_0\gamma_s)\hat{\boldsymbol{k}}. \tag{4-14}$$

Clearly, all the components of the split-quaternion are dimensionless. This really is not too wild of an idea: after all, we are translating the matrix itself, and *not* the two-dimensional vector space that it acts on. The units are inherited from the vector space, so we should only add them when returning from the split-quaterions back to the realm of the matrices.

The preceding argument only explains why we can work around this issue without performing illegal operations, but it does not give a satisfactory answer as to why we would be interested to add numbers that are seemingly incompatible. Indeed, observe that γ_s and γ_p have the same units, whereas $\frac{1}{m}$ and Ω_n^2 do not. So, in which sense can the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$-components be of any signficance? To answer this question, we first note that 'rescaling of units' is a linear operation on the vector space given by a diagonal matrix (with nonzero diagonal entries):

$$N = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} \qquad \nu_1, \nu_2 \in \mathbb{R}^*,$$

which form the group isomorphic to $(\mathbb{R}^*)^2$. This transformation of the vector space manifests itself on the A-matrix as: $A' = N^{-1}AN$. It is easy to see that the basis matrices (or vector fields) for '1' and $\hat{\boldsymbol{k}}$ are invariant under this transformation, while the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$-matrices are not (that is, without making use of the reference quantities). A geometric explanation is that the eigenvectors of the identity matrix and the $\hat{\boldsymbol{k}}$-matrix point along the axes; and are therefore invariant under rescaling of these axes. As a result of this fact, the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components will not transform properly under a unit transformation. It is common practice in physics to rescale the state space of the undamped harmonic oscillator as follows [18, 19]

$$p \mapsto \frac{p}{m} \qquad q \mapsto m\Omega q,$$

such that the Hamiltonian reverts to a particularly convenient form. We can see that this is precisely the transformation that kills the $\hat{\boldsymbol{\jmath}}$-component of the split-quaternion. This would essentially resolve this 'unit problem', because it only arises when we attempt to make the *distinction* between the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$-component.

In contrast to common practice in physics, we are interested in the full range of geometrical properties that the trajectories in the phase plane can exhibit, including those that are not invariant under the action of the structure group $(\mathbb{R}^*)^2$ that contains the changes of units. Furthermore, many invariants, such as the split-quaternion (vector) norm, scalar part, etc. that we use to draw conclusions about the nature of the system *do* commute with this group action, and are therefore remain valid. It is even possible to effect unit transformations within the split-quaternion transformations by translating the matrix N to the appropriate split-quaternion using the isomorphism. We can indeed observe that the action of $n^{-1}an$ (where $n = \phi^{-1}(n)$) produces a split-quaternion with zero $\hat{\boldsymbol{\jmath}}$-component.

As a final argument, we can say that the 'rescaling of the axes', while common in physics and mathematically allowed, is of little use for engineers, since they tend to stick to SI units in the first place. The 'scale of the axes' is therefore a physical reality. This is why we choose not to discard the $\hat{\boldsymbol{\jmath}}$-component through a rescaling.

To conclude, it is not so much the case that unit transformations are not allowed in the split-quaternion space, but the question as to what the units of the $\hat{\boldsymbol{\jmath}}$-components are is moot. Unfortunately, the notation in Equation (4-14) is rather obfuscating. Hence, we take the freedom to choose $m_0 = 1(\mathrm{kg})$ and $t_0 = 1(\mathrm{s})$, and write the split-quaternion as follows:

$$a \;=\; -\tfrac{1}{2}\big(\gamma_s + \gamma_p\big) \;+\; \tfrac{1}{2}\big(\tfrac{1}{m} + m\Omega_n^2\big)\hat{\boldsymbol{\imath}} \;+\; \tfrac{1}{2}\big(\tfrac{1}{m} + m\Omega_n^2\big)\hat{\boldsymbol{\jmath}} \;+\; \tfrac{1}{2}\big(\gamma_p - \gamma_s\big)\hat{\boldsymbol{k}}. \tag{4-15}$$

This of course requires the implicit understanding that all the components are dimensionless, and that we are not just adding apples and pears.

4-3-2 Eigenvalues

Underdamped, overdamped critically damped.

4-3-3 Eigenvector geometry in SO(3, R)

We have learned in Section 4-1-2 that the eigenvalues of a 2×2-matrix can be expressed in terms of the scalar part and vector norm of the associated split-quaternion. Once we have removed the scalar part and normalized the vector part with its norm (if admissible), we are left with the unit vector part of the split-quaternion. Hence, this two-dimensional object contains the remaining information present in the matrix: the eigenvectors. The great advantage of split-quaternions is that the unit vector is entirely unambiguous, which is not the case for eigenvectors, for they represent only a direction, while the length of the eigenvector is arbitrary. The unit vector part of the split-quaternion encodes this information in a *single two-dimensional object*.

As mentioned in Section 4-1-1, the split-quaternion vectors live in the Lorentzian three-space $\mathbb{R}^{1,2}$. The *unit vectors* in this space live in a particular subspace, the Lorentzian equivalent of the unit sphere in this space. Due to the indefiniteness of the Lorentz norm, this subspace is disconnected: it consists out of three connected parts (shown in Figure 4-7):

- The *one-sheet hyperboloid*, which contains all the *spacelike* unit vectors (overdamped systems).
- The *light cone*, which contains all the *lightlike* vectors (the notion of a unit vector is not well defined in this region) (critically damped systems).
- The *two-sheet hyperboloid*; which contains all the *timelike* unit vectors (underdamped systems).

Given the regime of the mechanical systems, the eigenvectors of the state transition matrix determine the particular shape of the trajectories in the phase plane. We distinguish three types of shapes:

① Saddle point: two directions

② Stable line: two direction (?)

③ Pure translation: translation direction?

④ Node: two directions

⑥ A center/spiral, which has elliptic trajectories. These also include the spiral nodes, since the scalar part (contraction) is not part of the vector. We can characterize the shape of the elliptic trajectories by the *eccentricity* and their *tilt* (i.e. the rotation of the major axes of the ellipse with respect to the phase plane axes).

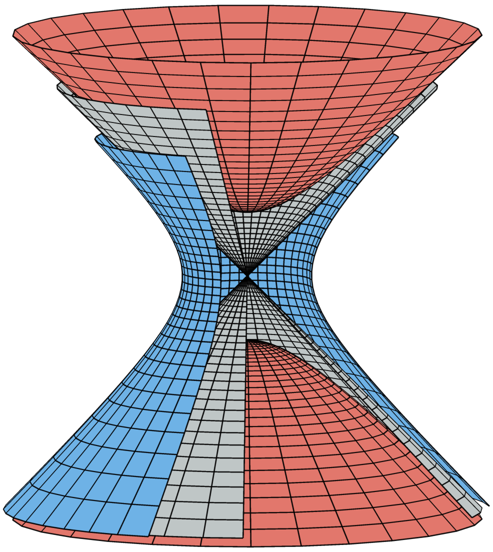

Figure 4-7: The disconnected 'unit sphere' in the Lorentzian 3-space. The blue surface is the one-sheet hyperboloid, containing all the spacelike unit vectors; the gray sheet is the light cone, that contains all the lightlike 'null' vectors with zero norm. Finally, the red surface is the two-sheet hyperboloid, which is the space of all timelike unit vectors.

Compute real eigenvectors by substituting the action of multiplying with a vector *in* the split-quaternion and solving appropriately.

Underdamped systems

TODO

Master figure with multiple trajectories and their respective locations on the poincaré disk, with poincaré disk as map

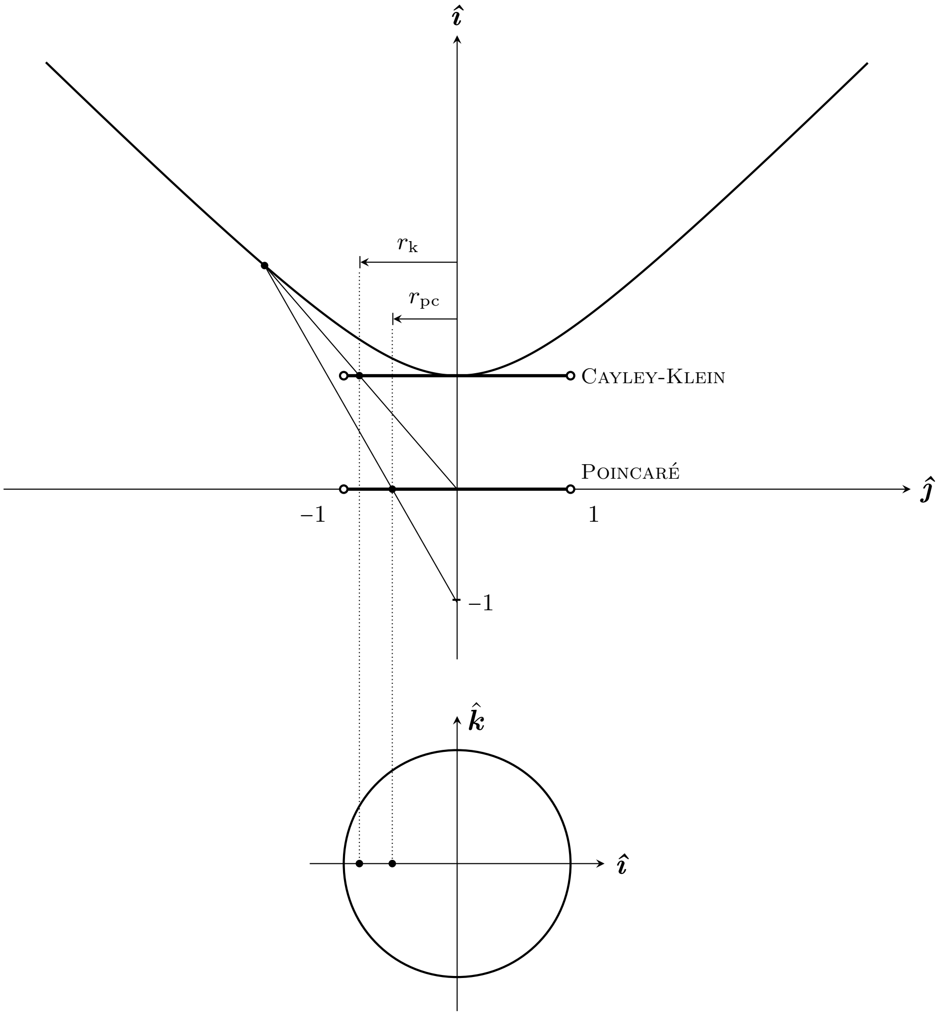

Figure 4-8: Illustration of the projection on the Poincaré disk and the Cayley-Klein disk.

4-4 Notes

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{\imath}} - u_2\hat{\boldsymbol{\jmath}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned} \boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}\hat{\boldsymbol{n}} \\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}), \end{aligned} \tag{4-16}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, we obtain:

$$\begin{aligned} \hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\ &= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}. \end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,\langle \boldsymbol{c}, \boldsymbol{a} \rangle - \boldsymbol{c}\,\langle \boldsymbol{a}, \boldsymbol{b} \rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b}\,\langle \boldsymbol{c}, \boldsymbol{a} \rangle_{\mathrm{L}} - \boldsymbol{c}\,\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$
\begin{aligned}
&\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}}\,\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} - \hat{\boldsymbol{n}}\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle_{\mathrm{L}})\mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}}\,\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}}\,\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} - \hat{\boldsymbol{n}}\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}\right)\mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}}\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}\right)\mathrm{i} \\
&= \boldsymbol{e}_3 - \boldsymbol{e}_2\mathrm{i}.
\end{aligned}
$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2+\boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ■

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

4-4-1 Relation with complex Hamiltonians

Chapter 5

Conclusion

Appendix A

Symplectic geometry in Analytical Mechanics

A-1 Lagrangian mechanics

Just like the cotangent bundle, the tangent bundle admits a canonical structure, which is called the *vertical endomorphism.* Its construction is slightly more convoluted than the canonical symplectic structure of the cotangent bundle, but nevertheless essential for a proper geometric interpretation of Lagrangian mechanics.

The vertical endomorphism The *double tangent bundle* is the tangent bundle to $\mathrm{T}M$, denoted by $\mathrm{T}(\mathrm{T}M)$. This space has not one but two canonical vector bundle structures, defined by projection maps from $\mathrm{T}(\mathrm{T}M) \to \mathrm{T}M$. First, there is the trivial projection $\pi_{\mathrm{T}M}$ thats 'forgets' about the tangent elements to $\mathrm{T}M$. Secondly, there is $(\pi_M)_*$ the pushforward (tangent map) of the projection map $\pi_M : \mathrm{T}M \to M$. [3]

$$\begin{tikzcd} & \mathrm{T}(\mathrm{T}M) \arrow[dl, "(\pi_M)_*"'] \arrow[dr, "\pi_{\mathrm{T}M}"] & \\ \mathrm{T}M \arrow[dr, "\pi_M"'] & & \mathrm{T}M \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd}$$

Vectors on the tangent bundle $\mathrm{T}M$ (they live in $\mathrm{T}(\mathrm{T}M)$) are called vertical if they vanish under the action of $(\pi_M)_*$. These vectors point entirely in the 'direction' of the fiber: in the Lagrangian formalism, they reflect a pure change in velocity, and no change in the generalized position. The *vertical lift* Ψ maps a vector on M to a vertical vector on $\mathrm{T}M$. [20]

$$\begin{aligned}&\Psi_{\boldsymbol{v}} : \mathrm{T}_q M \to \mathrm{T}_{\boldsymbol{v}}(\mathrm{T}_q M) : \\ &\Psi_{\boldsymbol{v}}(\boldsymbol{w})\, f = \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{v} + t\boldsymbol{w})\Big|_{t=0} \qquad q \in M,\ \boldsymbol{v}, \boldsymbol{w} \in \mathrm{T}_q M,\ f \in C^\infty(\mathrm{T}M).\end{aligned} \tag{A-1}$$

In components, the effect of the vertical lift is as follows:

$$\Psi_{\boldsymbol{v}} : \quad \boldsymbol{w} = w_i \frac{\partial}{\partial q_i}\Big|_q \quad \mapsto \quad \Psi_{\boldsymbol{v}}(\boldsymbol{w}) = w_i \frac{\partial}{\partial v_i}\Big|_{(q,\boldsymbol{v})} .$$

The vertical lift can also lift entire sections of $\mathrm{T}M$ by simply applying the vertical lift point-wise.

Using the concept of the vertical lift, we can define the *vertical isomorphism* S from the double tangent bundle to itself, first by projecting with $(\pi_M)_*$ and then lifting again:

$$S : \mathrm{T}(\mathrm{T}M) \to \mathrm{T}(\mathrm{T}M) : \quad S(q, \boldsymbol{v})\, u = (\Psi_{\boldsymbol{v}} \circ (\pi_M)_*)\, u \qquad u \in \mathrm{T}_{(q,\boldsymbol{v})}\mathrm{T}M.$$

The action of S can also be stated in the form of the following diagram:

$$\begin{tikzcd} \mathrm{T}(\mathrm{T}M) \arrow[r, "S"] \arrow[d, "(\pi_M)_*"'] & \mathrm{T}(\mathrm{T}M) \\ \mathrm{T}M \arrow[r, "\mathrm{id}_{\mathrm{T}M}"'] & \mathrm{T}M \arrow[u, "\Psi"'] \end{tikzcd} .$$

The action of the vertical endomorphism on the chart-induced basis is:

$$S : \quad \frac{\partial}{\partial q_i}\Big|_{(q,\boldsymbol{v})} \mapsto \frac{\partial}{\partial v_i}\Big|_{(q,\boldsymbol{v})} \qquad \frac{\partial}{\partial v_i}\Big|_{(q,\boldsymbol{v})} \mapsto 0.$$

The vertical isomorphism is therefore a tensor of valence (1, 1) — it takes a vector and produces another. Locally, S can be expressed as:

$$S = \frac{\partial}{\partial v_i} \otimes \mathrm{d}q_i \, .$$

with $\otimes$ being the tensor product. [20]

The Lagrangian formalism only applies to second-order vector fields. A second-order vector field is a vector field X such that $(\pi_M \circ X) = \mathrm{id}_{\mathrm{T}M}$; i.e. the following diagram commutes: [3]

$$\begin{tikzcd} & \mathrm{T}(\mathrm{T}M) \arrow[dl, "(\pi_M)_*"'] & \\ \mathrm{T}M \arrow[rr, "\mathrm{id}_{\mathrm{T}M}"'] & & \mathrm{T}M \arrow[ul, "X"'] \end{tikzcd} .$$

The identity on $\mathrm{T}M$ is $\mathrm{id}_{\mathrm{T}M} : (q, \boldsymbol{v}) \mapsto (q, \boldsymbol{v})$. Therefore, for a vector field X to be second order, we should have that the component in $\frac{\partial}{\partial q_i}$ that is picked out by $(\pi_M)_*$ should be equal to v_i; for example

$$X = \sum_{i=1}^{n} \Big[v_i \frac{\partial}{\partial q_i} + F_i \frac{\partial}{\partial v_i} \Big].$$

The corresponding differential equations are

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = v_i \qquad \frac{\mathrm{d}v_i}{\mathrm{d}t} = F_i,$$

which means that the second-order vector field coincides with the notion of a 'second-order differential equation' in q_i.

The Euler-Lagrange equations With the infrastructure set up in the preceding paragraph, we can now define the precise geometric setting of Lagrangian mechanics. Given a Lagrangian function $L \in C^\infty(\mathrm{T}M)$, define the *Lagrange 1-form*[1]

$$\vartheta_L \equiv \mathrm{d}L \circ S = \sum_{j=1}^{n} \frac{\partial L}{\partial v^j} \, \mathrm{d}q^j \,. \tag{A-2}$$

Observe that the Lagrange 1-form is also equal to the pullback of the Liouville form under the Legendre transformation: $\vartheta_L = (\mathbb{F}L)^* \vartheta$. [3] Secondly, we define the *Lagrange 2-form* as: [3, 20]

$$\omega_L \equiv -\,\mathrm{d}\vartheta_L = \frac{\partial^2 L}{\partial v^i \partial v^j} \, \mathrm{d}q^j \wedge \mathrm{d}v^i + \frac{\partial^2 L}{\partial q^i \partial v^j} \, \mathrm{d}q^j \wedge \mathrm{d}q^i \,. \tag{A-3}$$

Because the exterior derivative and the pullback commute, the Lagrange 2-form is equal to the pullback of the symplectic 2-form under the Legendre transform. If the rank of the Hessian $\frac{\partial^2 L}{\partial v^i \partial v^j}$ is full (and constant), then ω_L is nondegenerate and therefore defines a symplectic structure on $\mathrm{T}M$. However, observe that whether ω_L is symplectic or not depends on the nature of the Lagrangian, while the symplectic structure in the Hamiltonian setting is canonically derived from the cotangent bundle itself — there is no need for the Hamiltonian to be regular.

The final ingredient for the Euler-Lagrange equations is the *energy function*

$$E \equiv Z(L) - L,$$

where $Z = \sum v^i \frac{\partial}{\partial v^i}$ is the Liouville vector field on $\mathrm{T}M$.

The *Lagrangian vector field* X_L is then the unique vector field that satisfies the equation: [21]

$$X_L \lrcorner \, \omega_L = \mathrm{d}E \,, \tag{A-4}$$

In components, the right hand side of this equation is:

$$\begin{aligned} \mathrm{d}E &= \sum_{i,j} \left(\frac{\partial^2 L}{\partial v_j \partial q_i} v_j \, \mathrm{d}q_i + \frac{\partial^2 L}{\partial v_j \partial v_i} v_j \, \mathrm{d}v_i + \frac{\partial L}{\partial v_j} \, \mathrm{d}v_j \right) - \mathrm{d}L \,, \\ \mathrm{d}E &= \sum_{i,j} \left(\frac{\partial^2 L}{\partial v_j \partial q_i} v_j \, \mathrm{d}q_i + \frac{\partial^2 L}{\partial v_j \partial v_i} v_j \, \mathrm{d}v_i - \frac{\partial L}{\partial q_j} \, \mathrm{d}q_j \right). \end{aligned} \tag{A-5}$$

[1] Cariñena [20] calls ϑ the Euler-Poincaré 1-form.

Let $X_L = \sum_i \left(A_i \frac{\partial}{\partial q_i} + B_i \frac{\partial}{\partial v_i} \right)$; the left hand side can then be written as follows:

$$X_L \lrcorner \omega_L = -\sum_{i,j} A_i \frac{\partial^2 L}{\partial q_i \partial v_j} \mathrm{d}q_j + \sum_{i,j} A_j \frac{\partial^2 L}{\partial q_i \partial v_j} \mathrm{d}q_i - \sum_{i,j} B_i \frac{\partial^2 L}{\partial v_i \partial v_j} \mathrm{d}q_j + \sum_{i,j} A_j \frac{\partial^2 L}{\partial v_i \partial v_j} \mathrm{d}v_i \,. \quad \text{(A-6)}$$

Comparing this expression with Equation (A-5), it is immediately clear that

$$A_j \frac{\partial^2 L}{\partial v_i \partial v_j} = v_j \frac{\partial^2 L}{\partial v_i \partial v_j}.$$

We therefore have that $A_j = v_j$, but *only* if the Hessian of L with respect to the velocities is nonsingular. If this is indeed the case (i.e. L is regular), and the condition implies that the vector field X_L is second-order. We can use this knowledge to obtain a second condition (since the terms in $\mathrm{d}q_i$ cancel):

$$\sum_i B_i \frac{\partial^2 L}{\partial v_i \partial v_j} = \frac{\partial L}{\partial q_j} - \sum_i v_i \frac{\partial^2 L}{\partial q_i \partial v_j}.$$

The Hessian of L in the velocities $M_{ij} = \frac{\partial^2 L}{\partial v_i \partial v_j}$ is also called the mass matrix of the system. We have already assumed that this matrix is invertible (i.e. L is regular). As such, we have that

$$\sum_i \frac{\partial^2 L}{\partial v_i \partial v_j} \frac{\mathrm{d}^2 q_j}{\mathrm{d}t^2} + \sum_i \frac{\partial^2 L}{\partial q_i \partial v_j} \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial L}{\partial q_j},$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_j} \right) - \frac{\partial L}{\partial q_j} = 0,$$

which is the traditional form of the Euler-Lagrange equations.

Provided that X_L is a second-order vector field, the equation Equation (A-4) is equivalent to the following statement:

$$\pounds_{X_L} \vartheta_L = \mathrm{d}L \,. \quad \text{(A-7)}$$

The equivalence is easily shown using the Cartan formula:

$$\begin{aligned} \pounds_{X_L} \vartheta_L &= \mathrm{d}L \\ \mathrm{d}(X_L \lrcorner \vartheta_L) + X_L \lrcorner \mathrm{d}\vartheta_L &= \mathrm{d}L \\ \mathrm{d}(X_L \lrcorner \vartheta_L) - X_L \lrcorner \omega_L &= \mathrm{d}L \end{aligned}$$

The fact that X_L is second-order implies that $X_L \lrcorner \vartheta_L = Z(L)$. Therefore

$$\begin{aligned} \mathrm{d}(Z(L)) - X_L \lrcorner \omega_L &= \mathrm{d}L \\ X_L \lrcorner \omega_L &= \mathrm{d}Z(L) - L \\ X_L \lrcorner \omega_L &= \mathrm{d}E \,. \end{aligned}$$

Lagrangians are not unique: from Equation (A-4) we can deduce that the addition of a closed 1-form (as a map from $\mathrm{T}M \to \mathbb{R}$) to the Lagrangian will not alter the Euler-Lagrange equations. The closed 1-forms on M therefore constitute the *gauge group* of Lagrangian mechanics. An equivalent statement is that the Euler-Lagrange equations remain invariant if a total derivative is added to the Lagrangian function. [3]

Appendix B

Contact geometry

This appendix provides a short introduction to the basic concepts of contact geometry that are relevant in this thesis, particularly Chapter 3.

B-1 Contact structures

A *contact element* on a manifold M is a point $m \in M$ combined with a tangent hyperplane $\xi_m \subset \mathrm{T}_m M$ (a subspace of the tangent space with codimension 1). The term 'contact' refers to the intuitive notion that if two submanifolds 'touch', they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [1] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

A *contact manifold* is a manifold M (of dimension $2n+1$) with a *contact structure*, which is a smooth field (or distribution) of contact elements on M. Locally, any contact element determines a 1-form α (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{B-1}$$

This α is called the (local) *contact form*, and it acts like a 'normal (co-)vector' to the hyperplane. For the field hyperplanes to be a constact structure, it must satisfy a nondegeneracy condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [1, 2, 3]

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of M such that ξ is everywhere tangent to it, or (ii) that $\mathrm{d}\alpha|_\xi$ is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient TM/ξ is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [22].

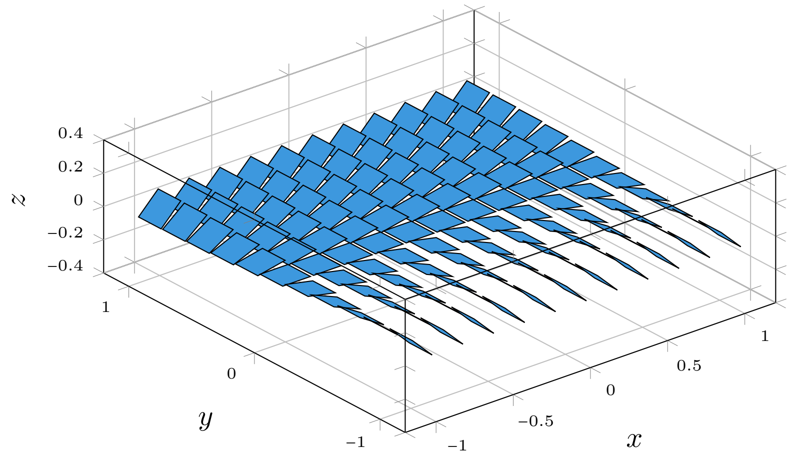

Figure B-1: The standard contact structure on $\mathbb{R}^3$, given by the contact form $\mathrm{d}z - y\,\mathrm{d}x$; the hyperplanes tilt more in the increasing y-direction.

The *Darboux theorem* for contact manifolds states that it is always possible to find coordinates z, x_i, y_i such that locally, the contact form is equal to

$$\mathrm{d}z - \sum y_i\,\mathrm{d}x_i\,,$$

which is also called the standard or natural contact structure. The standard contact structure on $\mathbb{R}^3$ is illustrated in Figure B-1. Finally, it is clear that the contact form singles out a 'special direction' in the tangent space at every point of the manifold. This direction is given by the unique *Reeb vector field*,

$$R_\alpha \in \mathfrak{X}(M): \quad R_\alpha \lrcorner\, \mathrm{d}\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner\, \alpha = 1. \tag{B-2}$$

The special direction identified by the Reeb vector field is referred to as the *vertical* direction. Likewise, vector field components in the direction of the Reeb vector field are vertical. A vector field with no vertical component is called *horizontal*.

B-2 The manifold of contact elements

A contact manifold is a manifold with a contact structure. One can, however, associate a *canonical* $(2n-1)$-dimensional contact manifold to *any* n-dimensional manifold Q, just like one can always find a canonical symplectic structure on T^*Q. Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold Q. As it turns out, this 'manifold of contact elements' has a natural contact structure.

The *manifold of contact elements* of an n-dimensional manifold is [1]

$$\mathrm{C}Q = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } \mathrm{T}_q Q\}.$$

This manifold $\mathrm{C}Q$ has dimension $2n-1$. It is clear that C has a natural bundle structure, i.e. $C \xrightarrow{\pi} Q$ where the bundle projection 'forgets' the contact element, that is

$$\pi : \mathrm{C}Q \to Q : (q, \xi_q) \mapsto q.$$

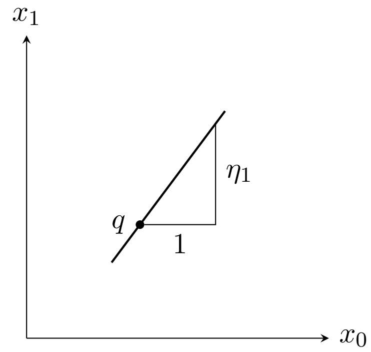

Figure B-2: A point in the manifold of contact elements on $Q = \mathbb{R}^2$. A coordinate system for CQ consists of (x_0, x_1) to indicate a point on Q, and projective coordinates $[\eta_0 : \eta_1]$, which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$, and the remaining coordinate η_1 covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the 'hyperplane' and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$.

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to Q, denoted by $\mathbb{P}(\mathrm{T}^*Q)$. This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in \mathrm{T}^*_q Q \setminus \{\mathbf{0}\} : \quad (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in Equation (B-1), in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure. $\mathbb{P}(\mathrm{T}^*Q)$ is then the quotient set of T^*Q (without zero section) with respect to the equivalence relation $\sim$. Visually, the projectivization of an n-dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension $n - 1$. It can be shown that this space is bundle-isomorphic to the manifold $\mathrm{C}Q$. [1]

As shown in Figure B-2, coordinates of the equivalence class of 1-forms are 'projective coordinates', $[\eta_0 : \eta_1 : \ldots : \eta_{n-1}]$, where η_i are coordinates for $\mathrm{T}^*_q Q$. The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes η_0 to be nonzero, the tuple $(1, \eta_1, \ldots, \eta_n)$ provides coordinates that cover most of $\mathbb{P}(\mathrm{T}^*Q)$.

Now, it remains to be explained why the 'manifold of contact elements' is itself a contact manifold. Indeed, there is a canonical field of hyperplanes *on* $\mathrm{C}Q$, which lifts the hyperplane tangent to Q to a hyperplane tangent to $\mathrm{C}Q$ (this is akin to the 'tautological' trick played in the symplectic structure of the cotangent bundle). The contact structure distinguishes the curves in $\mathrm{C}Q$ that are lifted versions from curves in Q. This is illustrated in Figure B-3. [23] Said otherwise, a tangent vector on $\mathrm{C}Q$ lies in the hyperplane defined by the contact structure if its projection down on Q lies in the hyperplane on Q defined by the given point on the $\mathrm{C}Q$.

This contact structure is associated with the 1-form:

$$\alpha = \mathrm{d}x_0 + \sum_{i=1}^{n-1} \eta_i \, \mathrm{d}x_i \,,$$

given that the η_0 is the 'special' coordinate wich is chosen to be 1.

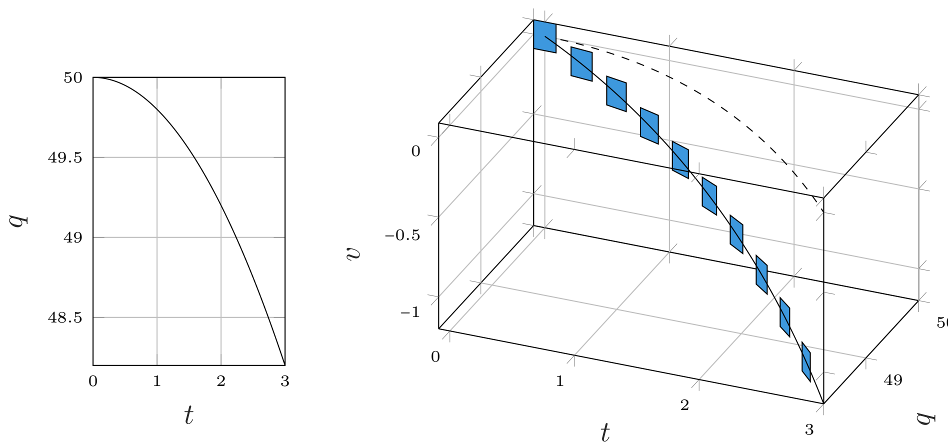

Figure B-3: Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let $(t, q) \in Q$, and let v be a coordinate for the contact (line) element. The standard contact form is then $\mathrm{d}q - v\,\mathrm{d}t$. On the left, the curve corresponding to a falling object is shown in Q. When this curve is 'lifted' to $\mathrm{C}Q$, the contact structure imposes that it be locally tangent to the contact structure, or that $v = \frac{\mathrm{d}q}{\mathrm{d}t}$. If the vertical direction is projected down onto the $(q - t)$-plane ($\mathrm{C}(Q) \to Q$), the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making v the actual velocity of the curve.

B-3 Contact Hamiltonian systems

TODO

Introduction

B-3-1 Contact Hamiltonian vector fields

Just like in the symplectic case, the contact Hamiltonian formalism defines an automorphism between a function on the contact manifold, $H \in C^\infty(M)$, and an associated 'Hamiltonian' vector field $X_H \in \mathfrak{X}(M)$. While the isomorphism is rather straightforward for symplectic manifolds, the contact counterpart is not so perspicuous: this is the prime reason behind the computational advantage of symplectification, as opposed to performing the calculations directly on the contact manifold.

Coordinate-free derivation Given a contact manifold (M, ξ) with contact form α (i.e. $\xi \in \ker \alpha$), the tangent bundle M can be decomposed into two subbundles: [1, 24]

$$\mathrm{T}M = \ker \alpha \oplus \ker \mathrm{d}\alpha \,,$$

where $\oplus$ denotes the Whitney sum. The first subbundle is referred to as the *horizontal* bundle, the second as the *vertical* bundle. The vertical subbundle is of rank 1 and its fiber is spanned by the Reeb vector field (cf. Equation (B-2)). As mentioned to in Appendix B-1, *any* vector field $X \in \mathfrak{X}(M)$ may be decomposed accordingly. This decomposition is unique and given by

$$X = \underbrace{(X \lrcorner\, \alpha)R_\alpha}_{X^{\text{ver}}} + \underbrace{[X - (X \lrcorner\, \alpha)R_\alpha]}_{X^{\text{hor}}}. \tag{B-3}$$

Observe that indeed $X^{\text{ver}} \in \ker \mathrm{d}\alpha$ and $X^{\text{hor}} \in \ker \alpha$. [1, 24, 25]

We now wish to find the relation between the contact Hamiltonian $H \in C^\infty(M)$ and the associated Hamiltonian vector field $X_H \in \mathfrak{X}(M)$. This one-to-one relation is uniquely determined by two conditions. Firstly, we impose that[1]

$$H \equiv -X_H \lrcorner\, \alpha.$$

This condition already provides us with the vertical component of the Hamiltonian vector field, namely

$$X_H^{\text{ver}} = -HR_\alpha.$$

Secondly, the automorphism generated by the Hamiltonian vector field must be a *contact automorphism*: it must preserve the contact structure. This condition is encoded in terms of the Lie derivative:[2]

$$X_H \text{ is an infinitesimal contact automorphism} \Leftrightarrow \pounds_{X_H}\alpha = s\alpha,$$

where $s \in C^\infty(M)$. The function s is there because $s\alpha$ and α give rise to the same hyperplane distribution. Using Cartan's 'magic' formula, the Lie derivative can be expanded as follows:

$$\begin{aligned} \pounds_{X_H}\alpha &= s\alpha \\ \mathrm{d}(X_H \lrcorner\, \alpha) + X_H \lrcorner\, \mathrm{d}\alpha &= s\alpha \\ -\mathrm{d}H + X_H \lrcorner\, \mathrm{d}\alpha &= s\alpha \end{aligned}$$

Contracting both sides with the Reeb vector field yields:

$$\begin{aligned} R_\alpha \lrcorner\, (-\mathrm{d}H + X_H \lrcorner\, \mathrm{d}\alpha) &= R_\alpha \lrcorner\, (s\alpha) \\ -R_\alpha \lrcorner\, \mathrm{d}H + R_\alpha \lrcorner\, X_H \lrcorner\, \mathrm{d}\alpha &= s\, R_\alpha \lrcorner\, \alpha \\ -R_\alpha \lrcorner\, \mathrm{d}H - X_H \lrcorner\, R_\alpha \lrcorner\, \mathrm{d}\alpha &= s. \end{aligned}$$

Hence, we have $s = -R_\alpha \lrcorner\, \mathrm{d}H$. Because the vertical component of X_H is spanned by the Reeb vector field, its contraction with $\mathrm{d}\alpha$ vanishes. As a result, we can rewrite the previous expression in terms of the *horizontal* component of X_H:

$$X_H \lrcorner\, \mathrm{d}\alpha = X_H^{\text{hor}} \lrcorner\, \mathrm{d}\alpha = [\mathrm{d}H - (R_\alpha \lrcorner\, \mathrm{d}H)\alpha], \tag{B-4}$$

[1] This is the sign convention observed by Bravetti et al. [26] en van der Schaft [27], as opposed to Libermann and Marle [24].

[2] Terminology differs somewhat in literature on this point: some authors, such as de León and Lainz [25] only refer to contactomorphisms as the special case where $g = 0$; while the more general case is called *conformal* contactomorphisms.

We must therefore recover X_H^{hor} from the above expression. Define the mapping

$$\alpha^\flat : \mathrm{T}M \to \mathrm{T}^*M : X \mapsto X \lrcorner \, \mathrm{d}\alpha \,,$$

when restricted to the space of horizontal vector fields, this mapping is an isomorphism onto the 'semi-basic' forms[3]. Define the inverse mapping of $\alpha^\flat$ by $\alpha^\sharp$, such that

$$X_H^{\mathrm{hor}} = \alpha^\sharp(\mathrm{d}H - (R_\alpha \lrcorner \, \mathrm{d}H)\,\alpha).$$

As such, the Hamiltonian vector field associated to the contact Hamiltonian H is

$$X_H = HR_\alpha + \alpha^\sharp(\mathrm{d}H - (R_\alpha \lrcorner \, \mathrm{d}H)\,\alpha). \tag{B-5}$$

Coordinate expression Given the contact manifold (M, ξ) with contact form

$$\mathrm{d}q_0 - \sum_{i=1}^n p_i \, \mathrm{d}q_i \,,$$

and define the contact Hamiltonian $H = H(q_0, q_1, \ldots, q_n, p_1, \ldots, p_n)$. The vertical component of the Hamiltonian vector field is straightforward (cf. Equation (B-2)):

$$X_H^{\mathrm{ver}} = -H \frac{\partial}{\partial q_0}.$$

For the horizontal component, first evaluate the right hand side of Equation (B-4) in coordinates:

$$X_H^{\mathrm{hor}} \lrcorner \, \mathrm{d}\alpha = \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0} \right) \mathrm{d}q_i + \frac{\partial H}{\partial p_i} \, \mathrm{d}p_i \,.$$

In terms of the basis vectors, the mapping $\alpha^\flat$ is

$$\frac{\partial}{\partial q_i} \mapsto \mathrm{d}p_i \qquad \frac{\partial}{\partial p_i} \mapsto -\mathrm{d}q_i \qquad \frac{\partial}{\partial q_0} \mapsto 0 \qquad i = 1, \ldots, n.$$

The inverse transformation is slightly ambiguous at first sight, since any $\frac{\partial}{\partial q_0}$ cannot be recovered directly from the 'forward' mapping. However, we know that $\alpha^\sharp$ must produce a horizontal vector field. Therefore, first perform the inverse mapping in the q_i, p_i-components to obtain

$$-\sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}.$$

Contracting this expression with α produces $-\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i}$. Hence, we can use this knowledge to find the actual horizontal component:

$$X_H^{\mathrm{hor}} = \sum_{i=1}^n p \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_0} - \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}.$$

[3]Semi-basic forms are forms that vanish when contracted with a vertical vector field. [24]

As such, the coordinate expression of Equation (B-5) is

$$X_H = \left(\sum_{i=1}^{n} p \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial q_0} - \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial q_0}\right)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \tag{B-6}$$

Furthermore, we have

$$\pounds_{X_H}\alpha = -\frac{\partial H}{\partial q_0}\alpha,$$

and

$$\pounds_{X_H}H = -H\frac{\partial H}{\partial q_0}.$$

B-3-2 Jacobi brackets

Just like the Poisson brackets define a Poisson algebra of the smooth functions on a symplectic manifold, there is a bracket operation on contact manifolds that serves (about) the same purpose. These brackets do not define a Poisson structure, but rather a *Jacobi structure*, which is a more general notion that includes the Poisson structure as a particular instance. In this treatment we will only focus on the associated *Jacobi bracket* for contact Hamiltonian systems. For more details regarding Jacobi manifolds, the reader is referred to [24, chap. V] and [25].

For two smooth functions $f, g \in C^\infty(M)$, and M a contact manifold with contact form α, the *Jacobi bracket* is defined as

$$\{\!\!\{\,,\,\}\!\!\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) : \ \{\!\!\{f, g\}\!\!\} = -\big[X_f, X_g\big] \lrcorner\, \alpha, \tag{B-7}$$

where $X_f, X_g \in \mathfrak{X}(M)$ are the contact Hamiltonian vector fields of f and g respectively, and $[\cdot, \cdot]$ is the Lie bracket (i.e. the commutator of vector fields). Equivalent expressions for the Jacobi bracket are: [24]

$$\begin{aligned}\{\!\!\{f, g\}\!\!\} &= -X_f \lrcorner\, \mathrm{d}g + g(R_\alpha \lrcorner\, \mathrm{d}f)\\ &= X_g \lrcorner\, \mathrm{d}f - f(R_\alpha \lrcorner\, \mathrm{d}g)\\ &= -\mathrm{d}\alpha\,(X_f, X_g) - f(R_\alpha \lrcorner\, \mathrm{d}g) + g(R_\alpha \lrcorner\, \mathrm{d}f).\end{aligned} \tag{B-8}$$

From these expressions, it is also clear that the Jacobi bracket is antisymmetric, i.e. $\{\!\!\{f, g\}\!\!\} = -\{\!\!\{g, f\}\!\!\}$ and $\{\!\!\{f, f\}\!\!\} = 0$. As a time evolution operator (with respect to the Hamiltonian H), we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{\!\!\{f, H\}\!\!\} + f(R_\alpha \lrcorner\, \mathrm{d}H) = \{\!\!\{f, H\}\!\!\} - fs.$$

Using the same coordinates as in Appendix B-3-1, the Jacobi bracket is equal to:

$$\{\!\!\{f, g\}\!\!\} = \left(\sum_{i=1}^{n} p_i \frac{\partial g}{\partial p_i} - g\right)\frac{\partial f}{\partial q_0} - \left(\sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial g}{\partial q_0} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right).$$

TODO

Check signs of Jacobi bracket, sign convention is again different from Libermann and Marle + mistake?

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Glossary

Economic symbols

$\dot{q}$	Quantity demanded; quantity supplied; flow of goods
p	Price
q	Quantity; amount of goods

Physical symbols

β	Work 1-form
γ	Damping coefficient
η	Heat 1-form
E	(Mechanical) energy
n_s	Amount of substance
P	Pressure
p	Momentum
q	Position
R_g	Universal gas constant
S	Entropy
T	Temperature
U	Internal energy
V	Volume

Mathematical symbols

α	General contact 1-form
ω	Symplectic 2-form

ϑ	Liouville 1-form
M	Phase space; general manifold
Q	Configuration space
Z	Liouville vector field
$\mathbb{R}^n$	Real coordinate space of dimension n
$\wedge$	Wedge (or exterior) product
$\lrcorner$	Interior product
d	Exterior derivative
$\pounds_X$	Lie derivative with respect to the vector field X
$\oplus$	Whitney sum; direct sum
$\times$	Cartesian product; cross product (depending on context)
$\otimes$	Tensor product
$E \xrightarrow{\pi} B$	Bundle with total space E, projection map π and base space B
$\mathrm{T}_x M$	Tangent space to the manifold M at the point x
$\mathrm{T}_x^* M$	Cotangent space of the manifold M at the point x
$\mathrm{T}M$	Tangent bundle of the manifold M
T^*M	Cotangent bundle of the manifold M
$\mathfrak{X}(M)$	Set of vector fields (smooth sections of $\mathrm{T}M$) on the manifold M
$C^\infty(M)$	Set of smooth functions on the manifold M
$\Lambda^n(M)$	Set of differential n-forms on the manifold M