Thesis Title

Optional Subtitle

E. B. Legrand

Master of Science Thesis

Delft University of Technology

Delft Center for Systems and Control

Thesis Title

Optional Subtitle

Master of Science Thesis

For the degree of Master of Science in Systems and Control at Delft University of Technology

E. B. Legrand

June 14, 2022

Faculty of Mechanical, Maritime and Materials Engineering · Delft University of Technology

Delft University of Technology
Department of
Delft Center for Systems and Control (dcsc)

The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

Thesis Title

by

E. B. Legrand

in partial fulfillment of the requirements for the degree of

Master of Science Systems and Control

Dated: June 14, 2022

Supervisor(s):

prof.em.dr.ir. M. Mendel

Reader(s):

prof.dr.ir. M.Y. First Reader

dr.ir. F.S.T. Reader-two

ir. Th. Reader-three

Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis…

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology E. B. Legrand
June 14, 2022

> If one were to bring ten of the wisest men in the world together and ask them what was the most stupid thing in existence, they would not be able to discover anything so stupid as astrology.

— *David Hilbert*

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator
- (Optimal) control of the distributions using the Brockett approach
- (Stochastic) inputs, link with Langevin equations
- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.
- Bayesian inversion of chaotic systems; guess the initial state by sampling after a certain time
- Define as streamtube, continuity equation asserts that streamlines cannot cross; i.e. streamtubes are conserves. To reduce computational complexity, define level sets (curves in 2-D) and check how they deform through the evolution of the phase space fluid; should always contain the same amount of probability troughout the evolution of the system.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\mathit{\Gamma\Delta\Upsilon\Omega\Theta}$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathit{\Gamma\Delta\Upsilon\Omega\Theta}}$
Tensor	$\boldsymbol{\mathsf{abcde}}$	$\boldsymbol{\mathsf{ABCDE}}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathsf{\Gamma\Delta\Upsilon\Omega\Theta}}$

Table 1-1: Caption

Math constants: $\mathrm{ie\pi}$
Variation: $\updelta S$

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner\, \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$
$\Gamma(\mathrm{T}M)$
$\mathfrak{X}(\mathrm{T}M)$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by **?**] and **?**] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.
- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

Symplectic and Contact Geometry in Economic Engineering

Chapter 3

Liouville Geometry for Dissipative Systems

The contact-geometric counterpart of Hamiltonian and Lagrangian mechanics has been the subject of increasing academic interest in recent years, see for example **? ? ? ? ?**], etc. The conception of the idea arguably traces back to the work of Herglotz [**?**], who derived it using the variational principle, and the developments in differential geometry, by e.g. **?**] and **?**].

In this chapter, the direct connection is made between the Caldirola-Kanai Hamiltonian given by **??** and the contact Hamiltonian described by **?**], using Liouville geometry[1]. We then proceed to *contact Lagrangian mechanics*, strongly related to the Herglotz' work. Finally, the whole theory is explained from a thermodynamic perspective as well. While it was already known for some time (dating back to Arnol'd) that contact geometry is the preferred geometry for thermodynamics, its equivalence to contact geometry in (dissipative) classical mechanics has not been desribed in past literature. This underpins a famous statement by Vladimir Arnol'd that 'contact geometry is all geometry', in the sense that conservative mechanical systems can be considered as part of a larger class of systems for which energy dissipation *is* allowed. [**?**]

The traditional picture is that Hamiltonian mechanics takes place in the space of generalized positions and momenta, colloquialy denoted by q's and p's. The generalized positions form coordinates of the configuration manifold, which encodes all the possible positions that the system can find itself in. The momenta, on the other hand, are *cotangent variables*: they live in the cotangent space of linear functions acting on tangent (velocity) vectors to the configuration manifold Q. We say that the Hamiltonian is a function on the *cotangent bundle* to the configuration manifold T^*Q. This cotangent bundle has a canonical 'symplectic' structure, given by its symplectic form ω, that pairs every position direction with its corresponding momenta:

$$\omega = \sum \mathrm{d}q^i \wedge \mathrm{d}p_i \,.$$

[1]It is interesting to note that Bravetti gives the Caldirola-Kanai method as an example of dissipative Hamiltonians in his paper, but fails to make the connection with his own method.

A vital property of symplectic manifolds (which include the cotangent bundle) is the fact that they are always even-dimensional: every position coordinate has a corresponding momentum and vice versa. Likewise, in an economic context this asserts that every product has its own price. It is precisely this symmetry that is broken by the introduction of contact manifolds.

The symplectic structure of Hamiltonian mechanics is related to the conservation of energy principle, The Hamiltonian function is conserved along the integral curves of the Hamiltonian vector field that it generates. For dissipative mechanics, this strict reciprocity between position and momentum is broken. Either, one constructs an explicit time-dependence and acknowledges the special nature of time, or one can introduce another coordinate that acts as 'reservoir' to facilitate the dissipation in the system. These Hamiltonian systems are known as *contact Hamiltonian systems*. One of the contributions of this thesis is to show how both methods are essentially equivalent, by connecting the most famous time-dependent model by Caldirola and Kanai to the contact Hamiltonian by **?**].

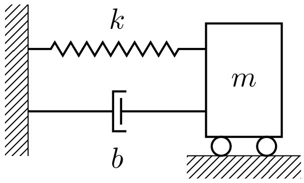

Figure 3-1: Schematic of the mass-spring-damper system.

This chapter (and the application in the following chapter) is primarily concerned with the prototypical dissipative mechanical system: the linearly damped harmonic oscillator depicted in **??**, with the governing second-order differential equation being

$$m\ddot{q} + b\dot{q} + kq = 0. \tag{3-1}$$

The choice for this system is rather perspicuous, since it is arguably the 'easiest' dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of choice in the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [**? ?**]. However, the method described in this section can be generalized directly to a general (possibly time-dependent) potential function $V = V(q,t)$. To make calculations and notation easier, some special parameters are frequently used throughout this chapter, they are summarized in **??**.

3-1 The Caldirola-Kanai method

A traditional, engineering-inclined method to incorporate damping in the framework is to include a Rayleigh damping term in the Lagrangian to emulate linear damping forces, and this works 'mathematically' to derive the correct equations of motion [**?**]. Although frequently used for practical problems, this damping term is not really part of the *actual* Lagrangian — rather, it simply makes use of the notion of a generalized force that is not inherently part of the system. As such, this method only works on a superficial level: the pristine differential geometric foundations of mechanics do not leave room for such ad hoc tricks. There is, as a result, also no Hamiltonian counterpart for this method.

Table 3-1: Parameter conventions of the damped harmonic oscillator. To avoid confusion with the symplectic form ω, angular frequencies are denoted by $\varOmega$ instead of the conventional lower case Greek letter.

Name	**Symbol**	**Value**	**Units**
Damping coefficient	γ	b/m	s^{-1}
Undamped frequency	$\varOmega_o$	$\sqrt{k/m}$	s^{-1}
Damped frequency	$\varOmega_d$	$\sqrt{\varOmega_0^2 - \left(\frac{\gamma}{2}\right)^2}$	s^{-1}
Damping ratio	ζ	$\frac{b}{2\sqrt{mk}}$	–

The historical attempts to do better than the Rayleigh method were primarily motivated by the application of the (dissipative) Hamiltonian formalism in quantum mechanics through discretization. For this application, a sound mathematical structure is of the essence, which calls for a more rigorous approach. A celebrated paper by **?**] provides an excellent summary of many attempts up to 1981. Indeed, the well-studied approach developed by **?**] and **?**] was intended exactly for this purpose. This method features an explicit time-dependence both in the Lagrangian function

$$L_{\mathrm{CK}}(q,\dot{q},t) = \mathrm{e}^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq_1^2\right), \tag{3-2}$$

and the corresponding Hamiltonian function:

$$H_{\mathrm{CK}}(q,\rho,t) = \frac{\rho^2}{2m}\mathrm{e}^{-\gamma t} + \frac{1}{2}kq^2\mathrm{e}^{\gamma t}. \tag{3-3}$$

In latter equation, ρ refers to a special 'canonical momentum', that is

$$\rho \equiv \frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}, \tag{3-4}$$

which is related to the 'true' kinematic momentum by the relation $\rho = p\mathrm{e}^{\gamma t} = m\dot{q}\mathrm{e}^{\gamma t}$. As such, it is also clear that the Caldirola-Kanai Lagrangian and Hamiltonian functions are related by the Legendre transformation *with respect to the canonical momentum*:

$$H_{\mathrm{CK}} = \rho\dot{q} - L_{\mathrm{CK}}.$$

From either **??** or **??**, the equations of motion are readily derived (for the Hamiltonian case with respect to ρ after which the transformation to p can be effected). Indeed, after taking the appropriate derivatives, one obtains:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}\right) - \frac{\partial L_{\mathrm{CK}}}{\partial q} = 0$$

$$\Rightarrow \mathrm{e}^{\gamma t}(m\ddot{q} + m\gamma\dot{q} + kq) = 0$$

for the Lagrangian case. Likewise, Hamilton's equations yield: [**?**]

$$\dot{q} = \frac{\partial H_{\text{CK}}}{\partial \rho} = \frac{\rho}{m}\mathrm{e}^{-\gamma t} = \frac{p}{m},$$

$$\dot{\rho} = -\frac{\partial H_{\text{CK}}}{\partial q} = -kq\mathrm{e}^{\gamma t}.$$

The relation between the time derivatives of the momenta $\dot{p}$ and $\dot{\rho}$ is slightly more involved since one must invoke the product rule as a result of their time-dependent relation:

$$\dot{\rho} = \mathrm{e}^{\gamma t}(\dot{p} + \gamma p). \tag{3-5}$$

Substition yields the correct equation for p, though the equation is again multiplied by $\mathrm{e}^{\gamma t}$. Because the latter is sufficiently well-behaved (that is, it has no zeros), it can be removed without any problems.

Geometric perspective To put the above derivation in a geometric setting, define the Liouville 1-form as

$$\alpha = \rho\, \mathrm{d}q \quad \Rightarrow \quad \omega = -\,\mathrm{d}\alpha = \mathrm{d}q \wedge \mathrm{d}\rho\,,$$

where the symplectic 2-form will be used to obtain Hamilton's equations. The Hamiltonian **??** is explicitly time-dependent. This will give rise to a time-dependent vector field governing the solution curves.[2] The construction of the vector field associated with a time-dependent Hamiltonian follows the same construction rules as a normal Hamiltonian (using the isomorphism given by ω), but 'frozen' at each instant of t. Even more bluntly speaking, one simply ignores the t-coordinate during the derivation, only to acknowledge the dependence at the very end. This leads to the following vector field, 'suspended' on the $\mathbb{R} \times Q$ space:

$$\tilde{X}_{H_{\text{CK}}} = -\mathrm{e}^{\gamma t}kq\frac{\partial}{\partial \rho} + \mathrm{e}^{-\gamma t}\frac{\rho}{m}\frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

The suspension is important to make the final coordinate transformation from ρ to p work properly. Indeed, effecting the transformation $(q, \rho, t) \mapsto (q, \mathrm{e}^{-\gamma t}\rho, t)$, one obtains

$$\tilde{X}_{H_{\text{CK}}} = (-kq - \gamma p)\frac{\partial}{\partial p} + \frac{p}{m}\frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

It is worthwile to ponder on some apparent peculiarities in the Caldirola-Kanai method, for they will be explained elegantly by the contact-Hamiltonian formalism. Firstly, the role of the two-different momenta is not very clear from the get-go, apart from being a consequence of the way the Caldirola-Kanai Lagrangian is formulated. This has also been the reason for considerable confusion in the academic community (see **?**]). Furthermore, there is the special role of the time coordinate, which is merely a parameter in the Hamiltonian function; for it does not partake in the dynamics of the system. Finally, there is the special role of the factor $\mathrm{e}^{\gamma t}$ through which the time-dependence makes its appearance both in the Lagrangian and the Hamiltonian.

[2] A *time-dependent vector field* on a manifold M is a mapping $X : M \times \mathbb{R} \to \mathrm{T}M$ such that for each $t \in \mathbb{R}$, the restriction X_t of X to $M \times \{t\}$ is a vector field on M. [**?**] An additional construction of importance, called the *suspension* of the vector field, is a mapping

$$\tilde{X} : \mathbb{R} \times M \to \mathrm{T}(\mathbb{R} \times M) \quad (t, m) \mapsto ((t, 1), (m, X(t, m))),$$

that is to say, it lifts the vector field to the extended space that also includes t and assigns the time coordinate with a trivial velocity of 1. [**?**]

3-2 From time-dependent to contact Hamiltonian systems

In the subsequent discussion the original Hamiltonian will be progressively lifted to higher-dimensional spaces in order to include dissipation in the Hamiltonian formalism. First of all, to a contact manifold, which is odd-dimensional: as mentioned, we need an additional degree of freedom — which has no momentum conjugate to it — to keep track of the dissipation in the system. This degree of freedom is sometimes referred to as a *gauge variable*, denoted by q_0. However, performing calculations in contact geometry directly is cumbersome and uninsightful: to quote Vladimir Arnol'd once more, 'one is advised to calculate symplectically but to think rather in contact geometry terms'. Hence, we make use of the *symplectization* of the contact structure, which gives rise to a so-called *Liouville structure*, and 'pretend' that we are dealing with the symplectic case. This symplectization will add yet another dimension to the system. [**?** **?**]

3-2-1 Symplectization & Liouville structures

The contact manifold of our system is three-dimensional, with coordinates p, q and q_0 – the latter is the gauge variable for the dissipation in the system. It can be viewed as the manifold of contact elements associated with the extended configuration manifold M for which q and q_0 are coordinates, denoted by $\mathbb{P}(\mathrm{T}^*M)$. The contact form on $\mathbb{P}(\mathrm{T}^*M)$ is given by

$$\alpha = \mathrm{d}q_0 - p\,\mathrm{d}q\,, \tag{3-6}$$

which accentuates the special role of the q_0 in the system dynamics. Contact forms are, by their very nature, ambiguous: they represent a distribution of hyperplanes, which coincides with the kernel of the contact form. Multiplication with a nonzero factor yields a different contact form with the same kernel, that is to say, they represent the same contact structure. This is the reason behind the 'projective' nature of contact mechanics[3]. Hence, one may just as well multiply the 1-form with a nonzero factor λ:

$$\lambda(\mathrm{d}q_0 - p\,\mathrm{d}q) \quad \lambda \in \mathbb{R}_0.$$

The factor λ can be considered to be an extra degree of freedom (leaving the contact structure unaffected), which provides a 'lift' from the odd-dimensional manifold to an even-dimensional one, which is called the symplectification of the contact manifold. [**?**]

To restate the above in canonical coordinates, choose[4]

$$\rho_0 = \lambda \quad \text{and} \quad \rho = -\lambda p \tag{3-7}$$

such that

$$\theta = \rho_0\,\mathrm{d}q_0 + \rho\,\mathrm{d}q\,, \tag{3-8}$$

which is the Liouville form on T^*M. [**?** , p. 308] The Liouville form defines a symplectic structure given by[5]

$$\omega = -\,\mathrm{d}\theta = \mathrm{d}q_0 \wedge \mathrm{d}\rho_0 + \mathrm{d}q \wedge \mathrm{d}\rho\,.$$

[3] As explained in **??**, the manifold of contact elements is bundle-isomorphic to the projectivization of the cotangent bundle.

[4] The minus sign is there to obtain the convential form of the Liouville form in symplectic geometry.

[5] The nondegeneracy condition on the contact structure guarantees that this structure is indeed symplectic.

Principal bundles Let us now formalize the Liouville structure in the language of principal bundles. The projectivized cotangent bundle $\mathbb{P}(\mathrm{T}^*M)$ has as its fiber the space of lines passing through the origin. These lines are also the orbits of the multiplicative group $\mathbb{R}_\times$ acting through dilations on the fiber of a bundle with two-dimensional fibers: the cotangent bundle of M without zero section, denoted by T_0^*M. Using the canonical coordinates defined previously, coordinates for T_0^*M are (q_0, q, ρ_0, ρ), where ρ and ρ_0 cannot vanish at the same time (the zero section).

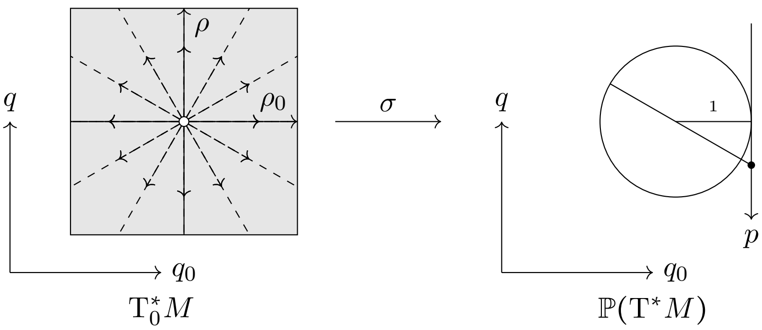

Figure 3-2: Illustration of the principal $\mathbb{R}_\times$-bundle $\mathrm{T}_0^*M \xrightarrow{\pi} \mathbb{P}(\mathrm{T}^*M)$. The total space T_0^*M is the cotangent bundle to M with zero section removed, which is shown on the left. The action by the multiplicative group $\mathbb{R}_\times$ is illustrated by the arrows, for it acts as a scaling (dilation) on all the cotangent variables. The origin is not part of the fiber, for it is part of the zero section. The bundle projection π projects all points that are on the same orbit (straight lines through the origin) to a single point on the base manifold: the projectivized cotangent bundle $\mathbb{P}(\mathrm{T}^*M)$. The former space has a symplectic structure while the latter space has a contact structure. Observe from **??** that $p = \rho/\rho_0$, i.e. such that p is a coordinate for the projectivization by stereographic projection, as shown on the right.

Define the $\mathbb{R}_\times$-*action* $\blacktriangleleft$ on T_0^*M as:

$$\blacktriangleleft : \mathrm{T}_0^*M \times \mathbb{R}_\times \to \mathrm{T}_0^*M : \quad (q_0, q, \rho_0, \rho) \blacktriangleleft \lambda = (q_0, q, \lambda\rho_0, \lambda\rho) \qquad \lambda \in \mathbb{R}_\times,$$

which are referred to as dilations of the fiber.

As illustrated in **??**, the *orbit space* of T_0^*M with respect to the group action $\blacktriangleleft$ is the space of all points in T_0^*M with all points on the same line through the origin (in the fiber) identified. This space is precisely equal to the projectivization of the cotangent bundle $\mathbb{P}(\mathrm{T}^*M)$. Hence, consider the *principal* $\mathbb{R}_\times$-bundle $\mathrm{T}_0^*M \xrightarrow{\sigma} \mathbb{P}(\mathrm{T}^*M)$

$$\begin{tikzcd}
\mathrm{T}_0^*M \\
\mathrm{T}_0^*M \arrow[u, "\blacktriangleleft \mathbb{R}_\times"] \arrow[d, "\sigma"] \\
\mathbb{P}(\mathrm{T}^*M) \cong \mathrm{C}M
\end{tikzcd}$$

The removal of the zero section is required for the group action to be free. The principal bundle $\mathrm{T}_0^*M \xrightarrow{\pi} \mathbb{P}(\mathrm{T}^*M)$ admits a *fibered symplectic Liouville structure*, given by the Liouville form

[**?**]

$$\theta = \rho_0 \,\mathrm{d}q_0 + \rho \,\mathrm{d}q \,,$$

and the associated two-form $\omega = -\,\mathrm{d}\theta$. The distinctive feature of these forms that makes this a Liouville structure is that they both commute with the group action $\blacktriangleleft$: [**?**]

$$(\blacktriangleleft \lambda)^{*}\theta = \lambda\theta \qquad \lambda \in \mathbb{R}_{*},$$

which makes them homogeneous forms of degree 1.

The projection map σ of the principal bundle is locally defined as

$$\sigma : \mathrm{T}_0^{*}M \to \mathbb{P}(\mathrm{T}^{*}M) : (q_0, q, \rho_0, \rho) \mapsto (q_0, q, -\rho/\rho_0), \tag{3-9}$$

with $p \equiv -\rho/\rho_0$ a coordinate for the projectivized fiber. This coordinate does not cover the entire fiber: the points for which $\rho_0 = 0$ is missing (in **??**, this point is the only point on the circle that cannot be projected on the p-axis). However, we will make the deliberate assumption that in our application, ρ_0 is never equal to zero.

Finally, the *Liouville vector field* Z associated with the Liouville structure is the vector field that represents the dilation of the fiber in the symplectization. It is defined as

$$Z = \omega^{\sharp}(\theta) = \rho_0 \frac{\partial}{\partial \rho_0} + \rho \frac{\partial}{\partial \rho}. \tag{3-10}$$

Vector field (components) colinear with the Liouville vector fields are called *vertical*; they represent dissipative action in the system. After the vertical components are removed, they remaining vector field is called *horizontal*.

[Liouville automorphisms, commute with the Liouville vector field -> very important in the chapter about split-quaternions]

To summarize, we lifted the original system with symplectic structure $\mathrm{d}q \wedge \mathrm{d}p$ to a contact manifold through the addition of a gauge variable q_0. We then symplectified the contact manifold to a four-dimensional system, with 'positions' (q_0, q) and 'momenta' (ρ_0, ρ).

3-2-2 Homogeneous Hamiltonian systems

The theoretical construction of the past section serves an important purpose, because it is the symplectified space which is the proper setting for the Caldirola-Kanai Hamiltonian discussed in **??**. Along with the symplectification of the contact structure described in the past section, we can do the same with a contact Hamiltonian system.

There is a one-to-one correspondence between contact Hamiltonians on $\mathbb{P}(\mathrm{T}^{*}M)$ and a special class of Hamiltonians on the symplectified space $\mathrm{T}_0^{*}M$. These are the Hamiltonians which are *homogeneous* in the cotangent variables with degree 1:[6]

$$\mathcal{H}(q_0, q, \lambda\rho_0, \lambda\rho) = \lambda\,\mathcal{H}(q_0, q, \rho_0, \rho) \quad \text{or} \quad \pounds_Z \mathcal{H} = \mathcal{H}, \tag{3-11}$$

with $\lambda \in \mathbb{R}_0$, $H \in C^\infty(\mathrm{T}_0^* M)$ and Z defined according to **??**. Given that H is indeed homogeneous of degree 1, this correspondence is in canonical coordinates:

$$\mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0\, H\left(q_0, q, -\frac{\rho}{\rho_0}\right) \tag{3-12}$$

where $\mathcal{H} \in C^\infty(\mathrm{T}_0^* M)$, $H \in C^\infty(\mathbb{P}(\mathrm{T}^* M))$ and $p = -\rho/\rho_0$ is a coordinate for the projectivized fiber. Likewise, there is also a direct correspondence between the vector fields generated by these Hamiltonians, and therefore the system dynamics. This is the reason why we go through the trouble of symplectification in the first place, it offers significant computational advantages. It is possible to derive the contact equations directly (as **?**] does), but it does not offer the same amount of insight as its symplectified counterpart. [**?** **?**]

Now, recall the Caldirola-Kanai Hamiltonian in **??**. Instead of assuming a direct time-dependence, we will think of the time-dependence as the gauge momentum, i.e. $\rho_0 \text{ “ = ” } -\mathrm{e}^{\gamma t}$. However, we will now consider ρ_0 to be a coordinate in its own right, instead of directly using the expression above — the equality sign must therefore not be taken too literally. The Caldirola-Kanai Hamiltonian is then written as (cf. **??**):

$$-\rho_0\left[\frac{1}{2m}\left(-\frac{\rho}{\rho_0}\right)^2 + \frac{1}{2}kq^2\right].$$

The motivation to make this particular choice is twofold: first, observe that $\mathcal{H}$ is homogeneous in the cotangent variables ρ_0, ρ, and second, that their fraction yields the *real* momentum: $p = -\rho/\rho_0$. However, we must acknowledge a potential dependence on q_0, since we want to convert the explicitly time-dependent Hamiltonian into a contact Hamiltonian. We therefore add the arbitrary function $f = f(q_0)$, whose value is to be determined later, and also multiply by ρ_0 to maintain homogeneity. The homogeneous Hamiltonian $\mathcal{H}$ is then:

$$\mathcal{H} : \mathrm{T}_0^* M \to \mathbb{R} : \quad \mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0\left[\frac{1}{2m}\left(-\frac{\rho}{\rho_0}\right)^2 + \frac{1}{2}kq^2 + f(q_0)\right]. \tag{3-13}$$

Using the correspondence given by **??**, the homogeneous Hamiltonian may be 'projected' to the contact Hamiltonian H:

$$H : \mathbb{P}(\mathrm{T}^* M) \to \mathbb{R} : \quad H(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 + f(q_0). \tag{3-14}$$

Numerically, this contact Hamiltonian is the same as the Hamiltonian for an undamped mass-spring system but it is defined on the contact manifold that also takes into account the gauge variable q_0.

[6] This is a consequence of the Euler theorem for homogeneous functions. If $\mathcal{H} = \mathcal{H}(\boldsymbol{q}, \boldsymbol{\rho})$ is homogeneous of degree r in $\boldsymbol{\rho}$, then

$$\sum_{i=1}^{n} \rho_i \frac{\partial \mathcal{H}}{\partial \rho_i} = r\mathcal{H}.$$

Therefore, for homogeneity of degree 1, we have:

$$\pounds_Z \mathcal{H} = Z(\mathcal{H}) = \sum_{i=1}^{n} \rho_i \frac{\partial \mathcal{H}}{\partial \rho_i} = \mathcal{H} \quad \text{with} \quad Z \equiv \sum \rho_i \frac{\partial}{\partial \rho_i}.$$

The correspondence between the

Equations of motion Now to derive the equations of motion. As mentioned, this is easiest in the symplectified space because Hamilton's equations can be readily applied (the reader can consult **??** for the direct derivation). Because we are using canonical coordinates, the Hamiltonian vector field

$$X_{\mathcal{H}} = \omega^{\sharp}(\mathrm{d}\mathcal{H}) \tag{3-15}$$

corresponds to Hamilton's equations in the familiar form:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q}, \quad \frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_0}, \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \rho}, \quad \frac{\mathrm{d}q_0}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \rho_0}. \tag{3-16}$$

Observe that this motivates why one has to take the partial with respect to the 'other' momentum in the Caldirola-Kanai momentum: we are dealing with a specific instance of a more general class of homogeneous coordinates of the cotangent variables. Of course, the variable of interest is the actual momentum p, not the scaled version ρ. The time-derivative of p can be written in terms of ρ and ρ_0, completely analogous to **??**:

$$p = -\rho/\rho_0 \quad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho}{\rho_0^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{p}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}t}. \tag{3-17}$$

Given **??**, the partial derivatives of $\mathcal{H}$ and H are related through the by relations: [**?**]

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial q} &= -\rho_0 \frac{\partial H}{\partial q}, \\
\frac{\partial \mathcal{H}}{\partial q_0} &= -\rho_0 \frac{\partial H}{\partial q_0}, \\
\frac{\partial \mathcal{H}}{\partial \rho} &= -\rho_0 \frac{\partial H}{\partial p}\frac{\partial p}{\partial \rho} = \frac{\partial H}{\partial p}, \\
\frac{\partial \mathcal{H}}{\partial \rho_0} &= -H - \rho_0 \frac{\partial H}{\partial p}\frac{\partial p}{\partial \rho_0} = -H - \frac{\partial H}{\partial p}\frac{\rho}{\rho_0} = \frac{\partial H}{\partial p}p - H.
\end{aligned} \tag{3-18}$$

Hence, the *contact* equations of motion can be found by combining of **??** and **??**:

$$\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{\partial H}{\partial p} \\
\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{1}{\rho_0}\frac{\partial \mathcal{H}}{\partial q} + \frac{p}{\rho_0}\frac{\partial \mathcal{H}}{\partial q_0} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial q_0} \\
\frac{\mathrm{d}q_0}{\mathrm{d}t} &= \frac{\partial H}{\partial p}p - H.
\end{aligned} \tag{3-19}$$

Some observations are important to note:

- The evolution of the position q remains the same as for the 'normal' (undamped) case.
- The evolution of the momentum operator picks up a term that is depends on presence of the gauge variable in the contact Hamiltonian.
- The evolution of the gauge variable is equal to the Legendre transformation of the contact Hamiltonian with respect to p.

We are now ready to determine the nature of the as of yet unknown function f to obtain the correct equations of motion. By comparing **??** and **??**, the following relation must hold:

$$\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = \frac{\partial H}{\partial q_0} = \frac{\mathrm{d}f}{\mathrm{d}q_0}.$$

Furthermore, since we initial 'substituted' $\mathrm{e}^{\gamma t}$ in favor of ρ_0, the left hand side of the equation should be equal to γ for it to be consistent with the Caldirola-Kanai Hamiltonian. Hence, we know that $\frac{\mathrm{d}f}{\mathrm{d}q_0} = \gamma$, or $f(q_0) = \gamma q_0$ up to a constant, which we choose to be zero. Although this may seem like an odd construction, the only thing we did is made the contact equation of motions equivalent with the time-dependent equations of motion. Now, the fact that $\rho_0 = \mathrm{e}^{\gamma t}$, ceases to be an a priori assumption, and is derivable through Hamilton's equations:

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_0} = \gamma\rho_0 \quad\Rightarrow\quad \rho_0 = \mathrm{e}^{\gamma t} + C$$

As such, we have for the contact Hamiltonian:

$$H : \mathbb{P}(\mathrm{T}^*M) \to \mathbb{R} : \quad H(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 + \gamma q_0, \tag{3-20}$$

and this is precisely Bravetti's result. For the homogenous Hamiltonian on the symplectified space, we have:

$$\mathcal{H} : \mathrm{T}_0^*M \to \mathbb{R} : \quad \mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0\left[\frac{1}{2m}\left(-\frac{\rho}{\rho_0}\right)^2 + \frac{1}{2}kq^2 + \gamma q_0\right]. \tag{3-21}$$

The Hamiltonian vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathrm{T}_0^*M)$ is then, using **??** and **??**:

$$X_{\mathcal{H}} = -\frac{1}{m}\frac{\rho}{\rho_0}\frac{\partial}{\partial q} + \left[\frac{1}{2m}\left(\frac{\rho}{\rho_0}\right)^2 - \frac{1}{2}kq^2 - \gamma q_0\right]\frac{\partial}{\partial q_0} + \rho_0 kq\frac{\partial}{\partial \rho} + \gamma\rho_0\frac{\partial}{\partial \rho_0}.$$

The contact Hamiltonian vector field $X_H \in \mathfrak{X}(\mathbb{P}(\mathrm{T}^*M))$ is obtained either by using the contact Hamilton equations given by **??**, or by using the pushforward of the projection map σ:

$$X_H = \sigma_* X_{\mathcal{H}} = \frac{p}{m}\frac{\partial}{\partial q} + \left(\frac{p^2}{2m} - \frac{1}{2}kq^2 - \gamma q_0\right)\frac{\partial}{\partial q_0} - (kq + \gamma p)\frac{\partial}{\partial p}$$

This is essentially the equivalent of the time-dependent transformation performed in **??**. Clearly, these yield the correct equations of motion for the damped harmonic oscillator.

Mechanical energy One of the computational advantages of the symplectified space is the fact that 'regular' Poisson brackets can be used in contrast to their slightly unwieldy contact counterparts.[7] The mechanical energy, denoted by E, is equal to

$$E : \mathbb{P}(\mathrm{T}^*M) \to \mathbb{R} : \quad E(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2,$$

[7] The Poisson bracket is defined as

$$\{f, g\} = \omega(X_f, X_g) = \pounds_{X_g} f,$$

where X_f and X_g are the Hamiltonian vector fields of f and g. [?]

With some abuse of notation, we will denote both functions on the contact space and their lifted version to the symplectified space by the same symbol. That is to say, the function E both refers to $E \in C^\infty(\mathbb{P}(\mathrm{T}^*M))$ and $(E \circ \sigma) \in C^\infty(\mathrm{T}^*_0)$.

The change of mechanical energy in the system is then readily determined using the Poisson brackets:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \{E, \mathcal{H}\} = \pounds_{X_\mathcal{H}} E = -\frac{\gamma}{2m}\left(\frac{\rho}{\rho_0}\right)^2 = -\frac{\gamma}{2m}p^2,$$

which is precisely the dissipative power in the damping element.

TODO

Check minus signs of the change in mechanical energy, should be opposite to heat generated.

TODO

Finish Poincaré lemma discussion.

The importance of the zero section

It may be tempting to disregard the removal of the zero section from the cotangent bundle as a mathematical technicality. It has, however, deep implications for the nature of the Hamiltonian systems that can be defined on it, encoded in the so-called de Rham cohomology groups.

Suppose that Y is a symplectic vector field with ω the symplectic form. The 1-form

$$\xi = Y \lrcorner\, \omega$$

is necessarily closed, because

$$\mathrm{d}Y \lrcorner\, \omega = \underbrace{\cancel{\pounds_Y \omega}}_{Y \text{ symp.}} - \underbrace{\cancel{Y \lrcorner\, \mathrm{d}\omega}}_{\omega \text{ closed}} = 0.$$

The Poincaré lemma (a specific instance of the de Rham cohomology) says that on a *contractible domain*, all closed forms are necessarily also exact (the converse is true on any manifold, for $\mathrm{d}^2 = 0$). This would mean that, if ξ were to be defined on a contractible manifold, it would automatically be an exact form (this is the same as saying that on these types of manifolds, all symplectic vector fields are Hamiltonian). In other words, there must be a function $\mathcal{H}$ such that $\xi = \mathrm{d}\mathcal{H}$.

Integration around curve to show that ξ in our case is not exact. Hard, because over two charts.

Also show that the region of interest is not simply connected in order to use de Rham in the first place. $\mathbb{R}^4/\{(q_0, q, 0, 0)\}$ not simply connected

3-2-3 Physical interpretation of the gauge variables

Another advantage of using the purely symplectic formalism on the lifted space is the fact that the homogeneous Hamiltonian is invariant under the flow it generates, since the explicit time-dependence has been removed:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \{\mathcal{H}, \mathcal{H}\} = \pounds_{X_{\mathcal{H}}}\mathcal{H} = 0.$$

We may therefore associate the homogeneous Hamiltonian with a constant. By inspection of **??**,

$$\mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0 \underbrace{\left(C\rho_0^{-1}\right)}_{H} \qquad C \in \mathbb{R}.$$

Recall that we made the assumption earlier that ρ_0 is a function *without zeros*.

The constant C is a degree of freedom in the system that we are free to choose, for the equations of motion will be consistent with any chosen value. This called a *gauge* of the system, and its choice will influence the value of the gauge variable directly. The contact Hamiltonian is not a constant of motion however (at least, for dissipative systems). With some abuse of notation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = C\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -C\frac{\partial\mathcal{H}}{\partial q_0}.$$

Until now, there were no assumptions regarding the value of the damping constant γ: indeed, the 'normal' equations of motion are readily derived when γ is set to zero. We will now add the assumption that there is at least some dissipation present in the system ($\gamma \neq 0$) to assign further intrepretation to the gauge variable. The evolution of q_0 is directly related to the evolution of H by **??**

$$q_0 = \frac{1}{\gamma}\left(\rho_0 C - \underbrace{\frac{p^2}{2m} - \frac{1}{2}kq^2}_{E}\right)$$

Because we are free to choose the value of C, let us now make a choice of particular interest; namely $C = 0$. In that case, both $\mathcal{H}$ and H vanish *weakly*;[8] this choice rids us from the additional freedom in C that would also show up in the equation of motion for q_0. Instead, $q_0 = -E/\gamma$; which can be interpreted as the heat dissipated by the system (one can add a suitable initial condition for $q_0(0) = E(0)$ to make this also numerically correct). This 'heat' function will be called Q; we therefore have

$$q_0 = Q/\gamma.$$

The vanishing of the Hamiltonians reflects the energy balance that is maintained throughout the evolution of the system:

$$\begin{aligned} H &= E + Q \\ &= \text{MECHANICAL ENERGY} + \text{DISSIPATED HEAT}. \end{aligned}$$

[8]The weak equality, as opposed to the strong equality, is not maintained under variations. Hence, although the numerical value of the function is zero, its partial derivatives do not necessarily vanish. The reader is referred to **?**] for a more elaborate discussion.

The fact that H vanishes makes it a constant of motion on par with the symplectified Hamiltonian $\mathcal{H}$. This careful choice of for the gauge variable removes a lot of the ambiguity that is naturally present in contact systems; this rather subtle point is a bit overlooked by past research. [**?**] Furthermore, the evolution of the dissipated heat is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \gamma\frac{\mathrm{d}q_0}{\mathrm{d}t} = \gamma\left[\frac{1}{2m}\left(\frac{\rho}{\rho_0}\right)^2 - \frac{1}{2}kq^2 - \gamma q_0\right] = \gamma\frac{1}{2m}\left(\frac{\rho}{\rho_0}\right)^2,$$

which is the expected result.

This very particular choice for the gauge variable may seem a little arbitrary. However, *in general*, the time-rate of change

$$[\![f, g]\!]$$

TODO

Canonical transformation

TODO

Action-angle coordinates

TODO

Generalization using Jacobi problems

3-3 Legendre involution

In the classic, symplectic case, the Legendre transformation is used to pass from the Hamiltonian to the Lagrangian formalism and vice versa. This is because the Legendre transform facilitates a mapping between the tangent and cotangent bundle. If the Lagrangian (or Hamiltonian) is (hyper)regular (i.e. the mass matrix is invertible), this mapping is a diffeomorphism. [**?**]

One would be tempted to use the normal Legendre transformation on the symplectified Hamiltonian $\mathcal{H}$. This approach will meet some problems though:

- A homogeneous function is not regular in the homogeneous variables — naturally, a degree of freedom still resides in the action of the multiplicative group. Therefore, the mapping from the cotangent to the tangent bundle is not a diffeomorphism. Said otherwise, there is not a one-to-one correspondence between the homogeneous momenta and the associated velocities in the Lagrangian description.

- As a consequence of Euler's theorem for homogeneous functions, the Legendre transformation for a homogeneous function is necessarily equal to zero. For any homogeneous function H (of degree 1), Euler's theorem states that

$$\sum_{i=1}^{n} \rho_i \frac{\partial \mathcal{H}}{\partial \rho_i} = \mathcal{H},$$

i.e. the function is equal to its associated 'action', and therefore the expression for the Legendre transformation vanishes. [? ?]

There is a better path to take. In essence the Legendre transform is (and was originally meant to be) a *contact transformation*.

3-4 Notes

Lie derivatives & Max' question

The Lie derivative of the tautological form $\alpha = p\,\mathrm{d}q$ with respect to the Hamiltonian vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial}{\partial p}$$

is denoted by

$$\pounds_{X_H}\alpha.$$

Using Cartan's magic formula ($\pounds_V\theta = \mathrm{d}(V \lrcorner \theta) + V \lrcorner \mathrm{d}\theta$), this expression can be written as

$$\begin{aligned}
\pounds_{X_H} p\,\mathrm{d}q &= \mathrm{d}(X_H \lrcorner p\,\mathrm{d}q) + X_H \lrcorner \mathrm{d}(p\,\mathrm{d}q) \\
&= \mathrm{d}(X_H \lrcorner p\,\mathrm{d}q) - X_H \lrcorner \omega \\
&= \mathrm{d}(X_H \lrcorner p\,\mathrm{d}q) - \mathrm{d}H \\
&= \mathrm{d}\left(\frac{\partial H}{\partial p}p\right) - \mathrm{d}H \\
&= \mathrm{d}(\dot{q}p) - \mathrm{d}H \\
&= \mathrm{d}(\dot{q}p - H) \\
&= \mathrm{d}L
\end{aligned}$$

Explicitly in components:

$$\pounds_{X_H}\alpha = [X_H^\nu(\partial_\nu\alpha_\mu) + (\partial_\mu X_H^\nu)\alpha_\nu]\,\mathrm{d}x^\mu$$

$$\begin{aligned}
\pounds_{X_H}\alpha = &\left[\frac{\partial H}{\partial p}\left(\frac{\partial}{\partial q}p\right) + \left(\frac{\partial}{\partial q}\frac{\partial H}{\partial p}\right)p - \frac{\partial H}{\partial q}\left(\frac{\partial}{\partial p}p\right) - \left(\frac{\partial}{\partial q}\frac{\partial H}{\partial q}\right)0\right]\mathrm{d}q \\
&+\left[\frac{\partial H}{\partial p}\left(\frac{\partial}{\partial q}0\right) + \left(\frac{\partial}{\partial p}\frac{\partial H}{\partial p}\right)p - \frac{\partial H}{\partial q}\left(\frac{\partial}{\partial q}0\right) - \left(\frac{\partial}{\partial p}\frac{\partial H}{\partial q}\right)0\right]\mathrm{d}p \\
&= \left[\left(\frac{\partial}{\partial q}\frac{\partial H}{\partial p}\right)p - \frac{\partial H}{\partial q}\right]\mathrm{d}q + \left[\left(\frac{\partial}{\partial p}\frac{\partial H}{\partial p}\right)p\right]\mathrm{d}p
\end{aligned}$$

Compare this with the expression using the Cartan equation:

$$\begin{aligned}
\mathrm{d}\left(\frac{\partial H}{\partial p} p - H\right) &= \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p} p\right) \mathrm{d}q + \frac{\partial}{\partial p}\left(\frac{\partial H}{\partial p} p\right) \mathrm{d}p - \frac{\partial H}{\partial q} \mathrm{d}q - \frac{\partial H}{\partial p} \mathrm{d}p \\
&= \left[p \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial p} \frac{\partial p}{\partial q} - \frac{\partial H}{\partial q}\right] \mathrm{d}q + \left[p \frac{\partial}{\partial p}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial p} \frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\right] \mathrm{d}p
\end{aligned}$$

which coincides with the previous expression.

Chapter 4

Split-Quaternions as Dynamical Systems

In this chapter, the geometric connection is made between the algebra of split-quaternions and the qualitative behavior of two-dimensional linear dynamical systems.

4-1 Split-quaternion algebra

4-1-1 Basic properties

Like the conventional quaternions, the split-quaternions are a number system that consists of linear combinations of four basis elements, which will be denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The algebra of split-quaternions is associative but not commutative — formally speaking, the algebraic structure is a *noncommutative ring*. The multiplication table for the split-quaternion algebra is shown in **??**. The set of split-quaternions is denoted by $\hat{\mathbb{H}}$ (since $\mathbb{H}$ is reserved for conventional quaternions).

Table 4-1: Multiplication table for the split-quaternion algebra.

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	-$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	-$\mathbf{k}$	1	-$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

The important distinction from conventional quaternions resides in the diagonal elements of **??**. Whereas for quaternions all the nonreal basis elements square to -1, this is not the case for the split-quaternions (only $\mathbf{i}$ does). This is precisely the reason why split-quaternions are 'split', for this difference in sign gives rise to an indefinite quadratic form when computing the norm of the split-quadratic form. Because the quadratic form is indefinite, it classifies the set of split-quaternions into several subsets, which is to be discussed later.

Dihedral group he basis elements of the split-quaternions form a group under multiplication, namely the *dihedral group* D_4, which represents all the symmetries of a square: the identity, a 90-degree rotation and two reflections (cf. **??**).

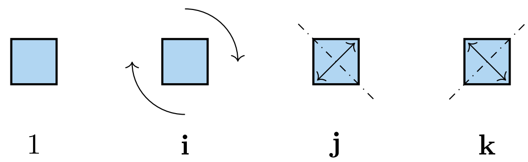

Figure 4-1: The dihedral group D_4 is the symmetry group of a square. This group is isomorphic to the group formed by $1, \mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ under multiplication.

The structure of the dihedral group can be visualized by its *cycle graph* in **??**. Many important properties of the split-quaternion algebra and the applications in this chapter can be traced back to the topology of this cycle graph. One example is the split nature of the quaternions: the $\mathbf{i}$-element generates an order four cycle, while $\mathbf{j}$ and $\mathbf{k}$ generate order two cycles (in contrast, the cycle graph for conventional quaternions is entirely symmetric for all these elements).

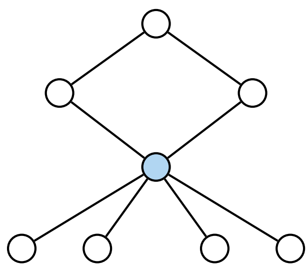

Figure 4-2: Cycle graph of the dihedral group. There are five cycles: one of order four which represents the rotations (or the element $\mathbf{i}$), and four order 2 cycles, which are all the possible reflections. The colored element represents the identity.

Split-quaternion norm Complex numbers have a real and imaginary part. Likewise, (split)-quaternions have a similar notion: a *scalar* (or real) and *vector* (or imaginary) components. For an arbitrary split-quaternion $q \in \hat{\mathbb{H}}$, [**?**]

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

the real part is $\mathrm{sca}(h) = a_0$ and the vector part is $\mathrm{vec}(a) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. For convenience, the vector part will be referred to in traditional bold vector notation:

$$\boldsymbol{a} = \mathrm{vec}(a) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Furthermore, for every split-quaternion there is a unique *conjugate*

$$a^* = \mathrm{sca}(a) - \mathrm{vec}(a) = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k},$$

through which the *split-quaternion norm* is defined:

$$\mathcal{N}(a) = ah^* = a_0^2 + a_1^2 - a_2^2 - a_3^2. \tag{4-1}$$

As mentioned, this norm is not positive definite, in stark contrast to quaternions or complex numbers. Split-quaternions can be categorized into three classes based on their norm being negative, zero or positive. In the tradition of special relativity, these classes are named *spacelike*, *lightlike* and *timelike* respectively: [**?** **?**]

- **Timelike**: $\mathcal{N}(a) > 0$, with real length $\|a\| = \sqrt{aa^*}$.
- **Lightlike**: $\mathcal{N}(a) = 0$, with zero length $\|a\| = 0$.
- **Spacelike**: $\mathcal{N}(a) < 0$, with imaginary length $\|a\| = \mathrm{i}\sqrt{|aa^*|}$.

Even though they behave similarly, the imaginary unit i is not to be confused with the split-quaternion basis element **i**, because they belong to different number systems.

Vector norm Apart from the split-quaternion norm, we can also define a norm that only considers the vector part of the split-quaternion. This norm is defined in accordance with the overall quaternion norm given by **??**:

$$\mathcal{N}_v(\boldsymbol{a}) = a_1^2 - a_2^2 - a_3^2.$$

The above expression is equivalent the Lorentz norm applied to a vector in $\mathbb{R}^3$; we will denote $\mathbb{R}^3$ equipped with the Lorentz norm by $\mathbb{R}^{2,1}$. [**?**] Observe that this quadratic form is not positive-definite either; in the the same veign as before, we can therefore classify quaternions by the 'sign' of their vector part again. We refer to these classes as *timelike (vectors)*, *spacelike (vectors)* and *lightlike (vectors)* in the same way.

Observe that $\mathcal{N}(a) < 0 \Rightarrow \mathcal{N}_v(\boldsymbol{a}) < 0$; that is to say, a spacelike split-quaternion always has a spacelike vector part. The converse is not necessarily true. Along the same line, a lightlike split-quaternion can only have a lightlike or spacelike vector part. All possible combinaions are listed in **??**. This classification is important because, as discussed in **??**, this classification is precisely equivalent to the qualitative classification of dynamic systems.

4-1-2 Relation with two-dimensional matrix algebra

The algebra of split-quaternions is isomorphic to the algebra of real two-dimensional matrices. This fact underlies this entire chapter, for it allows us to find an alternative for the traditional matrix description of linear dynamical systems.

Formally, an algebra is a vector space combined with a vector space V over a field $\mathbb{F}$, combined with an addition operation, scalar multiplication, and an $\mathbb{F}$-bilinear product operation $V \times V \to V$. [**?**]

Table 4-2: All the possible combinations of the class of a split-quaternion and its vector part. Spacelike split-quaternions can only have a spacelike vector, while lightlike split-quaternions can only have lightlike or spacelieke vector parts.

		$\mathcal{N}_v(\mathbf{a})$		
		spacelike	*lightlike*	*timelike*
$\mathcal{N}(a)$	*spacelike*	①	—	—
	lightlike	②	③	—
	timelike	④	⑤	⑥

- The split-quaternion algebra is an algebra over the field real numbers ($\mathbb{F} = \mathbb{R}$), where the multiplication is governed by the split-quaternion multiplication rules (see **??**).
- The set of 2×2-matrices also constitutes an $\mathbb{R}$-vector space; matrix multiplication makes it into an algebra.

An algebra isomorphism is an isomorphism between vector spaces that also commutes with the respective product operations in both vector spaces. If $(V, \bullet)$ and $(W, \diamond)$ are vector spaces equipped with their product operations, then $\phi : V \to W$ is an algebra isomorphism if (i) ϕ is a vector space isomorphism between V and W, and (ii)

$$\phi(v_1 \bullet v_2) = \phi(v_1) \diamond \phi(v_2) \qquad v_1, v_2 \in V.$$

In the case of the split-quaternions and the matrices, it is sufficient to map the basis elements of the split-quaternions to four linearly independent 'basis' matrices, and show that the resulting matrices observe the same multiplication rules as defined in **??**. Indeed, define the mapping ϕ by

$$\begin{aligned} \phi : \hat{\mathbb{H}} \to \mathbb{R}^{2\times2} : \quad 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{i} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \mathbf{j} &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{k} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4-2}$$

It is easily verified that (i) these matrices span $\mathbb{R}^{2\times2}$ and (ii) that the multiplication rules for split-quaternions are in accordance when translated to the respective matrices under matrix multiplication. Due to the bilinearity of the product, any linear combination of the basis elements will therefore satisfy the rules as well. Hence, we have established an algebra isomorphism between the split-quaternions and the 2×2-matrices. Based on this mapping for the basis vectors, a general quaternion maps to

$$\phi(a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \quad = \quad \begin{pmatrix} a_0 + a_3 & a_1 + a_2 \\ a_2 - a_1 & a_0 - a_3 \end{pmatrix}.$$

Likewise, the inverse mapping on an arbitrary matrix yields

$$\phi^{-1}\begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} \quad = \quad \frac{b_0 + b_3}{2} + \left(\frac{b_1 - b_2}{2}\right)\mathbf{i} + \left(\frac{b_1 + b_2}{2}\right)\mathbf{j} + \left(\frac{b_0 - b_3}{2}\right)\mathbf{k}.$$

One of the powerful features of the mapping ϕ is that it maps natural properties of the split-quaternion to natural properties of the associated matrix. Hence, given that $A = \phi(a)$ with $a \in \hat{\mathbb{H}}$ and $A \in \mathbb{R}^{2\times 2}$, we have the following properties:

- The *conjugate* of the split-quaternion maps to the *adjugate* of the matrix:[1]

$$\phi(a^*) = \operatorname{adj}(A).$$

- The *trace* of the matrix coincides with the *real or scalar part* of the split-quaternion:

$$\operatorname{sca}(a) = \frac{\operatorname{tr}(A)}{2}.$$

- The *determinant* of the matrix is equal to *norm* of the split-quaternion:

$$\mathcal{N}(a) = \det(A).$$

- The equivalence of the determinant and the split-quaternion norm hints at the fact that the multiplicative inverse of a split-quaternion does not always exist: only when its norm is nonzero. In that case, it is clear that

$$\phi\left(a^{-1}\right) = A^{-1} \qquad \mathcal{N}(A) \neq 0.$$

  The determinant properties also learns us something about the behavior of the classification under the split-quaternion multiplication

Table 4-3: Propagation of the class of split-quaternions when multiplied with a split-quaternion of another class. The timelike split-quaternions form a subgroup under multiplication, the timelike and spacelike split-quaternions do not: timelike split-quaternions do not have an inverse and the spacelike split-quaternions are not closed.

×	*space*	*light*	*time*
space	time	light	space
light	light	light	light
time	space	light	time

- The eigenvalues of a 2×2-matrix can be expressed in terms of its trace and its determinant:

$$\lambda_A = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det(A)}}{2}.$$

  The argument of the square root is equal to the *negative of the vector norm* of a. We therefore have:

$$\lambda_A = \frac{2a_0 \pm \sqrt{4a_0^2 - 4\mathcal{N}(a)}}{2} = a_0 \pm \mathrm{i}\|\mathbf{a}\|. \tag{4-3}$$

  Hence, the 'real' (scalar) and the magnitude of the 'imaginary' (vector) parts of the quaternion coincide with the real and imaginary part of the eigenvalues of the matrix.

[1] The adjugate of a matrix is the transpose of its cofactor matrix.

The algebra of 2×2-matrices (or equivalently, of the split-quaternions) also consitute the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ of the two-dimensional general linear group $\mathrm{GL}(2, \mathbb{R})$. Furthermore, the traceless matrices, or equivalently, the split-quaternions with zero real part form the subalgebra $\mathfrak{sl}(2, \mathbb{R})$ of the special linear group $\mathrm{SL}(2, \mathbb{R})$. These are the volume-preserving automorphisms on $\mathbb{R}^2$. Because in $\mathbb{R}^2$, volume and area coincide, the special linear group and the symplectic group $\mathrm{Sp}(1)$ are equivalent. For higher dimensions, this is not the case: area preservation is generally a stronger condition than volume preservation. The Lie algebra elements of the symplectic group are called Hamiltonian matrices; therefore, split-quaternions without real part are referred to as *Hamiltonian*.

4-2 Classification of dynamical systems

The classification of two-dimensional linear dynamical systems is important, or they also locally represent the fixed points of general nonlinear systems. Traditionally, this decomposition is done according to the eigenvalues of the state transition matrix matrix A, or equivalently, through a Poincaré diagram as shown in **??**.

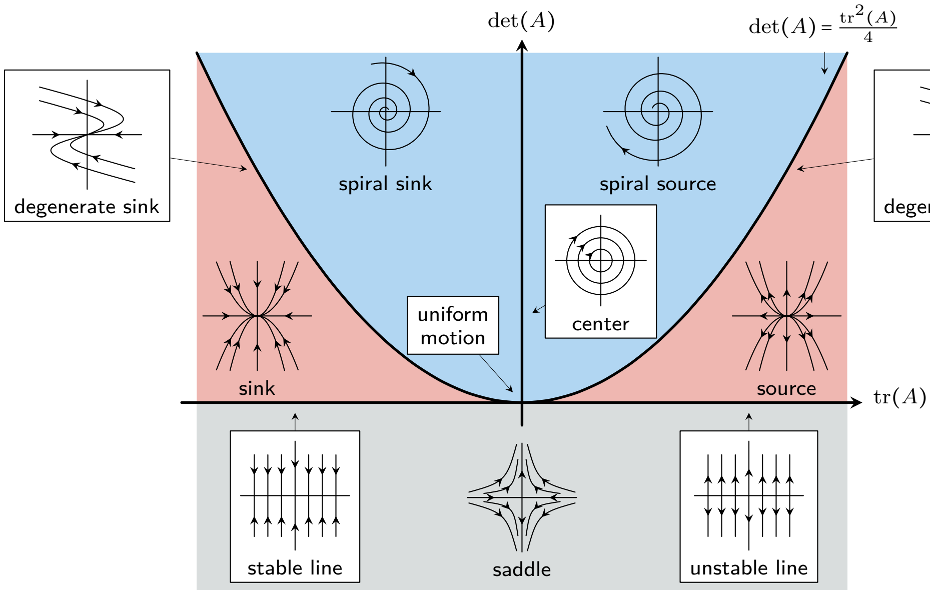

Figure 4-3: The classic Poincaré diagram, based on the conventional classification of fixed points based on the trace and determinant of the state transition matrix A.

Because the split-quaternion norms are directly related to the real and imaginary part of the eigenvalues of the associated matrix, this classification is more naturally done in the realm of split-quaternions.

Spacelike split-quaternion norm

① For spacelike split-quaternions, there is only one possibility: a negative split-quaternion norm corresponds to a negative determinant, which means that the fixed point is a *saddle*. We can distinguish one particular case: if the scalar part of the split-quaternion is zero ($a_0 = 0$), the saddle is 'balanced', and generates a proper *squeeze mapping*, which is a symplectomorphism of the phase space. The split-quaternion is therefore Hamiltonian. An example of the latter is the linearization of the unstable fixed point of a rotational pendulum.

Lightlike split-quaternion norm

② *Spacelike vector norm*: in this case, there is not just a fixed point but a fixed line in the phase space. This fixed line is stable or unstable depending on the sign of the scalar part of the quaternion.

③ *Lightlike vector norm*: this case is degenerate of the second degree; it coincides with the origin in the Poincaré diagram. The associated vector field is purely translational. An example is an object in uniform motion.

Timelike split-quaternion norm

④ *Spacelike vector norm*: this case gives rise to eigenvalues that are purely real; the fixed point is called a *node*. Depending on the sign of the scalar part, the fixed point can be an unstable node or *source* ($a_0 > 0$) or a stable node or *sink* ($a_0 < 0$). An example of such a system is the overdamped harmonic oscillator.

⑤ *Lightlike vector norm*: the eigenvalues of the associated matrix are real and equal; this type of fixed point is named a *degenerate node*. More specifically, in the unstable case ($a_0 > 0$) it is called a *degenerate source*, while in the stable case it is referred to as a *degenerate sink*. An example is a critically damped harmonic oscillator.

⑥ *Timelike vector norm*: this really is the only general case where the eigenvalues of A are complex. If $a_0 = 0$, the eigenvalues are imaginary and the fixed point is a *center*. Likewise, for $a_0 > 0$ it is an *unstable spiral node* and for $a_0 < 0$ a *stable spiral node*. An example is an underdamped (or even undamped) harmonic oscillator.

TODO

Connection with root locus

TODO

Connection with Jordan decomposition

TODO

Basis vector fields

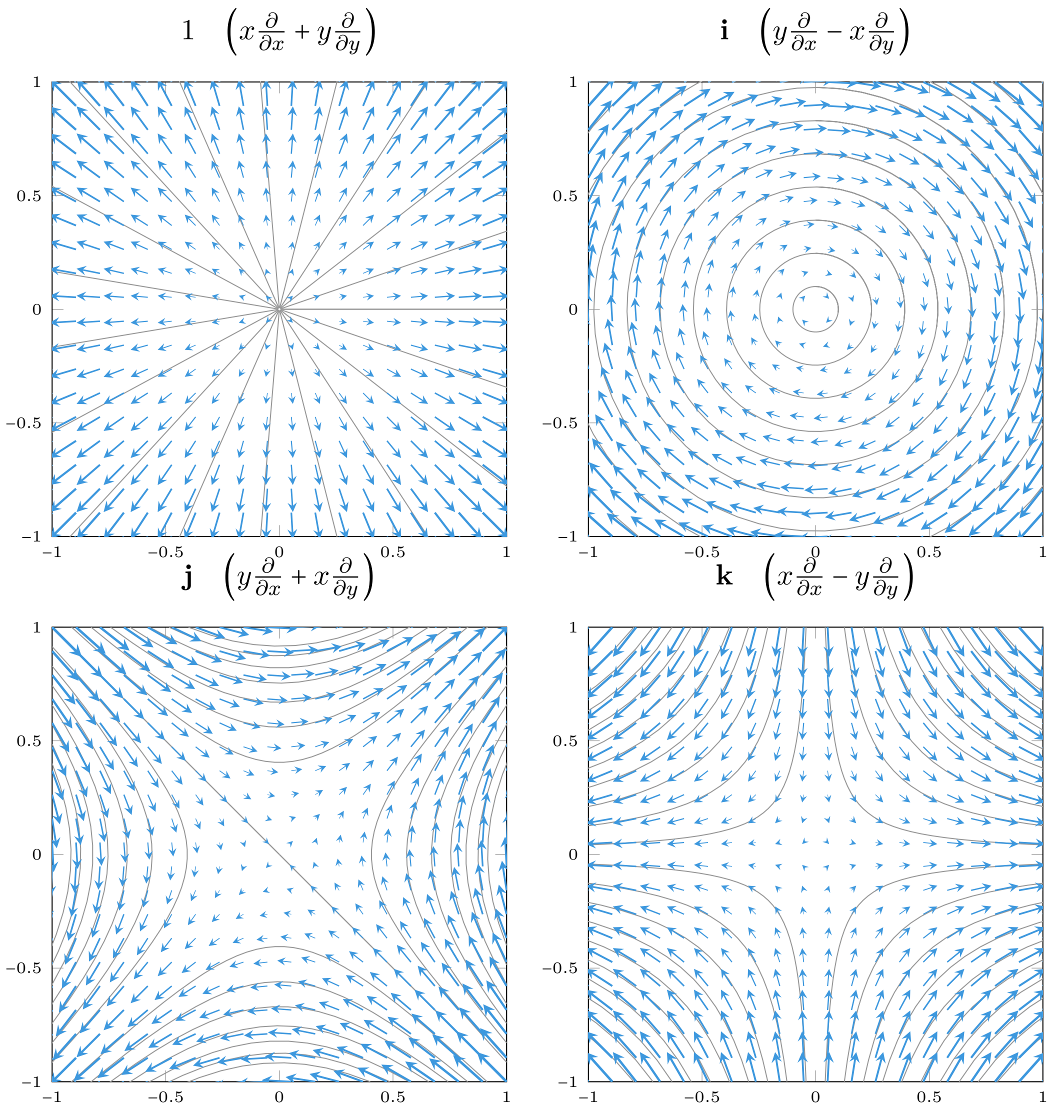

Figure 4-4: Basis vector fields corresponding to the basis elements of the split-quaternions.

4-3 Notes

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{\imath}} - u_2\hat{\boldsymbol{\jmath}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned}\boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle\hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}\hat{\boldsymbol{n}}\\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle\hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}),\end{aligned} \tag{4-4}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, we obtain:

$$\begin{aligned}\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i}\\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i}\\ &= \boldsymbol{e}_3 - \frac{\langle\hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}.\end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,\langle\boldsymbol{c}, \boldsymbol{a}\rangle - \boldsymbol{c}\,\langle\boldsymbol{a}, \boldsymbol{b}\rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b}\,\langle\boldsymbol{c}, \boldsymbol{a}\rangle_{\mathrm{L}} - \boldsymbol{c}\,\langle\boldsymbol{a}, \boldsymbol{b}\rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$
\begin{aligned}
&\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} - \hat{\boldsymbol{n}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle_{\mathrm{L}}) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \boldsymbol{e}_2 \mathrm{i}.
\end{aligned}
$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ■

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

Chapter 5

Conclusion

Appendix A

Symplectic geometry

Bla bla symplectic

Appendix B

Contact geometry

This appendix provides a short introduction to the basic concepts of contact geometry that are relevant in this thesis, particularly **??**.

B-1 Contact structures

A *contact element* on a manifold M is a point $m \in M$ combined with a tangent hyperplane $\xi_m \subset \mathrm{T}_m M$ (a subspace of the tangent space with codimension 1). The term 'contact' refers to the intuitive notion that if two submanifolds 'touch', they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [**?**] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

A *contact manifold* is a manifold M (of dimension $2n+1$) with a *contact structure*, which is a smooth field (or distribution) of contact elements on M. Locally, any contact element determines a 1-form α (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{B-1}$$

This α is called the (local) *contact form*, and it acts like a 'normal (co-)vector' to the hyperplane. For the field hyperplanes to be a constact structure, it must satisfy a nondegeneracy condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [**? ? ?**]

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of M such that the ξ is everywhere tangent to it, or (ii) that $\mathrm{d}\alpha|_\xi$ is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient TM/ξ is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [**?**].

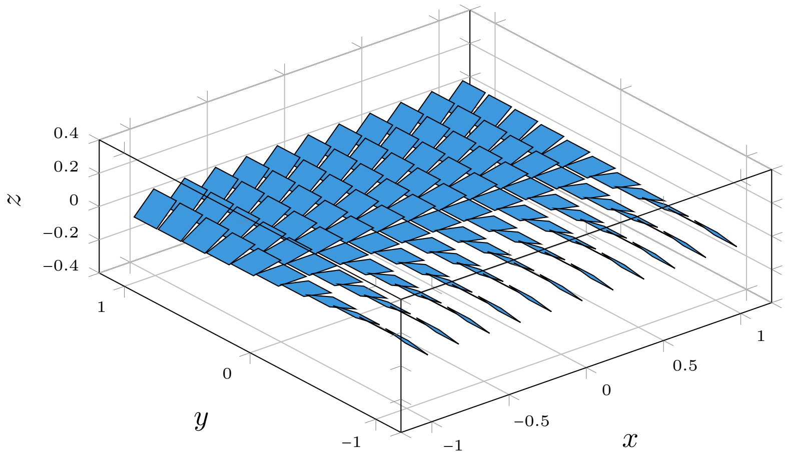

Figure B-1: The standard contact structure on $\mathbb{R}^3$, given by the contact form $\mathrm{d}z - y\,\mathrm{d}x$; the hyperplanes tilt more in the increasing y-direction.

The *Darboux theorem* for contact manifolds states that it is always possible to find coordinates z, x_i, y_i such that locally, the contact form is equal to

$$\mathrm{d}z - \sum y_i\,\mathrm{d}x_i\,,$$

which is also called the standard or natural contact structure. The standard contact structure on $\mathbb{R}^3$ is illustrated in **??**. Finally, it is clear that the contact form singles out a 'special direction' in the tangent space at every point of the manifold. This direction is given by the unique *Reeb vector field*,

$$R_\alpha \in \mathfrak{X}(M): \quad R_\alpha \lrcorner\, \mathrm{d}\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner\, \alpha = 1. \tag{B-2}$$

The special direction identified by the Reeb vector field is referred to as the *vertical* direction. Likewise, vector field components in the direction of the Reeb vector field are vertical. A vector field with no vertical component is called *horizontal*.

B-2 The manifold of contact elements

A contact manifold is a manifold with a contact structure. One can, however, associate a *canonical* $(2n-1)$-dimensional contact manifold to *any* n-dimensional manifold Q, just like one can always find a canonical symplectic structure on T^*Q. Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold Q. As it turns out, this 'manifold of contact elements' has a natural contact structure.

The *manifold of contact elements* of an n-dimensional manifold is [**?**]

$$\mathrm{C}Q = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } \mathrm{T}_qQ\}.$$

This manifold $\mathrm{C}Q$ has dimension $2n-1$. It is clear that C has a natural bundle structure, i.e. $C \xrightarrow{\pi} Q$ where the bundle projection 'forgets' the contact element, that is

$$\pi : \mathrm{C}Q \to Q : (q, \xi_q) \mapsto q.$$

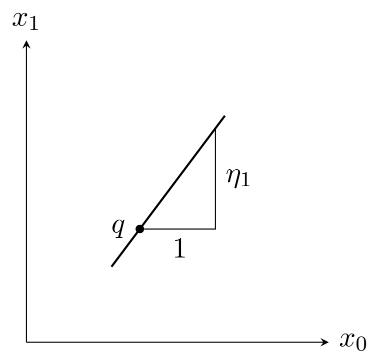

Figure B-2: A point in the manifold of contact elements on $Q = \mathbb{R}^2$. A coordinate system for $\mathrm{C}Q$ consists of (x_0, x_1) to indicate a point on Q, and projective coordinates $[\eta_0 : \eta_1]$, which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$, and the remaining coordinate η_1 covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the 'hyperplane' and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$.

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to Q, denoted by $\mathbb{P}(\mathrm{T}^*Q)$. This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in \mathrm{T}_q^*Q \setminus \{\mathbf{0}\} : \quad (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in **??**, in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure. $\mathbb{P}(\mathrm{T}^*Q)$ is then the quotient set of T^*Q (without zero section) with respect to the equivalence relation $\sim$. Visually, the projectivization of an n-dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension $n - 1$. It can be shown that this space is bundle-isomorphic to the manifold $\mathrm{C}Q$. [**?**]

As shown in **??**, coordinates of the equivalence class of 1-forms are 'projective coordinates', $[\eta_0 : \eta_1 : \ldots : \eta_{n-1}]$, where η_i are coordinates for T_q^*Q. The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes η_0 to be nonzero, the tuple $(1, \eta_1, \ldots, \eta_n)$ provides coordinates that cover most of $\mathbb{P}(\mathrm{T}^*Q)$.

Now, it remains to be explained why the 'manifold of contact elements' is itself a contact manifold. Indeed, there is a canonical field of hyperplanes *on* $\mathrm{C}Q$, which lifts the hyperplane tangent to Q to a hyperplane tangent to $\mathrm{C}Q$ (this is akin to the 'tautological' trick played in the symplectic structure of the cotangent bundle). The contact structure distinguishes the curves in $\mathrm{C}Q$ that are lifted versions from curves in Q. This is illustrated in **??**. [**?**] Said otherwise, a tangent vector on $\mathrm{C}Q$ lies in the hyperplane defined by the contact structure if its projection down on Q lies in the hyperplane on Q defined by the given point on the $\mathrm{C}Q$.

This contact structure is associated with the 1-form:

$$\alpha = \mathrm{d}x_0 + \sum_{i=1}^{n-1} \eta_i \, \mathrm{d}x_i \,,$$

given that the η_0 is the 'special' coordinate wich is chosen to be 1.

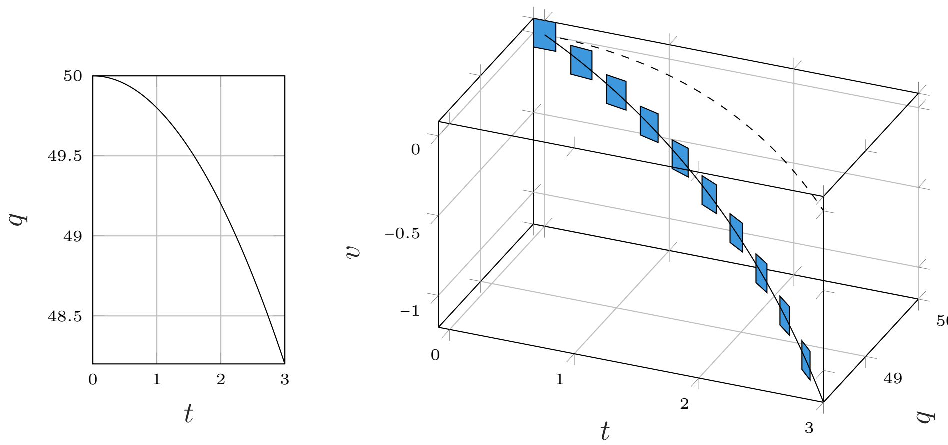

Figure B-3: Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let $(t, q) \in Q$, and let v be a coordinate for the contact (line) element. The standard contact form is then $\mathrm{d}q - v\,\mathrm{d}t$. On the left, the curve corresponding to a falling object is shown in Q. When this curve is 'lifted' to $\mathrm{C}Q$, the contact structure imposes that it be locally tangent to the contact structure, or that $v = \frac{\mathrm{d}q}{\mathrm{d}t}$. If the vertical direction is projected down onto the $(q - t)$-plane $(\mathrm{C}(Q) \to Q)$, the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making v the actual velocity of the curve.

B-3 Contact Hamiltonian systems

TODO

Introduction

B-3-1 Contact Hamiltonian vector fields

Just like in the symplectic case, the contact Hamiltonian formalism defines an automorphism between a function on the contact manifold, $H \in C^\infty(M)$, and an associated 'Hamiltonian' vector field $X_H \in \mathfrak{X}(M)$. While the isomorphism is rather straightforward for symplectic manifolds, the contact counterpart is not so perspicuous: this is the prime reason behind the computational advantage of symplectification, as opposed to performing the calculations directly on the contact manifold.

Coordinate-free derivation Given a contact manifold (M, ξ) with contact form α (i.e. $\xi \in \ker \alpha$), the tangent bundle M can be decomposed into two subbundles: [? ?]

$$\mathrm{T}M = \ker \alpha \oplus \ker \mathrm{d}\alpha\,,$$

where $\oplus$ denotes the Whitney sum. The first subbundle is referred to as the *horizontal* bundle, the second as the *vertical* bundle. The vertical subbundle is of rank 1 and its fiber is spanned by the Reeb vector field (cf. **??**). As mentioned to in **??**, *any* vector field $X \in \mathfrak{X}(M)$ may be decomposed accordingly. This decomposition is unique and given by

$$X = \underbrace{(X \lrcorner\, \alpha) R_\alpha}_{X^{\mathrm{ver}}} + \underbrace{\left[X - (X \lrcorner\, \alpha) R_\alpha\right]}_{X^{\mathrm{hor}}}. \tag{B-3}$$

Observe that indeed $X^{\mathrm{ver}} \in \ker \mathrm{d}\alpha$ and $X^{\mathrm{hor}} \in \ker \alpha$. [? ? ?]

We now wish to find the relation between the contact Hamiltonian $H \in C^\infty(M)$ and the associated Hamiltonian vector field $X_H \in \mathfrak{X}(M)$. This one-to-one relation is uniquely determined by two conditions. Firstly, we impose that[1]

$$H \equiv -X_H \lrcorner\, \alpha.$$

This condition already provides us with the vertical component of the Hamiltonian vector field, namely

$$X_H^{\mathrm{ver}} = -H R_\alpha.$$

Secondly, the automorphism generated by the Hamiltonian vector field must be a *contact automorphism*: it must preserve the contact structure. This condition is encoded in terms of the Lie derivative:[2]

$$X_H \text{ is an infinitesimal contact automorphism} \Leftrightarrow \pounds_{X_H} \alpha = s\alpha,$$

where $s \in C^\infty(M)$. The function s is there because $s\alpha$ and α give rise to the same hyperplane distribution. Using Cartan's 'magic' formula, the Lie derivative can be expanded as follows:

$$\begin{aligned} \pounds_{X_H} \alpha &= s\alpha \\ \mathrm{d}(X_H \lrcorner\, \alpha) + X_H \lrcorner\, \mathrm{d}\alpha &= s\alpha \\ -\mathrm{d}H + X_H \lrcorner\, \mathrm{d}\alpha &= s\alpha \end{aligned}$$

Contracting both sides with the Reeb vector field yields:

$$\begin{aligned} R_\alpha \lrcorner\, (-\mathrm{d}H + X_H \lrcorner\, \mathrm{d}\alpha) &= R_\alpha \lrcorner\, (s\alpha) \\ -R_\alpha \lrcorner\, \mathrm{d}H + R_\alpha \lrcorner\, X_H \lrcorner\, \mathrm{d}\alpha &= s\, R_\alpha \lrcorner\, \alpha \\ -R_\alpha \lrcorner\, \mathrm{d}H - X_H \lrcorner\, R_\alpha \lrcorner\, \mathrm{d}\alpha &= s. \end{aligned}$$

[1] This is the sign convention observed by **?**] en **?**], as opposed to **?**].

[2] Terminology differs somewhat in literature on this point: some authors, such as **?**] only refer to contactomorphisms as the special case where $g = 0$; while the more general case is called *conformal* contactomorphisms.

Hence, we have $s = -R_\alpha(\mathrm{d}H)$. Because the vertical component of X_H is spanned by the Reeb vector field, its contraction with $\mathrm{d}\alpha$ vanishes. As a result, we can rewrite the previous expression in terms of the *horizontal* component of X_H:

$$X_H \lrcorner \, \mathrm{d}\alpha = X_H^{\mathrm{hor}} \lrcorner \, \mathrm{d}\alpha = [\mathrm{d}H - (R_\alpha \lrcorner \, \mathrm{d}H)\alpha], \tag{B-4}$$

We must therefore recover X_H^{hor} from the above expression. Define the mapping

$$\alpha^\flat : \mathrm{T}M \to \mathrm{T}^*M : X \mapsto X \lrcorner \, \mathrm{d}\alpha \,,$$

when restricted to the space of horizontal vector fields, this mapping is an isomorphism onto the 'semi-basic' forms[3]. Define the inverse mapping of $\alpha^\flat$ by $\alpha^\sharp$, such that

$$X_H^{\mathrm{hor}} = \alpha^\sharp(\mathrm{d}H - (R_\alpha \lrcorner \, \mathrm{d}H)\,\alpha).$$

As such, the Hamiltonian vector field associated to the contact Hamiltonian H is

$$X_H = HR_\alpha + \alpha^\sharp(-\,\mathrm{d}H + (R_\alpha \lrcorner \, \mathrm{d}H)\,\alpha). \tag{B-5}$$

Coordinate expression Given the contact manifold (M, ξ) with contact form

$$\mathrm{d}q_0 - \sum_{i=1}^{n} p_i \, \mathrm{d}q_i \,,$$

and define the contact Hamiltonian $H = H(q_0, q_1, \ldots, q_n, p_1, \ldots, p_n)$. The vertical component of the Hamiltonian vector field is straightforward (cf. **??**):

$$X_H^{\mathrm{ver}} = -H\frac{\partial}{\partial q_0}.$$

For the horizontal component, first evaluate the right hand side of **??** in coordinates:

$$X_H^{\mathrm{hor}} \lrcorner \, \mathrm{d}\alpha = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} + p_i\frac{\partial H}{\partial q_0}\right)\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\,\mathrm{d}p_i \,.$$

In terms of the basis vectors, the mapping $\alpha^\flat$ is

$$\frac{\partial}{\partial q_i} \mapsto \mathrm{d}p_i \qquad \frac{\partial}{\partial p_i} \mapsto -\,\mathrm{d}q_i \qquad \frac{\partial}{\partial q_0} \mapsto 0 \qquad i = 1, \ldots, n.$$

The inverse transformation is slightly ambiguous at first sight, since any $\frac{\partial}{\partial q_0}$ cannot be recovered directly from the 'forward' mapping. However, we know that $\alpha^\sharp$ must produce a horizontal vector field. Therefore, first perform the inverse mapping in the q_i, p_i-components to obtain

$$-\sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} + p_i\frac{\partial H}{\partial q_0}\right)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}.$$

Contracting this expression with α produces $-\sum_{i=1}^{n} p\frac{\partial H}{\partial p_i}$. Hence, we can use this knowledge to find the actual horizontal component:

$$X_H^{\mathrm{hor}} = \sum_{i=1}^{n} p\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_0} - \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} + p_i\frac{\partial H}{\partial q_0}\right)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}.$$

[3]Semi-basic forms are forms that vanish when contracted with a vertical vector field. [**?**]

As such, the coordinate expression of **??** is

$$X_H = \left(\sum_{i=1}^{n} p\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial q_0} - \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} + p_i\frac{\partial H}{\partial q_0}\right)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \tag{B-6}$$

Furthermore, we have

$$\pounds_{X_H}\alpha = -\frac{\partial H}{\partial q_0}\alpha,$$

and

$$\pounds_{X_H}H = -H\frac{\partial H}{\partial q_0}.$$

B-3-2 Jacobi brackets

Just like the Poisson brackets define a Poisson algebra of the smooth functions on a symplectic manifold, there is a bracket operation on contact manifolds that serves (about) the same purpose. These brackets do not define a Poisson structure, but rather a *Jacobi structure*, which is a more general notion that includes the Poisson structure as a particular instance. In this treatment we will only focus on the associated *Jacobi bracket* for contact Hamiltonian systems. For more details regarding Jacobi manifolds, the reader is referred to [**?** , chap. V] and [**?**].

For two smooth functions $f, g \in C^\infty(M)$, and M a contact manifold with contact form α, the *Jacobi bracket* is defined as

$$[\![\,,\,]\!] : C^\infty(M) \times C^\infty(M) \to C^\infty(M) : \; [\![f, g]\!] = -\big[X_f, X_g\big] \lrcorner\, \alpha, \tag{B-7}$$

where $X_f, X_g \in \mathfrak{X}(M)$ are the contact Hamiltonian vector fields of f and g respectively, and $[\cdot, \cdot]$ is the Lie bracket (i.e. the commutator of vector fields). Equivalent expressions for the Jacobi bracket are: [**?**]

$$\begin{aligned} [\![f, g]\!] &= -X_f \lrcorner\, \mathrm{d}g + g(R_\alpha \lrcorner\, \mathrm{d}f) \\ &= X_g \lrcorner\, \mathrm{d}f - f(R_\alpha \lrcorner\, \mathrm{d}g) \\ &= -\mathrm{d}\alpha\,(X_f, X_g) - f(R_\alpha \lrcorner\, \mathrm{d}g) + g(R_\alpha \lrcorner\, \mathrm{d}f). \end{aligned} \tag{B-8}$$

From these expressions, it is also clear that the Jacobi bracket is antisymmetric, i.e. $[\![f, g]\!] = -[\![g, f]\!]$ and $[\![f, f]\!] = 0$. As a time evolution operator (with respect to the Hamiltonian H), we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [\![f, H]\!] + f(R_\alpha \lrcorner\, \mathrm{d}H) = [\![f, H]\!] - fs.$$

Using the same coordinates as in **??**, the Jacobi bracket is equal to:

$$[\![f, g]\!] = \left(\sum_{i=1}^{n} p_i\frac{\partial g}{\partial p_i} - g\right)\frac{\partial f}{\partial q_0} - \left(\sum_{i=1}^{n} p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial g}{\partial q_0} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right).$$

TODO

Check signs of Jacobi bracket, sign convention is again different from Libermann and Marle + mistake?

Glossary

List of Acronyms