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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis...

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology
May 29, 2022

E. B. Legrand

Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator
- (Optimal) control of the distributions using the Brockett approach
- (Stochastic) inputs, link with Langevin equations
- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.
- Bayesian inversion of chaotic systems; guess the initial state by sampling after a certain time
- Define as streamtube, continuity equation asserts that streamlines cannot cross; i.e. streamtubes are conserves. To reduce computational complexity, define level sets (curves in 2-D) and check how they deform through the evolution of the phase space fluid; should always contain the same amount of probability troughout the evolution of the system.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\Gamma\Delta\Upsilon\Omega\Theta$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\Gamma\Delta\Upsilon\Omega\Theta}$
Tensor	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\Gamma\Delta\Upsilon\Omega\Theta}$

Table 1-1: Caption

Math constants: ieπ
Variation: δS

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$
$\Gamma(TM)$
$\mathfrak{X}(TM)$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by Abraham and Marsden [1] and Cannas da Silva [2] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.
- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

Symplectic and Contact Geometry in Economic Engineering

Chapter 3

Liouville Geometry for Dissipative Systems

The contact-geometric counterpart of Hamiltonian and Lagrangian mechanics has been the subject of increasing academic interest in recent years, see for example van der Schaft [3], van der Schaft and Maschke [4], Maschke and van der Schaft [5], Bravetti et al. [6], de León and Lainz [7], etc. The conception of the idea arguably traces back to the work of Herglotz [8], who derived it using the variational principle, and the developments in differential geometry, by e.g. Arnol'd [9] and Libermann and Marle [10].

In this chapter, the direct connection is made between the Caldirola-Kanai Hamiltonian given by eq. (3-3) and the contact Hamiltonian described by Bravetti et al. [6], using Liouville geometry[1]. We then proceed to *contact Lagrangian mechanics*, strongly related to the Herglotz' work. Finally, the whole theory is explained from a thermodynamic perspective as well. While it was already known for some time (dating back to Arnol'd) that contact geometry is the preferred geometry for thermodynamics, its equivalence to contact geometry in (dissipative) classical mechanics has not been desribed in past literature. This underpins a famous statement by Vladimir Arnol'd that 'contact geometry is all geometry', in the sense that conservative mechanical systems can be considered as part of a larger class of systems for which energy dissipation *is* allowed. [11]

The traditional picture is that Hamiltonian mechanics takes place in the space of generalized positions and momenta, colloquialy denoted by q's and p's. The generalized positions form coordinates of the configuration manifold, which encodes all the possible positions that the system can find itself in. The momenta, on the other hand, are *cotangent variables*: they live in the cotangent space of linear functions acting on tangent (velocity) vectors to the configuration manifold Q. We say that the Hamiltonian is a function on the *cotangent bundle* to the configuration manifold T^*Q. This cotangent bundle has a canonical 'symplectic' structure, given by its symplectic form ω, that pairs every position direction with its corresponding

[1] It is interesting to note that Bravetti gives the Caldirola-Kanai method as an example of dissipative Hamiltonians in his paper, but fails to make the connection with his own method.

momenta:

$$\omega = \sum \mathrm{d}q^i \wedge \mathrm{d}p_i \,.$$

A vital property of symplectic manifolds (which include the cotangent bundle) is the fact that they are always even-dimensional: every position coordinate has a corresponding momentum and vice versa. Likewise, in an economic context this asserts that every product has its own price. It is precisely this symmetry that is broken by the introduction of contact manifolds.

The symplectic structure of Hamiltonian mechanics is related to the conservation of energy principle, The Hamiltonian function is conserved along the integral curves of the Hamiltonian vector field that it generates. For dissipative mechanics, this strict reciprocity between position and momentum is broken. Either, one constructs an explicit time-dependence and acknowledges the special nature of time, or one can introduce another coordinate that acts as 'reservoir' to facilitate the dissipation in the system. These Hamiltonian systems are known as *contact Hamiltonian systems*. One of the contributions of this thesis is to show how both methods are essentially equivalent, by connecting the most famous time-dependent model by Caldirola and Kanai to the contact Hamiltonian by Bravetti et al. [6].

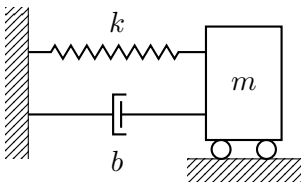

Figure 3-1: Schematic of the mass-spring-damper system.

This chapter (and the application in the following chapter) is primarily concerned with the prototypical dissipative mechanical system: the linearly damped harmonic oscillator depicted in fig. 3-1, with the governing second-order differential equation being

$$m\ddot{q} + b\dot{q} + kq = 0. \tag{3-1}$$

The choice for this system is rather perspicuous, since it is arguably the 'easiest' dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of choice in the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [12, 13]. However, the method described in this section can be generalized directly to a general (possibly time-dependent) potential function $V = V(q,t)$. To make calculations and notation easier, some special parameters are frequently used throughout this chapter, they are summarized in table 3-1.

3-1 The Caldirola-Kanai method

A traditional, engineering-inclined method to incorporate damping in the framework is to include a Rayleigh damping term in the Lagrangian to emulate linear damping forces, and this works 'mathematically' to derive the correct equations of motion [14]. Although frequently used for practical problems, this damping term is not really part of the *actual* Lagrangian — rather, it simply makes use of the notion of a generalized force that is not inherently part of

Table 3-1: Parameter conventions of the damped harmonic oscillator. To avoid confusion with the symplectic form ω, angular frequencies are denoted by Ω instead of the conventional lower case Greek letter.

Name	Symbol	Value	Units
Damping coefficient	γ	b/m	s^{-1}
Undamped frequency	Ω_o	$\sqrt{k/m}$	s^{-1}
Damped frequency	Ω_d	$\sqrt{\Omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$	s^{-1}
Damping ratio	ζ	$\frac{b}{2\sqrt{mk}}$	–

the system. As such, this method only works on a superficial level: the pristine differential geometric foundations of mechanics do not leave room for such ad hoc tricks. There is, as a result, also no Hamiltonian counterpart for this method.

The historical attempts to do better than the Rayleigh method were primarily motivated by the application of the (dissipative) Hamiltonian formalism in quantum mechanics through discretization. For this application, a sound mathematical structure is of the essence, which calls for a more rigorous approach. A celebrated paper by Dekker [12] provides an excellent summary of many attempts up to 1981. Indeed, the well-studied approach developed by Caldirola [15] and Kanai [16] was intended exactly for this purpose. This method features an explicit time-dependence both in the Lagrangian function

$$L_{\mathrm{CK}}(q, \dot{q}, t) = \mathrm{e}^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq_1^2\right), \tag{3-2}$$

and the corresponding Hamiltonian function:

$$H_{\mathrm{CK}}(q, \rho, t) = \frac{\rho^2}{2m}\mathrm{e}^{-\gamma t} + \frac{1}{2}kq^2\mathrm{e}^{\gamma t}. \tag{3-3}$$

In latter equation, ρ refers to a special 'canonical momentum', that is

$$\rho \equiv \frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}, \tag{3-4}$$

which is related to the 'true' kinematic momentum by the relation $\rho = p\mathrm{e}^{\gamma t} = m\dot{q}\mathrm{e}^{\gamma t}$. As such, it is also clear that the Caldirola-Kanai Lagrangian and Hamiltonian functions are related by the Legendre transform *with respect to the canonical momentum*:

$$H_{\mathrm{CK}} = \rho\dot{q} - L_{\mathrm{CK}}.$$

From either eq. (3-2) or eq. (3-3), the equations of motion are readily derived (for the Hamiltonian case with respect to ρ after which the transformation to p can be effected). Indeed, after taking the appropriate derivatives, one obtains:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}\right) - \frac{\partial L_{\mathrm{CK}}}{\partial q} = 0 \\ \Rightarrow\ &\mathrm{e}^{\gamma t}(m\ddot{q} + m\gamma\dot{q} + kq) = 0 \end{aligned}$$

for the Lagrangian case. Likewise, Hamilton's equations yield: [17]

$$\dot{q} = \frac{\partial H_{\mathrm{CK}}}{\partial \rho} = \frac{\rho}{m} \mathrm{e}^{-\gamma t} = \frac{p}{m},$$
$$\dot{\rho} = -\frac{\partial H_{\mathrm{CK}}}{\partial q} = -kq\mathrm{e}^{\gamma t}.$$

The relation between the time derivatives of the momenta $\dot{p}_1$ and $\dot{\rho}$ is slightly more involved since one must invoke the product rule as a result of their time-dependent relation:

$$\dot{\rho} = \mathrm{e}^{\gamma t}(\dot{p}_1 + \gamma p).$$

Substition yields the correct equation for p, though the equation is again multiplied by $\mathrm{e}^{\gamma t}$. Because the latter is sufficiently well-behaved (that is, it has no zeros), it can be removed without any problems.

Geometric perspective To put the above derivation in a geometric setting, define the Liouville 1-form as

$$\alpha = \rho \, \mathrm{d}q \quad \Rightarrow \quad \omega = -\mathrm{d}\alpha = \mathrm{d}q \wedge \mathrm{d}\rho \,,$$

where the symplectic 2-form will be used to obtain Hamilton's equations. The Hamiltonian eq. (3-3) is explicitly time-dependent. This will give rise to a time-dependent vector field governing the solution curves.[2] The construction of the vector field associated with a time-dependent Hamiltonian follows the same construction rules as a normal Hamiltonian (using the isomorphism given by ω), but 'frozen' at each instant of t. Even more bluntly speaking, one simply ignores the t-coordinate during the derivation, only to acknowledge the dependence at the very end. This leads to the following vector field, 'suspended' on the $\mathbb{R} \times Q$ space:

$$\tilde{X}_{H_{\mathrm{CK}}} = -\mathrm{e}^{\gamma t} kq \frac{\partial}{\partial \rho} + \mathrm{e}^{-\gamma t} \frac{\rho}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

The suspension is important to make the final coordinate transform from ρ to p work properly. Indeed, effecting the transformation $(q, \rho, t) \mapsto (q, \mathrm{e}^{-\gamma t}\rho, t)$, one obtains

$$\tilde{X}_{H_{\mathrm{CK}}} = (-kq - \gamma p) \frac{\partial}{\partial p} + \frac{p_1}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

It is worthwile to ponder on some apparent peculiarities in the Caldirola-Kanai method, for they will be explained elegantly by the contact-Hamiltonian formalism. Firstly, the role of the two-different momenta is not very clear from the get-go, apart from being a consequence of the way the Caldirola-Kanai Lagrangian is formulated. This has also been the reason for considerable confusion in the academic community (see Schuch [18]). Furthermore, there is the special role of the time coordinate, which is merely a parameter in the Hamiltonian function; for it does not partake in the dynamics of the system. Finally, there is the special role of the factor $\mathrm{e}^{\gamma t}$ through which the time-dependence makes its appearance both in the Lagrangian and the Hamiltonian.

[2] A *time-dependent vector field* on a manifold M is a mapping $X : M \times \mathbb{R} \to TM$ such that for each $t \in \mathbb{R}$, the restriction X_t of X to $M \times \{t\}$ is a vector field on M. [10] An additional construction of importance, called the *suspension* of the vector field, is a mapping

$$\tilde{X} : \mathbb{R} \times M \to T(\mathbb{R} \times M) \quad (t, m) \mapsto ((t, 1), (m, X(t, m))),$$

that is to say, it lifts the vector field to the extended space that also includes t and assigns the time coordinate with a trivial velocity of 1. [1]

3-2 From time-dependent to contact Hamiltonian systems

In the subsequent discussion the original Hamiltonian will be progressively lifted to higher-dimensional spaces in order to include dissipation in the Hamiltonian formalism. First of all, to a contact manifold, which is odd-dimensional: as mentioned, we need an additional degree of freedom — which has no momentum conjugate to it — to keep track of the dissipation in the system. This degree of freedom is sometimes referred to as a *gauge variable*, denoted by q_0. However, performing calculations in contact geometry directly is cumbersome and uninsightful: to quote Vladimir Arnol'd once more, 'one is advised to calculate symplectically but to think rather in contact geometry terms'. Hence, we make use of the *symplectization* of the contact structure, which gives rise to a so-called *Liouville structure*, and 'pretend' that we are dealing with the symplectic case. This symplectization will add yet another dimension to the system. [3, 19]

3-2-1 Symplectization & Liouville structures

The contact manifold of our system is three-dimensional, with coordinates p, q and q_0 – the latter is the gauge variable for the dissipation in the system. It can be viewed as the manifold of contact elements associated with the extended configuration manifold M for which q and q_0 are coordinates, denoted by $\mathbb{P}(T^*M)$. The contact form on $\mathbb{P}(T^*M)$ is given by

$$\alpha = \mathrm{d}q_0 - p\,\mathrm{d}q\,, \tag{3-5}$$

which accentuates the special role of the q_0 in the system dynamics. Contact forms are, by their very nature, ambiguous, for they represent a distribution of hyperplanes, which coincides with the kernel of the 1-form. Multiplication with a nonzero factor yields a different form with the same kernel, that is to say, they represent the same contact structure. This is the reason behind the 'projective' nature of contact mechanics[3]. Hence, one may just as well multiply the 1-form with a nonzero factor λ:

$$\lambda(\mathrm{d}q_0 - p\,\mathrm{d}q) \quad \lambda \in \mathbb{R}_0.$$

The factor λ can be considered to be an extra degree of freedom (leaving the contact structure unaffected), which provides a 'lift' from the odd-dimensional manifold to an even-dimensional one, which is called the symplectification of the contact manifold. [9]

More formally, introduce the *principal* $\mathbb{R}_\times$-bundle $T_0^*M \xrightarrow{\pi} \mathbb{P}(T^*M)$, where $\mathbb{R}_\times$ acts in the fiber through 'dilations' of the fiber which amounts to an equal scaling of all the cotangent variables (this is the multiplication by λ in the above expression). Here, T_0^*M is the cotangent bundle to M, but with the zero section removed — this is required for the group to act freely). This principal bundle $T_0^*M \xrightarrow{\pi} \mathbb{P}(T^*M)$ admits what is called a *fibered symplectic Liouville structure*, given by the Liouville form [10]

$$\theta = \lambda(\mathrm{d}q_0 + p\,\mathrm{d}q).$$

[3] As explained in appendix A, the manifold of contact elements is bundle-isomorphic to the projectivization of the cotangent bundle.

To restate the above in canonical coordinates, choose

$$\rho_0 = \lambda \quad \text{and} \quad \rho = -\lambda p \tag{3-6}$$

such that

$$\theta = \rho_0 \, \mathrm{d}q_0 + \rho \, \mathrm{d}q \,, \tag{3-7}$$

which is the Liouville form on T^*M. [10, p. 308] The Liouville form defines a symplectic structure given by[4]

$$\omega = -\,\mathrm{d}\theta \,.$$

The manifold T_0^*M is called the symplectization of the contact manifolds, and consists of all four-tuples (q_0, q, ρ_0, ρ) for which ρ_0 and ρ do not simultaneously vanish. For now, we have not assigned any specific meaning to the coordinates given above, but they will turn out to match with the convention adhered to in eqs. (3-2) and (3-3), etc.

To summarize, we lifted the original system with symplectic structure $\mathrm{d}q \wedge \mathrm{d}p$ to a contact manifold through the addition of a gauge variable q_0. We then symplectified the contact manifold to a four-dimensional system, with 'positions' (q_0, q) and 'momenta' (ρ_0, ρ).

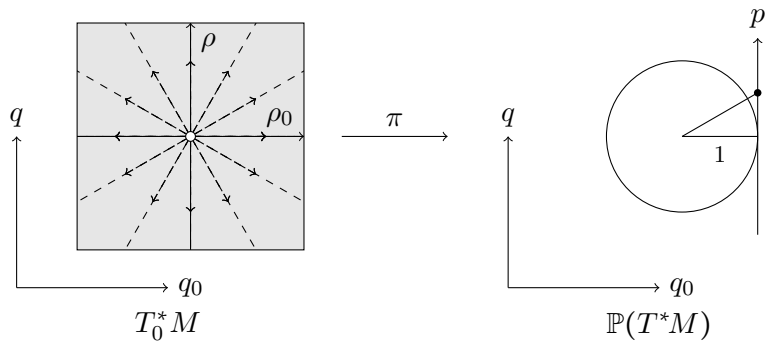

Figure 3-2: Illustration of the principal $\mathbb{R}_\times$-bundle $T_0^*M \xrightarrow{\pi} \mathbb{P}(T^*M)$. The total space T_0^*M is the cotangent bundle to M with zero section removed, which is shown on the left. The action by the multiplicative group $\mathbb{R}_\times$ is illustrated by the arrows, for it acts as a scaling (dilation) on all the cotangent variables. The origin is not part of the fiber, for it is part of the zero section. The bundle projection π projects all points that are on the same orbit (straight lines through the origin) to a single point on the base manifold: the projectivized cotangent bundle $\mathbb{P}(T^*M)$. The former space has a symplectic structure while the latter space has a contact structure. Observe from eq. (3-6) that $p = \rho/rho_0$, i.e. such that p is a coordinate for the projectivization by stereographic projection, as shown on the right.

3-2-2 Homogeneous Hamiltonian systems

The theoretical construction of the past section serves an important purpose, because it is the symplectified space which is the proper setting for the Caldirola-Kanai Hamiltonian discussed in section 3-1. Along with the symplectification of the contact structure described in the past section, we can do the same with a contact Hamiltonian system. This will result in a

[4]The nondegeneracy condition on the contact structure guarantees that this structure is indeed symplectic.

'horizontal' and 'vertical' part of the resulting vector field, which distinguishes the dissipative from the purely conservative dynamics.

Consider the bla bla

Chapter 4

Split-Quaternions as Dynamical Systems

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{i}} - u_2\hat{\boldsymbol{j}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned} \boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \hat{\boldsymbol{n}} \\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}), \end{aligned} \tag{4-1}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, one obtains:

$$\begin{aligned} \hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\ &= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}. \end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{c}, \boldsymbol{a} \rangle - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{c}, \boldsymbol{a} \rangle_{\mathrm{L}} - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$\begin{aligned}
&\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} - \hat{\boldsymbol{n}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle_{\mathrm{L}}) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \boldsymbol{e}_2 \mathrm{i}.
\end{aligned}$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ■

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

Chapter 5

Conclusion

Appendix A

Contact geometry

This appendix provides a short introduction to the basic concepts of contact geometry that are relevant in this thesis, particularly chapter 3.

A-1 Contact structures

A *contact element* on a manifold M is a point $m \in M$ combined with a tangent hyperplane $\xi_m \subset T_m M$ (a subspace of the tangent space with codimension 1). The term 'contact' refers to the intuitive notion that if two submanifolds 'touch', they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [2] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

A *contact manifold* is a manifold M (of dimension $2n+1$) with a *contact structure*, which is a smooth field (or distribution) of contact elements on M. Locally, any contact element determines a 1-form α (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{A-1}$$

This α is called the (local) *contact form*, and it acts like a 'normal (co-)vector' to the hyperplane. For the field hyperplanes to be a constact structure, it must satisfy a nondegeneracy condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [2, 1, 9]

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of M such that the ξ is everywhere tangent to it, or (ii) that $\mathrm{d}\alpha|_\xi$ is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient TM/ξ is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [11].

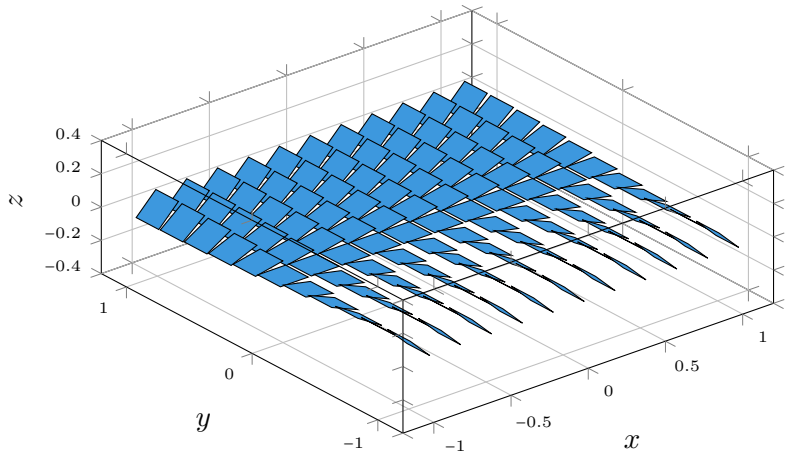

Figure A-1: The standard contact structure on $\mathbb{R}^3$, given by the contact form $\mathrm{d}z - y\,\mathrm{d}x$; the hyperplanes tilt more in the increasing y-direction.

The *Darboux theorem* for contact manifolds states that it is always possible to find coordinates z, x_i, y_i such that locally, the contact form is equal to

$$\mathrm{d}z - \sum y_i\,\mathrm{d}x_i\,,$$

which is also called the standard or natural contact structure. The standard contact structure on $\mathbb{R}^3$ is illustrated in fig. A-1. Finally, it is clear that the contact form singles out a 'special direction' in the tangent space at every point of the manifold. This direction is given by the unique *Reeb vector field*,

$$R_\alpha \in \mathfrak{X}(M): \quad R_\alpha \lrcorner\, \mathrm{d}\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner\, \alpha = 1,$$

that is, it locally points in the 'direction' of the contact form.

A-2 The manifold of contact elements

A contact manifold is a manifold with a contact structure. One can, however, associate a *canonical* $(2n-1)$-dimensional contact manifold to *any* n-dimensional manifold Q, just like one can always find a canonical symplectic structure on T^*Q. Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold Q. As it turns out, this 'manifold of contact elements' has a natural contact structure.

The *manifold of contact elements* of an n-dimensional manifold is [2]

$$CQ = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } T_qQ\}.$$

This manifold CQ has dimension $2n-1$. It is clear that C has a natural bundle structure, i.e. $C \xrightarrow{\pi} Q$ where the bundle projection 'forgets' the contact element, that is

$$\pi : CQ \to Q : (q, \xi_q) \mapsto q.$$

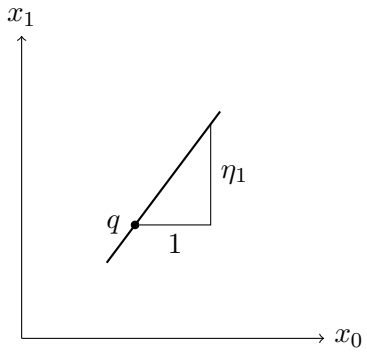

Figure A-2: A point in the manifold of contact elements on $Q = \mathbb{R}^2$. A coordinate system for CQ consists of (x_0, x_1) to indicate a point on Q, and projective coordinates $[\eta_0 : \eta_1]$, which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$, and the remaining coordinate η_1 covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the 'hyperplane' and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$.

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to Q, denoted by $\mathbb{P}(T^*Q)$. This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in T_q^*Q \setminus \{\mathbf{0}\}: \quad (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in eq. (A-1), in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure. $\mathbb{P}(T^*Q)$ is then the quotient set of T^*Q (without zero section) with respect to the equivalence relation $\sim$. Visually, the projectivization of an n-dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension $n - 1$. It can be shown that this space is bundle-isomorphic to the manifold CQ. [2]

As shown in fig. A-2, coordinates of the equivalence class of 1-forms are 'projective coordinates', $[\eta_0 : \eta_1 : \ldots : \eta_{n-1}]$, where η_i are coordinates for T_q^*Q. The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes η_0 to be nonzero, the tuple $(1, \eta_1, \ldots, \eta_n)$ provides coordinates that cover most of $\mathbb{P}(T^*Q)$.

Now, it remains to be explained why the 'manifold of contact elements' is itself a contact manifold. Indeed, there is a canonical field of hyperplanes *on* CQ, which lifts the hyperplane tangent to Q to a hyperplane tangent to CQ (this is akin to the 'tautological' trick played in the symplectic structure of the cotangent bundle). The contact structure distinguishes the curves in CQ that are lifted versions from curves in Q. This is illustrated in fig. A-3. [20] Said otherwise, a tangent vector on CQ lies in the hyperplane defined by the contact structure if its projection down on Q lies in the hyperplane on Q defined by the given point on the CQ.

This contact structure is associated with the 1-form:

$$\alpha = \mathrm{d}x_0 + \sum_{i=1}^{n-1} \eta_i \, \mathrm{d}x_i \,,$$

given that the η_0 is the 'special' coordinate wich is chosen to be 1.

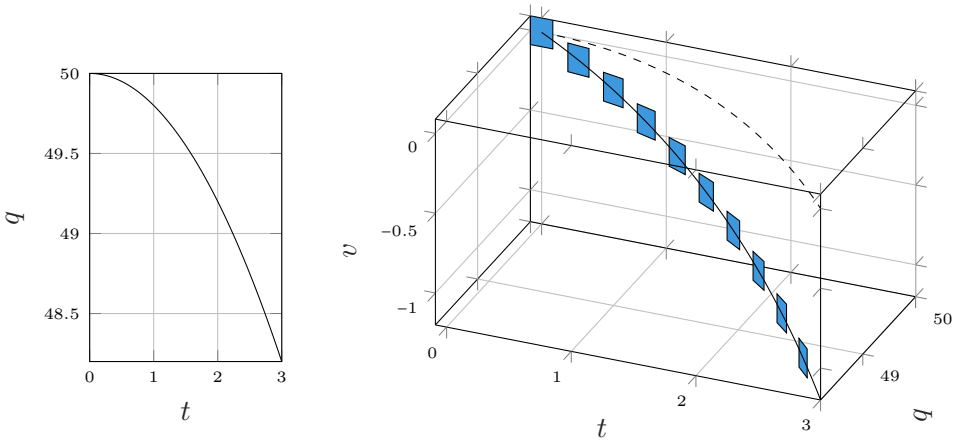

Figure A-3: Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let $(t, q) \in Q$, and let v be a coordinate for the contact (line) element. The standard contact form is then $\mathrm{d}q - v\,\mathrm{d}t$. On the left, the curve corresponding to a falling object is shown in Q. When this curve is 'lifted' to CQ, the contact structure imposes that it be locally tangent to the contact structure, or that $v = \frac{\mathrm{d}q}{\mathrm{d}t}$. If the vertical direction is projected down onto the $(q - t)$-plane $(C(Q) \to Q)$, the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making v the actual velocity of the curve.

Zoo of structures in literature

- Cosymplectic = structure on odd-dimensional manifold, closed 2-form ω and closed 1-form η, with $\eta \wedge \omega^n \neq 0$.
- Presymplectic = structure on an even-dimensional manifold, rank of 2-form not maximal but at least constant.
- Precontact system = odd-dimensional extension of presymplectic.

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Glossary

List of Acronyms

Mathematical notation

$\boldsymbol{v}$	A (tangent) vector