Thesis Title

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Faculty of Mechanical, Maritime and Materials Engineering · Delft University of Technology

Delft University of Technology
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The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

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by

E. B. Legrand

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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis...

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology E. B. Legrand
May 16, 2022

Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator
- (Optimal) control of the distributions using the Brockett approach
- (Stochastic) inputs, link with Langevin equations
- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.
- Bayesian inversion of chaotic systems; guess the initial state by sampling after a certain time
- Define as streamtube, continuity equation asserts that streamlines cannot cross; i.e. streamtubes are conserves. To reduce computational complexity, define level sets (curves in 2-D) and check how they deform through the evolution of the phase space fluid; should always contain the same amount of probability troughout the evolution of the system.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\mathit{\Gamma\Delta\Upsilon\Omega\Theta}$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathit{\Gamma\Delta\Upsilon\Omega\Theta}}$
Tensor	$\boldsymbol{\mathsf{abcde}}$	$\boldsymbol{\mathsf{ABCDE}}$	$\boldsymbol{\alpha\beta\gamma\delta\epsilon}$	$\boldsymbol{\mathsf{\Gamma\Delta\Upsilon\Omega\Theta}}$

Table 1-1: Caption

Math constants: $\mathrm{i}\mathrm{e}\pi$
Variation: δS

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner\, \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by Abraham and Marsden [1] and Cannas da Silva [2] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.
- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

A Differential-Geometric Perspective on Economic Engineering

Chapter 3

Symplectified Contact Mechanics for Dissipative Systems

The traditional view is that the methods of analytical mechanics, such as the Lagrangian and Hamiltonian formalisms, are only suited for conservative systems. However, several attempts, especially in the previous century, have been made to extend these principles to dissipative systems as well. This chapter (and the application in the following chapter) will be primarily concerned with the prototypical dissipative mechanical system: the linearly damped harmonic oscillator depicted in fig. 3-1, with the governing second-order differential equation being

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{3-1}$$

The choice for this system is rather perspicuous, since it is arguably the 'easiest' dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [3, 4]. However, the method described in this section can be generalized directly to a general (possibly time-dependent) potential function $V = V(x, t)$.

A traditional, engineering-inclined method to incorporate damping in the framework is to include a Rayleigh damping term in the Lagrangian to emulate linear damping forces, and this works 'mathematically' to derive the correct equations of motion [5]. Although frequently

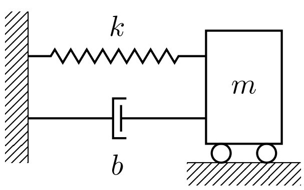

Figure 3-1: Schematic of the mass-spring-damper system.

used for engineering problems, this damping term is not really part of the *actual* Lagrangian — rather, it simply makes use of the notion of a generalized force that is not inherently part of the system. As such, this method only 'works' on a superficial level: the pristine differential geometric foundations of mechanics do not leave room for such ad hoc tricks.

The historical attempts to do better than the Rayleigh method were primarily motivated by the application of the (dissipative) Hamiltonian formalism in quantum mechanics through discretization. For this application, a sound mathematical structure is of the essence, which calls for a more rigorous approach. A celebrated paper by Dekker [3] provides an excellent summary of many attempts up to 1981. A brief summary (with some recent additions) is given below [motivate]. In this and the following chapter, some important connections between these methods that were not mentioned in Dekker's treatment.

- The **mirror system** or **Bateman** approach, doubles the number of system dimensions by including a mirror system that runs opposite in time. Arguably the most flexible of all methods, it is
- Expressing the Hamiltonian in **complex coordinates** has also produced promising results: notable are the attempts of Bopp [6], Dedene [7] and the very recent contribution by Hutters and Mendel [4] in the research group to which the author belongs as well.
- A different approach, related to the contact method, are the
- Contact mechanics
- Mathematical Hamiltonians

[Mention Max as contributor]

Chapter 4

Split-Quaternions as Dynamical Systems

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{i}} - u_2\hat{\boldsymbol{j}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned} \boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}\hat{\boldsymbol{n}} \\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}), \end{aligned} \tag{4-1}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, one obtains:

$$\begin{aligned} \hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\ &= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}. \end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,\langle \boldsymbol{c}, \boldsymbol{a}\rangle - \boldsymbol{c}\,\langle \boldsymbol{a}, \boldsymbol{b}\rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b}\,\langle \boldsymbol{c}, \boldsymbol{a}\rangle_{\mathrm{L}} - \boldsymbol{c}\,\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$\begin{aligned}
&\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}(\hat{\boldsymbol{u}}\,\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle_{\mathrm{L}} - \hat{\boldsymbol{n}}\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}\rangle_{\mathrm{L}})\mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}}\,\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle_{\mathrm{L}}\,\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle} - \hat{\boldsymbol{n}}\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}\right)\mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}}\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}\right)\mathrm{i} \\
&= \boldsymbol{e}_3 - \boldsymbol{e}_2\mathrm{i}.
\end{aligned}$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ■

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

Chapter 5

Conclusion

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Glossary

List of Acronyms

Mathematical notation

$\boldsymbol{v}$	A (tangent) vector