

Thesis Title

Optional Subtitle

E. B. Legrand

Master of Science Thesis



# **Thesis Title**

**Optional Subtitle**

MASTER OF SCIENCE THESIS

For the degree of Master of Science in Systems and Control at Delft  
University of Technology

E. B. Legrand

August 9, 2022



Copyright © Delft Center for Systems and Control (DCSC)  
All rights reserved.

DELFT UNIVERSITY OF TECHNOLOGY  
DEPARTMENT OF  
DELFT CENTER FOR SYSTEMS AND CONTROL (DCSC)

The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

THESIS TITLE

by

E. B. LEGRAND

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE SYSTEMS AND CONTROL

Dated: August 9, 2022

Supervisor(s): \_\_\_\_\_  
prof.em.dr.ir. M. B. Mendel

Reader(s): \_\_\_\_\_  
prof.dr.ir. T. Keviczky

\_\_\_\_\_ dr. D. Boskos

\_\_\_\_\_ dr. R. Swierstra

\_\_\_\_\_ ir. C. Hutters



---

## **Abstract**

This is an abstract.



---

# Table of Contents

<b>Preface</b>	<b>xi</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 A Geometric Perspective of Economic Engineering</b>	<b>3</b>
2-1 Symplectic manifolds in economic engineering . . . . .	3
2-2 Lagrangian mechanics in economic engineering . . . . .	4
2-2-1 Kinetic energy . . . . .	5
2-2-2 Potential energy . . . . .	6
2-2-3 Symplectic geometry in Lagrangian mechanics . . . . .	7
2-3 Hamiltonian mechanics in economic engineering . . . . .	8
2-3-1 The Legendre transform . . . . .	8
2-3-2 Hamilton's equations . . . . .	9
2-3-3 The role of dissipation . . . . .	9
<b>3 Geometric Structures in Dissipative Mechanics</b>	<b>11</b>
3-1 Symplectic mechanical systems . . . . .	11
3-1-1 Symplectic manifolds . . . . .	11
3-1-2 Hamiltonian mechanics . . . . .	12
3-2 Contact mechanical systems . . . . .	15
3-2-1 Contact manifolds . . . . .	16
3-2-2 Contact Hamiltonian systems . . . . .	17
3-2-3 Contact geometry in dissipative mechanics . . . . .	19
3-2-4 Symplectification of contact Hamiltonian systems . . . . .	28
3-2-5 The harmonic oscillator with serial damping . . . . .	38
3-3 Jacobi structures for general systems . . . . .	41
3-3-1 Jacobi structures . . . . .	43
3-3-2 Jacobi structure of mechanical systems . . . . .	45

<b>4 Split-Quaternion Representations of Dynamical Systems</b>	<b>51</b>
4-1 The algebra of split-quaternions . . . . .	51
4-1-1 Basic properties of split-quaternions . . . . .	51
4-1-2 Relation with two-dimensional matrix algebra . . . . .	55
4-2 Dynamical systems as split-quaternions . . . . .	58
4-2-1 The algebra of vector fields . . . . .	58
4-2-2 Classification of dynamical systems . . . . .	59
4-2-3 The exponential function of split-quaternions . . . . .	61
4-3 Application to mechanical systems . . . . .	66
4-3-1 System dynamics . . . . .	66
4-3-2 Physical interpretation of the split-quaternion representation . . . . .	68
4-3-3 Geometric analysis of underdamped systems . . . . .	73
<b>5 Conclusion and Recommendations</b>	<b>85</b>
<b>A Geometry of Lagrangian Mechanics</b>	<b>87</b>
A-1 The double tangent bundle . . . . .	87
A-2 The Euler-Lagrange equations . . . . .	89
<b>B Contact Geometry</b>	<b>91</b>
B-1 Contact structures . . . . .	91
B-2 The manifold of contact elements . . . . .	92
B-3 Contact Hamiltonian systems . . . . .	94
B-3-1 Contact Hamiltonian vector fields . . . . .	94
B-3-2 Jacobi brackets . . . . .	97
<b>Bibliography</b>	<b>99</b>
<b>Glossary</b>	<b>105</b>

---

# List of Figures

2-1	Kinetic energy and co-energy in terms of the relation between momentum and velocity. Figure courtesy of B. Krabbenborg [1]. . . . .	6
3-1	On the left, a schematic of the mechanical harmonic oscillator is shown as a mass-spring system with mass $m$ and spring constant $k$ . On the right, the equivalent bond graph representation is shown. It consists of an inductive I-element (mass) and capacitive C-element (spring) connected through a 1-junction, indicating that the 'flow' (i.e. velocity) is constant across the connection, which is to say that both are connected to the same mass. . . . .	14
3-2	Splitting of the tangent and cotangent bundle using the structure provided by the contact form, with $\dim M = 3$ . The left figure shows the cotangent space to $M$ at some point $x$ , where the blue plane contains the semi-basic forms. The other subbundle is of rank 1 and consists of all the scalar multiples of the contact form $\alpha$ . On the right, the tangent space to $M$ at $x$ is depicted, with the red plane containing the horizontal vector fields. The blue line is spanned by the vertical vector fields. . . . .	18
3-3	The left figure shows a schematic of the mechanical damped harmonic oscillator with mass $m$ , spring constant $k$ and damping constant $b$ . The bond graph representation is shown on the right. In addition to the I- and R-element in Figure 3-1, there is now an R-element as well. . . . .	21
3-4	System boundaries of the damper-oscillator system. The mechanical subsystem stores mechanical energy $E$ in the form of kinetic and potential energy, while the heat bath stores internal energy in the form of heat. They interface through the action of the damper. . . . .	22
3-5	Integral curves of $X_K$ for $b = 0.3 \frac{\text{kg}}{\text{s}}$ , $m = 1\text{kg}$ and $k = 10 \frac{\text{kg}}{\text{s}^2}$ . The left plot shows the physical trajectory ( $K = 0$ ) in black, together with some neighboring non-physical trajectories that approach the black trajectory with increasing time. The trajectories on the right are all unphysical, but show the case where we would choose a zero initial value for $U$ (also with some perturbations). The 'wobble' is caused by the exponentially decaying value of the Hamiltonian being counterbalanced with the nonuniform decrease of the mechanical energy in the system. In this case, $U$ clearly does <i>not</i> represent the internal energy of the heat bath, or any other physical variable. . . . .	27

3-6	A contact element on the manifold $Q_e$ is a line through the origin in the tangent space. The manifold of contact elements of $Q_e$ is the space of all contact elements at every tangent space to $Q_e$ . . . . .	29
3-7	Illustration of the principal $\mathbb{R}_x$ -bundle $\dot{T}^*M \xrightarrow{\pi} \mathbb{P}T^*M$ . $x$ is a point in the extended configuration space $Q_e$ , where we attach fibers $\dot{T}_x^*Q_e$ and $\mathbb{P}T_x^*Q_e$ . The orbits of the group action $\mathbb{R}_x$ on $\dot{T}^*Q_e$ are identified by $\sigma$ and mapped to the orbit space $\mathbb{P}T^*M$ . . . . .	32
3-8	Schematic of the harmonic oscillator with two dampers: one in series and one in parallel. The corresponding bond graph representation is shown on the right. . . .	38
3-9	Multi-degree of freedom mechanical system with two masses, two dampers and three springs. The corresponding bond graph representation is shown below. . . .	42
4-1	The dihedral group $D_4$ is the symmetry group of a square. This group is isomorphic to the group formed by $1, \hat{i}, \hat{j}$ and $\hat{k}$ under multiplication. . . . .	52
4-2	Cycle graph of the dihedral group. There are five cycles: one of order four which represents the rotations (or the element $\hat{i}$ ), and four order-2 cycles, which are all the possible reflections. The colored element represents the identity. . . . .	52
4-3	Basis vector fields corresponding to the basis elements of the split-quaternions. . .	59
4-4	The classic Poincaré diagram, based on the conventional classification of fixed points based on the trace and determinant of the state transition matrix $A$ . The corresponding split-quaternion regimes defined in Table 4-2 are displayed as well. The determinant axis coincides with the squared norm of the split quaternion being 0, while all the points on the parabolic line correspond to split-quaternions with zero vector norm. A further distinction is made with the scalar part of the split-quaternion, which, for each of the regimes, determines (asymptotic) (in)stability. . .	60
4-5	Generalized version of a root locus plot in terms of hypercomplex numbers. The traditional root locus is set in the complex $s$ -plane (shown in blue), but we added a third axis for the hyperbolic part of the eigenvalue. When the gain is increased, the initially complex pole pair ventures towards the real axis. If the pole pair is critically damped, both poles are separated from the real axis by an infinitesimal distance of $\varepsilon$ . Increasing the gain even more pushes the pole pair into the hyperbolic regime (the associated split-quaternion vector is now spacelike). Observe that in this picture, the symmetry with respect to the real axis is preserved. In the traditional root locus, these points are projected onto the real axis, indicated by the dashed lines. . . . .	64
4-6	Comparison between the influence of the $\hat{j}$ -component (left) and $\hat{k}$ -component on the integral curves of $X_{\hat{i}}$ . The $\hat{j}$ component stretches the elliptic trajectory along the horizontal axis and squeezes it by the same amount in the vertical direction, while the $\hat{k}$ component does the same but along axes that are rotated by 45 degrees. As shown by the figures, this is a consequence of the constructive or destructive interference of those vector fields in these particular directions. . . . .	70
4-7	The disconnected ‘unit sphere’ in the Lorentzian 3-space. The red surface is the one-sheet hyperboloid, containing all the spacelike unit vectors. The gray sheet is the light cone, that contains all the lightlike ‘null’ vectors with zero norm. Finally, the two blue surfaces constitute the two-sheet hyperboloid. . . . .	72
4-8	Solution trajectories and vector field of an exemplary underdamped system. The red trajectories are the actual solution of the system, while the blue trajectories are the ‘conervative’ version, i.e. with the real part of the split-quaternion removed. . . . .	73

---

4-9	Geometry of the ellipse. Its shape is characterized by two lengths: the semi-major axis $r_+$ and the semi-minor axis $r_-$ . The various measures of the aspect ratio of the ellipse given in Table 4-7 express the relative size of $r_+$ with respect to $r_-$ in various ways. The (first) eccentricity $e$ parameter locates the focal points $F_1, F_2$ as a proportion of the semi-major axis. The angle $\vartheta$ by which the semi-major axis is rotated with respect to the horizontal is called the <i>tilt</i> . . . . .	74
4-10	Illustration of the projection on the Poincaré disk and the Cayley-Klein disk. . . . .	77
4-11	The Cayley-Klein disk and the hyperboloid model are related through gnomonic projection. A geodesic of on the hyperboloid model is shown as well, its projected image in the Cayley-Klein disk (also a geodesic) is a straight line. Illustration adapted from Balazs and Voros [2]. . . . .	78
4-12	The Poincaré disk and the hyperboloid model are related through stereographic projection. A geodesic of on the hyperboloid model is shown as well, its projected image in the Poincaré disk (also a geodesic) is part of a circle that intersects the boundary of the disk at right angles. Illustration adapted from Balazs and Voros [2].	79
4-13	Vector generated by the Lorentzian cross product by $a$ on the vectors in the plane that is Lorentz-orthogonal to $a$ . The positive sheet of the hyperboloid is shown as well. . . . .	81
B-1	The standard contact structure on $\mathbb{R}^3$ , given by the contact form $dz - y dx$ ; the hyperplanes tilt more in the increasing $y$ -direction. . . . .	92
B-2	A point in the manifold of contact elements on $Q = \mathbb{R}^2$ . A coordinate system for $CQ$ consists of $(x_0, x_1)$ to indicate a point on $Q$ , and projective coordinates $[\eta_0 : \eta_1]$ , which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$ , and the remaining coordinate $\eta_1$ covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the ‘hyperplane’ and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$ . . . . .	93
B-3	Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let $(t, q) \in Q$ , and let $v$ be a coordinate for the contact (line) element. The standard contact form is then $dq - v dt$ . On the left, the curve corresponding to a falling object is shown in $Q$ . When this curve is ‘lifted’ to $CQ$ , the contact structure imposes that it be locally tangent to the contact structure, or that $v = \frac{dq}{dt}$ . If the vertical direction is projected down onto the $(q - t)$ -plane ( $C(Q) \rightarrow Q$ ), the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making $v$ the actual velocity of the curve. . . . .	94



---

# List of Tables

2-1	Overview of important analogies between economic engineering and mechanics. . .	10
3-1	Substitution parameters for the harmonic oscillator with serial and parallel damping, shown in Figure 3-8. . . . .	39
4-1	Multiplication table for the split-quaternion algebra. . . . .	51
4-2	All the possible combinations of the regime of a split-quaternion and its vector part. Spacelike split-quaternions can only have a spacelike vector, while lightlike split-quaternions can only have lightlike or spacelike vector parts. . . . .	54
4-3	Overview of the split-quaternion properties that are often used in this chapter. . .	54
4-4	Regime transition under the action of split-quaternion multiplication. The timelike split-quaternions form a group under multiplication, the timelike and spacelike split-quaternions do not: timelike split-quaternions do not have an inverse and the spacelike split-quaternions are not closed. . . . .	56
4-5	Overview of the correspondence between the algebra of split-quaternions and the algebra of $2 \times 2$ matrices, given that $A = \phi(a)$ and $B = \phi(b)$ . In the top section we compare matrices to split-quaternions, i.e. which means that they are related by the isomorphism $\phi$ . The bottom section compares scalar properties: these are numerically equal. . . . .	57
4-6	Overview of the classification of fixed points based on the regime of the associated split-quaternion and its vector part. . . . .	62
4-7	Commonly used parameters to measure the eccentricity/flattening of an ellipse in terms of its semi-major axis $r_+$ and semi-minor axis $r_-$ [3]. . . . .	75



---

# Preface



Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*



---

# Chapter 1

---

## Introduction

Idea: introduction using Arnol'ds thermodynamics quote.



---

## Chapter 2

---

# A Geometric Perspective of Economic Engineering

The discipline of ‘economic engineering’ is a very new one. The theoretical foundations have been developed over the past years at the Delft Center of Systems and Control primarily by prof. em. Mendel, combined with the contributions of several theses that have been recently written about the subject. The purpose of economic engineering is to use tools from various engineering disciplines and physics to improve the predictive power of (macro)economic models.

### 2-1 Symplectic manifolds in economic engineering

The basic elements of an economy are a collection of goods. These can be physical goods (e.g. bushels of wheat) but also more abstract goods such as capital in the financial analogy [4]. The *economic configuration space*  $Q$  is analogous to the configuration space of mechanics, and consists of all the possible combinations of goods in the economy. This may simply be a ‘flat’ space, i.e. a vector space of all the goods (or *stock levels*), but can also be subject to certain (holonomic) constraints. Hence, in economic engineering, the goods are analogous to the (*generalized*) *positions* in classical mechanics [5].

The natural coordinates for this space are the number of each of the goods in the space, denoted by  $q = (q^1, q^2, \dots, q^n)$ . We say that each of these coordinates is measured in units of ‘quantity’, denoted by ‘[#]’. In many cases, the economic configuration manifold is simply a vector space containing all the goods, subject to a total constraint on the total *endowment* for each of the goods (i.e. the total number of goods available). In the two-dimensional case, this is also referred to as the *Edgeworth box*.

The tangent space  $T_x Q$  to a point  $x$  in  $Q$  is the vector space of differential changes in goods, also called the *flow* of goods. A vector in the tangent space is denoted by  $\dot{q} = (\dot{q}^1, \dot{q}^2, \dots, \dot{q}^n)$ . It is very important to make the distinction between a flow of goods and an absolute amount

of goods: from a mathematical perspective, the former is a vector in the tangent space, and the latter is a point in the economic configuration manifold. This distinction is often not given much attention in economics, but it is fundamental in economic engineering.

The *dual space* of  $T_x Q$  is the cotangent space  $T_x^* Q$  containing all the linear functions on vectors in the tangent space to produce a number. This is the natural setting of the *prices* in the economy: as a function, they assign a *value to a change in goods*. These prices are measured in units of currency per quantity, i.e.  $\left[\frac{\text{€}}{\#}\right]$ . A covector in the cotangent space is denoted by  $\mathbf{p} = (p_1, p_2, \dots, p_n)$ . The action of a price covector on a vector measuring change in goods is: (using the index summation convention)

$$\mathbf{p}(\dot{\mathbf{q}}) = p_i \dot{q}^i, \quad (2-1)$$

i.e. it produces the total value associated to this change in goods.

The *cotangent bundle*  $T^* Q$  associated to the economic configuration manifold is the space of prices and quantities: we call it the *economic phase space*. Just like in mechanics, a point in this space determines the state of the economic system: it gives the level of all the goods and their price point.

The cotangent bundle of any manifold has a 1-form canonically defined on it: the Liouville form (also called tautological 1-form or canonical 1-form), defined as

$$\vartheta := p_i dq^i. \quad (2-2)$$

Even more so than in mechanics, this Liouville 1-form has a very intuitive interpretation in economic engineering: it relates the goods and their prices. When integrated over a curve in  $T^* Q$ , it measures the total accumulation of value along that curve.

The (negative of) the exterior derivative of the Liouville form is a *symplectic 2-form*  $\omega$ :

$$\omega := -d\vartheta = dq^i \wedge dp_i. \quad (2-3)$$

When integrated over, this form measures the *oriented area* in the economic phase space, with units of currency [€]. The symplectic 2-form relates every price to the associated quantity and thereby represents the fundamental structure of the economic phase space.

The link between symplectic geometry and economics has been recognized by (more mathematically inclined) researchers outside the field of economic engineering as well, most notably Russell [6] and Swierstra [7].

The role of symplectic geometry in economics begs the question whether the concepts of Hamiltonian mechanics can be applied to economic dynamical systems as well. In the field of economic engineering, we argue that this is indeed the case. To explain how this works, we start from Lagrangian mechanics, which arguably have a slightly more intuitive interpretation, after which we make the transition to Hamiltonian mechanics.

## 2-2 Lagrangian mechanics in economic engineering

The *Lagrangian* is a function on the tangent bundle of the configuration manifold:

$$L : TQ \rightarrow \mathbb{R}.$$

Lagrangian mechanics is based on *Hamilton's principle* which states that the physical motion  $\gamma : [t_0, t_1] \rightarrow TM$  is the one for which the action functional

$$\mathcal{A}[\gamma] = \int_{t_0}^{t_1} (L \circ \gamma) dt, \quad (2-4)$$

is stationary<sup>1</sup>. A necessary and sufficient condition for a curve to satisfy Hamilton's equations is given by the set of  $n$  *Euler-Lagrange equations*:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. \quad (2-5)$$

In economic engineering, Hamilton's principal is interpreted as an economic agent minimizing her *disutility*, in practice often taken to be the *cost*. The interpretation of the Lagrangian function is then the *running cost* or *running disutility*. This Lagrangian therefore has units of currency per time, e.g.  $\left[ \frac{\text{€}}{\text{yr}} \right]$ .

As mentioned, the Lagrangian is a function of the flow of goods  $\dot{q}$  and the stock levels  $q^i$ . The partial derivative of the Lagrangian with respect to the flow of goods is equal to the *price vector*:

$$p_i := \frac{\partial L}{\partial \dot{q}^i}. \quad (2-6)$$

This means that prices and flows of goods are *conjugate variables*.

### 2-2-1 Kinetic energy

When the Lagrangian depends quadratically on the flow of goods (analogous to kinetic energy), the price and the flow of goods are linearly related: this is the typical picture of a supply and demand curve. We have:

$$p_i = m_{ij} \dot{q}^j,$$

where the  $m_{ij}$  are referred to as *price elasticities*. In the simplest of cases, the  $m_{ij}$  form a diagonal matrix and every good simply has its own price. When cross terms appear however, they refer as the marginal elasticity of one good relative to another. These cross terms encode the effect of *substitution* of one good with respect to another.

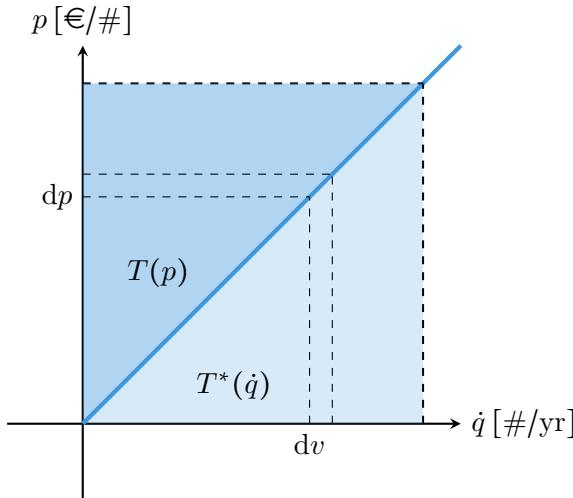
In mechanics, the part of the Lagrangian that depends on  $\dot{q}$  is called the *kinetic co-energy*<sup>2</sup>  $T^*$ :

$$T^* = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j. \quad (2-7)$$

The kinetic co-energy is equivalent to the *cost* associated with a flow of goods: the factor of one half is there to subtract the market surplus from the total expenditure (which is simply equal to  $p(\dot{q})$ ).

<sup>1</sup> Although often referred to as the 'Principle of *minimum action*', this is inaccurate: Hamilton's principle only asserts that the first variation of the functional vanishes, which does not necessarily imply a minimum.

<sup>2</sup> We make the distinction between the kinetic energy  $T$  and co-energy  $T^*$ : they dual are representations related through the Legendre transform. Because work is the integral of force over distance, kinetic energy is naturally represented in terms of *momentum*, rather than velocity, for which we need to take the Legendre transform [8].



**Figure 2-1:** Kinetic energy and co-energy in terms of the relation between momentum and velocity. Figure courtesy of B. Krabbenborg [1].

Traditionally, microeconomics makes the distinction between *firm theory* and *consumer theory*. In economic engineering, we dispense with this explicit distinction, for we argue that they are fundamentally based on the same mathematical principles. However, it is important to keep in mind that the situation for firms and consumers is typically ‘mirrored’: the cost of one represents the surplus of the other, and vice versa. Also, a flow of goods that a consumer buys from a firm has an opposite direction depending on the perspective of the firm or the consumer: this is the reason why a demand curve typically slopes downwards and the supply curve upwards, because their associated surplus is ‘flipped’. In the grand scheme of things, both perspectives are two sides of the same coin. The ‘minimization principle’ can therefore just as well be considered to be a ‘maximization principle’ if the signs flip due to a change in perspective.

If the matrix of elasticities (analogous to the mass matrix in economics) is positive definite, it serves as a *Riemannian metric* on the economic configuration manifold. Level lines of the metric are *indifference curves*: they represent constant level of (dis)utility for different combinations of flows of goods. Furthermore, in more complicated situations, the price elasticities are allowed to vary with the stock levels, i.e.  $m = m(q^1, q^2, \dots, q^n)$ , which would (under appropriate conditions) imply that the configuration manifold is *curved* in the Riemannian sense.

In the language of bond graphs, we say that the kinetic energy (or market surplus) is stored in an I-element. As such, in economic engineering, I-elements represent a local ‘piece’ of demand or a ‘market’ where exchange of goods can occur.

### 2-2-2 Potential energy

In mechanics, the part of the Lagrangian that does not depend on  $\dot{q}$  is called the *potential energy*  $V = V(q^1, q^2, \dots, q^n)$ . In economic engineering, we say that potential energy either represents the *benefits* of holding a good, also called *convenience yield*. From the perspective of the firm, the benefits of holding are typically opposite (e.g. storage costs). When the

potential energy is quadratic in the  $q^i$ 's, we have something like a spring force acting on the system. In a bond graph, this is called a C-element: it measures or stores<sup>3</sup> an amount of goods.

### 2-2-3 Symplectic geometry in Lagrangian mechanics

We will now give a brief description of the differential geometric infrastructure underpinning Lagrangian mechanics and its economic interpretation. For a more detailed account of the geometry of Lagrangian mechanics, the reader is referred to Appendix A.

The geometry of Lagrangian mechanics is based on the geometry of the so-called *double tangent bundle*  $TQ$ . This is because we consider the amount of goods  $q^i$  and the flow of goods  $\dot{q}^i$  to be separate coordinates, but of course the flow of goods  $\dot{q}^i$  should *also* be the time rate of change of the associated amount of goods<sup>4</sup>. This extra constraint pairing the  $q^i$ 's with their associated  $\dot{q}^i$  is embedded in the canonical structure of the double tangent bundle. Said otherwise, the significance of this structure is equivalent to Newton's law being second-order.

The pairing of flows and amounts of goods is specified by the *vertical isomorphism*, which is a tensor of valence  $(1, 1)$  on  $TQ$ : [9]

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i. \quad (2-8)$$

Using the vertical isomorphism, the *Lagrange 1-form* is defined as

$$\vartheta_L := dL \circ S = \frac{\partial L}{\partial \dot{q}^i} dq^i. \quad (2-9)$$

The Lagrange 1-form plays an equivalent role to the Liouville form in the previous section. Recall that  $\frac{\partial L}{\partial \dot{q}^i}$  are equal to the prices, so  $\vartheta_L$  computes the 'valuation' of a flow of goods based on the price levels dictated by the Lagrangian.

Similarly, the negative of the exterior derivative of  $\vartheta_L$  yields the *Lagrange 2-form*  $\omega_L$ :

$$\omega_L := -d\vartheta_L = \frac{\partial^2 L}{\partial v^i \partial v^j} dq^j \wedge dv^i + \frac{\partial^2 L}{\partial q^i \partial v^j} dq^j \wedge dq^i. \quad (2-10)$$

This 2-form has a compelling economic interpretation: the first term contains the *price elasticities* associated to each of the goods in the economy. The second term contains the dependency of the price elasticities on the amounts of goods. We propose here that this 2-form can be seen as an alternative statement to the *Slutsky matrix* in microeconomics, which relates the total elasticity to two factors: [10]

- (i) the elasticity due to *substitution* (i.e. the local elasticities), represented by the first term of the above equation,

---

<sup>3</sup>With 'measuring' we mean that this element makes a particular quantity part of the dynamics of the system, which is to say that they are measurable. For example, if there are no C-elements, one can still conceptualize the 'amount of goods', but they do not influence the system and can therefore not be measured.

<sup>4</sup>In this context, the 'dot' simply distinguishes the coordinates, but does not intrinsically imply that one is the time derivative of the other.

- (ii) the *wealth effects*: these are the changes in the elasticities due to the changing state of the economic system (i.e. the accumulation of goods).

We will, however, not go into the implications of this (potential) relation, and reserve them as a recommendation for future research in the theory of economic engineering.

The Lagrange 2-form can be used to state the Euler-Lagrange equations in a geometric language. Define the *energy function*  $E$  as<sup>5</sup>

$$E := \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L. \quad (2-11)$$

From the perspective of the firm, the energy function is analogous to *profit*, since it is equal to the Lagrangian (being the cost function) subtracted from the total revenue of the firm  $\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i$ .

The dynamics of the mechanical system are then given by the *Lagrangian vector field*  $X_L$ , which is defined

$$X_L \lrcorner \omega_L = dE, \quad (2-12)$$

or alternatively  $X_L = \omega_L^\sharp(dE)$ .

The Lagrange 2-form is a *symplectic form* if the matrix of price elasticities is regular: in economics this means that there are no ‘zero price directions’; i.e. there every change in the amounts of goods is associated with some change in value. Usually, an even stronger condition is assumed on  $L$ , namely that it is *convex* in the flows of products, which is sometimes known as the *Legendre condition*. In economics, this means that the prices are in the same direction as the goods<sup>6</sup>. If this is indeed the case, the *Legendre transform* can be used to pass from the Lagrangian representation to the Hamiltonian representation and back, this is the subject of the next section.

## 2-3 Hamiltonian mechanics in economic engineering

### 2-3-1 The Legendre transform

When the economic Lagrangian function is convex in product flow variables,  $\dot{q}$  we can use the *Legendre transform* to pass to the Hamiltonian formalism as follows [11]

$$H = E \circ (\mathcal{F}L)^{-1}, \quad (2-13)$$

where  $E$  is the previously defined energy function and  $\mathcal{F}L$  is the *fiber derivative* of  $L$ : [12]

$$\mathcal{F}L : TQ \rightarrow T^*Q : [\mathcal{F}L(\mathbf{v})](\mathbf{w}) = \left. \frac{d}{ds} \right|_{s=0} L(\mathbf{v} + s\mathbf{w}), \quad (2-14)$$

where  $\mathbf{v}, \mathbf{w} \in T_q Q$ . The fiber derivative maps a flow of goods to the associated price vector. As a result, Hamiltonian function is simply equal to the energy function  $E$ , but expressed in

<sup>5</sup>The coordinate-free definition involves the Liouville vector field, and is given in Appendix A.

<sup>6</sup>This makes intuitive sense, although there are exceptions; for example, the negative yield on some government bonds in recent years.

terms of the price coordinates  $p_i$  rather than  $\dot{q}^i$ . The fiber derivative is a local diffeomorphism only if the Lagrangian is regular.

Since the energy function and the Hamiltonian are numerically equal, the economic interpretation of the Hamiltonian is also *profit* (from the perspective of the firm). This fact has been recognized by economists as well, in the so-called *duality theory*. The duality theory states that *profit function* and *cost function* are dual representations of the same firm (in literature, this is called a *technology*) [10, 13].

### 2-3-2 Hamilton's equations

In their traditional form, Hamilton's equations are:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (2-15)$$

It is easy to see that these equations of motion are equivalent to the Euler-Lagrange equations, given that  $p_i = \frac{\partial L}{\partial \dot{q}^i}$ . We will go a lot more into detail on the (geometric) nature of Hamilton's equations in Chapter 3; the discussion here is limited to their interpretation in economic engineering.

The first Hamilton equation is equivalent to *Hotelling's lemma*, which states that the demand for a particular good is that the supply (i.e.  $\dot{q}^i$ ) of a good  $i$  is equal to the derivative of the profit function with respect to the price  $p_i$ . It is usually emphasized that this is only the case when the price is the *optimal* price for profit, but this is built into the definition of  $p_i$ , since  $p_i = \frac{\partial L}{\partial \dot{q}^i}$  is exactly the value for  $p_i$  that maximizes the profit function

$$p_i \dot{q}^i - L,$$

if  $L$  is convex in  $\dot{q}^i$ .

The second Hamilton equation provides to the change in momentum or *force*. In economic engineering, these forces are analogous to *economic wants*, i.e. what drives a price up or down. The equation can be interpreted as the *law of scarcity*, i.e. economic wants arise as a consequence of the stock levels / amount of goods in the system, pushing the price of the goods up or down.

### 2-3-3 The role of dissipation

Traditional Hamiltonian and Lagrangian mechanics deal with conservative systems only; those that conserve the value of the total Hamiltonian. In economics, these would be systems that would, without external sources, always maintain the same level of profit. These type of conservative economic systems are just as hypothetical as their mechanical counterpart, since there *always* are elements present that limit the 'efficiency' of an economic system, such as depreciation of goods, consumption behavior, 'transaction costs', etc.

Given the prominent role that the Hamiltonian and Lagrangian formalism play in the field of economic engineering, it is therefore essential to find a suitable way to include dissipative effects in the Hamiltonian and Lagrangian formalism (or an extension thereof), while still

maintaining their economic interpretation. This is the subject of Chapter 3, and although it is formulated in terms of classical mechanics, the analogies described in this chapter may be readily used to substitute the mechanical quantities for their economic engineering counterpart. To that end, Table 2-1 provides an overview of the most important analogies used in economic engineering.

**Table 2-1:** Overview of important analogies between economic engineering and mechanics.

Symbol	Mechanical variable	Units	Economic variable	Units
$q$	Displacement	[m]	Quantity of goods	[#]
$\dot{q}$	Velocity	[ $\frac{m}{s}$ ]	Flow of goods	[ $\frac{\#}{yr}$ ]
$p$	Momentum	[ $\frac{kg\ m}{s}$ ]	Quantity of goods	[ $\frac{\epsilon}{\#}$ ]
$F$	Force	[ $\frac{kg\ m}{s^2}$ ]	Economic want	[ $\frac{\epsilon}{\#\ yr}$ ]
$m$	Mass	[kg]	Price elasticity	[ $\frac{\epsilon_{yr}}{\#^2}$ ]
$H$	Hamiltonian / total energy	[J]	Profit	[ $\frac{\epsilon}{yr}$ ]
$L$	Lagrangian	[J]	Running cost / disutility	[ $\frac{\epsilon}{yr}$ ]

---

# Chapter 3

---

# Geometric Structures in Dissipative Mechanics

[intro here]

## 3-1 Symplectic mechanical systems

### 3-1-1 Symplectic manifolds

In the traditional sense, Hamiltonian mechanics takes place on *symplectic manifolds*. A symplectic manifold  $(M, \omega)$  is a smooth manifold  $M$  equipped with a *closed, nondegenerate* 2-form  $\omega$ . Because  $\omega$  must be nondegenerate, symplectic manifolds are necessarily even-dimensional.

The celebrated Darboux theorem asserts that locally, all symplectic manifolds of the same dimension (say  $2n$ ) are all symplectomorphic to each other. As a result, we define the prototypical symplectic 2-form that serves as a representative for *all* symplectic structures of that dimension as

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i , \quad (3-1)$$

where  $p_i$  and  $q_i$  are coordinates for the manifold  $M$ . A coordinate chart in which the symplectic 2-form has the above form is called a *Darboux charts*, and the associated coordinates *Darboux coordinates* [14, 15].

In mechanics, the *configuration manifold*  $Q$ : is the manifold specified by all the possible generalized positions  $q_i$  (or configurations) of the mechanical system. The *generalized momenta* associated with each of the generalized positions live in the collection of cotangent spaces to the configuration manifold. This is because from the Lagrangian viewpoint the generalized

momenta are given by<sup>1</sup>

$$p_i = \frac{\partial L}{\partial q_i},$$

which indicates that the vector of  $p_i$ 's is a cotangent (covariant) vector to  $Q$ . Hence, the *cotangent bundle* of the configuration manifold contains all the possible position and momentum pairs; it is colloquially called the *phase space* [11, 15, 16].

The structure that associates each position with its corresponding momentum is given by the *Liouville 1-form*<sup>2</sup>  $\vartheta$  on  $T^*Q$ . The Liouville form is defined at every point  $(q_1, \dots, q_n, p_1, \dots, p_n) \in T^*M$  as

$$\vartheta = \sum_{i=1}^n p_i dq_i. \quad (3-2)$$

Hence, the Liouville form tells us which momentum coordinate corresponds to a given position coordinate and vice versa. This turns out to be an essential piece of the geometric structure that underpins classical mechanics.

Every cotangent bundle is canonically endowed with a Liouville form. The exterior derivative of the Liouville form produces a *symplectic 2-form*. By convention, we define this symplectic form as follows:<sup>3</sup>

$$\omega = -d\vartheta = \sum_{i=1}^n dq_i \wedge dp_i.$$

Hence, the space of generalized positions and momenta (i.e. the cotangent bundle the configuration manifold  $Q$ ) is canonically symplectic. The symplectic structure pairs the corresponding position and momentum coordinates in a skew-symmetric fashion.

In the context of bond graphs, the symplectic form represent the dual nature of a bond. That is to say, a bond represents an exchange of both an effort and a flow, and they are inherently tied to each other. The flow is a change in position or generalized velocity, and the effort is a change in momentum. The effort and flow associated to a bond are conjugate: the symplectic form provides precisely the structure that is visually present in a bond graph (e.g. Figure 3-1).

### 3-1-2 Hamiltonian mechanics

The idea of Hamiltonian mechanics is that the equations of motion are generated by *Hamilton's equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

given the Hamiltonian function  $H$ , which is equal to the mechanical energy in the system. Observe that the above equation assumes that the pairing between the positions and momenta is known a priori.

---

<sup>1</sup>In coordinate-free language, the Liouville 1-form is defined pointwise through its action on a tangent vector  $\xi$  to  $T^*Q$  as follows

$$\vartheta(\xi) = (x \circ \pi_*)(\xi).$$

Here  $\pi_*$  is the pushforward of the bundle projection map  $\pi : T^*Q \rightarrow Q : (\mathbf{q}, \mathbf{p}) \mapsto \mathbf{q}$ . The point  $x \in T^*Q$  is interpreted as a map on the tangent space to  $\pi(x)$  on the base manifold.

<sup>2</sup>The Liouville 1-form makes its appearance in literature under a myriad of names, such as the canonical 1-form, tautological 1-form, Poincaré 1-form or the symplectic potential.

<sup>3</sup>In this text, the 'q-first' sign convention used by Abraham and Marsden [11] and Cannas da Silva [14] is observed and maintained in the following sections concerning contact and Jacobi manifolds.

In the language of differential geometry, Hamilton's equations are specified by a symplectic structure on  $T^*Q$  and an appropriate Hamiltonian function on that manifold: the pairing between the positions and momenta is therefore built in. A generic Hamiltonian system is a triple  $(M, \omega, H)$ , where  $(M, \omega)$  is a symplectic manifold. In mechanics, we have that  $M = T^*Q$ .

### The Hamiltonian isomorphism for symplectic manifolds

To produce the equations of motion, the symplectic structure provides a mapping between the smooth functions on the manifold and the Hamiltonian vector fields on the manifold<sup>4</sup>. First, define the mapping

$$\omega^\flat : TM \rightarrow T^*M : X \mapsto X \lrcorner \omega. \quad (3-3)$$

Because  $\omega$  is nondegenerate by definition, the mapping  $\omega^\flat$  is an isomorphism. Thus, the inverse mapping is well-defined, and is denoted by  $\omega^\sharp$  [16].

In the notation used here, the difference between the manifolds  $Q$  and  $M$  is crucial. In the context of mechanics, we have that the symplectic manifold  $M$  is the cotangent bundle of  $Q$ . Hence, since the Hamiltonian is a function on  $M = T^*Q$ ,  $dH$  and  $X_H$  are sections of  $T^*(T^*Q)$  and  $T(T^*Q)$  respectively. This is illustrated by the diagram below (the projection arrows from  $T^*(T^*Q)$  and  $T(T^*Q)$  indicate the bundle structure but are left unnamed).

$$\begin{array}{ccc} T^*(T^*Q) & \xrightleftharpoons[\omega^\flat]{\omega^\sharp} & T(T^*Q) \\ & \searrow & \swarrow \\ & T^*Q & \\ & \downarrow \pi & \\ & Q & \end{array}$$

This isomorphism specified by  $\omega$  allows us to find the corresponding Hamiltonian vector field  $X_H$  to a Hamiltonian function, given by:

$$X_H = \omega^\sharp(dH). \quad (3-4)$$

In (Darboux) coordinates, the action of  $\omega^\sharp$  on the basis 1-forms is

$$dp_i \mapsto \frac{\partial}{\partial q_i} \quad dq_i \mapsto -\frac{\partial}{\partial p_i}.$$

The minus sign arises as a consequence of the anticommutativity of the wedge product in appearing in  $\omega$ .

A classical example of this formalism is the harmonic oscillator (undamped) shown in Figure 3-1. The Hamiltonian function is the sum of the potential and kinetic energy in the system

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2,$$

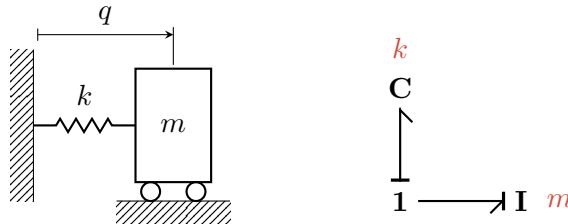
<sup>4</sup>These are a special class of vector fields on  $M$  that arise as a result of this mapping.

where  $m$  is the mass and  $k$  the spring constant. The Hamiltonian vector field is then

$$X_H = \omega^\sharp(dH) = \omega^\sharp\left(\frac{p}{m}dp + kqdq\right) = \frac{p}{m}\frac{\partial}{\partial q} - kq\frac{\partial}{\partial p},$$

or stated as a system of differential equations

$$\dot{q} = \frac{p}{m} \quad \dot{p} = -kq.$$



**Figure 3-1:** On the left, a schematic of the mechanical harmonic oscillator is shown as a mass-spring system with mass  $m$  and spring constant  $k$ . On the right, the equivalent bond graph representation is shown. It consists of an inductive I-element (mass) and capacitive C-element (spring) connected through a 1-junction, indicating that the ‘flow’ (i.e. velocity) is constant across the connection, which is to say that both are connected to the same mass.

## Poisson brackets

The symplectic form endows the manifold  $M$  also with a *Poisson structure*, i.e. a Lie algebra structure on the vector space space of functions on  $M$ . The commutator of this algebra structure is the *Poisson bracket*,

$$\begin{aligned} \{ , \} : C^\infty(M) \times C^\infty(M) &\rightarrow C^\infty(M) : \quad \{f, g\} = \omega(\omega^\sharp df, \omega^\sharp dg) \\ &= \omega(X_f, X_g) \\ &= \mathcal{L}_{X_f} g \end{aligned} \tag{3-5}$$

$$\text{(Darboux coordinates)} \quad = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Poisson brackets are anticommutative, bilinear and satisfy the Jacobi identity. Additionally, they also satisfy the *Leibniz property*,

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

The Poisson brackets defined in terms of the symplectic structure make symplectic manifolds into Poisson manifolds. In Sections 3-2 and 3-3 the notion of Poisson manifolds is generalized to *Jacobi manifolds* to cover more general mechanical systems. In contrast to the Poisson structure, the Jacobi structure does not have the Leibniz property [15, 16].

The usefulness of Poisson brackets is due to the fact that they provide a convenient way to calculate the time-rate of change of an observable  $f$ :

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

If the Hamiltonian does not depend on time, it is conserved under its own Hamiltonian vector field. This is easily seen from the anticommutativity of Poisson brackets ( $\{H, H\} = 0$ ). Therefore, Hamiltonian systems conserve energy; they do not allow for dissipative (friction) forces in a straightforward manner, unless included in the form of an explicit time dependence. This is a direct consequence of the symplectic structure: *because* the  $X_H$  is generated by the symplectic 2-form, the Hamiltonian vector field conserves its generating function. The fact that the system is conservative should therefore be seen as something that is built into the structure of the symplectic Hamiltonian system itself, and not as emergent.<sup>5</sup>

To conclude, the overall structure that constitutes a conservative mechanical system is three-fold: first, there is the configuration manifold and its cotangent bundle. Secondly, the symplectic structure on that manifold, and thirdly, we have the Hamiltonian function. In principle, the symplectic structure is a canonical consequence of the cotangent bundle structure, but we wish to emphasize that the system dynamics are also symplectic. That is to say, the Hamiltonian vector field is symplectic: it leaves  $\omega$  invariant. The symplectic nature of the dynamics does not persist to the extension for systems with dissipation. This is in contrast to the fact that even for the most general systems, we still require the pairing of conjugate variables to be encoded into the geometric structure of the system. Hence, we can expect the symplectic structure to remain important even in the upcoming generalizations.

In the next section we will extend the Hamiltonian formalism to *contact manifolds* to incorporate dissipation in the Hamiltonian system.

## 3-2 Contact mechanical systems

[Part of the intro] In this section, the thermodynamic principles and their relation with contact geometry are used to establish a contact Hamiltonian system for the damped harmonic oscillator. In contrast to the conservative system discussed in Section 3-1 (cf. Figure 3-1), a dissipative element is now present in the system. This precludes the damped harmonic oscillator from being modeled by a symplectic Hamiltonian system that is not explicitly time-dependent. Time-dependence indicate a nonautonomous system, and they are typically reserved for either external control inputs or disturbance inputs. The control inputs are meant to drive the system to a specific state. The disturbance input are (potentially stochastic) external inputs that cannot be controlled but influence the system in some specified manner.

What both disturbances and control inputs have in common, is that they are inherently *exogenous*: they are not part of the system itself. In contrast, the dissipative element in the form of the damper *is* part of the system (endogenous). From both conceptual and

---

<sup>5</sup>This is especially clear from the explicit coordinate expressions of the dynamics. The mapping from  $H$  to  $X_H$  takes the partial derivatives, and switches around them between the associated  $q$ 's and  $p$ 's, while one of them picks up a minus sign. As such, it is very clear that  $\mathcal{L}_{X_H} H = X_H(H) = 0$ , given that  $H$  is not explicitly time-dependent.

practical standpoint, modeling dissipation as a time-dependence, and therefore and exogenous phenomenon, is not desirable. This is why, in this section, we aim to use contact geometry to include the dissipation as an intrinsic component of the overall system.

### 3-2-1 Contact manifolds

In contrast to symplectic manifolds, contact manifolds are odd-dimensional. A contact manifold  $(M, \xi)$  is a smooth manifold  $M$  of dimension  $2n + 1$  equipped with a maximally non-integrable hyperplane distribution  $\xi$ . That is to say, at every point  $x \in M$  the contact structure specifies a  $2n$ -dimensional linear subspace (i.e. a hyperplane) of  $TM$ . Locally<sup>6</sup>, the hyperplane distribution is specified as the kernel of a 1-form on  $M$ , which must be nondegenerate:<sup>7</sup> [14, 15, 17]

$$\xi|_x = \ker \alpha|_x.$$

It is worth pointing out that the correspondence between a hyperplane and the kernel of a 1-form is not one-to-one. Indeed, multiplying  $\alpha$  by any nonzero function yields a different 1-form with the same kernel. The contact forms are different, but they give rise to the same contact structure. This ambiguity is very important, and will play a vital role in the process of symplectification discussed in Section 3-2-4.

Nonintegrability of the hyperplane distribution means that we cannot find codimension-1 foliations that are everywhere tangent to the distribution of hyperplanes. This is analogous to a nonholonomic constraint on a mechanical system: these constraints cannot be integrated to obtain a submanifold of the configuration space that contains all the allowable positions. Indeed, the condition for nonholonomicity applies here as well: for  $\xi$  to be nonintegrable, the associated contact form  $\alpha$  must satisfy the Frobenius condition

$$\alpha \wedge d\alpha \neq 0,$$

or equivalently, that  $\alpha \wedge (d\alpha)^n$  is a volume form on  $M$ .

Contact geometry is closely related to symplectic geometry, for the nonintegrability condition implies that  $d\alpha$  implies that  $d\alpha|_{\xi}$  is a symplectic form. There also an extension of the Darboux theorem to contact manifold, which says that locally, every contact form can be written as

$$dq_0 - \sum_i p_i dq_i, \tag{3-6}$$

the coordinates  $(q_0, q_1, \dots, q_n, p_1, \dots, p_n)$  are then called *Darboux coordinates*.

For a slightly more comprehensive introduction to contact geometry, the reader is referred to Appendix B. More extensive literature are, among others, the works of Geiges [17], Libermann and Marle [16], Arnol'd [15, 18] and Godbillon [19].

---

<sup>6</sup>Contact structures which are globally defined by a 1-form are called *exact* or *strictly* contact structures. This is the case when the quotient line bundle  $TM/\xi$  is orientable.

<sup>7</sup>Equations of the form  $\alpha = 0$ , where  $\alpha$  is a 1-form, determine so-called *Pfaffian equations* [16].

### 3-2-2 Contact Hamiltonian systems

Similar to symplectic Hamiltonian systems, a *contact Hamiltonian system* needs three ingredients: a smooth manifold  $M$ , a contact form  $\alpha$  on that manifold, and a Hamiltonian function  $K$  on the manifold. The contact structure then provides a mapping between the smooth functions on the manifold and the contact Hamiltonian vector fields on the manifold. As such, the contact structure generates the contact version of Hamilton's equations.

The mapping  $\Psi_\alpha$  that relates the smooth functions and contact Hamiltonian vector fields, given a contact 1-form  $\alpha$ , is defined as follows:

$$\Psi_\alpha : \mathfrak{X}_c(X)M \rightarrow C^\infty(M) : X_K \mapsto K = -X_K \lrcorner \alpha, \quad (3-7)$$

where  $\mathfrak{X}_c(X)M$  is the collection of infinitesimal strict contactomorphisms. These are vector fields that preserve the strictly contact structure specified by  $\alpha$ , and are subject to the following condition:

$$\mathcal{L}_{X_K} \alpha = s\alpha, \quad (3-8)$$

where  $s$  is an arbitrary smooth function on  $M$ . This condition is based on the fact any nonzero multiple of a given contact form determines the same contact structure.

#### Horizontal and vertical vector fields

To obtain the vector field from a Hamiltonian function, we are interested in the inverse mapping  $\Psi_\alpha^{-1}$ . This mapping is not quite straightforward, for it has to map the general class of smooth functions back to a very *specific* subclass of vector fields. The trick is to use a splitting of the (co)tangent bundle, decomposing the vector field into two components.

The nonintegrability condition on the contact structure ensures that  $\alpha \wedge d\alpha \neq 0$ . As such, at every point, we have at any  $x \in M$ :

$$\ker \alpha|_x \cap \ker d\alpha|_x = \{\mathbf{0}\},$$

i.e. they can only coincide at the origin. Consequently, we can define the following splitting of the tangent bundle:

$$TM = \ker \alpha \oplus \ker d\alpha,$$

where  $\oplus$  denotes the Whitney sum<sup>8</sup>. Vector fields that are in the kernel of  $\alpha$  are called *horizontal*; they form a subbundle of rank  $2n$ , which coincides with the hyperplane distribution specified by the contact structure. In contrast, *vertical* vector fields are in the kernel of  $d\alpha$ , which is a subbundle of rank 1 [16].

The *Reeb vector field*  $R_\alpha$  associated with  $\alpha$  is a ‘unit’ vertical vector field, defined by the conditions<sup>9</sup>

$$R_\alpha \lrcorner \alpha = 1 \quad R_\alpha \lrcorner d\alpha = 0.$$

<sup>8</sup>The Whitney sum applies to vector bundles, roughly speaking, a ‘fibered’ version of the direct sum for vector spaces. The Whitney sum of two vector bundles (over the same base space) is a vector bundle over that base space, with every fiber being the direct sum of the fibers of the original vector bundles.

<sup>9</sup>The Reeb vector field is not uniquely associated with a contact structure, and depends on the particular choice of contact form.

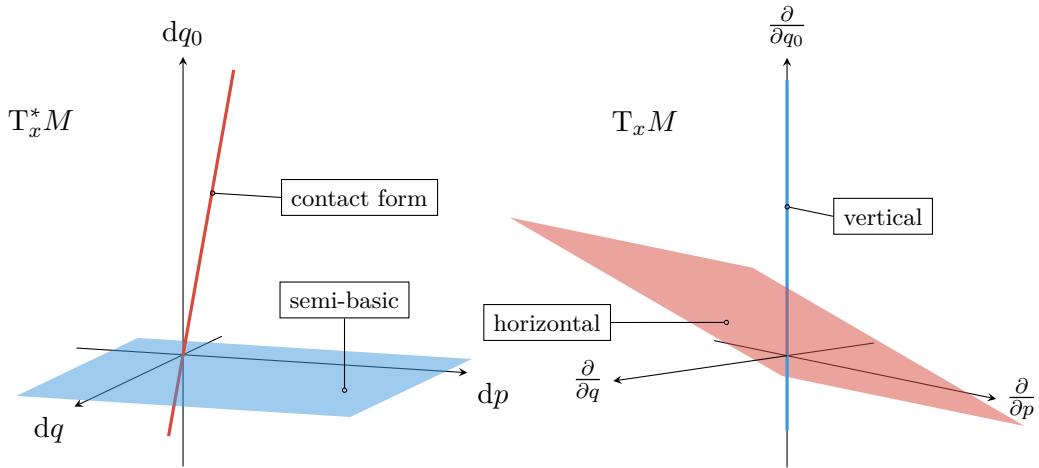
In the Darboux coordinates as given in Equation (3-6), the Reeb vector field is

$$R_\alpha = \frac{\partial}{\partial q_0}.$$

Because the space of vertical vector fields is of rank 1, every vertical vector field is colinear with the Reeb vector field.

An arbitrary vector field  $X \in \mathfrak{X}(M)$  can be canonically decomposed into a horizontal and a vertical component as follows:

$$X = \underbrace{(X \lrcorner \alpha)R_\alpha}_{\text{vertical}} + \underbrace{X - (X \lrcorner \alpha)R_\alpha}_{\text{horizontal}}. \quad (3-9)$$



**Figure 3-2:** Splitting of the tangent and cotangent bundle using the structure provided by the contact form, with  $\dim M = 3$ . The left figure shows the cotangent space to  $M$  at some point  $x$ , where the blue plane contains the semi-basic forms. The other subbundle is of rank 1 and consists of all the scalar multiples of the contact form  $\alpha$ . On the right, the tangent space to  $M$  at  $x$  is depicted, with the red plane containing the horizontal vector fields. The blue line is spanned by the vertical vector fields.

### Semi-basic forms

Similar to the splitting the tangent bundle, the cotangent bundle may be decomposed into two subbundles as well. The first subbundle consists of the annihilators of the horizontal vector fields. This is a subbundle of rank 1 that contains all the multiples of the contact form; they generate the contact structure. The other subbundle contains the *semi-basic forms*, which annihilate the vertical vector fields [16].

Any 1-form  $\zeta \in \Gamma(T^*M)$  can be canonically decomposed into a semi-basic component and a multiple of  $\alpha$  as follows:

$$\zeta = \underbrace{(R_\alpha \lrcorner \zeta)\alpha}_{\text{contact struct.}} + \underbrace{\zeta - (R_\alpha \lrcorner \zeta)\alpha}_{\text{semi-basic}}. \quad (3-10)$$

In Darboux coordinates, semi-basic forms are forms that have no component in  $dq_0$ .

The decompositions of  $T^*M$  and  $TM$  for the three-dimensional case are shown in Figure 3-2. They also play an important role in Section 3-3.

### The Hamiltonian isomorphism for contact manifolds

To find the Hamiltonian vector field  $X_K$  associated to a Hamiltonian  $K$ , we decompose  $X_K$  into a horizontal and a vertical component:

$$X_K = X_K^{\text{hor}} + X_K^{\text{ver}}.$$

The vertical component of  $X_K$  is easily obtained from the definition of  $\Psi_\alpha$ :

$$X_K^{\text{ver}} = -KR_\alpha, \quad (3-11)$$

where  $R_\alpha$  is the *Reeb vector field*<sup>10</sup> associated to the contact form  $\alpha$ .

Finding the horizontal component is more involved; a detailed account of the required technicalities is given in Appendix B. In short, we again need a mapping similar to the one defined in Equation (3-3), but now defined in terms of  $d\alpha$  instead:

$$d\alpha^\flat(X) := X \lrcorner d\alpha. \quad (3-12)$$

However, this is not an isomorphism between  $TM$  and  $T^*M$ , for it will annihilate any vertical component  $X$ . However, it *is* an isomorphism from the horizontal vector fields to the semi-basic forms (respectively the red and blue planes in Figure 3-2). Likewise, the inverse mapping  $d\alpha^\sharp$  takes a semi-basic form as an argument and produces a vertical vector field.

The horizontal component of the Hamiltonian vector field is equal to this mapping applied to  $dK$ , canonically projected to the space of semi-basic forms (cf. Equation (3-10)):

$$X_K^{\text{hor}} = d\alpha^\sharp(dK - (R_\alpha \lrcorner dK)\alpha). \quad (3-13)$$

Hence, the Hamiltonian vector field is equal to

$$X_K = \Psi_\alpha^{-1}(K) = KR_\alpha + d\alpha^\sharp(dK - (R_\alpha \lrcorner dK)\alpha). \quad (3-14)$$

To apply the contact Hamiltonian formalism to dissipative mechanical systems, we first require a manifold with a suitable contact structure. This contact structure is derived from the principles of thermodynamics in the next section. Subsequently, the contact Hamiltonian system and the equations of motion are set up in Section 3-2-3.

#### 3-2-3 Contact geometry in dissipative mechanics

As mentioned the contact structure will be derived based on thermodynamic reasoning. Therefore, the next section first discusses the traditional role of contact geometry in thermodynamics, after it will be applied to dissipative mechanics.

---

<sup>10</sup>The Reeb vector field is defined by two conditions: [16]

$$R_\alpha \lrcorner \alpha = 1 \quad R_\alpha \lrcorner d\alpha = 0.$$

In the Darboux coordinates as given in Equation (3-6), the Reeb vector field has the form

$$R_\alpha = \frac{\partial}{\partial q_0}.$$

## Contact geometry in classical thermodynamics

It has been argued in the past by several authors that contact geometry is the natural framework for thermodynamics by i.a. Arnol'd [15, 18, 20, 21], Bamberg and Sternberg [22], Burke [23] and Hermann [24], ultimately leading back to the seminal work of Gibbs [25]. It is commonly seen as a testament to the brilliance of Gibbs' work that he managed to recognize and describe the correct geometric framework well before the required mathematical infrastructure came to invention [26]. In recent years, the contact Hamiltonian formalism has been successfully applied to thermodynamic theory by e.g. Mrugała et al. [27], Mrugała [28, 29, 30, 31, 32], Balian and Valentin [33], van der Schaft [34], van der Schaft and Maschke [35], Maschke and van der Schaft [36], Bravetti et al. [37], and Simoes et al. [38].

Contact geometry arises in thermodynamics as a consequence of the first law, which asserts that the change in internal energy of the system is equal to the difference between the heat added *to* the system and the work performed *by* the system.

**The first law of thermodynamics** To state the first law in the language of exterior forms, define the 1-forms  $\eta$  and  $\beta$  as the differential amounts of heat and work (in respective order) added to the system.  $\eta$  and  $\beta$  are 1-forms that are generally *not* closed [22, 39]. However, the first law states that the difference between them *is* a closed form. Locally, this closed form can be written as the gradient of a function called the *internal energy*  $U$ . Hence, we state the first law as<sup>11</sup>:

$$dU = \eta - \beta. \quad (3-15)$$

This equation can be equivalently expressed as the fact that the 1-form

$$\alpha = dU - \eta + \beta \quad (3-16)$$

should pull back to zero over the physical trajectories of the systems.

**The Gibbs 1-form** For the purposes of illustration, we now apply this concept to what is arguably the most simple thermodynamic system: the ideal gas in a piston.

The ideal gas is characterized by five thermodynamic properties: the temperature  $T$ , entropy  $S$ , volume  $V$ , pressure  $P$  and the internal energy  $U$ . We call the five-dimensional space containing all the possible states the *thermodynamic phase space*.

For the ideal gas, we have that the work done by the system is equal to the pressure multiplied with the change in volume, i.e.  $\beta = P dV$ . Furthermore, heat added to the system is given by the temperature multiplied by the change of energy:  $\eta = T dS$  [21, 22, 26]. Therefore, Equation (3-16) becomes

$$\alpha_G := dU - T dS + P dV, \quad (3-17)$$

which is called the *Gibbs form* (hence the subscript). It is clear that the Gibbs form defines a contact structure on the thermodynamic phase space (these are Darboux coordinates). Along the physically allowable trajectories, the Gibbs form must pull back to zero.

---

<sup>11</sup>By using differential forms, the inexactness of the heat and work differentials need not be explicitly emphasized using additional notation such as  $\delta$  or  $d$ .

**Legendre submanifolds** In contact geometry, submanifolds on which the contact form vanishes everywhere are called *Legendre submanifolds*. As such, these submanifolds are vital in thermodynamics, because they contain the allowable states (Balian and Valentin [33] call them thermodynamic manifolds). Due to the nonintegrability condition on the contact structure, Legendre submanifolds have at most dimension  $n$ , if the overall contact manifold is of dimension  $2n + 1$ .

For the ideal gas, the Legendre submanifolds are two-dimensional. They can be computed explicitly by integrating the Gibbs form. To do so, we need two additional *equations of state*,

$$U = c n_s R_g T \quad P V = n_s R_g T, \quad (3-18)$$

where  $n_s$  is the amount of substance,  $R_g = 8.314 \frac{\text{J}}{\text{molK}}$  is the ideal gas constant and  $c$  is another constant dependent on the molecular nature of the gas<sup>12</sup>.

In addition, the internal energy is, by definition, a function of the extensive state properties: in this case, the entropy and the volume. We can therefore integrate the Gibbs form by rearranging the equations of state to express  $T$  and  $P$  in terms of  $S$  and  $V$  as well. Integrating Equation (3-17) yields

$$U = \log(C_0) e^{\frac{S}{cn_s R_g}} V^{\frac{-1}{c}},$$

where  $C_0$  is an integration constant. Since  $U = U(S, V)$ , we have that

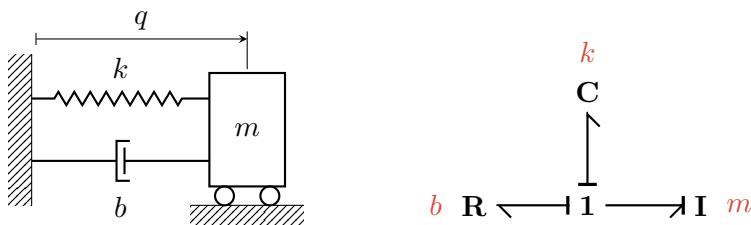
$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV.$$

Hence, we can fully specify a Legendre submanifold by the integrated equation and following conditions

$$T = \frac{\partial U}{\partial S} \quad P = -\frac{\partial U}{\partial V}.$$

### Contact geometry of the damped harmonic oscillator

The damped harmonic oscillator is shown in Figure 3-3, together with its bond graph representation. We assume here that this system is completely isolated: there is no exchange of energy nor matter with the environment.



**Figure 3-3:** The left figure shows a schematic of the mechanical damped harmonic oscillator with mass  $m$ , spring constant  $k$  and damping constant  $b$ . The bond graph representation is shown on the right. In addition to the I- and R-element in Figure 3-1, there is now an R-element as well.

<sup>12</sup>For a monatomic gas,  $c = \frac{3}{2}$ .

**Energy balance of the damped oscillator** We distinguish two types of energy that can be stored in the damped oscillator system: microscopic and macroscopic energy.

Microscopic energy consists of the kinetic energy of particles that does not result in an overall observable motion of the system. This energy is called internal energy  $U$  and manifests itself as temperature.

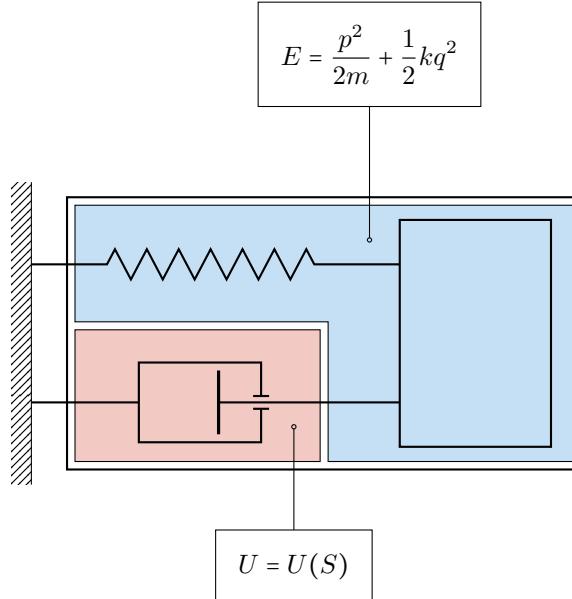
Internal energy is stored in a ‘heat bath’. This is to be interpreted loosely: it can be the damper fluid, but also the surrounding air (although a heterogeneous medium will not allow for an unambiguous notion of temperature). We will not be concerned with all these possibilities and consider a single heat bath with a single temperature: generalizations to more complex thermodynamic systems are immediate.

In contrast, macroscopic energy *is* observable, either due to an observable motion of the system (kinetic energy) or the energy resulting from external force potentials (potential energy). Their sum is called the mechanical energy  $E$ : in the damped harmonic oscillator, it is the sum of the kinetic energy stored in the mass (I-element) and the potential energy stored in the spring (C-element).

Since the system is isolated, the first law states that

$$d(E + U) = 0. \quad (3-19)$$

Let us now decompose the system into two subsystems, one containing the mass and the spring and one the heat bath, as illustrated in Figure 3-4.



**Figure 3-4:** System boundaries of the damper-oscillator system. The mechanical subsystem stores mechanical energy  $E$  in the form of kinetic and potential energy, while the heat bath stores internal energy in the form of heat. They interface through the action of the damper.

Through the dissipative action of the damper, energy flows from the mechanical subsystem to the heat bath. We can apply the first law to the subsystems separately, too: the first

subsystem performs work on the damper, which manifests itself as the heat added to the heat bath. We therefore have

$$\begin{aligned} dE &= -\beta, \\ dU &= \eta, \end{aligned} \tag{3-20}$$

where  $\beta$  is the (differential) work done *by* the mechanical subsystem on the damper and  $\eta$  is the (differential) heat added *to* the second subsystem as a result of this.

As a consequence of Equation (3-19), we have that  $\beta = \eta$ ; i.e. all the work done by the damper enters the fluid as heat. For a linearly damped system, the work form is by definition equal to

$$\beta := \gamma p dq, \tag{3-21}$$

with  $\gamma := b/m$  being the damping coefficient of the damped oscillator.

**Contact structure for the damped oscillator** We now define the phase space of the system to be equal to  $M = \mathbb{R} \times T^*Q$ , where  $Q$  is the configuration space considered in Section 3-1. For the damped harmonic oscillator,  $Q = \mathbb{R}$ . With coordinates for  $M$  being  $U, q$  and  $p$ , we can define a *contact form* on this space by combining Equation (3-20) and Equation (3-21):

$$\alpha = dU - \gamma p dq. \tag{3-22}$$

This contact form specifies precisely how energy is dissipated in the system and enters the ‘reservoir’ that is the heat bath, characterized by its internal energy  $U$ .

Observe from Equation (3-22) that  $d\alpha = \gamma dq \wedge dp$ , i.e. a multiple of the symplectic form used in Section 3-1. As such, the contact form contains *both* information about the rate of dissipation present in the system, and about the pairing of the conjugate variables  $p$  and  $q$ . We get the latter ‘for free’ in this particular instance, since the pairing in this simple three-dimensional is rather trivial.

It is important to note that, in the general case, a 1-form that describes the dissipation in the system is *under no obligation* to be of this very specific form (that is, one that pairs the conjugate variables). As such, we can not expect this situation to occur in general: this is subject of Section 3-3.

### Contact Hamiltonian system for the damped harmonic oscillator

With the contact structure defined, we can now establish the contact Hamiltonian system for the damped harmonic oscillator using Equations (3-11), (3-13) and (3-14).

Recall that the Hamiltonian vector field is split into a horizontal and vertical component, which belong respectively to the kernel of  $\alpha$  and  $d\alpha$ .

**Vertical component of the Hamiltonian vector field** The Reeb vector field  $R_\alpha$  associated with the contact form given in Equation (3-22) is

$$R_\alpha = \frac{\partial}{\partial U}.$$

As such, the vertical component of the Hamiltonian vector field is, in accordance with Equation (3-11):

$$X_K^{\text{ver}} = -K \frac{\partial}{\partial U}. \quad (3-23)$$

However, we can only guess as to what the Hamiltonian function might be. Indeed, its definition is rather circular, since the vertical part of the vector field is *defined* in terms of the vertical part and vice versa.

**Horizontal component of the Hamiltonian vector field** The horizontal part is obtained by projecting  $dK$  (an arbitrary exact form) to a *semi-basic* form, and mapping it to a vector field using the isomorphism  $d\alpha^\sharp$  like so:

$$X_K^{\text{hor}} = d\alpha^\sharp(dK - (R_\alpha \lrcorner dK)\alpha).$$

The Hamiltonian is a function on the contact manifold  $M$ , i.e.  $K = K(U, q, p)$ . In coordinates, the projected form is

$$\frac{\partial K}{\partial q} dq + \frac{\partial K}{\partial p} dp + \frac{\partial K}{\partial U} \gamma p dq. \quad (3-24)$$

The projection thus removes any term in  $dU$  (which makes it semi-basic).

Recall that  $d\alpha = \gamma\omega$ . Therefore, we can compare the above equation to the purely symplectic case without dissipation, where the isomorphism is provided by  $\omega$  (cf. Equation (3-4)). The difference here is (apart from the factor  $\gamma$ ) that we have to project  $dK$  by means of the term  $(R_\alpha \lrcorner dK)\alpha$ .

In the conservative case, we have that  $H = E = \frac{p^2}{2m} + \frac{1}{2}kq^2$ . It is therefore reasonable to expect that the form in Equation (3-24) contains the differential of  $E$  (representing the conservative side, or the I- and C-element) plus an extra term that enforces the dissipation (R-element).

Clearly, the rightmost term in Equation (3-24) is the work form of the damper, i.e. the amount of energy escaping from  $E$ . We can thus conjecture that the first two terms in Equation (3-24) amount to  $dE$ .

However, there is one complication:  $d\alpha$  contains the factor  $\gamma$ . To cancel this factor out, we include  $\gamma$  in the Hamiltonian as well. That is

$$\frac{\partial K}{\partial q} dq + \frac{\partial K}{\partial p} dp = \gamma \left( \frac{\partial E}{\partial q} dq + \frac{\partial E}{\partial p} dp \right)$$

and

$$\frac{\partial K}{\partial U} = \gamma.$$

As a result, the gradient of the Hamiltonian is equal to

$$dK = \gamma(dE + dU),$$

and we obtain the correct Hamiltonian up to a closed form

$$K = \gamma(E + U) = \gamma\left(\frac{p^2}{2m} + \frac{1}{2}kq^2 + U\right). \quad (3-25)$$

Hence, the Hamiltonian function is equal to the total amount of energy in the system, both mechanical and internal, multiplied by the damping coefficient.

Now to derive the horizontal component of the vector field. The interior product of  $\omega$  with the basis vectors yields:

$$\frac{\partial}{\partial q} \lrcorner d\alpha = \gamma dp \quad \frac{\partial}{\partial p} \lrcorner d\alpha = -\gamma dq \quad (3-26)$$

Clearly, the image of this mapping for any vector field is a semi-basic form. The inverse mapping must, to qualify as an isomorphism, map a semi-basic form back to a horizontal vector field (i.e. one that is in the kernel of  $\alpha$ ). Hence, we have that

$$d\alpha^\sharp(dq) = -\frac{1}{\gamma} \frac{\partial}{\partial p} \quad d\alpha^\sharp(dp) = \frac{1}{\gamma} \frac{\partial}{\partial q} + p \frac{\partial}{\partial U}. \quad (3-27)$$

The term in  $dU$  ensures that the vector field is horizontal. Using this mapping, and expression for the Hamiltonian in Equation (3-25), we obtain the horizontal component of the Hamiltonian vector field:

$$X_K^{\text{hor}} = \frac{p}{m} \frac{\partial}{\partial q} - (\gamma p + kq) \frac{\partial}{\partial p} + \gamma \frac{p^2}{m} \frac{\partial}{\partial U}. \quad (3-28)$$

**Equations of motion** Combining Equation (3-23) and Equation (3-28), the Hamiltonian vector field is

$$X_K = \frac{p}{m} \frac{\partial}{\partial q} - (\gamma p + kq) \frac{\partial}{\partial p} + \left( \gamma \frac{p^2}{m} - K \right) \frac{\partial}{\partial U}.$$

The corresponding equations of motion are

$$\begin{aligned} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -kq - \gamma p \\ \dot{U} &= \gamma \frac{p^2}{m} - K(q, p, U) = \gamma p \dot{q} - K(q, p, U). \end{aligned} \quad (3-29)$$

The correct dynamics are certainly obtained for  $p$  and  $q$ . However, from a physical standpoint, we expect  $\dot{U}$  to be the rate of energy (i.e. the power) dissipated by the damper, i.e. equal to  $\gamma p \dot{q}$ . However, the additional term  $-K(q, p, U)$  (a result of the vertical component of the vector field) contributes to the rate of change of  $U$  as well, ‘spoiling’ the physical dynamics.

If we wish to impose that  $U$  indeed be the internal energy of the heat bath, the *vertical vector field must vanish*. This is the case only if the Hamiltonian is numerically equal to zero, i.e.  $K = 0$ . This equation is a so-called *weak equality*, as opposed to a *strong* or *identical* equality. In the former case, the Hamiltonian is numerically equal to zero, but its partial derivatives do not vanish. That is to say, there is a specific submanifold of  $M$  on which  $K$  vanishes, but we are allowed to make variations that are not necessarily tangent to this

submanifold (see Dirac [40] for further details). On this submanifold, the equations of motion read

$$\begin{aligned}\dot{q} &= \frac{p}{m} \\ \dot{p} &= -kq - \gamma p \\ \dot{U} &= \gamma p \dot{q},\end{aligned}\tag{3-30}$$

which indeed represent the damped harmonic oscillator with  $U$  being the dissipated energy. From a thermodynamic standpoint, energy is only determined up to an additive constant, so this assertion would be admissible from a conceptual standpoint. Additionally, a value of 0 for the total energy is a common convention in literature, see for example Fermi [41].

It is here that our result differs from the existing literature on this subject. The applicability of contact Hamiltonian systems has already been recognized from a mathematical standpoint by Bravetti et al. [42], resulting in the equations of motion including the vertical vector field. However, when the variables are given a physical interpretation (in particular, the ‘extra dimension’ represented by  $U$ ) as we do here, the vanishing of the Hamiltonian is crucial. Leaving the vertical vector field in leads to extra ‘parasitic’ dynamics that are unphysical and delude us from the intended meaning of the variable  $U$ .

**Why the Hamiltonian must vanish** The assumption that the contact Hamiltonian should be equal to zero is rather striking, and the preceding arguments do not provide a sound mathematical basis for it. Indeed, we could (and should) be quite leery of cancelling terms using zero factors, for it often leads to unanticipated consequences or even downright contradictions. This is why we provide some more mathematically oriented arguments to show that this is indeed allowed.

Recall from Equation (3-7) that, by definition,  $K = -X_K \lrcorner \alpha$ . In the previous section we defined Legendre submanifolds as manifolds on which the contact form pulls back to zero: in other words, tangent vectors to a Legendre manifold produce zero when contracted with the contact form. So,  $K$  measures in essence how ‘non-Legendrian’ an integral manifold of  $X_K$  is. We have stipulated earlier that Legendre submanifolds contain the physically meaningful trajectories. That is, the dynamics must take place on a Legendre submanifold to be physical, which is why  $K = 0$ .

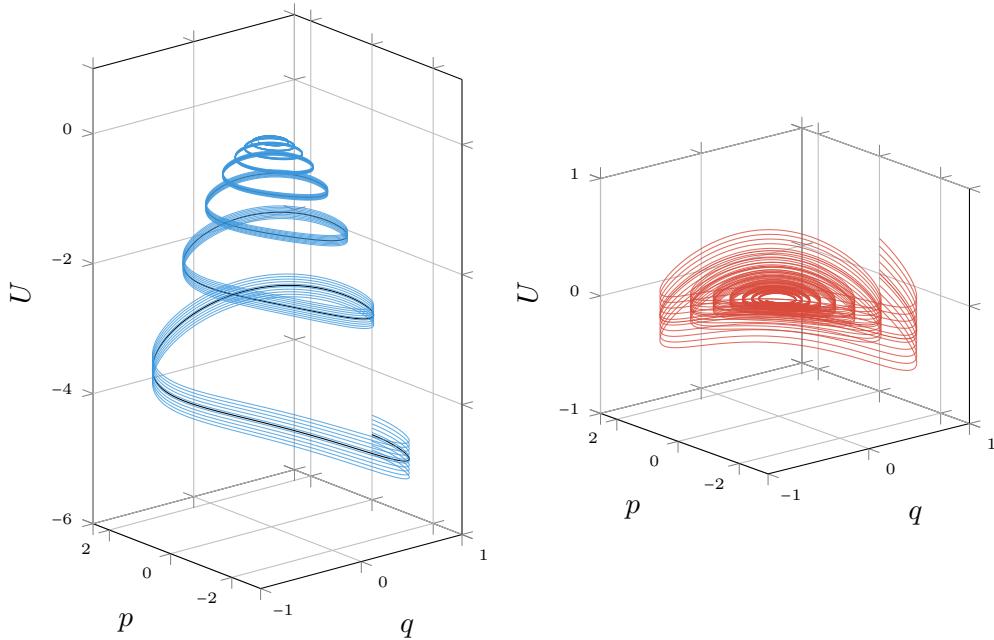
As an additional argument, we can show that

$$\mathcal{L}_{X_K^{\text{hor}}} K = 0 \quad \Rightarrow \quad \frac{dK}{dt} = \mathcal{L}_{X_K^{\text{ver}}} K = -K \frac{\partial K}{\partial U}.$$

Hence, if the Hamiltonian does not vanish, it changes exponentially over time (for its change is proportional to its own value). If  $\gamma > 0$ , the Hamiltonian decays exponentially from its initial value:

$$K(t) = K_0 e^{-\gamma t}.\tag{3-31}$$

As a result, any nonphysical trajectories will approach a Legendre submanifold as time proceeds; the associated vector fields also become ever more tangent to the Legendre submanifold.



**Figure 3-5:** Integral curves of  $X_K$  for  $b = 0.3 \frac{\text{kg}}{\text{s}}$ ,  $m = 1 \text{ kg}$  and  $k = 10 \frac{\text{kg}}{\text{s}^2}$ . The left plot shows the physical trajectory ( $K = 0$ ) in black, together with some neighboring non-physical trajectories that approach the black trajectory with increasing time. The trajectories on the right are all unphysical, but show the case where we would choose a zero initial value for  $U$  (also with some perturbations). The ‘wobble’ is caused by the exponentially decaying value of the Hamiltonian being counterbalanced with the nonuniform decrease of the mechanical energy in the system. In this case,  $U$  clearly does *not* represent the internal energy of the heat bath, or any other physical variable.

Based on the expression for  $K$ , an expression for  $U$  may also be derived:

$$U(t) = \frac{1}{\gamma} (K_0 e^{-\gamma t} - E).$$

If  $K = 0$ , the internal energy is equal to the negative of the mechanical energy in the system. These findings are illustrated by Figure 3-5: the left plot shows perturbations of the ‘ideal’ physical trajectory. The right plot shows the trajectories for an initial value of  $U = 0$  (also with perturbations); as a result of the above equation,  $U$  ‘wobbles’ around its zero point; but it is clearly not a physical trajectory.

Finally, recall that the contact form  $\alpha$  is not uniquely determined with respect to the associated contact structure. It can be multiplied with any nonzero function and still represent the same contact structure.

We can regard this ambiguity as a gauge transformation of the system (cf. Balian and Valentin [33]). However, the Hamiltonian is not intrinsically invariant under these transformations; if  $\alpha' = f\alpha$  ( $f$  being a function without zeros), then the mapping  $\Psi'_\alpha$  and the corresponding Hamiltonians are also different: [16, p. 321]

$$\Psi_{\alpha'}^{-1}(K) = \Psi_\alpha^{-1}\left(\frac{1}{f} K\right). \quad (3-32)$$

The vertical component *is* directly dependent on the numerical value of the Hamiltonian. As a result, the *only* way to maintain invariance under the gauge transformation (which we assert to be crucial for it to be of physical significance) is to set  $K = 0$ .

[intro to symplectification]

### 3-2-4 Symplectification of contact Hamiltonian systems

In this section, we use a procedure called *symplectification* to cast the contact manifold of the previous section on a symplectic manifold in a canonical fashion. The advantage of this method is that the calculations for the dynamics are considerably simplified, since we can use the theory outlined in Section 3-1. To quote Vladimir Arnold's, who originally came up with concept of symplectification, 'one is advised to calculate symplectically but to think rather in contact geometry terms [18, 34].'

In addition, we show that the resulting 'symplectified' Hamiltonian system explains the particular form of a popular existing model for the damped harmonic oscillator, using an explicit time-dependence: the Caldirola-Kanai Hamiltonian.

#### Symplectification of contact manifolds

To make the process of symplectization mathematically precise, we first need to move to a slightly different space than the manifold  $M$  used in the previous section.  $M$  is the product manifold of the cotangent bundle of  $Q$  (the space of the position  $q$ ) with the real line to accommodate  $U$ . In contrast, we now start with the *extended configuration manifold*  $Q_e \cong \mathbb{R}^2$ , equipped with an extra position coordinate  $q_0$ . That is,

$$(q_0, q) \in Q_e.$$

**The manifold of contact elements** Consider now the *manifold of contact elements* of  $Q_e$ . This is a fiber bundle with base  $Q_e$ , and the fibers at each point are the space of lines (through the origin) in the tangent space to  $Q_e$  at that specific point. For more details about the manifold of contact elements, the reader is referred to Appendix B.

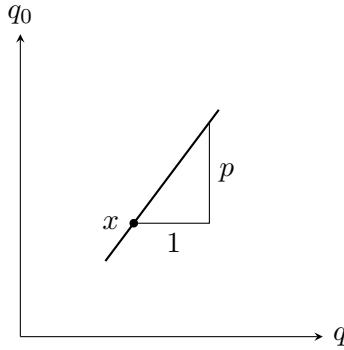
The fibers of this bundle are projective spaces; more specifically, they are diffeomorphic to the one-dimensional real projective line  $\mathbb{PR}$ . The fibers are therefore compact. It can be shown that the manifold of contact elements of  $Q_e$  (or any manifold) is diffeomorphic to the projectivization of its tangent bundle: we denote this by  $\mathbb{PT}^*Q_e$  [14, 15, 16].

The manifold  $\mathbb{PT}^*Q_e$  is three-dimensional. Consider a following chart for  $\mathbb{PT}^*Q_e$  with following coordinates:

$$(q_0, q, p)$$

where  $p$  represents the *slope* of the line in the tangent space, illustrated in Figure 3-6. This chart cannot cover the entire manifold, for the fiber is compactified at the point  $p = \infty$  (in this specific chart).

Because  $p$  is meant to represent momentum just like in the previous section, and infinite momentum lies outside our realm of interest, this space can be thought of as roughly equivalent



**Figure 3-6:** A contact element on the manifold  $Q_e$  is a line through the origin in the tangent space. The manifold of contact elements of  $Q_e$  is the space of all contact elements at every tangent space to  $Q_e$ .

(for practical purposes) to  $M$ . That is to say, by disregarding the point  $p = \infty$ , we end up with a manifold that is diffeomorphic to  $M$ .

The manifold of contact elements  $\mathbb{P}T^*Q_e$  is equipped with a natural contact structure [15], represented by

$$\alpha_e = dq_0 - p dq. \quad (3-33)$$

Observe the similarity with the contact form  $\alpha$  defined in Equation (3-22):

$$\alpha_e = \frac{1}{\gamma} \alpha, \quad q_0 = \frac{U}{\gamma}.$$

Because the contact forms differ simply by multiplication, they represent the same contact structure (provided that  $\gamma$  is nonzero).

**Contact Hamiltonian system in Darboux coordinates** The contact Hamiltonian system  $(M, \alpha, K)$  damped harmonic oscillator can be defined on the manifold of contact elements as well, with the silent understanding that  $M$  and  $\mathbb{P}T^*Q_e$  are slightly different from a topological perspective. Since the new contact form is scaled by  $\frac{1}{\gamma}$ , we can use relation Equation (3-32) to find the new Hamiltonian  $K_e$ :

$$K_e = \frac{1}{\gamma} K.$$

Furthermore, because  $U = \gamma q_0$ , the contact Hamiltonian in the new coordinates is

$$K_e(q_0, q, p) = \frac{p^2}{2m} + \frac{1}{2} k q^2 + \gamma q_0. \quad (3-34)$$

Numerically,  $K_e$  is equal to the total energy in system (the scaling factor is removed). Observe also that the units of the Hamiltonian have been changed from power to energy.

The corresponding Hamiltonian vector field is then

$$X_{K_e} = \frac{p}{m} \frac{\partial}{\partial q} + (\gamma p - kq) \frac{\partial}{\partial p} + \left[ \frac{p^2}{m} - K_e(q_0, q, p) \right] \frac{\partial}{\partial q_0}. \quad (3-35)$$

If we enforce that  $K = 0$  (as motivated in the previous section), then  $K_e$  vanishes as well and the vector field becomes

$$X_{K_e}|_{K_e=0} = \frac{p}{m} \frac{\partial}{\partial q} + (\gamma p - kq) \frac{\partial}{\partial p} + \frac{p^2}{m} \frac{\partial}{\partial q_0}. \quad (3-36)$$

This contact Hamiltonian system can be lifted to the symplectification of the contact manifold  $\mathbb{P}\mathrm{T}^*Q_e$ .

**Symplectification of a contact Hamiltonian system** The procedure known as symplectification of a contact manifold turns a contact manifold in a symplectic manifold, thereby raising its dimension by one. The power of this method resides in the fact that this can be done in a canonical fashion: it is uniquely determined by the contact structure of the contact manifold [15].

The symplectification procedure exploits the natural ambiguity that contact forms have, and that has been pointed out already. Multiplying the contact form  $\alpha_e$  with any nonzero real number<sup>13</sup>  $\lambda \in \mathbb{R}^\times$

$$\lambda(dq_0 - p dq). \quad (3-37)$$

The above expression gives a representation of *all* the contact forms that give rise to the same contact structure as  $\alpha_e$ . Hence, if  $\lambda$  is considered to be an additional coordinate in its own right, we move to a four-dimensional space with coordinates  $(q_0, q, p, \lambda)$ ; this is the space of all contact forms on the contact manifold.

We now effect the following coordinate transform:

$$\rho := -\lambda p \quad \rho_0 := \lambda.$$

Hence, Equation (3-37) becomes

$$\rho_0 dq_0 + \rho dq =: \vartheta_e \quad (3-38)$$

which is the Liouville form on the cotangent bundle  $\mathrm{T}^*Q_e$  in Darboux coordinates, denoted by  $\vartheta_e$ . From this form, we obtain the canonical symplectic structure on  $\mathrm{T}^*Q_e$  as follows

$$\omega_e := -d\vartheta = dq_0 \wedge d\rho_0 + dq \wedge d\rho. \quad (3-39)$$

The coordinate  $\lambda$ , and therefore  $\rho_0$  and  $\rho$ , are not canonical coordinates, for they depend on the particular choice of the contact form to begin with. Indeed, in this particular choice of  $\alpha_e$  one point in the fiber is left out ( $p = \infty$ ), which effectively rules all the points for which  $\rho_0 = 0$ .

In reality, only the points for which *both*  $\rho$  and  $\rho_0$  vanish should be taken out of the manifold, because the other cases can be covered by picking a different coordinate chart (this will be made clear later). The resulting space is the cotangent bundle of  $Q_e$  without its zero section, denoted by  $\dot{\mathrm{T}}^*Q_e$  [16, 34].

As illustrated by the point above, the former discussion relies heavily on the choice of the particular coordinate chart. Therefore, we wish to make the symplectization procedure more mathematically precise using the language of principal bundles.

---

<sup>13</sup> $\mathbb{R}^\times$  denotes both the real multiplicative group and the underlying set, being the real line excluding zero.

### Liouville geometry and principal G-bundles

A *principal G-bundle* is a smooth bundle  $P \xrightarrow{\sigma} B$  if  $P$  is equipped with a *free* right  $G$ -action  $\blacktriangleleft$ , where  $G$  is a Lie group. Furthermore, let

$$\begin{array}{ccc} P & & P \\ \downarrow \sigma & \cong_{\text{bundle}} & \downarrow \sigma' \\ B & & P/G \end{array}$$

where  $\sigma'$  is the quotient map that sends each point in  $P$  to the corresponding point in the orbit space  $P/G$ . In other words, if we define the equivalence relation between two points  $x_1, x_2 \in P$  as

$$x_1 \sim x_2 \iff \exists g \in G : x_2 = x_1 \blacktriangleleft g,$$

then

$$\sigma' : P \rightarrow P/G : x \mapsto [x]_{\sim},$$

with  $[x]_{\sim}$  being the equivalence class with respect to  $\sim$  and  $P/G = P/\sim$ .<sup>14</sup>

**Principal  $\mathbb{R}^{\times}$ -bundles** In our context of symplectization, the Lie group in question is the real multiplicative group  $\mathbb{R}^{\times}$ . The group acts on the cotangent bundle of  $Q_e$  without zero section<sup>15</sup> (i.e.  $\dot{T}^*Q_e$ ) through *dilation of its fibers*. In the coordinates defined above, we define the group action  $\blacktriangleleft \mathbb{R}^{\times}$  as:

$$\blacktriangleleft : \dot{T}^*Q_e \times \mathbb{R}^{\times} \rightarrow \dot{T}^*Q_e : (q_0, q, \rho_0, \rho) \blacktriangleleft \lambda = (q_0, q, \lambda\rho_0, \lambda\rho) \quad \lambda \in \mathbb{R}^{\times}.$$

The orbit space of  $\dot{T}^*Q_e$  with respect to  $\blacktriangleleft \mathbb{R}^{\times}$  is precisely equal to the projectivization of the cotangent bundle  $\mathbb{P}\dot{T}^*Q_e$ . As a result, we have the principal  $\mathbb{R}^{\times}$ -bundle structure given by the following diagram:

$$\begin{array}{c} \dot{T}^*M \\ \blacktriangleleft \mathbb{R}^{\times} \uparrow \\ \dot{T}^*M \\ \downarrow \sigma \\ \mathbb{P}\dot{T}^*M. \end{array}$$

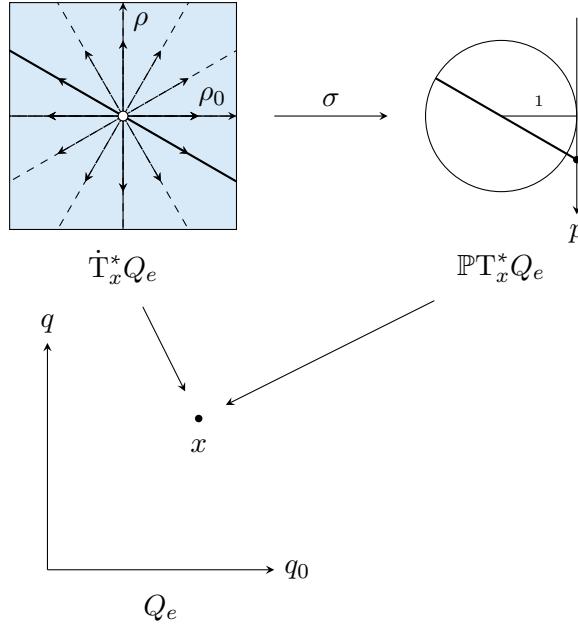
The symplectification  $\dot{T}^*Q_e$  is therefore a principal  $\mathbb{R}^{\times}$ -bundle, with as base manifold the contact manifold  $\mathbb{P}\dot{T}^*Q_e$ . The projection map  $\sigma$  is equal to

$$\sigma : \dot{T}^*Q_e \rightarrow \mathbb{P}\dot{T}^*Q_e : (q_0, q, \rho_0, \rho) \mapsto (q_0, q, -\rho/\rho_0).$$

<sup>14</sup>This is the definition used in the lectures of F. P. Schuller, see [43].

<sup>15</sup>The zero section must be removed from the cotangent bundle, because otherwise the group action defined above is not free (the origin of any cotangent space is stabilized by the entire group). If the group action is free, the orbits are diffeomorphic to the group itself. If this is not the case, not all the orbits are diffeomorphic to each other, and the ‘bundle of orbits’ would not be a fiber bundle. In this case, the origin (being the orbit of the origin) is of course not diffeomorphic to the other orbits (lines with a point removed).

A geometric perspective of this construction is given in Figure 3-7. Both  $\dot{T}^*Q_e$  and  $\mathbb{P}T^*Q_e$  are also bundles over the extended configuration space  $Q_e$ . At a specific point of  $Q_e$ , say  $x$ , we have two fibres. The fiber  $\dot{T}_x^*Q_e$  is the cotangent space to  $x$  without the origin. The group action  $\mathbb{R}_\times$  manifests itself as dilations of the fiber: this is indicated by the arrows. The *orbits* of this group are lines through the origin, with the origin removed (which is topologically the same space as  $\mathbb{R}_\times$  itself, since the action is free).



**Figure 3-7:** Illustration of the principal  $\mathbb{R}_\times$ -bundle  $\dot{T}^*M \xrightarrow{\pi} \mathbb{P}T^*M$ .  $x$  is a point in the extended configuration space  $Q_e$ , where we attach fibers  $\dot{T}_x^*Q_e$  and  $\mathbb{P}T_x^*Q_e$ . The orbits of the group action  $\mathbb{R}_\times$  on  $\dot{T}^*Q_e$  are identified by  $\sigma$  and mapped to the orbit space  $\mathbb{P}T^*M$ .

The space of all orbits is a circle with antipodal points identified, which also has the topology of a circle: this is the space  $\mathbb{PR}$ , and it is the fiber  $\mathbb{P}T_x^*Q_e$  of  $\mathbb{P}T^*Q_e$  at the point  $x$ . The projection map that takes a point in  $\mathbb{P}T^*xQ_e$  to its associated orbit in the orbit space is  $\sigma$ .

In Figure 3-7, the coordinate chart used for  $\mathbb{P}T^*Q_e$  is indicated as well:  $p = -\rho/\rho_0$ , which is as the negative of the slope of that line. This coordinate chart covers almost the entire fiber, apart from one ‘point’ (i.e. orbit): the north and south pole of the circle on the right.

From the perspective of  $\mathbb{P}T^*Q_e$ ,  $\rho_0$  and  $\rho$  can also be seen as *homogeneous coordinates* for this space.

**Principal bundles in system theory** To illustrate the concept of principal bundles and their relevance, we give an instructive example of principal bundles in control theory. For more information, the reader is referred to Hermann [44].

LTI systems can be represented both as a collection of state-space matrices, or in the frequency domain using a transfer matrix. The state-space representation is typically given in the form specified by a collection of four matrices:  $A, B, C, D$ . For an LTI system with  $n$  states,  $m$  inputs

and  $o$  outputs, we have:

$$A \in \mathbb{R}^{n \times n} \quad B \in \mathbb{R}^{n \times m} \quad C \in \mathbb{R}^{o \times n} \quad D \in \mathbb{R}^{o \times m}.$$

Hence, the ‘manifold of LTI systems’ given these dimensions is diffeomorphic to [45]

$$\mathbb{R}^\ell, \quad \ell = n^2 + nm + on + om.$$

This is the total space of the principal bundle.

A state space representation of a transfer matrix is not unique: any similarity transform of the state space yields different state space matrices that correspond to the same transfer matrix. Hence, the *structure group* is in this case the general linear group of dimension  $n$ ,  $\text{GL}(n, \mathbb{R})$ , which contains all the similarity transforms. The group action  $\blacktriangleleft \text{GL}(n, \mathbb{R})$  is defined as follows:

$$(A, B, C, D) \blacktriangleleft T = (TAT^{-1}, TB, CT^{-1}, D).$$

The orbit space  $\mathbb{R}^\ell/\text{GL}(n, \mathbb{R})$  can be identified with the space of transfer matrices. The projection map that takes the state space representation to a transfer matrix is given by

$$\sigma(A, B, C, D) = C(sI - A)^{-1}B + D,$$

which is invariant with respect to group action.

The topology of the orbit space, and therefore of the space of transfer matrices, is highly nontrivial. This makes the process of system identification very challenging, for there are usually no easy coordinate charts of this space [44, 45].

### Homogeneous Hamiltonian systems

In this section, we will lift the contact Hamiltonian system defined in Section 3-2-3 to the symplectified manifold, resulting in a symplectic Hamiltonian system with an *Liouville structure* additional structure.

**Liouville structures** The symplectified space  $\dot{T}^*Q_e$  has a symplectic structure because the cotangent bundle (with zero section removed) is canonically equipped with one. Moreover, the group action that makes it into a principal bundle provides an additional structure: a *symplectic Liouville structure*, which requires that the symplectic 2-form is *homogeneous of degree 1* with respect to the group action  $\blacktriangleleft \mathbb{R}_\times$ . That is,

$$(\blacktriangleleft \lambda)^* \omega_e = \lambda \omega_e, \quad \lambda \in \mathbb{R}_\times,$$

which is indeed the case for  $\omega_e$  as defined in Equation (3-39) [16]. Because the group action  $\blacktriangleleft \mathbb{R}_\times$  is free, the symplectic Liouville structure is said to be *fibered* [16].

It can be shown that there is again an mapping between the smooth functions on the manifold with Liouville structure and vector fields that preserve this structure<sup>16</sup>, along the same line as for the symplectic manifolds in Section 3-1 and the contact manifolds earlier in this section.

<sup>16</sup>For a vector field  $X$  to preserve the Liouville structure means (i) that it preserves  $\omega_e$ , i.e.  $\mathcal{L}_X \omega_e = 0$ , and (ii) that it is invariant under the group action:  $(\blacktriangleleft \lambda)_* X = X$  ( $\lambda \in \mathbb{R}_\times$ ).

The smooth functions in this case are not completely arbitrary, since they must also comply with the Liouville structure. More precisely, they must be *homogeneous* of degree 1 with respect to the group action  $\blacktriangleleft \mathbb{R}_\times$ . For a function  $\mathcal{H}$  on  $T^*Q_e$  to be homogeneous means it must satisfy the following condition

$$(\blacktriangleleft \lambda)^*\mathcal{H} = \lambda\mathcal{H}.$$

In the coordinates defined above, this is equivalent to:

$$\mathcal{H}(q_0, q, \lambda\rho_0, \lambda\rho) = \lambda\mathcal{H}(q_0, q, \rho_0, \rho) \quad \lambda \in \mathbb{R}_\times,$$

which is to say that  $\mathcal{H}$  *commutes* with the group action.

Thus, we have an isomorphism between the vector fields preserving the Liouville structure and the homogeneous functions on the manifold. This manifold is symplectic, so isomorphism is defined in terms of the symplectic form  $\omega_e$  like so (and in identical fashion to Equation (3-3)),

$$\omega_e^\sharp(d\mathcal{H}) = X_{\mathcal{H}}.$$

This gives rise to the notion of *homogeneous Hamiltonian systems*, consisting of a manifold with fibered symplectic Liouville structure and a homogeneous Hamiltonian function  $\mathcal{H}$ .

**Equations of motion for the symplectified system** The contact Hamiltonian system for the damped harmonic oscillator, as defined by Equation (3-34), can now be lifted to a homogeneous Hamiltonian system on the symplectified space. The relation between the contact Hamiltonian and the corresponding homogeneous Hamiltonian is defined as [15, 16, 34]

$$K_e(q_0, q, p) = \mathcal{H}(q_0, q, -1, \rho),$$

or equivalently

$$K_e\left(q_0, q, -\frac{\rho}{\rho_0}\right) = \mathcal{H}(q_0, q, \rho_0, \rho). \quad (3-40)$$

Based on Equation (3-34), we obtain the following expression for the homogeneous Hamiltonian:

$$\mathcal{H}(q_0, q, \rho_0, \rho) = -\rho_0 \left[ \frac{1}{2m} \left( -\frac{\rho}{\rho_0} \right)^2 + \frac{1}{2} kq^2 + \gamma q_0 \right]. \quad (3-41)$$

The Hamiltonian vector field is easily obtained, for we can use the mapping  $\omega_e^\sharp$ . As already mentioned this is a major advantage of performing calculations in the symplectified space. We have

$$X_{\mathcal{H}} = \omega_e^\sharp(d\mathcal{H}),$$

with

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q_0} dq_0 + \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial \rho_0} d\rho_0 + \frac{\partial \mathcal{H}}{\partial \rho} d\rho.$$

It is instructive to first specify the partial derivatives of  $\mathcal{H}$  in terms of  $K_e$ , so as to compare the generic equations of motion obtained from  $\mathcal{H}$  to those obtained from  $K_e$  (cf. Equation (3-35)). Using Equation (3-40), the partial derivatives can be expressed in terms of the contact

Hamiltonian  $K_e$ :

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial q} &= -\rho_0 \frac{\partial K_e}{\partial q}, \\ \frac{\partial \mathcal{H}}{\partial q_0} &= -\rho_0 \frac{\partial K_e}{\partial q_0}, \\ \frac{\partial \mathcal{H}}{\partial \rho} &= -\rho_0 \frac{\partial K_e}{\partial p} \frac{\partial p}{\partial \rho} = \frac{\partial K_e}{\partial p}, \\ \frac{\partial \mathcal{H}}{\partial \rho_0} &= -H - \rho_0 \frac{\partial K_e}{\partial p} \frac{\partial p}{\partial \rho_0} = -K_e - \frac{\partial H}{\partial p} \frac{\rho}{\rho_0} = \frac{\partial H}{\partial p} p - K_e.\end{aligned}\tag{3-42}$$

The homogeneous Hamiltonian vector field is then

$$X_{\mathcal{H}} = \left( \frac{\partial K_e}{\partial p} p - K_e \right) \frac{\partial}{\partial q_0} + \frac{\partial K_e}{\partial p} \frac{\partial}{\partial q} + \rho_0 \frac{\partial K_e}{\partial q_0} \frac{\partial}{\partial \rho_0} + \rho_0 \frac{\partial K_e}{\partial q} \frac{\partial}{\partial \rho}.\tag{3-43}$$

The equations of motion  $q_0$  and  $q$  remain identical to those obtained earlier; this is to be expected since otherwise the dynamics would not correspond.

For  $\rho_0$ , we have

$$\dot{\rho}_0 = \rho_0 \frac{\partial K_e}{\partial q_0} = \rho_0 \frac{\partial K_e}{\partial q_0} = \gamma \rho_0 \quad \Rightarrow \quad \rho_0 = C e^{\gamma t},$$

where  $C$  is an integration constant which we can choose to be 0, so  $\rho_0(t) = e^{\gamma t}$ .

In addition,  $\dot{\rho} = \rho_0 k q = e^{\gamma t} k q$ . Since  $p = -\rho/\rho_0$ , the dynamics of  $p$  can be obtained from  $\dot{\rho}_0$  and  $\dot{\rho}$  using the product rule:

$$\dot{p} = -\frac{\dot{\rho}}{\rho_0} + \frac{\rho}{\rho_0} \frac{\dot{\rho}_0}{\rho_0} = -k q - \gamma p,\tag{3-44}$$

which is equivalent to the expression obtained in Section 3-2-3.

Observe that these equations of motion are invariant under the earlier defined group action  $\blacktriangleleft \mathbb{R}_x$ , which means that the vector field indeed preserves the Liouville structure. The other condition is that it preserves  $\omega_e$ , but this is satisfied rather trivially as a result of the mapping  $\omega_e^\sharp$ .

**Liouville submanifolds** In Section 3-2-3 we devoted considerable attention to the fact that the contact Hamiltonian should numerically be equal to zero for the equations of motion to represent a physical trajectory. This is equivalent to stating that the trajectories lie in Legendre submanifolds.

As pointed out by van der Schaft [34] and Libermann and Marle [16], because of the equivalence between contact and Liouville structures, the notion of Legendre submanifolds can be lifted to the symplectified space as well. Indeed, from Equation (3-38) we can observe that if  $\alpha_e$  pulls back to zero on the trajectories in the contact manifold, so should  $\vartheta_e$  on the lifted trajectories, for they only differ by multiplication with  $\rho_0$ . These are called *Liouville submanifolds*, and are a special subclass of Lagrangian submanifolds<sup>17</sup>

<sup>17</sup>Lagrangian submanifolds satisfy the weaker condition that the symplectic 2-form  $\omega_e$  vanishes when restricted to them. This is also the case when  $\vartheta_e$  vanishes, but the converse is not necessarily true.

Using Equation (3-43), we find that

$$X_{\mathcal{H}} \lrcorner \vartheta_e = -K_e,$$

which means that the contact Hamiltonian (i.e. the total energy in the system) must be equal to zero for the lifted trajectories to lie in a Liouville submanifold.

For yet another perspective regarding this point, we can make use of the symplectic nature of the homogeneous Hamiltonian. Indeed, no matter what the value of  $K_e$ , the homogeneous Hamiltonian  $\mathcal{H}$  *must* be constant over time, because it does explicitly depend on it. Since the dynamics are symplectic, we can simply use Poisson brackets support this fact:

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} + \frac{\partial \mathcal{H}}{\partial t} = 0.$$

Hence, we can set the Hamiltonian equal to a constant, say  $\mathcal{H}(t) = \mathcal{H}_0$ . But, we also know from  $\mathcal{H} = \rho_0 K_e = e^{\gamma t} K_e$ . Hence, it is now very easy to see that  $K_e$  either decays exponentially (if  $\gamma > 0$ ), for it then cancels exactly the exponential growth of  $\rho_0$ , or it equal to zero. This is equivalent to Equation (3-31). On Liouville submanifolds, both the homogeneous Hamiltonian and the contact Hamiltonian vanish, which amounts to the particular choice that  $\mathcal{H}_0 = 0$ .

If we assume that the dynamics take place on a Liouville submanifold, the Hamiltonian vector field becomes

$$X_{\mathcal{H}}|_{\mathcal{H}_0=0} = -\frac{1}{m} \frac{\rho}{\rho_0} \frac{\partial}{\partial q} + \frac{\partial}{\partial q_0} + \rho_0 k q \frac{\partial}{\partial \rho} + \gamma \rho_0 \frac{\partial}{\partial \rho_0}.$$

If this vector field is be projected to  $\mathbb{P}T^*Q_e$  using the pushforward of the projection map  $\sigma$ , we obtain Equation (3-36).<sup>18</sup>

### Relation with the Caldirola-Kanai Hamiltonian

In this section, we show that the homogeneous Hamiltonian is equivalent to a well-known existing model for the damped harmonic oscillator, the Caldirola-Kanai Hamiltonian (and Lagrangian).

The Caldirola-Kanai Hamiltonian, commonly attributed to Caldirola [46] and Kanai [47], is a method to describe the linearly damped harmonic oscillator using a Lagrangian or Hamiltonian function that explicitly depend on time. It was originally motivated for the purposes of quantum mechanics.

We depart from the Lagrangian function for it depends directly on physical coordinates  $q$  and  $\dot{q}$ , as opposed to the Hamiltonian. The Caldirola-Kanai Lagrangian is

$$L_{CK}(q, \dot{q}, t) = e^{\gamma t} \left( \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 \right). \quad (3-45)$$

---

<sup>18</sup>This is the ‘differential geometric’ way of deriving the equation for  $\dot{p}$  as we did above.

The correct equations of motion are readily derived through the Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L_{\text{CK}}}{\partial \dot{q}} \right) - \frac{\partial L_{\text{CK}}}{\partial q} &= 0, \\ \frac{d}{dt} (e^{\gamma t} m \dot{q}) + e^{\gamma t} k q &= 0, \\ e^{\gamma t} (m \ddot{q} + m \gamma \dot{q} + k q) &= 0, \\ m \ddot{q} + m \gamma \dot{q} + k q &= 0. \end{aligned} \tag{3-46}$$

The Caldirola-Kanai Hamiltonian is obtained from the Lagrangian by means of a Legendre transform. The Legendre transform is effected with respect to the *canonical momentum*

$$\rho = \frac{\partial L_{\text{CK}}}{\partial \dot{q}} = e^{\gamma t} m \dot{q},$$

which is manifestly different from the kinematic momentum  $p = m \dot{q} = \rho e^{-\gamma t}$ .

The Hamiltonian is then equal to

$$H_{\text{CK}} = \rho \dot{q} - L_{\text{CK}} = \frac{\rho^2}{2m} e^{-\gamma t} + \frac{1}{2} k q^2 e^{\gamma t}.$$

Because the Hamiltonian is explicitly time-dependent, the associated Hamiltonian vector field will be time-dependent as well<sup>19</sup>.

The construction of the vector field associated with a time-dependent Hamiltonian follows the same construction rules as a normal Hamiltonian using the isomorphism given by  $\omega^\sharp$ , but ‘frozen’ at each instant of  $t$ . The Hamiltonian vector field on  $T^*Q$  is

$$X_{H_{\text{CK}}} = -e^{\gamma t} k q \frac{\partial}{\partial \rho} + e^{-\gamma t} \frac{\rho}{m} \frac{\partial}{\partial q}.$$

The suspension of this vector field on  $T^*Q \times \mathbb{R}$  is

$$\tilde{X}_{H_{\text{CK}}} = -e^{\gamma t} k q \frac{\partial}{\partial \rho} + e^{-\gamma t} \frac{\rho}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

The suspension is important because it allows us to perform the time-dependent transformation from the canonical momentum  $\rho$  to the kinematic momentum  $p$ :  $\phi: (q, \rho, t) \mapsto (q, e^{\gamma t} p, t)$ . The transformed vector field in terms of the physical coordinates  $p, q, t$  is

$$\phi_*(\tilde{X}_{H_{\text{CK}}}) = (-kq - \gamma p) \frac{\partial}{\partial p} + \frac{p}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

<sup>19</sup>A *time-dependent vector field* on a manifold  $N$  is a mapping  $X: N \times \mathbb{R} \rightarrow TN$  such that for each  $t \in \mathbb{R}$ , the restriction  $X_t$  of  $X$  to  $N \times \{t\}$  is a vector field on  $N$ . [16] An additional construction of importance, called the *suspension* of the vector field, is a mapping

$$\tilde{X}: \mathbb{R} \times N \rightarrow T(\mathbb{R} \times N) \quad (t, n) \mapsto ((t, 1), (n, X(t, n))),$$

that is to say, the suspension lifts the vector field to the extended space that also includes  $t$  and assigns the time coordinate with a trivial velocity of 1 [11].

The extra term in  $p$  arises as a consequence of the fact that the mapping from  $\rho$  to  $p$  depends also on  $t$ .

The similarity between the derivation of the equations of motion — in particular, the crucial role of the product rule — and the one given by Equation (3-44) is striking. Indeed, if we substitute into the Caldirola-Kanai Hamiltonian  $-\rho_0 = e^{\gamma t}$ , we obtain

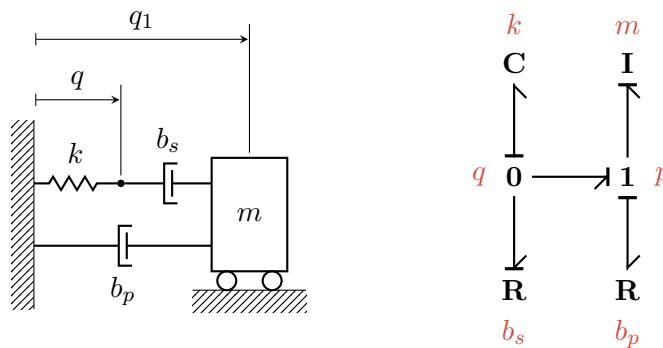
$$H_{\text{CK}} = -\frac{\rho^2}{2m\rho_0} - \rho_0 \frac{1}{2} k q^2 = -\rho_0 \left( \frac{1}{2m} \left( \frac{\rho}{\rho_0} \right)^2 + \frac{1}{2} k q^2 \right),$$

which is precisely equal to the homogeneous Hamiltonian given in Equation (3-41) excluding the term in  $q_0$ . The dependence on  $q_0$  is not required, since it is replaced by an explicit dependence on time that would otherwise generate the exponential factor  $e^{\gamma t}$ .

Many interpretations have already been given for the particular form of  $H_{\text{CK}}$ ; for example through time-dependent canonical transformations, or by a rescaling of time itself (see i.a. Tokieda and Endo [48], Caldirola [46] and Bravetti et al. [42]). Here we can see that the Caldirola-Kanai can be regarded as directly equivalent to the homogeneous Hamiltonian system, where the dynamics of the additional coordinates  $\rho_0$  and  $q_0$  are replaced by their explicit solution in time. Additionally, the role of the ‘mysterious’ canonical momentum  $\rho$  is explained as being a coordinate of the symplectified space, or homogeneous coordinate for the underlying contact space<sup>20</sup>.

### 3-2-5 The harmonic oscillator with serial damping

In this section, we extend the method outlined in Section 3-2-3 to a harmonic oscillator with two dampers: one in series and one in parallel. This system will play an important role in Chapter 4.



**Figure 3-8:** Schematic of the harmonic oscillator with two dampers: one in series and one in parallel. The corresponding bond graph representation is shown on the right.

### System dynamics

The harmonic oscillator with two dampers is shown in Figure 3-8, together with the corresponding bond graph representation. Comparing this to Figure 3-3, there is another 0-

<sup>20</sup>This has caused considerable confusion in literature, as stated by Schuch [49].

junction present in the system that compares flows (velocities) rather than efforts (forces). The equations of motion can be readily derived:

$$\begin{aligned} m\ddot{q}_1 &= -kq - b_p\dot{q}_1 \\ kq &= b_s(\dot{q}_1 - \dot{q}) \end{aligned} \tag{3-47}$$

Due to the presence of the serial damper, the situation is somewhat curious, since there are two positions in the system; one measuring the spring deflection  $q$  and the position of the mass  $q_1$ . The subscript ‘1’ refers to the fact that  $q_1$  is the position measured at the 1-junction in the bond graph shown in Figure 3-3. However, the node connecting the serial damper and the spring has no mass, and therefore no second-order dynamics: as such, the overall order of the system is two.

In accordance with the economic analogy, we will say that position is stored in the spring, but momentum is stored in a mass. Hence, we let  $p = mq_1$  — but  $\dot{q} \neq p/m$  in general. That is to say, the spring is naturally associated with a position coordinate, while the mass has a momentum, though its position does not partake in the dynamics directly.

Using the damping coefficients defined in Table 3-1, the equations of motion become:

$$\begin{aligned} \dot{q} &= -\gamma_s q + p/m \\ \dot{p} &= -\gamma_p p - kq. \end{aligned} \tag{3-48}$$

**Table 3-1:** Substitution parameters for the harmonic oscillator with serial and parallel damping, shown in Figure 3-8.

Name	Symbol	Value	Units
Serial damping coefficient	$\gamma_s$	$k/b_s$	$\frac{1}{s}$
Parallel damping coefficient	$\gamma_p$	$b_p/m$	$\frac{1}{s}$
Natural frequency	$\Omega_n$	$\sqrt{k/m}$	$\frac{1}{s}$
Damped frequency	$\Omega_d$		$\frac{1}{s}$

### Contact Hamiltonian system

In order to establish the contact structure for the harmonic oscillator with two dampers, we must find the expression for the work done by the system on the dampers. We will do so using the structure of the bond graph shown in Figure 3-8.

Bonds carry two signals: an effort and a flow. Both can be assigned a ‘direction’; they are always opposite. The direction indicates whether either the effort or flow should be regarded as the ‘input’ of the model. For example, *traditionally* (though this is a matter of convention), an I-element takes efforts as an input, and returns a flow. That is to say, one applies a force to a mass, with a change in velocity as a result. Conversely, when a spring is stretched along a certain distance to return a force proportional to it; it takes a flow and returns an effort [50]. In a bond graph, this is indicated by a causality stroke, which is placed at the side of the bond that determines the flow.

If a causality convention is chosen, all the I- and C-elements in the bond graph should conform to this convention<sup>21</sup>. This is *not* the case for R-elements; they are *indifferent to causality*. The reason for this is that there is no integral/derivative present in the mathematical description of their dynamic behavior: they relate an effort and flow, which are both time derivatives. So, depending on the system architecture, a particular R-element may receive an effort and return a flow, or vice versa [50].

This can be observed from Figure 3-8: the serial damper (on the 0-junction) receives an effort and returns a flow, while the parallel damper receives a flow and returns an effort (1-junction). This distinction is reflected in the work form associated to the damper. For the parallel damper, we have

$$\beta_p = \underbrace{\gamma_p p}_{\text{EFFORT}} \times \underbrace{dq}_{\text{FLOW}}.$$

The variable that is ‘varied’ externally is the flow, hence  $dq$

For the serial damper, we have the opposite situation. Here, the effort is varied externally, which is equal to  $-dp$ . The minus sign is a consequence of the power direction in the bond connecting the 1-junction and 0-junction<sup>22</sup>. The flow is equal to  $\dot{q}_1 - \dot{q} = kq/b_s = \gamma_s q$  (using Equation (3-47)). Hence, we have

$$\beta_s = \underbrace{\gamma_s q}_{\text{FLOW}} \times \underbrace{-dp}_{\text{EFFORT}}.$$

Combining these work forms, we find the contact form for the system with two dampers:

$$\alpha = dU - \gamma_p p dq + \gamma_s q dp. \quad (3-49)$$

The exterior derivative of the contact form  $\alpha$  is then equal to

$$d\alpha = (\gamma_s + \gamma_p) dq \wedge dp,$$

and the Reeb vector field is simply

$$R_\alpha = \frac{\partial}{\partial U}.$$

Now to find the Hamiltonian and the system dynamics. In the following, we will only consider the horizontal component of the Hamiltonian vector field, for the various reasons pointed out in Sections 3-2-3 and 3-2-4. The horizontal component is given by (cf. Equation (3-13)):

$$\begin{aligned} X_K^{\text{hor}} &= d\alpha^\sharp(dK - (R_\alpha \lrcorner dK)\alpha) \\ &= d\alpha^\sharp \left( \left( \frac{\partial K}{\partial q} + \gamma_p p \frac{\partial K}{\partial U} \right) dq + \left( \frac{\partial K}{\partial p} - \gamma_s q \frac{\partial K}{\partial U} \right) dp \right). \end{aligned} \quad (3-50)$$

The mapping  $d\alpha^\sharp$  acts on the basis 1-forms as follows:

$$d\alpha^\sharp(dp) = \frac{1}{\gamma_s + \gamma_p} \left( \frac{\partial}{\partial q} + \gamma_p p \frac{\partial}{\partial U} \right) \quad d\alpha^\sharp(dq) = \frac{1}{\gamma_s + \gamma_p} \left( -\frac{\partial}{\partial p} + \gamma_s q \frac{\partial}{\partial U} \right).$$

<sup>21</sup>Not doing so leads to a differential algebraic system (DAE).

<sup>22</sup>Intuitively, it is clear from Figure 3-8 that the force acting *on* the damper is proportional to the *decrease* in momentum. Also, in Equation (3-48) we can observe that the damping force is negatively proportional to  $\dot{q}_1$ , which is itself positively proportional to the momentum of the mass.

The extra terms in  $\frac{\partial}{\partial U}$  appear again to ensure that the vector field be horizontal.

Using the same reasoning applied in Section 3-2-3, we can observe that the contact Hamiltonian must be proportional to the sum of the mechanical and internal energy of the system. In addition, we wish to cancel the factor  $(\gamma_s + \gamma_p)$  present in the mapping  $d\alpha^\sharp$  by multiplying the contact Hamiltonian with the same factor. Hence we, have

$$K = (\gamma_s + \gamma_p) \left( \frac{p^2}{2m} + \frac{1}{2} kq^2 + U \right).$$

Assuming again that  $K = 0$ , the contact Hamiltonian vector field is then:

$$X_K = X_K^{\text{hor}} = \left( \frac{p}{m} - \gamma_s q \right) \frac{\partial}{\partial q} + (-kq - \gamma_p p) \frac{\partial}{\partial p} + \left( \gamma_p \frac{p^2}{m} + \gamma_s kq^2 \right) \frac{\partial}{\partial U},$$

since the cross terms in  $\frac{\partial}{\partial U}$  cancel out. Next to the familiar dissipated power for the parallel damper, we also have the dissipated power of the serial damper

$$\gamma_s kq^2 = \underbrace{\gamma_s q}_{\text{FLOW}} \times \underbrace{kq}_{\text{EFFORT}},$$

in the dynamics of  $U$ . Hence, this vector field yields the correct dynamics for  $p$  and  $q$  as given by Equation (3-48), in addition to the internal energy  $U$ .

### 3-3 Jacobi structures for general systems

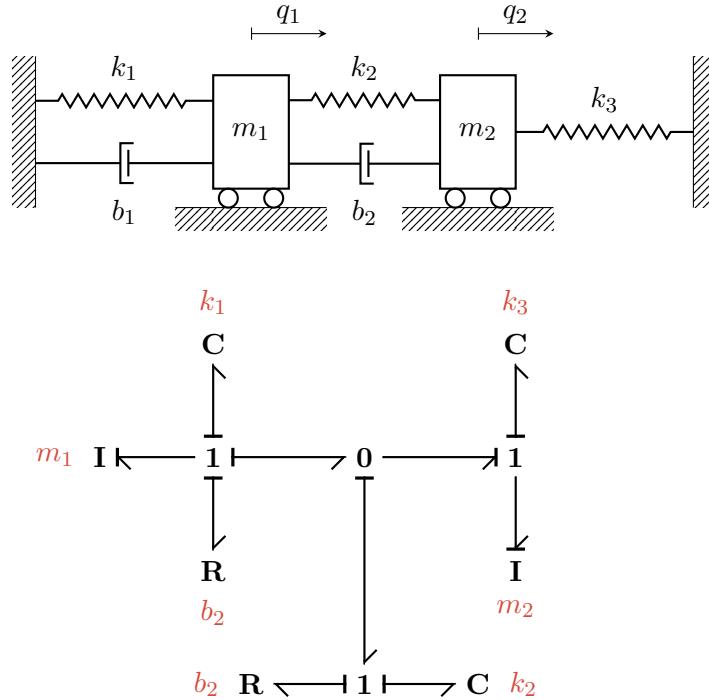
In this section we take the ideas outlined in Sections 3-1 and 3-2 one step further to more general mechanical systems. In particular, we will focus on multi-degree of freedom (MDOF) systems, and systems with exogeneous inputs (external forces). As it turns out, a contact structure is not sufficient to describe such systems. Instead, we use a generalization of contact and symplectic structures called a *Jacobi structure*.

To illustrate the need for a Jacobi structure, we use the mechanical MDOF system shown in Figure 3-9. The corresponding equations of motion are

$$\begin{aligned} \dot{q}_1 &= \frac{p_1}{m_1}, \\ \dot{q}_2 &= \frac{p_2}{m_2}, \\ \dot{p}_1 &= -\frac{b_1}{m_1}p_1 - \frac{b_2}{m_1}p_1 + \frac{b_2}{m_2}p_2 - k_1q_1 - k_2q_1 + k_2q_2, \\ \dot{p}_2 &= -\frac{b_2}{m_2}p_2 + \frac{b_2}{m_1}p_1 - k_3q_2 - k_2q_2 + k_2q_1. \end{aligned} \tag{3-51}$$

To proceed with the method discussed in Section 3-2-3, we have to find the work form that specifies the work done by the system of on the dampers. The work done the first damper ( $b_1$ ) is

$$\beta_1 = \left( \frac{b_1}{m_1} \right) p_1 dq_1.$$



**Figure 3-9:** Multi-degree of freedom mechanical system with two masses, two dampers and three springs. The corresponding bond graph representation is shown below.

The second damper ( $b_2$ ) is placed between the two masses; the flow is relative. The effort is proportional to this flow; i.e.

$$\beta_2 = b_2 \left( \frac{p_2}{m_2} - \frac{p_1}{m_1} \right) d(q_2 - q_1).$$

Hence, the contact 1-form that specifies the dissipation is

$$\begin{aligned} \alpha &= dU - \left( \frac{b_1}{m_1} \right) p_1 dq_1 - b_2 \left( \frac{p_2}{m_2} - \frac{p_1}{m_1} \right) d(q_2 - q_1) \\ &= dU - \left[ \left( \frac{b_1}{m_1} + \left( \frac{b_2}{m_1} \right) \right) p_1 - \left( \frac{b_2}{m_2} \right) p_2 \right] dq_1 - \left[ \left( \frac{b_2}{m_2} \right) p_2 - \left( \frac{b_2}{m_1} \right) p_1 \right] dq_2. \end{aligned} \quad (3-52)$$

From this expression, we can observe a crucial difference with the contact forms of the single degree of freedom systems (cf. Equations (3-22) and (3-49)). In contrast to the single-degree of freedom case given in the previous section,  $\alpha$  is here *not* of the form

$$dU - \gamma \vartheta,$$

where  $\vartheta$  is the Liouville form on the cotangent bundle of the configuration manifold  $T^*Q$ . This has important ramifications, for the Liouville form (and its exterior derivative) facilitates the ‘pairing’ between the position and momentum coordinates. In case of a single degree of freedom system, the pairing is trivial because there is only one momentum and one position coordinate. For more complicated systems this is no longer the case, as illustrated Equation (3-52).

The exterior derivative of  $\alpha$  is

$$d\alpha = \left( \frac{b_1}{m_1} + \left( \frac{b_2}{m_1} \right) \right) dq_1 \wedge dp_1 - \left( \frac{b_2}{m_2} \right) dq_1 \wedge dp_2 + \left( \frac{b_2}{m_2} \right) dq_2 \wedge dp_2 - \left( \frac{b_1}{m_1} \right) dq_2 \wedge dp_1,$$

which indicates indeed that there is also a ‘mixing’ of  $p_1$  and  $q_2$  and  $p_2$  and  $q_1$  in the resulting 2-form. As a result, the mapping  $d\alpha^\sharp$  will not produce the mapping that we would expect in the purely symplectic case.

From a conceptual standpoint, this is not quite surprising: there is no inherent reason why the form that describes dissipation should somehow also include the ‘pairing’ structure: they are fundamentally different, and both required for the geometric description of the mechanical system. In the previous section, we were indeed rather ‘lucky’ to find that, *that particular case*, the dissipation form  $\alpha$  also included the pairing structure. This severely limits the applicability of a contact Hamiltonian systems to dissipative mechanical systems.

The multi-degree of freedom systems for which  $\alpha$  is of the form  $dU - \gamma\vartheta$  are those that do exhibit not damping on the relative velocities of the masses, and for which all dampers have the same damping coefficient. This is a very restrictive requirement, and we wish to do better. To do so, we introduce a generalization of contact and symplectic structures called a *Jacobi* structure in the next section, and subsequently apply it to the system shown in Figure 3-9.

### 3-3-1 Jacobi structures

A *Jacobi structure* on a manifold  $M$  is a bilinear mapping on the functions on  $M$  [51]

$$\{ , \} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) : (f, g) \mapsto \{f, g\}$$

called the *Jacobi bracket*. This mapping needs to satisfy three properties:

(i) it must be *skew-symmetric*

$$\{f, g\} = -\{g, f\},$$

(ii) it satisfies the *Jacobi identity*

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\},$$

(iii) it is *local*

$$\text{supp } \{f, g\} \subset \text{supp } f \cap \text{supp } g,$$

where  $\text{supp}$  denotes the support of a function.

Manifolds equipped with a Jacobi structure are called *Jacobi manifolds*.

It can be shown that any Jacobi structure can be uniquely defined in terms of a bivector field<sup>23</sup>  $\Lambda$  and a vector field  $R$ . The corresponding Jacobi bracket is then given by: [16, 51]

$$\{f, g\} = \Lambda(df, dg) + f(R \lrcorner dg) - g(R \lrcorner df).$$

---

<sup>23</sup>A *bivector* is the contravariant counterpart of a 2-form: it is a skew-symmetric tensor with valence  $(2, 0)$  [52].

Not just any choice of bivector field and vector field give rise to a Jacobi structure. As shown by Lichnerowicz [53],  $\Lambda$  and  $R$  must satisfy two conditions:

$$[\![\Lambda, \Lambda]\!] = 2R \wedge \Lambda \quad [\![R, \Lambda]\!] = 0,$$

where  $[\![ , ]!]$  is the *Schouten bracket*<sup>24</sup>. A Jacobi manifold is therefore a triple  $(M, \Lambda, R)$  [16].

A Jacobi structure induces a mapping from the functions on the manifold to the vector fields on the manifold (sometimes called the *Hamiltonian correspondence*) [55, 56] defined as follows:

$$\Psi : C^\infty(M) \rightarrow \mathfrak{X}(M) : \quad X_f = \Lambda^\sharp(df) + fR, \quad (3-53)$$

where  $f$  is the Hamiltonian function,  $X_f$  the associated Hamiltonian vector fields. The sharp mapping  $\Lambda^\sharp$  is defined as:

$$\Lambda^\sharp : T^*M \rightarrow TM : \quad \Lambda^\sharp(\eta) = \Lambda \lrcorner \eta,$$

or equivalently

$$\Lambda(\eta, \chi) = \Lambda^\sharp(\eta) \lrcorner \chi.$$

We will now see that both symplectic and contact manifolds are particular instances of a Jacobi structure, as well as the Hamiltonian systems defined on them.

### Symplectic manifolds are Jacobi

For a symplectic manifold  $(M, \omega)$  with dimension  $2n$ , the vector field  $R$  is simply zero and the bivector  $\Lambda$  field is defined by:

$$\Lambda(\eta, \chi) = \omega(\omega^\sharp(\eta), \omega^\sharp(\chi)) \quad \eta, \chi \in \Omega^1(M),$$

with  $\omega^\sharp$  defined as in Equation (3-3).

If  $\omega$  is expressed in Darboux coordinates, i.e.

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i,$$

then the associated bivector can be found to be Equation (3-53):

$$\Lambda = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

The associated Jacobi bracket reverts to the familiar Poisson bracket on the symplectic manifold. A Poisson structure is a particular instance of a Jacobi structure where the vector field  $R$  vanishes. This makes the Poisson/Jacobi bracket into a *derivation* on the algebra of smooth functions (over the real numbers): consequently, Poisson brackets satisfy the Leibniz property in addition to the conditions for Jacobi brackets given above [51].

---

<sup>24</sup>The Schouten bracket of an  $r$ -vector field  $A$  and an  $s$ -vector field  $B$  on a manifold is a  $(r+s-1)$ -vector field  $[\![A, B]\!]$ , defined by its action on a closed  $(r+s-1)$ -form  $\beta$  as follows:

$$[\![A, B]\!](\beta) = (-1)^{rs+s} A \lrcorner d(B \lrcorner \beta) + (-1)^r B \lrcorner d(A \lrcorner \beta).$$

For  $r = s = 1$ , the Schouten bracket simply reverts to the ordinary Lie bracket [54].

### Contact manifolds are Jacobi

A strictly contact manifold<sup>25</sup>  $(M, \alpha)$  with dimension  $2n + 1$  is also a Jacobi manifold. The vector field  $R = R_\alpha$  is the Reeb vector field. and the bivector  $\Lambda$  is equal to

$$\Lambda(\eta, \chi) = d\alpha(d\alpha^\sharp(\eta), d\alpha^\sharp(\chi)),$$

where  $d\alpha^\sharp$  is defined as in Equation (3-12) and  $\eta, \chi$  are semi-basic 1-forms on  $M$ .

If  $\alpha$  is expressed in Darboux coordinates:

$$\alpha = dq_0 - \sum_{i=1}^n p_i dq_i,$$

then

$$R = \frac{\partial}{\partial q_0}.$$

The expression for the bivector can be found as follows (by comparison with Equation (3-53)):

$$\begin{aligned} \Lambda^\sharp(\zeta) &= d\alpha^\sharp(\zeta - (R \lrcorner \zeta)\alpha) \\ \Lambda^\sharp(\zeta) &= \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) \lrcorner (\zeta - (R \lrcorner \zeta)\alpha) \\ \Lambda^\sharp(\zeta) &= \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) \lrcorner \zeta - \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) \lrcorner ((R \lrcorner \zeta)\alpha) \\ \Lambda^\sharp(\zeta) &= \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) \lrcorner \zeta - \sum_{i=1}^n p_i \frac{\partial}{\partial p_i} (R \lrcorner \zeta) \\ \Lambda^\sharp(\zeta) &= \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) \lrcorner \zeta - \left( \sum_{i=1}^n p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_0} \right) \lrcorner \zeta. \end{aligned} \tag{3-54}$$

From this expression, we gather that

$$\Lambda = \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) + \left( \sum_{i=1}^n p_i \frac{\partial}{\partial q_0} \wedge \frac{\partial}{\partial p_i} \right).$$

We will now apply the Jacobi structure to general mechanical systems.

### 3-3-2 Jacobi structure of mechanical systems

The geometric structure of a mechanical system has four components:

- An odd-dimensional manifold  $M = T^*Q \times \mathbb{R}$ , where  $Q$  is the configuration manifold. It is extended by one dimension to incorporate the dissipated (internal) energy  $U$  and therefore always odd-dimensional. In the following, we assume the ‘Darboux’ coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n, U)$ . This manifold has a bundle structure  $M \xrightarrow{\pi} T^*Q$ , where  $\pi$  is the projection map that ‘forgets’ the  $U$ -coordinate.

---

<sup>25</sup>For contact structures that are not globally determined by a single contact form, Marle [51] introduced the concept of a *Jacobi bundle*.

- A closed 2-form with constant rank  $2n$ , defined as the negative of the exterior derivative of the Liouville form on  $T^*Q$ :

$$\omega = -d\vartheta = \sum_{i=1}^n dq_i \wedge dp_i,$$

i.e.  $\omega$  is the canonical symplectic 2-form on  $T^*Q$ .

- A *dissipation form*  $\alpha$  that encodes the work done by the system on its environment:

$$\alpha = dU - \beta,$$

where  $\beta = \pi^*\beta_{T^*Q}$  is a pullback of a form on  $T^*Q$ , i.e. it does not depend on  $U$ . When there is no dissipation,  $\beta = 0$ .

- A *Hamiltonian function*  $H \in C^\infty(M)$ , equal to the sum of the mechanical energy of the system and the internal energy:

$$H = E + U,$$

with  $E = E(q_1, \dots, q_n, p_1, \dots, p_n)$  the mechanical energy of the system.

In the purely conservative case discussed in Section 3-1, there is no dissipation, so the extra dimension in  $U$  does not play a role, and the system may be completely described on  $T^*Q$  with its symplectic structure.

For the simple dissipative mechanical systems in Section 3-2, the form  $\alpha$  would both encode the pairing structure *and* the dissipation form, since  $d\alpha$  would be of the form  $dU - \gamma\vartheta$ . We now separate both functionalities (i.e. pairing and dissipation) to distinct components, for which the symplectic and contact systems are particular cases.

The Jacobi structure for general mechanical systems is constructed in an analogous manner to the one for contact manifolds, apart from the fact that we now have a separate 2-form  $\omega$ , instead of using  $d\alpha$ . We can already expect that this will work given the right conditions, for the derivations in Section 3-2 did not use the fact that  $d\alpha$  is indeed the exterior derivative of  $\alpha$ .

However, not just any  $\omega$  and  $\alpha$  will make this work. Recall that the maximum nonintegrability of  $\alpha$  is equivalent to  $\alpha \wedge (d\alpha)^n$  being a volume form on the contact manifold. Along the same line, we require the following condition on  $\omega$  and  $\alpha$ :

$$\alpha \wedge (\omega)^n \neq 0 \tag{3-55}$$

everywhere on  $M$ ; that is to say, it is a volume form on  $M$  [55]. If  $M$ ,  $\omega$  and  $\alpha$  are defined as given above, this condition is clearly satisfied:

$$\alpha \wedge (\omega)^n = n! dU \wedge \left( \bigwedge_{i=1}^n dq_i \wedge dp_i \right).$$

If Equation (3-55) is satisfied we can — similarly to the discussion in Section 3-2-2 — define the splitting of the tangent bundle as follows:

$$T^*M = \ker \alpha \oplus \ker \omega.$$

Vector fields in the kernel of  $\alpha$  are called *horizontal*, while vector fields in the kernel of  $\omega$  are *vertical*. Define the *Reeb vector field*  $R$  (for this Jacobi structure) as the unique vector field that satisfies the following conditions:

$$R \lrcorner \alpha = 1 \quad R \lrcorner \omega = 0.$$

In Darboux coordinates, we have

$$R = \frac{\partial}{\partial U}.$$

The vector field of the Jacobi structure is equal to the Reeb vector field.

In addition, define *semi-basic forms* as the forms that annihilate the vertical vector fields; in Darboux coordinates, they are forms that have no component in  $dU$ .

The decompositions of vector fields into horizontal and vertical components, and of forms into semi-basic components and multiples of  $\alpha$  are analogous to Equations (3-9) and (3-10) respectively.

The construction of  $\Lambda$  is analogous to the case of contact manifolds: the sharp mapping first isolates the semi-basic component of the argument, after it is mapped to a horizontal vector field through  $\omega^\sharp$ :

$$\Lambda^\sharp(\zeta) = d\alpha^\sharp(\zeta - (R \lrcorner \zeta)\alpha).$$

Using the coordinates defined above, we find:

$$\Lambda = \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) - \frac{\partial}{\partial q_0} \wedge \left[ \sum_{i=1}^n \left( \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right) \lrcorner \beta \right]. \quad (3-56)$$

The dynamics of the general mechanical system are then equal to

$$X_H = \Lambda^\sharp(dH), \quad (3-57)$$

assuming again that  $H$  is numerically equal to zero, so as to make the vertical component of the Hamiltonian vector field disappear.

For computational convenience, this mapping can also be represented by a matrix:

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{U} \end{pmatrix} &= \left[ \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_q \\ \beta_p \\ 0 \end{pmatrix} \right] (\nabla H) \\ &= \left[ \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -\beta_p \\ 0 & 0 & \beta_q \\ \beta_p & -\beta_q & 0 \end{pmatrix} \right] (\nabla H) \\ &= \begin{pmatrix} 0 & I_n & \beta_p \\ -I_n & 0 & -\beta_q \\ -\beta_p & \beta_q & 0 \end{pmatrix} (\nabla H), \end{aligned} \quad (3-58)$$

where  $\beta_q$  and  $\beta_p$  represent the  $q$ - and  $p$ -components of the form  $\beta$ , and  $\nabla H$  is the gradient of  $H$ . The minus sign in front of the  $\beta_p$ -components is usually cancelled because those components often already carry a minus sign as a consequence of the power direction of the bond connecting the 0- and 1-junctions on which they are defined (e.g. in the case of the serial damper in Section 3-2-5).

### Application to 2-DOF mechanical system

We now revisit the mechanical system shown in Figure 3-9. The four structure components are

- The manifold  $M = \mathbb{R}^5 = \mathbb{R} \times T^*Q$ , for which we choose coordinates  $(U, q_1, q_2, p_1, p_2)$ .
- The 2-form  $\omega$  is the canonical symplectic structure on  $T^*Q$ :

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

- The dissipation form is given by Equation (3-52):

$$\beta = \left[ \left( \frac{b_1}{m_1} + \frac{b_2}{m_1} \right) p_1 - \left( \frac{b_2}{m_2} \right) p_2 \right] dq_1 + \left[ \left( \frac{b_2}{m_2} \right) p_2 - \left( \frac{b_2}{m_1} \right) p_1 \right] dq_2.$$

- The Hamiltonian is equal to the sum of kinetic, potential and internal energy in the system:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_3q_2^2 + \frac{1}{2}k_2(q_2 - q_1)^2 + U.$$

The expression for  $\beta$  is given by Equation (3-52) as a part of the dissipation form  $\alpha$ . The exterior derivative of  $H$  is

$$dH = \frac{p_1}{m_1} dp_1 + \frac{p_2}{m_2} dp_2 + [k_1q_1 + k_2(q_1 - q_2)] dq_1 + [k_3q_2 + k_2(q_2 - q_1)] dq_2 + dU.$$

Using either Equations (3-56) and (3-57) or Equation (3-58), we obtain the correct equations of motion for the system:

$$\begin{aligned} \dot{q}_1 &= \frac{p_1}{m_1}, \\ \dot{q}_2 &= \frac{p_2}{m_2}, \\ \dot{p}_1 &= -\frac{b_1}{m_1}p_1 - \frac{b_2}{m_1}p_1 + \frac{b_2}{m_2}p_2 - k_1q_1 - k_2q_1 + k_2q_2, \\ \dot{p}_2 &= -\frac{b_2}{m_2}p_2 + \frac{b_2}{m_1}p_1 - k_3q_2 - k_2q_2 + k_2q_1, \\ \dot{U} &= b_1 \frac{p_1^2}{m_1^2} + b_2 \frac{p_2^2}{m_1^2} + b_2 \frac{p_2^2}{m_2^2} - 2b_2 \frac{p_1 p_2}{m_1 m_2}. \end{aligned} \tag{3-59}$$

The reason why the equation for  $U$  is always correct is because we force the vector field to be annihilate the dissipation form  $\alpha$ ; as such, any work done by the dampers must constitute to the change in  $U$ . Because the Hamiltonian is equal to zero, there are no other contributions to  $\dot{U}$ .

Observe from Equation (3-59) that the rate of change of  $U$  can be written in terms of the *Rayleigh dissipation matrix*:

$$\dot{U} = \begin{pmatrix} \frac{p_1}{m_1} & \frac{p_2}{m_2} \end{pmatrix} \begin{pmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 \end{pmatrix} \begin{pmatrix} \frac{p_1}{m_1} \\ \frac{p_2}{m_2} \end{pmatrix}.$$

It is important to point out though that our method does not rely on the fact that the damping force relies on the momenta/velocities in a linear fashion: we made no assumptions on  $\beta$ , apart from the fact that it does not depend on  $U$ . Hence, any type of force that performs work in the direction of the generalized coordinates can be incorporated in this fashion. Consequently, we may use this method just as well to include external forces for nonautonomous systems, which is the subject of the next section.

### Nonautonomous systems

For the purposes of control, the above formalism may also be extended with external inputs; i.e. flow sources, effort sources, or sources that contribute to the internal energy. In the port-Hamiltonian formalism proposed by Van Der Schaft [57], the external sources are simply added to their respective Hamiltonian functions. It is then possible to interconnect several mechanical systems by means of power-preserving connections.

When subject to external inputs, the vector field  $X$  governing the dynamics of the mechanical system is the superposition of the time-dependent ‘input vector field’  $X_u$  and the Hamiltonian vector field generated by the Jacobi-structure:

$$X = X_u + X_H. \quad (3-60)$$

In matrix form, the equations of motion given by Equation (3-58) become

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \\ \dot{U} \end{pmatrix} = \begin{pmatrix} 0 & I_n & \beta_p \\ -I_n & 0 & -\beta_q \\ -\beta_p & \beta_q & 0 \end{pmatrix} (\nabla H) + \begin{pmatrix} \mathbf{u}_q \\ \mathbf{u}_p \\ u_U \end{pmatrix}, \quad (3-61)$$

where  $\mathbf{u}_q$ ,  $\mathbf{u}_p$  and  $u_U$  represent the flow sources, effort sources and source of internal energy (i.e. a heat source).



---

# Chapter 4

---

## Split-Quaternion Representations of Dynamical Systems

In this chapter, the geometric connection is made between the algebra of split-quaternions and the qualitative behavior of two-dimensional linear dynamical systems.

### 4-1 The algebra of split-quaternions

#### 4-1-1 Basic properties of split-quaternions

Like conventional quaternions, the split-quaternions form a number system that consists of linear combinations of four basis elements, which will be denoted by  $1$ ,  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ .<sup>1</sup> The algebra of split-quaternions is associative but not commutative — formally speaking, we are dealing with an algebraic structure called a *noncommutative ring*. The multiplication table for the split-quaternion algebra is shown in Table 4-1. The set of split-quaternions is denoted by  $\hat{\mathbb{H}}$ , since  $\mathbb{H}$  is reserved for conventional quaternions.<sup>2</sup>

**Table 4-1:** Multiplication table for the split-quaternion algebra.

	$1$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$1$	$1$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	$\hat{i}$	$-1$	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$\hat{j}$	$-\hat{k}$	$1$	$-\hat{i}$
$\hat{k}$	$\hat{k}$	$\hat{j}$	$\hat{i}$	$1$

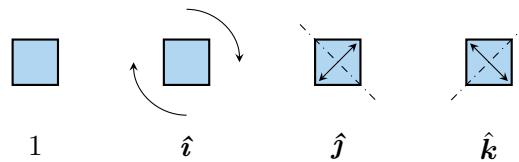
<sup>1</sup>Even though they behave similarly, the imaginary unit  $i$  is not to be confused with the split-quaternion basis element  $\hat{i}$ , because they belong to different number systems.

<sup>2</sup>The ‘ $\mathbb{H}$ ’ is in honour of sir William Rowan Hamilton, who also developed the Hamiltonian formalism: the fruits of his work truly form the central theme in this thesis. [58]

The distinctive feature that sets split-quaternions apart from conventional quaternions resides in the diagonal elements of Table 4-1. For quaternions, all the nonreal basis elements square to  $-1$ , which is not the case for the split-quaternions (only  $\hat{i}$  does). This is also the reason why split-quaternions are ‘split’, for this difference in sign makes their norm (to be defined later) into an indefinite quadratic form. That is to say, whereas quaternions have a ‘metric signature’ (in a very imprecise sense of the word metric) of  $(+, +, +, +)$ , for the split-quaternions we have  $(+, +, -, -)$ . The difference ‘metric’ signature makes the algebra of split-quaternions different from its conventional quaternion counterpart.

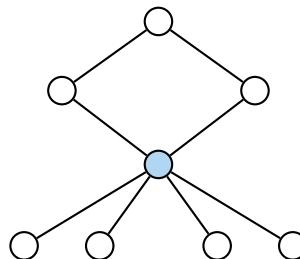
### The dihedral group

The basis elements of the split-quaternions  $\{1, \hat{i}, \hat{j}, \hat{k}\}$  form a finite group under multiplication, namely the *dihedral group*  $D_4$ , which represents all the symmetries of a square: the identity, a 90-degree rotation and two reflections, as illustrated in Figure 4-1. [59]



**Figure 4-1:** The dihedral group  $D_4$  is the symmetry group of a square. This group is isomorphic to the group formed by  $1, \hat{i}, \hat{j}$  and  $\hat{k}$  under multiplication.

The structure of the dihedral group can be visualized by its *cycle graph* in Figure 4-2. Many important properties of the split-quaternion algebra and the applications in this chapter can be traced back to the shape of this cycle graph. One example is the split nature of the quaternions: the  $\hat{i}$ -element generates an order-4 cycle, while  $\hat{j}$  and  $\hat{k}$  generate order-2 cycles (in contrast, the cycle graph for conventional quaternions is entirely symmetric for all these elements). [59]



**Figure 4-2:** Cycle graph of the dihedral group. There are five cycles: one of order four which represents the rotations (or the element  $\hat{i}$ ), and four order-2 cycles, which are all the possible reflections. The colored element represents the identity.

### The split-quaternion norm

Likewise, (split)-quaternions have a similar notion: a *scalar* (or real) and *vector* (or imaginary) components. For an arbitrary split-quaternion  $a \in \hat{\mathbb{H}}$ , [60]

$$a = a_0 + a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}$$

the real part is  $\text{sca}(a) = a_0$  and the vector part is  $\text{vec}(a) = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}$ . For convenience, the vector part will be referred to in traditional bold vector notation:

$$\mathbf{a} \equiv \text{vec}(a) \equiv a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}.$$

Furthermore, for every split-quaternion there is a unique *conjugate*

$$a^* = \text{sca}(a) - \text{vec}(a) = a_0 - a_1 \hat{\mathbf{i}} - a_2 \hat{\mathbf{j}} - a_3 \hat{\mathbf{k}},$$

which we require to define the *squared split-quaternion norm*:

$$\mathcal{N}: \hat{\mathbb{H}} \rightarrow \mathbb{R}: \mathcal{N}(a) \equiv aa^* = a_0^2 + a_1^2 - a_2^2 - a_3^2. \quad (4-1)$$

As mentioned, this norm is not positive definite, in stark contrast to quaternions (or complex numbers, for that matter).

Split-quaternions can be categorized into three regimes based on their norm being negative, zero or positive. In the tradition of special relativity, these regimes are named<sup>3</sup> [61, 62]

- *timelike* if  $\mathcal{N}(a) > 0$ ,
- *lightlike* if  $\mathcal{N}(a) = 0$ ,
- *spacelike* if  $\mathcal{N}(a) < 0$ .

The *split-quaternion norm* is then defined as

$$\|a\| \equiv \sqrt{|\mathcal{N}(a)|}.$$

### The vector norm

Apart from the split-quaternion norm, we may also define a (square) norm that only considers the *vector part* of the split-quaternion. This squared norm is defined in accordance with the overall squared split-quaternion norm given by Equation (4-1):

$$\mathcal{N}(\mathbf{a}) = a_1^2 - a_2^2 - a_3^2.$$

The notation used here is not abusive:  $\mathbf{a}$  simply refers to the split-quaternion with the same vector part as  $a$  but with zero scalar part. We can therefore use the same function with no ambiguity. Likewise, the vector norm is  $\|\mathbf{a}\| = \sqrt{|\mathcal{N}(\mathbf{a})|}$ .

---

<sup>3</sup>Spacetime is also four-dimensional, but the signature of the Minkowski metric is different from the split-quaternion signature: it is either  $(-, +, +, +)$  or equivalently  $(+, -, -, -)$  depending on the sign convention one chooses to observe. The terminology (i.e. spacelike, timelike, lightlike) applies nonetheless.

The quadratic form of the squared vector norm is not positive-definite either; in the same vein as before, we can therefore classify split-quaternions by the ‘sign’ of their vector part again. We refer to (vectors in) these regimes as *timelike (vectors)*, *spacelike (vectors)* and *lightlike (vectors)* in the same fashion.

In contrast to the larger space of split-quaternions, the space of vectors *does* have a traditional Lorentz (i.e. ‘special relativity’) structure, but in three dimensions instead of four. This is because the signature of the squared vector norm only has one minus sign instead of two. The above expression is equivalent the Lorentz norm applied to a vector in  $\mathbb{R}^3$ ; we will denote  $\mathbb{R}^3$  equipped with the Lorentz norm by  $\mathbb{R}^{1,2}$ , and call it the Lorentzian three-space. [60]

Observe that  $\mathcal{N}(a) < 0 \Rightarrow \mathcal{N}(\mathbf{a}) < 0$ ; that is to say, *a spacelike split-quaternion always has a spacelike vector part*. The converse is not necessarily true. Along the same line, a lightlike split-quaternion can only have a lightlike or spacelike vector part. All possible regime combinations for split-quaternions and their vector parts are listed in Table 4-2. This classification is important because, as discussed in Section 4-2, this classification is precisely equivalent to the qualitative classification of dynamical systems. It will play a central role throughout this chapter.

**Table 4-2:** All the possible combinations of the regime of a split-quaternion and its vector part. Spacelike split-quaternions can only have a spacelike vector, while lightlike split-quaternions can only have lightlike or spacelike vector parts.

		$\mathcal{N}(\mathbf{a})$		
		spacelike	lightlike	timelike
$\mathcal{N}(a)$	spacelike	(1)	—	—
	lightlike	(2)	(3)	—
	timelike	(4)	(5)	(6)

In the remainder of this chapter, the split-quaternion properties that are defined above are used profusely. Table 4-3 provides an overview of these properties and their definition.

**Table 4-3:** Overview of the split-quaternion properties that are often used in this chapter.

Terminology	Notation	Value / definition
Split-quaternion	$a$	$a_0 + a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$
Conjugate split-quaternion	$a^*$	$a_0 - a_1\hat{\mathbf{i}} - a_2\hat{\mathbf{j}} - a_3\hat{\mathbf{k}}$
Squared norm	$\mathcal{N}(a)$	$aa^* = a_0^2 + a_1^2 - a_2^2 - a_3^2$
Norm	$\ a\ $	$\sqrt{ \mathcal{N}(a) }$
Scalar part	$sca(a)$	$a_0$
Vector part	$vec(a) = \mathbf{a}$	$a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$
Squared vector norm	$\mathcal{N}(\mathbf{a})$	$\mathbf{a}\mathbf{a}^* = a_1^2 - a_2^2 - a_3^2$
Vector norm	$\ \mathbf{a}\ $	$\sqrt{ \mathcal{N}(\mathbf{a}) }$

### 4-1-2 Relation with two-dimensional matrix algebra

The utility of split-quaternions (for our purpose) originates from the fact that the algebra of split-quaternions is isomorphic to the algebra of real two-dimensional matrices. This underlies this entire chapter, for it allows us to find an alternative for the traditional matrix description of linear dynamical systems.

#### Algebra isomorphisms

Formally, an algebra is a vector space combined with a vector space  $V$  over a field  $\mathbb{F}$ , combined with an addition operation, scalar multiplication, and an  $\mathbb{F}$ -bilinear product operation  $V \times V \rightarrow V$ . [43]

- The split-quaternion algebra is an algebra over the field real numbers ( $\mathbb{F} = \mathbb{R}$ ), where the multiplication is governed by the split-quaternion multiplication rules (see Table 4-1).
- The set of  $2 \times 2$ -matrices also constitutes an  $\mathbb{R}$ -vector space; matrix multiplication makes it into an algebra.

An algebra isomorphism is an isomorphism between vector spaces that also commutes with the respective product operations in both vector spaces. If  $(V, \bullet)$  and  $(W, \diamond)$  are vector spaces equipped with their product operations, then  $\phi : V \rightarrow W$  is an algebra isomorphism if (i)  $\phi$  is a vector space isomorphism between  $V$  and  $W$ , and (ii) [63]

$$\phi(v_1 \bullet v_2) = \phi(v_1) \diamond \phi(v_2) \quad v_1, v_2 \in V.$$

In the case of the split-quaternions and two-dimensional matrices, it is sufficient to map the basis elements of the split-quaternions to four linearly independent ‘basis’ matrices, and show that the resulting matrices observe the same multiplication rules as defined in Table 4-1. Indeed, define the mapping  $\phi$  by

$$\begin{aligned} \phi : \hat{\mathbb{H}} \rightarrow \mathbb{R}^{2 \times 2} : \quad 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \hat{i} &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \hat{j} &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \hat{k} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4-2}$$

It is easily verified that (i) these matrices span  $\mathbb{R}^{2 \times 2}$  and (ii) that the multiplication rules for split-quaternions are in accordance when translated to the respective matrices under matrix multiplication. Due to the bilinearity of the product, any linear combination of the basis elements will therefore satisfy the rules as well. Hence, we have established an algebra isomorphism between the split-quaternions and the  $2 \times 2$ -matrices.

Based on the mapping  $\phi$  for the basis vectors, a general quaternion is sent to

$$\phi(a_0 + a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = \begin{pmatrix} a_0 + a_3 & a_1 + a_2 \\ a_2 - a_1 & a_0 - a_3 \end{pmatrix}. \tag{4-3}$$

Likewise, the inverse mapping on an arbitrary matrix yields

$$\phi^{-1} \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} = \frac{b_0 + b_3}{2} + \left( \frac{b_1 - b_2}{2} \right) \hat{i} + \left( \frac{b_1 + b_2}{2} \right) \hat{j} + \left( \frac{b_0 - b_3}{2} \right) \hat{k}. \tag{4-4}$$

### Relation between matrix and split-quaternion properties

One of the powerful features of the mapping  $\phi$  is that it maps natural properties of the split-quaternion to natural properties of the associated matrix. Hence, given that  $A = \phi(a)$  with  $a \in \hat{\mathbb{H}}$  and  $A \in \mathbb{R}^{2 \times 2}$ , we have the following correspondence:

- The *sum of matrices* is equivalent to the *sum of split-quaternion*. If  $A = \phi(a)$  and  $B = \phi(b)$ , then

$$a + b = \phi(A + B).$$

- The *matrix multiplication* is equivalent to *split-quaternion multiplication*. If  $A = \phi(a)$  and  $B = \phi(b)$ , then

$$ab = \phi(AB).$$

- The *conjugate* of the split-quaternion maps to the *adjugate* of the matrix:<sup>4</sup>

$$\phi(a^*) = \text{adj}(A).$$

- The *trace* of the matrix coincides with the *real or scalar part* of the split-quaternion:

$$\text{sca}(a) = a_0 = \frac{\text{tr}(A)}{2}.$$

- The *determinant* of the matrix is equal to the *squared norm* of the split-quaternion:

$$\mathcal{N}(a) = \det(A).$$

- The equivalence of the determinant and the split-quaternion norm suggests that the multiplicative inverse of a split-quaternion does not always exist: it must be nonzero. In that case, it is clear that

$$\phi(a^{-1}) = A^{-1} \quad \mathcal{N}(a) \neq 0.$$

The determinant property also shows us what the regime will be of the product of two split-quaternions; this is shown in Table 4-4.

**Table 4-4:** Regime transition under the action of split-quaternion multiplication. The timelike split-quaternions form a group under multiplication, the timelike and spacelike split-quaternions do not: timelike split-quaternions do not have an inverse and the spacelike split-quaternions are not closed.

×	space	light	time
space	time	light	space
light	light	light	light
time	space	light	time

<sup>4</sup>The adjugate of a matrix is the transpose of its cofactor matrix. [45]

**Table 4-5:** Overview of the correspondence between the algebra of split-quaternions and the algebra of  $2 \times 2$  matrices, given that  $A = \phi(a)$  and  $B = \phi(b)$ . In the top section we compare matrices to split-quaternions, i.e. which means that they are related by the isomorphism  $\phi$ . The bottom section compares scalar properties: these are numerically equal.

Matrix	Split-quaternion	
Sum	$A + B$	$a + b$
Matrix product	$AB$	$ab$
Inverse	$A^{-1}$	$a^{-1}$
Adjugate	$\text{adj}(A)$	$a^*$
Determinant	$\det(A)$	$\mathcal{N}(a)$
Trace	$\text{tr}(A)$	$\text{sca}(a) = a_0$
Eigenvalues	$\lambda_A$	$a_0 + ai$

- The eigenvalues of a  $2 \times 2$ -matrix can be expressed in terms of its trace and its determinant:

$$\lambda_A = \frac{\text{tr}(A) \pm \sqrt{\text{tr}^2(A) - 4 \det(A)}}{2}.$$

The argument of the square root is equal to the *negative of the squared vector norm* of  $a$ . We therefore have:

$$\lambda_A = \frac{2a_0 \pm \sqrt{4a_0^2 - 4\mathcal{N}(a)}}{2} = \begin{cases} a_0 \pm i\|a\| & \mathbf{a} \text{ timelike,} \\ a_0 \pm 0 & \mathbf{a} \text{ lightlike,} \\ a_0 \pm \|a\| & \mathbf{a} \text{ spacelike.} \end{cases} \quad (4-5)$$

Hence, the ‘real’ (scalar) and the magnitude of the ‘imaginary’ (vector) parts of the quaternion coincide with the real and imaginary part of the eigenvalues of the matrix.

**Split-quaternions as a Lie algebra** The algebra of  $2 \times 2$ -matrices (or equivalently, of the split-quaternions) also constitute the Lie algebra  $\mathfrak{gl}(2, \mathbb{R})$  of the two-dimensional general linear group  $\text{GL}(2, \mathbb{R})$ . Furthermore, the traceless matrices, or equivalently, the split-quaternions with zero real part form the subalgebra  $\mathfrak{sl}(2, \mathbb{R})$  of the special linear group  $\text{SL}(2, \mathbb{R})$ . These are the volume-preserving automorphisms on  $\mathbb{R}^2$ . Because in  $\mathbb{R}^2$ , volume and area coincide, the special linear group and the symplectic group  $\text{Sp}(1)$  are equivalent. For higher dimensions, this is not the case: area preservation is generally a stronger condition than volume preservation. The Lie algebra elements of the symplectic group are called Hamiltonian matrices; therefore, split-quaternions without real part are referred to as *Hamiltonian*.

## 4-2 Dynamical systems as split-quaternions

### 4-2-1 The algebra of vector fields

The isomorphism between the split-quaternions and the algebra of two-dimensional square matrices exposed in the preceding section can be used to develop an alternative representation of linear dynamical systems. Indeed, an autonomous dynamical system is defined by a *vector field* on the state space. If this vector field is a linear mapping from the state space into the tangent space, it can be represented by a matrix.

#### Basis vector fields

The vector fields corresponding to two-dimensional linear dynamical systems form a vector space on their own, spanned (for example) by the four basis elements shown in Equation (4-2). Each of the basis elements  $1, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$  corresponds to a specific ‘basis’ vector field, denoted by  $X_1, X_{\hat{\mathbf{i}}}, X_{\hat{\mathbf{j}}}$  and  $X_{\hat{\mathbf{k}}}$  respectively. The basis vector fields are shown in Figure 4-3.

The vector field element  $X_1$ , corresponding to the identity element is an infinitesimal dilation, while  $X_{\hat{\mathbf{i}}}$  represents an infinitesimal clockwise rotation, and  $X_{\hat{\mathbf{j}}}$  and  $X_{\hat{\mathbf{k}}}$  are infinitesimal ‘squeeze mappings’, hyperbolic rotations or *Lorentz transformations* along two different sets of principal axes. The binary operation of matrix multiplication translates to the composition of the vector fields.

Apart from the multiplication operation for split-quaternions and matrices, we can also define the *commutator* of a binary operation, which measures exactly by how much two elements fail to commute. For split-quaternions, matrices and vector fields the commutator is defined as (in that order):

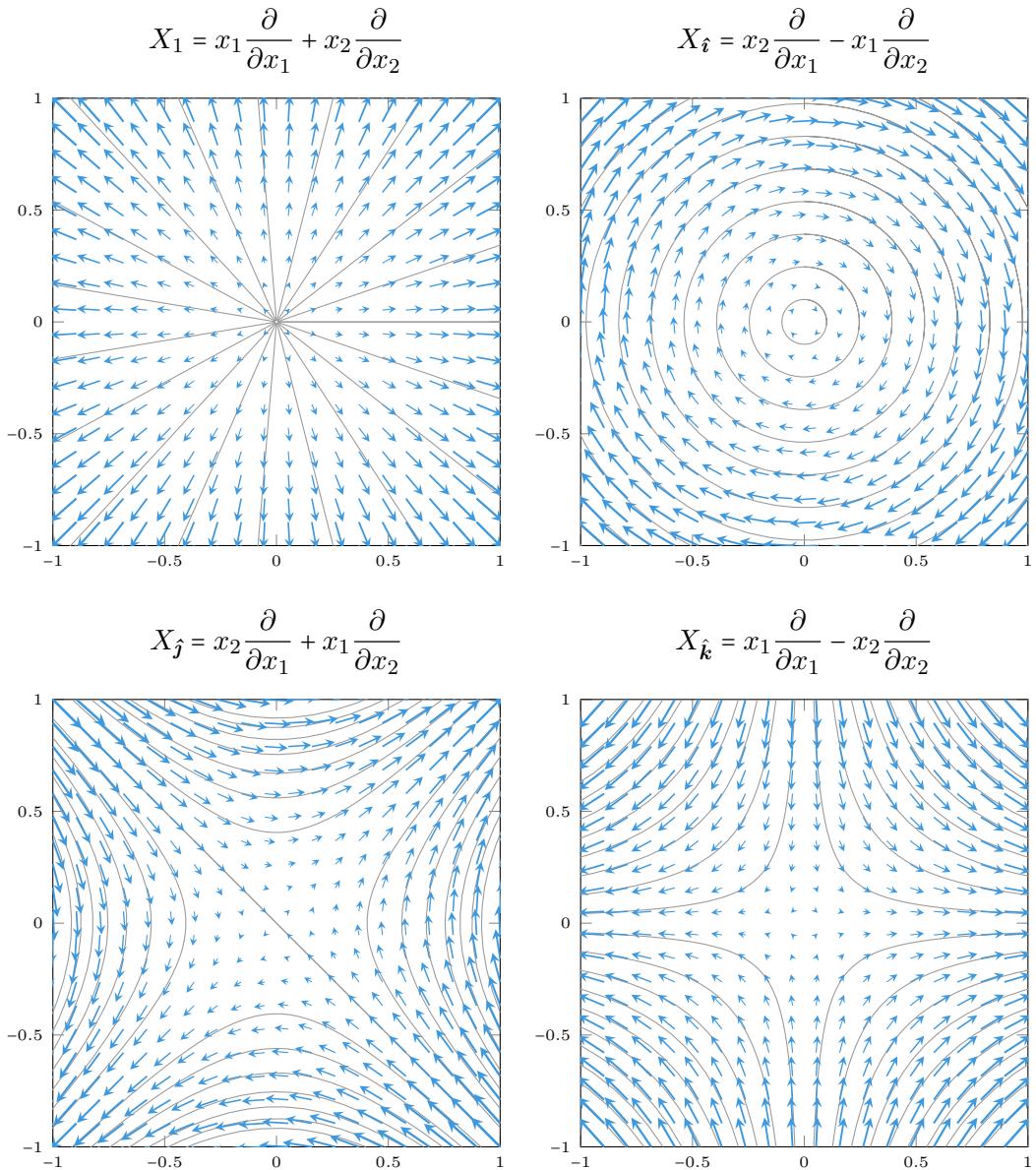
$$\begin{aligned} [a, b] &= ab - ba, \quad a, b \in \hat{\mathbb{H}}, \\ [A, B] &= AB - BA, \quad A, B \in \mathbb{R}^{2 \times 2}, \\ [X, Y] &= \mathcal{L}_X Y, \quad X, Y \in \mathfrak{X}(M), \end{aligned} \tag{4-6}$$

for some smooth manifold  $M$  of the appropriate dimension. For vector fields, the commutator is also referred to as the *Lie bracket* (it therefore defines the *Lie algebra*  $\mathfrak{sl}(2, \mathbb{R})$ ). The commutation relations (or *structure constants*) for the basis vector fields are [43]

$$[X_1, X_{\hat{\mathbf{i}}}] = [X_1, X_{\hat{\mathbf{j}}}] = [X_1, X_{\hat{\mathbf{k}}}] = 0, \tag{4-7}$$

$$[X_{\hat{\mathbf{i}}}, X_{\hat{\mathbf{j}}}] = 2X_{\hat{\mathbf{k}}}, \quad [X_{\hat{\mathbf{i}}}, X_{\hat{\mathbf{k}}}] = -2X_{\hat{\mathbf{j}}}, \quad [X_{\hat{\mathbf{j}}}, X_{\hat{\mathbf{k}}}] = -2X_{\hat{\mathbf{i}}}. \tag{4-8}$$

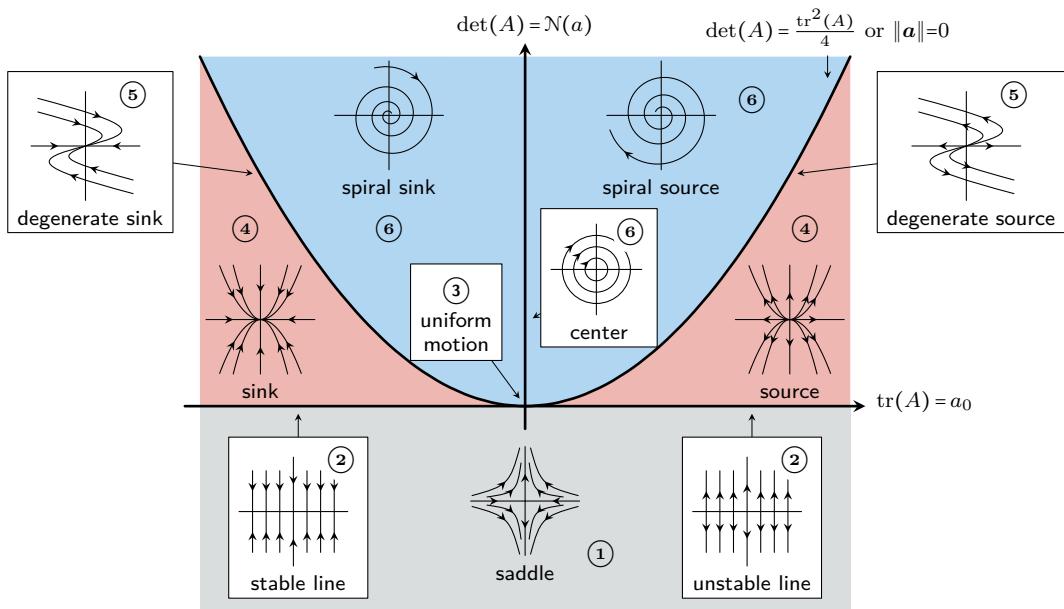
Of course, these commutation relations are exactly the same for the corresponding split-quaternion or matrix basis elements. Scalar multiples of the identity element commute with every other element of the algebra; they are in the *center* of the algebra. Importantly, the vector field  $X_1$  and all its multiples commute with *all* the other vector fields. This has important ramifications: it means that one can consider the action associated with those vector fields as completely separate from the action of the other vector field components, for they do not influence each other in the process. This fact will be used in Section 4-3.



**Figure 4-3:** Basis vector fields corresponding to the basis elements of the split-quaternions.

#### 4-2-2 Classification of dynamical systems

The classification of two-dimensional linear dynamical systems is important, for they also locally represent the fixed points of general nonlinear systems. Traditionally, this decomposition is done according to the eigenvalues of the state transition matrix matrix  $A$ , or equivalently, through a Poincaré diagram as shown in Figure 4-4. Because the split-quaternion norms are directly related to the real and imaginary part of the eigenvalues of the associated matrix, we propose that the split-quaternion representation offers a more convenient alternative, based on their squared (vector) norm, on par with the regimes defined in Table 4-2.



**Figure 4-4:** The classic Poincaré diagram, based on the conventional classification of fixed points based on the trace and determinant of the state transition matrix  $A$ . The corresponding split-quaternion regimes defined in Table 4-2 are displayed as well. The determinant axis coincides with the squared norm of the split quaternion being 0, while all the points on the parabolic line correspond to split-quaternions with zero vector norm. A further distinction is made with the scalar part of the split-quaternion, which, for each of the regimes, determines (asymptotic) (in)stability.

### Spacelike split-quaternion norm

- ① For spacelike split-quaternions, there is only one possibility: a negative split-quaternion norm corresponds to a negative determinant, which means that the fixed point is a *saddle*. We can distinguish one particular case: if the scalar part of the split-quaternion is zero ( $a_0 = 0$ ), the saddle is ‘balanced’, and generates a proper *squeeze mapping*, which is a symplectomorphism of the phase space. The split-quaternion is therefore Hamiltonian. An example of the latter is the linearization of the unstable fixed point of a rotational pendulum.

### Lightlike split-quaternion norm

- ② *Spacelike vector norm*: in this case, there is not just a fixed point but a fixed line in the phase space. This fixed line is stable or unstable depending on the sign of the scalar part of the quaternion.
- ③ *Lightlike vector norm*: this case is degenerate to the second degree; it coincides with the origin in the Poincaré diagram. The associated vector field is purely translational. An example is an object in uniform motion.

### Timelike split-quaternion norm

- ④ *Spacelike vector norm:* this case gives rise to eigenvalues that are purely real; the fixed point is called a *node*. Depending on the sign of the scalar part, the fixed point can be an unstable node or *source* ( $a_0 > 0$ ) or a stable node or *sink* ( $a_0 < 0$ ). An example of such a system is the overdamped harmonic oscillator.
- ⑤ *Lightlike vector norm:* In this case, the two eigenvalues are equal to each other. This can be seen from the expression of the vector norm in terms of the matrix properties:

$$\mathcal{N}(\mathbf{a}) = \mathcal{N}(a - a_0) = \det\left(A - \frac{\text{tr}(A)}{2}I\right).$$

When the two eigenvalues are identical,  $\text{tr}(A) = 2\lambda$ , the above expression necessarily vanishes and the split-quaternion therefore has a lightlike vector part. There are then two subcategories to be distinguished, depending on whether the matrix is diagonalizable or not:

- If the matrix is diagonalizable, it *must* be a scalar multiple of the identity [64]. This is because the rank of the matrix  $A - \lambda I$  must be equal to 0 for the eigenspace to be two-dimensional. The corresponding split-quaternion is then *purely real*; i.e.  $a = \lambda$ . Because it has no vector part, the vector is then lightlike in the most trivial fashion. The associate fixed point is then a *star*, which can either be stable or unstable depending on the sign of the real part (this particular case is not shown in Figure 4-4).
- Conversely, if the vector norm is zero but the vector part not, the matrix is not diagonalizable for the reason mentioned above: the null space of  $A - \lambda I$  is one-dimensional, and can therefore not be spanned by two independent eigenvectors. The fixed point is an *improper node*, which is again stable or unstable depending on the sign of the real part.

An example of this type of system is the critically damped harmonic oscillator.

- ⑥ *Timelike vector norm:* this really is the only general case where the eigenvalues of  $A$  are complex. If  $a_0 = 0$ , the eigenvalues are imaginary and the fixed point is a *center*. Again, for  $a_0 > 0$  it is an *unstable spiral node* and for  $a_0 < 0$  a *stable spiral node*. An example is an underdamped (or even undamped) harmonic oscillator.

It is clear from the present discussion that the split-quaternions offer a very natural representation of linear dynamical systems, and their natural properties translate directly to the classification of the qualitative behavior of these systems.

### 4-2-3 The exponential function of split-quaternions

Just like the concept of the exponential function was originally generalized for square matrices, we can do the same for split-quaternions in an analogous manner. As such, the *split-quaternion exponential function* is defined as

$$\exp(a) \equiv \sum_{k=0}^{\infty} \frac{a^k}{k!} \quad a \in \hat{\mathbb{H}}.$$

**Table 4-6:** Overview of the classification of fixed points based on the regime of the associated split-quaternion and its vector part.

#	$\mathcal{N}(a)$	$\mathcal{N}(\mathbf{a})$	Fixed point
①	spacelike	spacelike	Saddle
②	lightlike	spacelike	Line
③	lightlike	lightlike	Uniform motion
④	timelike	spacelike	Source / sink
⑤	timelike	lightlike	Degenerate node
⑥	timelike	timelike	Spiral node / center

Because this definition is identical to the one for matrices, we may, as a result of the isomorphism defined in Section 4-1-2, also expect the exact same result; i.e.

$$\exp(a) = \phi^{-1}(\exp(\phi(a)))$$

where the second exponential function refers to the matrix exponential instead.

To evaluate the exponential function of a split-quaternion, let us first use the following property of the matrix exponential [65]

$$AB = BA \quad \Rightarrow \quad \exp(A + B),$$

i.e. we can only ‘split’ the exponential of a sum if the two elements *commute*. We can regard an arbitrary split-quaternion

$$a = a_0 + a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

as the sum of  $a_0$  and  $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ . The real part is distinguished from the other three parts in the sense that it commutes with every other element (cf. Section 4-2-1). We may therefore use the former property and apply it to the split-quaternion exponential as well:

$$\exp(a) = e^{a_0} \exp(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}).$$

We therefore only have to be concerned with the evaluation exponential of  $\mathbf{a}$ . To do so, observe that we can consider the vector part of a split-quaternion to be a split-quaternion in its own right, but with zero real part. This means that  $\mathbf{a}^* = -\mathbf{a}$ , and the squared vector norm is simply the negative of the square of the vector part:

$$\mathcal{N}(\mathbf{a}) = \mathbf{a}\mathbf{a}^* = -\mathbf{a}^2.$$

Let us now introduce the concept of *unit split-quaternion vectors*, which are vector split-quaternions with a vector norm of  $\pm 1$ . The unit vector may be obtained by normalization of the vector part:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\sqrt{|\mathcal{N}(\mathbf{a})|}} \quad \mathcal{N}(\mathbf{a}) \neq 0,$$

which squares to

$$\hat{\mathbf{a}}^2 = -\mathcal{N}(\hat{\mathbf{a}}) = -\frac{\mathcal{N}(\mathbf{a})}{|\mathcal{N}(\mathbf{a})|} = -\text{sgn}(\mathcal{N}(\mathbf{a})).$$

Normalizing lightlike vectors is not possible, because they have all the same length of zero: there is no point in making the distinction between vector and unit vector. Based on the regime of the vector part, three possibilities arise: [66, 67]

- If  $\mathbf{a}$  is timelike, then  $\hat{\mathbf{a}}^2 = -1$ . We can therefore say that the unit vector ‘behaves’ like the imaginary unit  $i$  ( $i^2 = -1$ ). In general, we can identify the split-quaternion (with timelike vector part)  $a_0 + \|\mathbf{a}\|\hat{\mathbf{a}}$  with the *complex number*  $a_0 + \|\mathbf{a}\|i$ .
- If  $\mathbf{a}$  is lightlike, then  $\mathbf{a}^2 = 0$ , and the notion of the unit vector is not well-defined. Because the vector is nilpotent with degree 2, it is analogous to the nilpotent unit  $\varepsilon$  (for which we have that  $\varepsilon^2 = 0$ ). Split-quaternions with timelike vector part can be identified with the *dual number*  $a_0 + \varepsilon$ .
- Finally, if  $\mathbf{a}$  is spacelike, then  $\hat{\mathbf{a}}^2 = 1$ . The unit vector behaves like the idempotent unit  $j$ , with defining property  $j^2 = 1$  ( $j \notin \mathbb{R}$ ).<sup>5</sup> Likewise, a split-quaternion with spacelike vector part is analogous to the *split-complex number* (or hyperbolic number)  $a_0 + \|\mathbf{a}\|j$ .

The connection between split-quaternions and the generalized complex numbers<sup>6</sup> (i.e. complex, dual and split-complex) sheds some additional light on the behavior of the eigenvalues of the associated matrix  $A$  by means of the root locus plot (see Equation (4-5)). A typical branch of the root locus consists of a complex pole pair approaching the real axis when the gain is increased. When they finally collide on the real axis, they each go their opposite ways on the real axis, essentially breaking the symmetry with respect to the real axis. The split-quaternions and hypercomplex numbers paint a slightly more elegant picture, which is shown in Figure 4-5:

- As shown above, when the pole pair is complex, the associated split-quaternion vector is timelike. The eigenvalues are naturally conjugate with respect to the real axis, i.e.

$$\lambda_A = a_0 \pm \|\mathbf{a}\|i.$$

- When the pole pair collides on the real axis (often called the *branch point*), the imaginary part of the eigenvalue is zero, and the vector is timelike. Observe that we can make the case that, because the branches continue afterwards in a separate manner, they cannot be *exactly* the same. Indeed, the eigenvalues are

$$\lambda_A = a_0 \pm \varepsilon.$$

The nilpotent unit  $\varepsilon$  is often interpreted as a differential, or an infinitesimally small quantity.<sup>7</sup> We argue that in this case, the pole pair is still conjugate, but the poles differ only by an infinitesimal amount.

---

<sup>5</sup>Again, we must take care not to confuse the hyperbolic unit with the split-quaternion basis element  $\hat{j}$ . They behave the same, and are related in the sense that they give rise to ‘split’ behavior, but are part of a very different number systems.

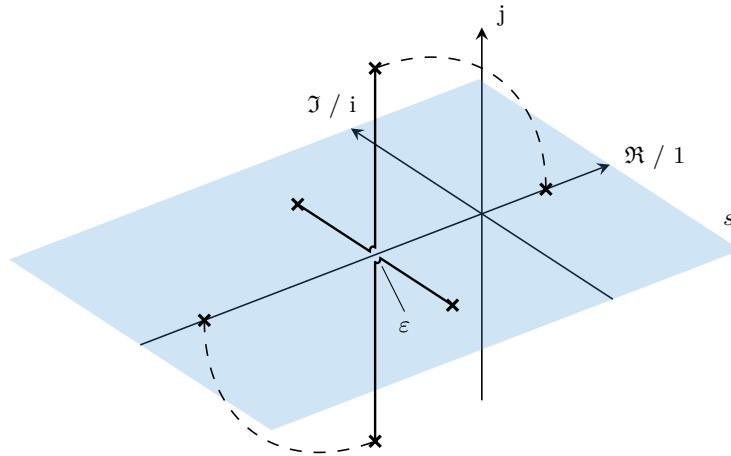
<sup>6</sup>For a more detailed account of generalized complex numbers, the reader is referred to Harkin and Harkin [67].

<sup>7</sup>A common application of dual numbers is automatic differentiation: because higher powers vanish, they can be used to generate first-order polynomial approximations. The unit ‘circle’ for dual numbers consists of two vertical lines crossing the horizontal axis at  $\pm 1$ . These lines can again be interpreted as linear approximations of the actual unit circle (or unit hyperbola) associated to (split-)complex numbers. The plane spanned by the  $j$ -axis and the real axis is the split-complex plane. The ‘projection’ to the real axis is in this plane a reflection with respect to the light cone (first diagonal).

- When the gain is increased further, the poles are real and the symmetry with respect to the real axis is broken. However, we can infer from the preceding discussion that the imaginary part is now hyperbolic instead, i.e.

$$\lambda_A = a_0 \pm \|\mathbf{a}\|j.$$

Of course, it is possible to project these points on the real axis, but this obscures the natural symmetry of the root locus branch. In Figure 4-5, we therefore put the hyperbolic part on a third axis.



**Figure 4-5:** Generalized version of a root locus plot in terms of hypercomplex numbers. The traditional root locus is set in the complex  $s$ -plane (shown in blue), but we added a third axis for the hyperbolic part of the eigenvalue. When the gain is increased, the initially complex pole pair ventures towards the real axis. If the pole pair is critically damped, both poles are separated from the real axis by an infinitesimal distance of  $\varepsilon$ . Increasing the gain even more pushes the pole pair into the hyperbolic regime (the associated split-quaternion vector is now spacelike). Observe that in this picture, the symmetry with respect to the real axis is preserved. In the traditional root locus, these points are projected onto the real axis, indicated by the dashed lines.

Let us now return to the exponential function. We can manipulate the definition of  $\exp(a)$  as follows:

$$\begin{aligned} \exp(a) &= e^{a_0} \left( \sum_{k=0}^{\infty} \frac{a^k}{k!} \right) \\ &= e^{a_0} \left[ \sum_{k=0}^{\infty} \frac{(a^2)^k}{(2k)!} + \sum_{k=0}^{\infty} \frac{a(a^2)^k}{(2k+1)!} \right]. \end{aligned}$$

Furthermore, if  $a$  is not lightlike, we have:

$$\exp(a) = e^{a_0} \left[ \sum_{k=0}^{\infty} \frac{\|\mathbf{a}\|^{2k} (\hat{\mathbf{a}}^2)^k}{(2k)!} + \hat{\mathbf{a}} \sum_{k=0}^{\infty} \frac{\|\mathbf{a}\|^{2k+1} (\hat{\mathbf{a}}^2)^k}{(2k+1)!} \right].$$

Once again, there are three possibilities, depending on the regime of  $a$ :

- If  $\mathbf{a}$  is *timelike*, then the above expression reverts to

$$\begin{aligned}\exp(a) &= e^{a_0} \left[ \sum_{k=0}^{\infty} \frac{\|\mathbf{a}\|^{2k} (-1)^k}{(2k)!} + \hat{\mathbf{a}} \sum_{k=0}^{\infty} \frac{\|\mathbf{a}\|^{2k+1} (-1)^k}{(2k+1)!} \right], \\ &= e^{a_0} [\cos(\|\mathbf{a}\|) + \hat{\mathbf{a}} \sin(\|\mathbf{a}\|)].\end{aligned}\quad (4-9)$$

This is roughly equivalent to the Euler identity for complex numbers, which is not at all surprising, since we found before that  $\hat{\mathbf{a}}$  can be associated with the imaginary unit if  $\mathbf{a}$  is timelike.

- Secondly, if  $\mathbf{a}$  is *lightlike*, we can simply use the definition of the exponential in its original form:<sup>8</sup>

$$\begin{aligned}\exp(a) &= e^{a_0} \sum_{k=0}^{\infty} \frac{\mathbf{a}^k}{k!}, \\ &= e^{a_0} \left[ 1 + \mathbf{a} + \sum_{k=2}^{\infty} \frac{\mathbf{a}^{k-2} \mathbf{a}^2}{k!} \right], \\ &= e^{a_0} (1 + \mathbf{a}).\end{aligned}\quad (4-10)$$

- Finally, if  $\mathbf{a}$  is *spacelike*, we have

$$\begin{aligned}\exp(a) &= e^{a_0} \left[ \sum_{k=0}^{\infty} \frac{\|\mathbf{a}\|^{2k}}{(2k)!} + \hat{\mathbf{a}} \sum_{k=0}^{\infty} \frac{\|\mathbf{a}\|^{2k+1}}{(2k+1)!} \right], \\ &= e^{a_0} [\cosh(\|\mathbf{a}\|) + \hat{\mathbf{a}} \sinh(\|\mathbf{a}\|)].\end{aligned}\quad (4-11)$$

The exponential map of a split-quaternion can be used to obtain the solution of the corresponding linear differential equation. For the linear ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

has the following solution [69]

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0,$$

where the one-parameter group of transformations generated by  $\exp(At)$  is referred to as the *flow* of the vector field  $A\mathbf{x}$ . Hence, for two-dimensional systems, the matrix  $A$  can be represented by a split-quaternion, and we have just derived easy and insightful ways to evaluate its exponential:

$$\exp(at) = \begin{cases} e^{a_0 t} [\cos(\|\mathbf{a}\|t) + \hat{\mathbf{a}} \sin(\|\mathbf{a}\|t)] & \mathbf{a} \text{ timelike,} \\ e^{a_0 t} (1 + \|\mathbf{a}\|t) & \mathbf{a} \text{ lightlike,} \\ e^{a_0 t} [\cosh(\|\mathbf{a}\|t) + \hat{\mathbf{a}} \sinh(\|\mathbf{a}\|t)] & \mathbf{a} \text{ spacelike.} \end{cases} \quad (4-12)$$

<sup>8</sup>We can also use the ‘split’ expression, defining that  $0^0 \equiv 1$ , a common convention in power series and algebra. Observe that the trigonometric functions associated to the dual numbers (i.e. the lightlike vectors) are then equal to the small-angle approximation for sin and cos. [67, 68]

Evaluating a matrix exponential by hand usually involves diagonalizing (strictly speaking, finding the Jordan form). The convenience of using split-quaternions instead resides in the fact that they resolve the ambiguity that is naturally present in the eigenvectors of the matrix  $A$ ; especially when they are complex. In the next section, the relation between the eigenvectors and the unit vector  $\hat{\mathbf{a}}$  are discussed in greater detail [70].

## 4-3 Application to mechanical systems

We will now proceed by using a mechanical ‘prototype’ example instead of the generic dynamical system of the preceding section. This mechanical system is the harmonic oscillator with two dampers: one in series and one in parallel, as already discussed in Section 3-2-5. The reason for this is twofold: first, it helps to establish the connection between this chapter and Chapter 3, and secondly, it allows us to gain a physical intuition behind the split-quaternion representation.

In particular, the oscillator with two dampers is chosen because the corresponding state transition matrix the extra damper ‘fills’ the one entry that would otherwise remain zero in the state transition matrix in case of the conventional damped harmonic oscillator. Therefore, this system can represent (in theory) all the possible two-dimensional systems discussed in the previous section.

### 4-3-1 System dynamics

The harmonic oscillator with two dampers is shown in Figure 3-8, and its equations of motion are given by Equation (3-47). In matrix form, the equations of motion are:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -\frac{k}{b_s} & \frac{1}{m} \\ -k & -\frac{b_p}{m} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

Alternatively, using the  $\gamma_p = b_p/m$ ,  $\gamma_s = k/b_s$ , and  $\Omega_n = \sqrt{k/m}$ , we have the following:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} -\gamma_s & \frac{1}{m} \\ -m\Omega_n^2 & -\gamma_p \end{pmatrix}}_A \begin{pmatrix} q \\ p \end{pmatrix}. \quad (4-13)$$

The split-quaternion associated with the  $A$ -matrix of the doubly damped system can easily be found using the mapping defined by Equation (4-4). We must, however, be careful when dealing with physical systems, because the entries of the  $A$ -matrix are not dimensionless. In a vector space, we associate the units with the basis vectors, not with the components. For example, in a two-dimensional vector space spanned by a axis for apples and an axis for pears, and we wish to represent that someone possesses three apples and four pairs, the *components* of that vector are  $(3, 4)$ , and the *unit vectors* are  $(1 \text{ apple}, 1 \text{ pear})$ . Along the same line, we must define the units in the  $A$ -matrix in the split-quaternion basis elements  $1$ ,  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ ,  $\hat{\mathbf{k}}$ . To do so, we define the reference quantities and  $m_0, t_0$ . The basis elements are then mapped in terms of these reference quantities:

$$\phi(1) = \begin{pmatrix} \frac{1}{t_0} & 0 \\ 0 & \frac{1}{t_0} \end{pmatrix}, \quad \phi(\hat{\mathbf{i}}) = \begin{pmatrix} 0 & \frac{1}{m_0} \\ -\frac{m_0}{t_0^2} & 0 \end{pmatrix}, \quad \phi(\hat{\mathbf{i}}) = \begin{pmatrix} 0 & \frac{1}{m_0} \\ \frac{m_0}{t_0^2} & 0 \end{pmatrix}, \quad \phi(\hat{\mathbf{k}}) = \begin{pmatrix} \frac{1}{t_0} & 0 \\ 0 & -\frac{1}{t_0} \end{pmatrix},$$

where, in case we would use SI units,  $m_0 = 1 \text{ kg}$  and  $t_0 = 1 \text{ s}$ . As a result, the split-quaternion associated with the  $A$ -matrix given in Equation (4-13) becomes

$$a = -\frac{1}{2}(t_0\gamma_s + t_0\gamma_p) + \frac{1}{2}\left(\frac{m_0}{m} + \frac{m\Omega_n^2 t_0^2}{m_0}\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{m_0}{m} + \frac{m\Omega_n^2 t_0^2}{m_0}\right)\hat{\mathbf{j}} + \frac{1}{2}(t_0\gamma_p - t_0\gamma_s)\hat{\mathbf{k}}. \quad (4-14)$$

Clearly, all the components of the split-quaternion are dimensionless. This really is not too wild of an idea: after all, we are translating the matrix itself, and *not* the two-dimensional vector space that it acts on. The units are inherited from the vector space, so we should only add them when returning from the split-quaternions back to the realm of the matrices.

The preceding argument only explains why we can work around this issue without performing illegal operations, but it does not give a satisfactory answer as to why we would be interested to add numbers that are seemingly incompatible. Indeed, observe that  $\gamma_s$  and  $\gamma_p$  have the same units, whereas  $\frac{1}{m}$  and  $\Omega_n^2$  do not. So, in which sense can the  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$ -components be of any significance? To answer this question, we first note that ‘rescaling of units’ is a linear operation on the vector space given by a diagonal matrix (with nonzero diagonal entries):

$$N = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} \quad \nu_1, \nu_2 \in \mathbb{R}^*,$$

which form the group isomorphic to  $(\mathbb{R}^*)^2$ . This transformation of the vector space manifests itself on the  $A$ -matrix as:  $A' = N^{-1}AN$ . It is easy to see that the basis matrices (or vector fields) for ‘1’ and  $\hat{\mathbf{k}}$  are invariant under this transformation, while the  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$ -matrices are not (that is, without making use of the reference quantities).

A geometric explanation is that the eigenvectors of the identity matrix and the  $\hat{\mathbf{k}}$ -matrix point along the axes; and are therefore invariant under rescaling of these axes. As a result of this fact, the  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  components will not transform properly under a unit transformation. It is common practice in physics to rescale the state space of the undamped harmonic oscillator as follows [71, 72]

$$p \mapsto \frac{p}{m} \quad q \mapsto m\Omega q,$$

such that the Hamiltonian reverts to a particularly convenient form. We can see that this is precisely the transformation that kills the  $\hat{\mathbf{j}}$ -component of the split-quaternion. This would essentially resolve this ‘unit problem’, because it only arises when we attempt to make the *distinction* between the  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$ -component.

In contrast the common practice of rescaling the units, we are interested in the full range of geometrical properties that the trajectories in the phase plane can exhibit, including those that are not invariant under the action of the structure group  $(\mathbb{R}^*)^2$ . Furthermore, many invariants, such as the split-quaternion (vector) norm, scalar part, etc. that we use to draw conclusions about the nature of the system *do* commute with this group action, and are therefore remain valid. It is even possible to effect unit transformations within the split-quaternion transformations by translating the matrix  $N$  to the appropriate split-quaternion using the isomorphism. We can indeed observe that the action of  $n^{-1}an$  (where  $n = \phi^{-1}(N)$ ) produces a split-quaternion with zero  $\hat{\mathbf{j}}$ -component.

As a final argument, we can say that the ‘rescaling of the axes’, while common in physics and mathematically allowed, is of little use for engineers, since they tend to stick to SI units in

the first place. The ‘scale of the axes’ is therefore a physical reality. This is why we choose not to discard the  $\hat{\mathbf{j}}$ -component through a rescaling.

To conclude, it is not so much the case that unit transformations are not allowed in the split-quaternion space, but the question as to what the units of the  $\hat{\mathbf{j}}$ -components are is moot. Unfortunately, the notation in Equation (4-14) is rather obfuscating. Hence, we take the freedom to choose  $m_0 = 1(\text{kg})$  and  $t_0 = 1(\text{s})$ , and write the split-quaternion as follows:

$$\mathbf{a} = -\frac{1}{2}(\gamma_s + \gamma_p) + \frac{1}{2}\left(\frac{1}{m} + m\Omega_n^2\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{1}{m} - m\Omega_n^2\right)\hat{\mathbf{j}} + \frac{1}{2}(\gamma_p - \gamma_s)\hat{\mathbf{k}}. \quad (4-15)$$

Leaving out the reference quantities requires the silent understanding that all the components are dimensionless, and that we are not just adding apples and pears.

In the next section, we investigate the influence that of each of the split-quaternion components has on the physical behavior of the mechanical system.

#### 4-3-2 Physical interpretation of the split-quaternion representation

As mentioned previously, the qualitative regime of the dynamical system can be deduced from the sign of its split-quaternion and vector norm. In the particular case of the harmonic oscillator with two dampers, we assume that the damping coefficients  $\gamma_s, \gamma_p$  are either positive or zero and that mass  $m$  and the spring constant  $k$  are both strictly positive. The norm of the split-quaternion given in Equation (4-15) is

$$\mathcal{N}(\mathbf{a}) = m\Omega_n^2 + \gamma_p\gamma_s.$$

From this expression we find that, given the assumptions, the norm of  $\mathbf{a}$  is always positive, and that we are therefore always dealing with a timelike split-quaternion.

##### Real part of the split-quaternion

The real part of the split-quaternion representation of the harmonic oscillator with two dampers is (cf. Equation (4-15)),

$$a_0 = -\frac{1}{2}(\gamma_s + \gamma_p).$$

This real part therefore to the combined effect represents the combined effect of both the dampers.

As can be observed from Equation (4-12) and Figure 4-3, the real part  $a_0$  appears in the system solution as part of the argument of the exponential that envelopes the inner part of the solution, but it does not influence it in any other fashion. This is a consequence of the fact that the vector fields proportional to  $X_1$  commute with all the other vector fields, as discussed in Section 4-2-1.

In the context of the mechanical system, this ‘commuting of vector fields’ means that the overall transformation of the state-space that is generated by the vector field can be seen as the composition of two separate transformations. The first is a purely exponential contraction of the phase space (assuming  $\gamma_s, \gamma_p > 0$ ), which is the overall effect of the dampers. Secondly, we have the remaining transformation that is generated by the combined effect of the vector

part  $\mathbf{a}$ . The latter cannot be decomposed into separate  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$  and  $\hat{\mathbf{k}}$  transformations, because the vector fields that generate them do not commute.

Because the transformations generated by the vector part of  $a$  are conservative (in contrast to the exponential contraction), this decomposition of the transformations is the amounts to separating the dissipative and conservative effects in present in the mechanical system. That is to say, we decompose the system solution into two parts:

$$\exp(at) = \underbrace{\exp(a_0 t)}_{\text{dissipative}} \underbrace{\exp(\mathbf{a}t)}_{\text{conservative}}.$$

From the above expression, it can be seen that the dissipative component of the solution is rather simple: it is simply a contraction by  $e^{a_0 t} = e^{-(\gamma_s + \gamma_p)t}$ . This is why we will now focus exclusively on the nature of the second transformation, being the conservative part. From the split-quaternion point of view, this means that we assume that the real part vanishes, and only take the vector part into account.

### Vector part of the split-quaternion

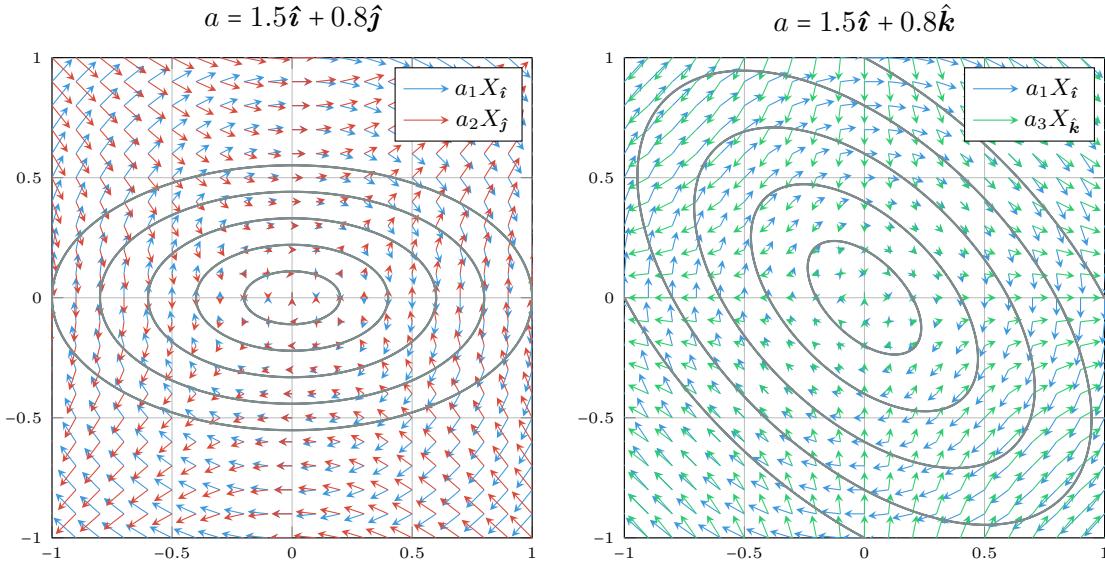
The vector part of  $a$  has three components. Based on the associated basis vector fields shown in Figure 4-3 and Equation (4-15), the following observations can be made

- The  $\hat{\mathbf{i}}$ -component of the split-quaternion gives rise to a rotational vector field that is rotationally symmetric, since the solution trajectories are concentric circles. A positive component gives rise to a clockwise rotation. For the mechanical system, this component is equal to  $\frac{1}{2}\left(\frac{1}{m} + m\Omega_n^2\right)$ ; it induces the *oscillatory* or *periodic motion* of the harmonic oscillator.
- The  $\hat{\mathbf{j}}$ -component of the split-quaternion is a hyperbolic (saddle) vector field, its solution trajectories are hyperbolae whose asymptotes are the diagonals of the phase plane. For the mechanical system, this component is equal to  $\frac{1}{2}\left(\frac{1}{m} - m\Omega_n^2\right)$ : it measures the imbalance between the two terms that constitute the rotation. As a result, the combination of the  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$ -components is an ellipse that is stretched or squeezed along its primary axes, but *not* rotated. The undamped harmonic oscillator is of this type.
- Finally, the  $\hat{\mathbf{k}}$ -component is also a hyperbolic (saddle) vector field whose asymptotes are the horizontal and vertical axis. For the mechanical system, the  $\hat{\mathbf{k}}$ -component is equal to the *imbalance between the two dampers*, i.e.  $\frac{1}{2}(\gamma_p - \gamma_s)$ . It also gives rise to a stretch and squeeze of the trajectory, but now along the diagonals of the phase plane. Therefore, it results in a *phase shift* or ‘wobble’ in the trajectory.

The influence of the vector components on the shape of the solution trajectories is visualized in Figure 4-6.

**Vector norm** As mentioned before, the vector norm of a split-quaternion is equal to the imaginary part of the eigenvalues of the associated matrix. For the harmonic oscillator with two dampers the squared vector norm is equal to

$$\mathcal{N}(\mathbf{a}) = m\Omega^2 - \frac{1}{4}\gamma_s - \frac{1}{4}\gamma_p + \frac{1}{2}\gamma_s\gamma_p.$$



**Figure 4-6:** Comparison between the influence of the  $\hat{j}$ -component (left) and  $\hat{k}$ -component on the integral curves of  $X_{\hat{i}}$ . The  $\hat{j}$  component stretches the elliptic trajectory along the horizontal axis and squeezes it by the same amount in the vertical direction, while the  $\hat{k}$  component does the same but along axes that are rotated by 45 degrees. As shown by the figures, this is a consequence of the constructive or destructive interference of those vector fields in these particular directions.

Provided that the full system exhibits dissipation, i.e.  $(\gamma_s + \gamma_p > 0)$ , the sign of the squared vector norm determines the regime of the mechanical system:

- if  $\mathcal{N}(\mathbf{a}) > 0$  (timelike vector), the system is underdamped (or undamped);
- if  $\mathcal{N}(\mathbf{a}) = 0$  (lightlike vector), the system is critically damped;
- if  $\mathcal{N}(\mathbf{a}) < 0$  (spacelike vector), the system is overdamped.

This can also be explained in terms of the *damping ratio*  $\zeta$  of the system. Recall that the eigenvalues of the matrix associated to the split quaternion are given by

$$\lambda_A = a_0 \pm \|\mathbf{a}\| \mathbf{i}. \quad (4-16)$$

The *damping ratio* is defined as

$$\zeta := \frac{\Re(\lambda_A)}{|\lambda_A|} = \frac{a_0}{\sqrt{a_0^2 + \mathcal{N}(\mathbf{a})}}. \quad (4-17)$$

Hence, if  $\mathcal{N}(\mathbf{a})$  vanishes (provided that the system is indeed damped, so  $a_0 \neq 0$ ), then  $\zeta = 1$  and the system is critically damped. Likewise, if  $\mathcal{N}(\mathbf{a}) > 0$ ,  $\zeta < 1$ , then the system is underdamped and vice versa.

More generally, one can see that the vector norm measures whether either the ‘circular’  $\hat{i}$ -component is dominant, or otherwise the combination of  $\hat{j}$  and  $\hat{k}$ . In the former case, the solution trajectory is elliptic, while in the latter case it is hyperbolic ()

We now wish to quantify precisely how the vector components of the split-quaternion representation influence the solution trajectories (with damping removed) by investigating the *eigenvectors* of  $A$ . We can do this in the split-quaternion representation by removing all the ‘eigenvalue’ information from  $a$ , by disregarding the real part and looking at the *normalized vector part*, i.e. removing the influence of both the real<sup>9</sup> and imaginary part of the eigenvalues. What remains is the information contained in the eigenvectors of  $A$ .

**The normalized vector in Lorentzian 3-space** The vector part of a split-quaternion live in a three-dimensional vector space with basis  $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ . This vector space is equipped with the *Lorentzian inner product*: given two vectors  $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$  and  $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ , their Lorentzian inner product is

$$\langle \mathbf{a}, \mathbf{b} \rangle_L := a_1b_1 - a_2b_2 - a_3b_3.$$

Because this inner product has a signature  $(+, -, -)$ , the Lorentzian 3-space is denoted by  $\mathbb{R}^{1,2}$ . The squared vector norm of the vector part of a split-quaternion is then equal to the Lorentzian inner product of that vector with itself:

$$\mathcal{N}(\mathbf{a}) = \langle \mathbf{a}, \mathbf{a} \rangle_L = a_1^2 - a_2^2 - a_3^2.$$

As discussed in Section 4-2-3, the squared vector/Lorentzian norm is indefinite, and we can distinguish three possible normalizations:

- *timelike* vectors have a positive squared norm, and their normalized length is 1;
- *lightlike* vectors have a squared norm of zero, they cannot be normalized;
- *spacelike* vectors have a negative squared norm, their normalized length is -1.

We can identify three subspaces based on these cases, in which the vectors have norm 1, 0 and -1 respectively:

- If  $\mathbf{a}$  is timelike, the unit vectors live on the *two-sheet unit hyperboloid*:

$$\left\{ \mathbf{a} \in \mathbb{R}^3 \mid a_1^2 - a_2^2 - a_3^2 = 1 \right\},$$

which consists of two separate sheets (referred to as the positive and negative sheet, depending on the sign of the  $\hat{\mathbf{i}}$ -component).

- If  $\mathbf{a}$  is lightlike, the vectors live on the *light cone* (normalizing these vector is not possible):

$$\left\{ \mathbf{a} \in \mathbb{R}^3 \mid a_1^2 - a_2^2 - a_3^2 = 0 \right\};$$

---

<sup>9</sup>The real part of the split-quaternion, or the trace of the matrix, does not influence the eigenvectors of the matrix. This can be observed from the following expression

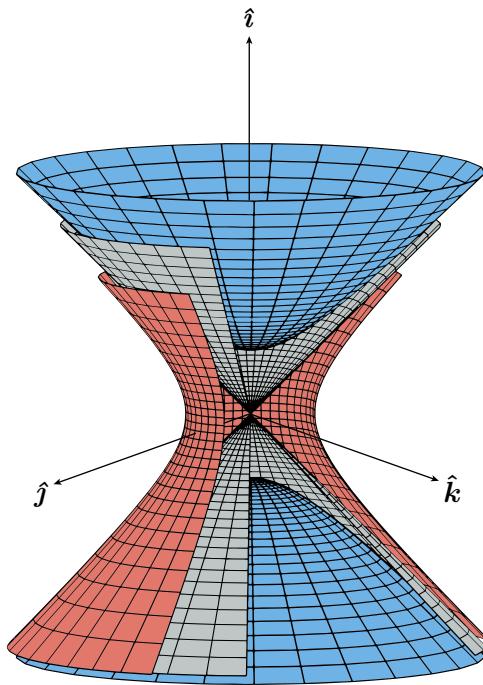
$$(A - \lambda I)\mathbf{v} = [(A + sI) - ((s + \lambda)I)\gamma I]\mathbf{v},$$

i.e. adding any scalar multiple of the identity to the matrix adds this scalar to the value of all the eigenvalues, and leaves the eigenvectors unaltered.

- If  $\mathbf{a}$  is spacelike, the vectors live on the *one-sheet unit hyperboloid*:

$$\{\mathbf{a} \in \mathbb{R}^3 \mid a_1^2 - a_2^2 - a_3^2 = -1\}.$$

These three subspaces are visualized in Figure 4-7. Because the vector norm determines with the regime of a damped mechanical system, we can state that the two-sheet hyperboloid contains all the normalized underdamped systems, the light cone contains all the critically damped systems and the one-sheet hyperboloid contains all the normalized overdamped systems.



**Figure 4-7:** The disconnected ‘unit sphere’ in the Lorentzian 3-space. The red surface is the one-sheet hyperboloid, containing all the spacelike unit vectors. The gray sheet is the light cone, that contains all the lightlike ‘null’ vectors with zero norm. Finally, the two blue surfaces constitute the two-sheet hyperboloid.

For the harmonic oscillator with two dampers, the unit vector is equal to:

$$\hat{\mathbf{a}} = \left( \frac{\frac{1}{m} + m\Omega_n^2}{2\Omega_d} \right) \hat{\mathbf{i}} + \left( \frac{\frac{1}{m} - m\Omega_n^2}{2\Omega_d} \right) \hat{\mathbf{j}} + \left( \frac{\gamma_p - \gamma_s}{2\Omega_d} \right) \hat{\mathbf{k}}, \quad (4-18)$$

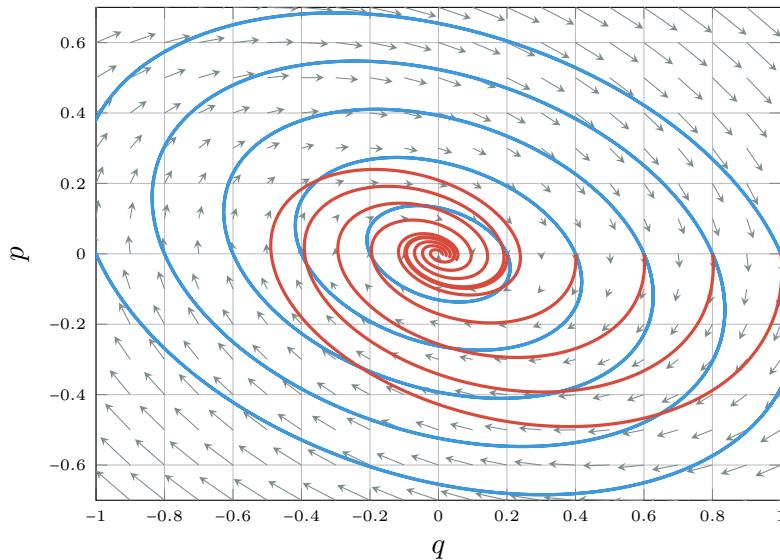
with  $\Omega_d$  being the *damped frequency* of the oscillator, equal to the vector norm:

$$\Omega_d = \|\mathbf{a}\| = \sqrt{m\Omega^2 - \frac{1}{4}\gamma_s - \frac{1}{4}\gamma_p + \frac{1}{2}\gamma_s\gamma_p}. \quad (4-19)$$

We will now investigate the relation between the normalized vector part and the shape of the solution trajectories in each of the three cases (underdamped, critically damped and overdamped) in more detail.

### 4-3-3 Geometric analysis of underdamped systems

For underdamped systems, the trajectories are spiral-shaped. Removing the real part from the split-quaternion is the same as removing the ‘contraction’ in the spiral: as a result, the conservative version of this trajectory is an elliptic trajectory. This ellipse is subject to the same stretch and tilt of the original spiral, as illustrated by Figure 4-8.



**Figure 4-8:** Solution trajectories and vector field of an exemplary underdamped system. The red trajectories are the actual solution of the system, while the blue trajectories are the ‘conservative’ version, i.e. with the real part of the split-quaternion removed.

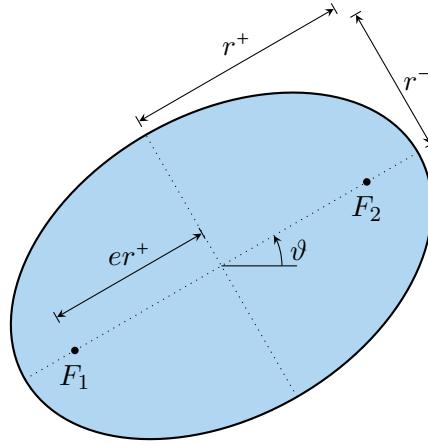
The purpose of this section is to relate the normalized vector part of the mechanical system given by Equation (4-18) to the shape of the associated elliptic trajectory. The geometry of the ellipse is governed by its aspect ratio and its tilt angle, as shown in Figure 4-9. For the aspect ratio many measures exist, of which the *eccentricity*  $e$  is the best known. The other measures are listed in Table 4-7; especially the third flattening and third eccentricity play an important role in the remainder of the discussion.

Instead of the ‘Cartesian’ representation of the normalized vector part (i.e. in terms the component  $a_1, a_2, a_3$ ) we will adopt a more parsimonious description in the form of the so-called *pseudospherical coordinates*. These consist of a *hyperbolic angle*  $\tau$  and a Euclidean angle  $\sigma$ , which specify a point on either one of the sheets of the two-sheet hyperboloid shown in Figure 4-7. The pseudospherical coordinates are related to the Cartesian coordinates through the following relations:

$$a_1 = \pm \|\mathbf{a}\| \cosh(\tau) \quad a_2 = \|\mathbf{a}\| \sinh(\tau) \cos(\sigma) \quad a_3 = \|\mathbf{a}\| \sinh(\tau) \sin(\sigma). \quad (4-20)$$

It is easily verified that the components indeed satisfy the relation

$$\mathcal{N}(\hat{\mathbf{a}}) = a_1^2 - a_2^2 - a_3^2 = 1, \quad (4-21)$$



**Figure 4-9:** Geometry of the ellipse. Its shape is characterized by two lengths: the semi-major axis  $r_+$  and the semi-minor axis  $r_-$ . The various measures of the aspect ratio of the ellipse given in Table 4-7 express the relative size of  $r_+$  with respect to  $r_-$  in various ways. The (first) eccentricity  $e$  parameter locates the focal points  $F_1, F_2$  as a proportion of the semi-major axis. The angle  $\vartheta$  by which the semi-major axis is rotated with respect to the horizontal is called the *tilt*.

given that  $\|\mathbf{a}\| = 1$ . In addition, we have:

$$\begin{aligned} \cosh(\tau) &= \frac{a_1}{\|\mathbf{a}\|}, & \sinh(\tau) &= \frac{\sqrt{a_2^2 + a_3^2}}{\|\mathbf{a}\|}, \\ \cos(\sigma) &= \frac{a_2}{\sqrt{a_2^2 + a_3^2}}, & \sin(\sigma) &= \frac{a_3}{\sqrt{a_2^2 + a_3^2}}. \end{aligned} \quad (4-22)$$

We will now try to relate the location of the normalized vector part on the two-sheet hyperboloid to the geometric features of the elliptic solution trajectories. To do so, we can use the solution trajectories of the basis vector fields, and build the solution for the system based on the vector components<sup>10</sup>. For the  $X_i$ , the solution curves have the implicit form

$$\frac{1}{2}(q^2 + p^2) = C,$$

i.e. they are concentric circles of some radius. Secondly, for  $X_j$ , we have the implicit form

$$\frac{1}{2}(p^2 - q^2) = C.$$

Thirdly, for  $X_k$ , the solution trajectories are of the form

$$pq = C.$$

From the above expression we can find the shape of the actual solution trajectory by weighing each of these curves with the respective components of the normalized vector part:

$$\frac{a_1}{2\|\mathbf{a}\|}(q^2 + p^2) + \frac{a_2}{2\|\mathbf{a}\|}(p^2 - q^2) + \frac{a_3}{\|\mathbf{a}\|}pq = C.$$

<sup>10</sup>These expressions can be easily derived by finding the ‘Hamiltonian function’ that generates that particular vector field, since it is necessarily constant along the trajectories.

**Table 4-7:** Commonly used parameters to measure the eccentricity/flattening of an ellipse in terms of its semi-major axis  $r_+$  and semi-minor axis  $r_-$  [3].

Parameter	Notation	Formula
Semi-major axis	$r^+$	—
Semi-minor axis	$r^-$	—
(First) eccentricity	$e$	$\frac{r_+^2 - r_-^2}{r_+^2}$
Second eccentricity	$e'$	$\frac{r_+^2 - r_-^2}{r_-^2}$
Third eccentricity	$e''$	$\frac{r_+^2 - r_-^2}{r_+^2 + r_-^2}$
(First) flattening	$f$	$\frac{r_+ - r_-}{r_+}$
Second flattening	$f'$	$\frac{r_+ - r_-}{r_-}$
Third flattening	$f''$	$\frac{r_+ - r_-}{r_+ + r_-}$

Because we are not interested in the specific radius of the ellipse, we may multiply this expression by any nonzero factor. We have the following expression

$$\underbrace{(a_1 - a_2) q^2}_R + \underbrace{2a_3 pq}_S + \underbrace{(a_1 + a_2) p^2}_T = 2\|a\|C, \quad (4-23)$$

which is the implicit equation for an ellipse in case  $S^2 - 4RT < 0$  [3]. Substitution shows that this condition is equivalent to  $a$  being timelike.

### Eccentricity

The eccentricity of the ellipse described by Equation (4-23) in terms of the parameters  $R, S, T$  can be shown to be [3]

$$e = \sqrt{\frac{2\sqrt{(R-T)^2 + S^2}}{(R+T) + \sqrt{(R-T)^2 + S^2}}}. \quad (4-24)$$

Since we have that  $R - T = -2a_2$ ,  $R + T = 2a_1$  and  $S = 2a_3$ , the eccentricity may be expressed as a function of the components of the split-quaternion as well:

$$e = \sqrt{\frac{2\sqrt{a_2^2 + a_3^2}}{q_1 + \sqrt{a_2^2 + a_3^2}}}. \quad (4-25)$$

Finally, using the relations stated in Equation (4-22), this expression can be written in terms of the pseudospherical coordinates:

$$e = \sqrt{\frac{2 \sinh(\tau)}{\cosh(\tau) + \sinh(\tau)}}. \quad (4-26)$$

### Tilt angle

The tilt angle  $\vartheta$  of the ellipse is defined as the angle between the positive horizontal ( $q$ )-axis and the major axis of the ellipse. Just like the eccentricity, we start from the general expression in terms of the parameters  $R, T, S$ :

$$\tan(\vartheta) = \frac{T - R - \sqrt{(R - T)^2 + S^2}}{S}. \quad (4-27)$$

Again, substituting the split-quaternion components, we have

$$\tan(\vartheta) = \frac{a_2 - \sqrt{a_2^2 + a_3^2}}{a_3}. \quad (4-28)$$

Rather unsurprisingly, the  $\hat{i}$ -component does not play a role in the tilt angle of the trajectory: from Figure 4-3, we can see that the associated vector field  $X_{\hat{i}}$  is indeed rotationally symmetric, in contrast to  $X_{\hat{j}}$  or  $X_{\hat{k}}$ .

Again, Equation (4-28) may be expressed in terms of the pseudospherical coordinates (cf. Equation (4-22)):

$$\tan(\vartheta) = \frac{\cos(\sigma) - 1}{\sin(\sigma)} = -\tan\left(\frac{\sigma}{2}\right).$$

Hence, we obtain that the angle

$$\vartheta = -\frac{\sigma}{2}.$$

### Orientation

Since the two-sheet hyperboloid consists of two disconnected surfaces, this begs the question what the significance is of either of those surfaces. The hyperboloids are directed along the  $\hat{i}$ -axis. Therefore, sign of the  $\hat{i}$ -component of  $a$  determines on which hyperboloid the vector is located. Physically, the sign of the  $\hat{i}$ -component determines whether the ‘rotational’ vector field has a clockwise or counterclockwise rotation. By default,  $X_{\hat{i}}$  is an infinitesimal clockwise rotation (cf. Figure 4-3): as a result, if  $a_0$  is positive the rotation is clockwise and vice versa.

The obtained relations show that the pseudospherical coordinates  $\sigma, \tau$  are directly related to respectively the tilt angle and the eccentricity of the elliptic trajectories. We will now investigate the role of these coordinates in two common projections of the two-sheet hyperboloid, being the *Poincaré model* and the *Cayley-Klein model*.

### Relation with models of the hyperbolic plane

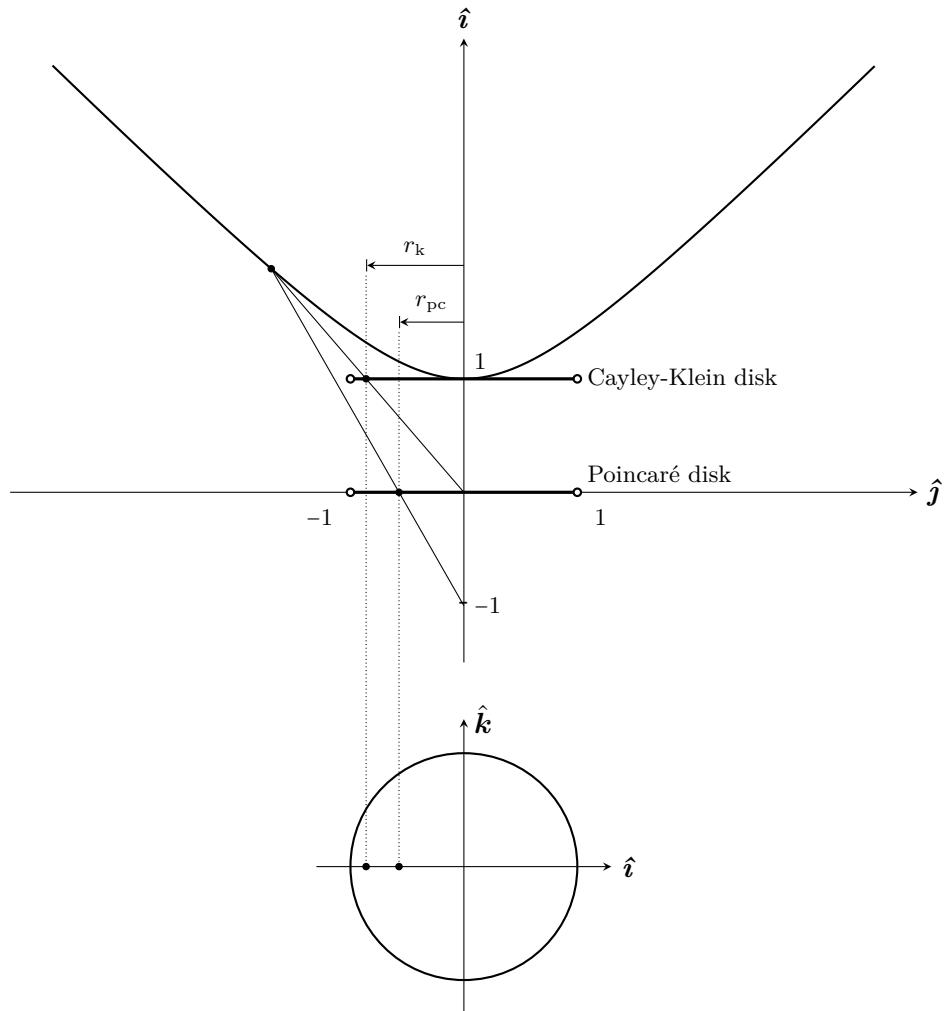
The *hyperbolic plane* is a plane that has a constant negative Gaussian curvature everywhere. Its geometry is known as *hyperbolic geometry*, as opposed to *Euclidean geometry* and *spherical geometry*, which are associated with surfaces of no curvature and constant positive curvature respectively.

In contrast to the Euclidean plane and the sphere (which exhibits spherical geometry), the hyperbolic plane cannot be isometrically embedded in three-dimensional space; this is formalized in Hilbert’s theorem on differential geometry [73]. This is why mathematicians must

resort to so-called *models* of the hyperbolic plane. Many are in existence, for example the hyperboloid model, the Poincaré disk, the Cayley-Klein disk, and the Poincaré half-plane. The former three will be of our interest in this treatment. For more information concerning hyperbolic geometry, the reader is referred to Needham [74, 75] and Thurston [73].

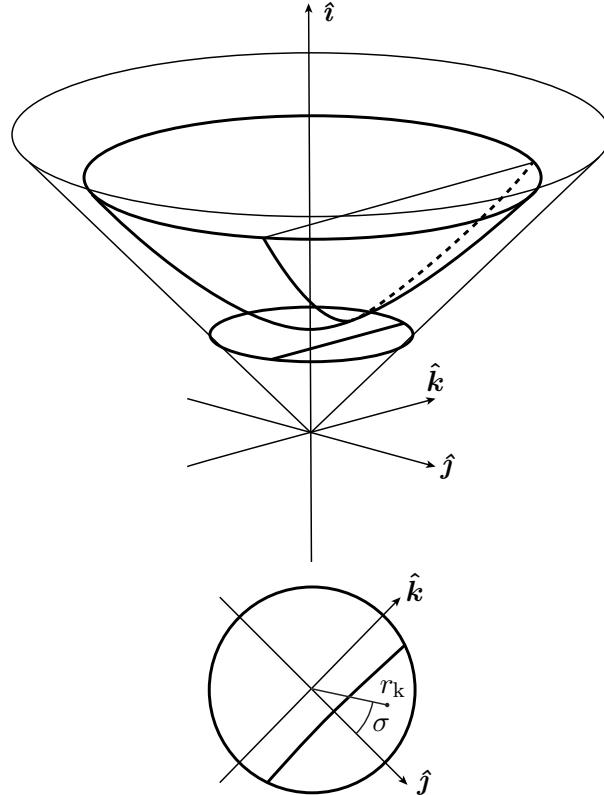
**Hyperboloid model** The *hyperboloid model* consists simply of the positive sheet of the two-sheet hyperboloid, embedded in the Lorentzian 3-space. In this space, this sheet of the hyperboloid has a constant negative Gaussian curvature equal to -1 [2]. Because of this constant negative curvature, the positive sheet of the two-sheet hyperboloid can serve as a model for the hyperbolic plane. On the hyperboloid, geodesics are any line that arises by intersecting the hyperboloid with a plane through the origin.

Both the Poincaré disk and the Cayley-Klein disk arise by projecting the hyperboloid surface to a disk with unit radius in a prescribed fashion. In the  $\hat{i}$ - $\hat{j}$ -plane, these projections are illustrated in Figure 4-10.



**Figure 4-10:** Illustration of the projection on the Poincaré disk and the Cayley-Klein disk.

**The Cayley-Klein disk** The Cayley-Klein is the result of *gnomonic projection* of the positive hyperboloid sheet. The idea of gnomonic projection is that geodesics on the hyperboloid map to straight lines in the disk; on the downside, this projection method is not conformal (it does not preserve angles). As shown in Figure 4-10 and Figure 4-11, the projection of a point on the hyperboloid is the intersection of the line segment of that point with the origin and the disk with unit radius, centered at the point  $\hat{i}$  (i.e. the point in Lorentzian 3-space with coordinates  $(1, 0, 0)$ ).



**Figure 4-11:** The Cayley-Klein disk and the hyperboloid model are related through gnomonic projection. A geodesic of on the hyperboloid model is shown as well, its projected image in the Cayley-Klein disk (also a geodesic) is a straight line. Illustration adapted from Balazs and Voros [2].

If polar coordinates are used to specify a point in the Cayley-Klein disk, the rotation angle  $\sigma$  remains identical to the corresponding pseudospherical coordinate, and from Figure 4-10 it can readily be deduced that the radial coordinate  $r_k \in [0, 1]$  is simply equal to

$$r_k = \tanh(\tau). \quad (4-29)$$

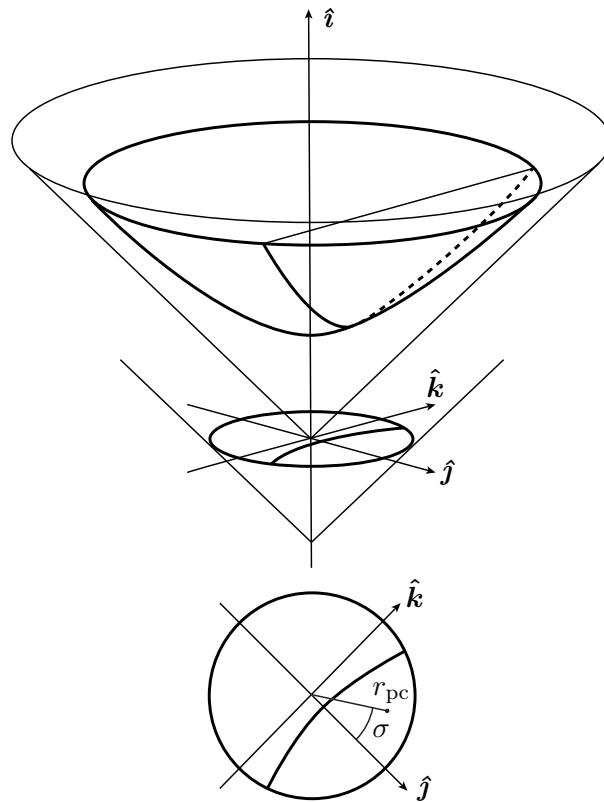
We can now use Equation (4-26) to relate the radial coordinate to the eccentricity of the elliptic trajectory (of course, the significance of the angular coordinate remains identical):

$$e = \sqrt{\frac{2 \sinh(\tau)}{\cosh(\tau) + \sinh(\tau)}} = \sqrt{\frac{2r_k}{1 + r_k}}. \quad (4-30)$$

This expression shows that the radius in the Cayley-Klein disk is equal to the *square of the third eccentricity*  $e''$ , also denoted by ‘ $m$ ’ in literature (we will not do so here, for it is already reserved for mass). In terms of the major and minor axes ( $r_+$  and  $r_-$  respectively), this we have: [3]

$$r_p = \frac{r_+^2 - r_-^2}{r_+^2 + r_-^2} = (e'')^2. \quad (4-31)$$

**Poincaré disk** The *Poincaré disk* is the image of the positive hyperboloid sheet under *stereographic projection* with respect to the point  $-\hat{\mathbf{i}}$ , i.e. the point in Lorentzian 3-space with coordinates  $(-1, 0, 0)$ . That is to say, the projection of a point on the hyperboloid on the Poincaré disk is equal to the intersection of the line segment connecting that point with  $-\hat{\mathbf{i}}$  and the unit disk centered at the origin. This is illustrated in Figure 4-12 and Figure 4-10.



**Figure 4-12:** The Poincaré disk and the hyperboloid model are related through stereographic projection. A geodesic of on the hyperboloid model is shown as well, its projected image in the Poincaré disk (also a geodesic) is part of a circle that intersects the boundary of the disk at right angles. Illustration adapted from Balazs and Voros [2].

If polar coordinates are used in the Poincaré disk, then the Euclidean rotation remains identical, and the radial coordinate is equal to

$$r_{pc} = \tanh\left(\frac{\tau}{2}\right) = \frac{\sinh(\tau)}{1 + \cosh(\tau)}. \quad (4-32)$$

Because stereographic projection projects each point on the positive hyperboloid uniquely onto the Poincaré disk, every point on the disk corresponds to two possible elliptic trajectories: one with clockwise and one with counterclockwise rotation.

As stated by Equation (4-31), the Cayley-Klein radius is equal to the square of the third eccentricity. Along the same line, the radius in the Poincaré disk may be related to a specific measure of elliptic eccentricity as well. Using the double angle formula for  $\tanh$ , we have that

$$r_k = \frac{2r_{pc}}{1 + r_{pc}^2}.$$

Hence, we gather that the Poincaré radius is equal to the *third flattening*, commonly denoted by  $n$  or  $f''$ . In terms of the major and minor axes of the elliptic trajectory, the Poincaré radius is equal to

$$r_{pc} = \frac{r_+ - r_-}{r_+ + r_-} = f''.$$

### Relation with eigenvectors

As mentioned, the normalized vector part  $a$  encodes the same information as the eigenvectors of the associated matrix  $A$ . For an underdamped system, both the eigenvectors and the eigenvalues of the matrix are complex. These complex eigenvectors are not directly interpretable; they can be multiplied by any complex number to obtain another eigenvector of the system. In one particular instance (i.e. a specific length in  $\mathbb{C}^2$ ), the real and imaginary part of the eigenvectors align with the major and minor axes ( $r^+, r^-$ ) of the elliptic trajectory, but this requires additional computation to find the correct multiplication factor for the eigenvectors [64].

Conversely, as demonstrated in the above discussion, the normalized split-quaternions are naturally parameterized in way that corresponds directly to the shape parameters of the solution trajectories. As vectors in the phase plane, the semi-major and semi-minor axes can be expressed in terms of  $\sigma$  and  $\tau$  as follows (where the semi-major axis has length 1):

$$\begin{aligned} \mathbf{r}^+ &= \begin{pmatrix} \cos\left(\frac{\sigma}{2}\right) & \sin\left(\frac{\sigma}{2}\right) \\ -\sin\left(\frac{\sigma}{2}\right) & \cos\left(\frac{\sigma}{2}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{r}^- &= \begin{pmatrix} \cos\left(\frac{\sigma}{2}\right) & \sin\left(\frac{\sigma}{2}\right) \\ -\sin\left(\frac{\sigma}{2}\right) & \cos\left(\frac{\sigma}{2}\right) \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1-\tanh\left(\frac{\tau}{2}\right)}{1+\tanh\left(\frac{\tau}{2}\right)} \end{pmatrix}, \end{aligned} \quad (4-33)$$

$$\text{since } \frac{r^-}{r^+} = \frac{1-f''}{1+f''}.$$

### The Lorentzian cross-product

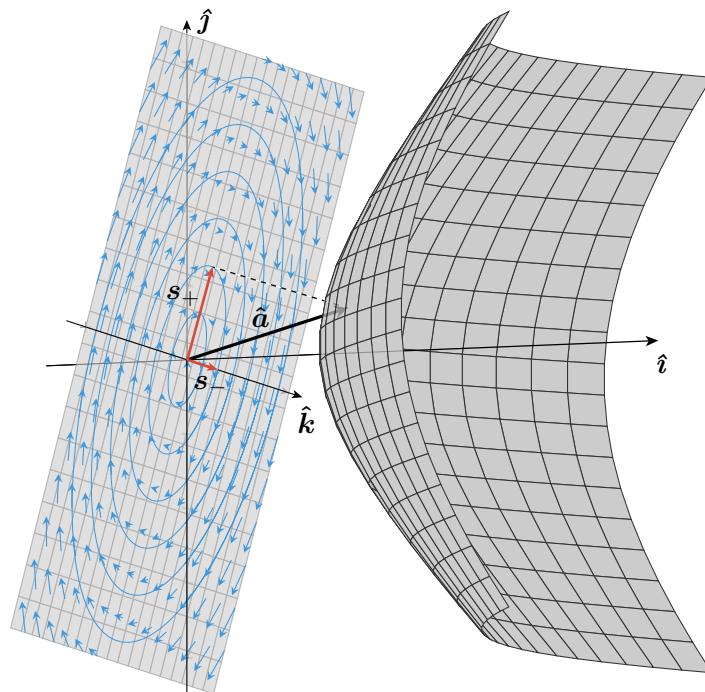
The action on the phase plane of the infinitesimal transformation specified by the matrix  $A$  can also be related by the vector  $a$  acting through the *Lorentzian cross product*  $\times_L$ . This is a binary operation similar to the conventional cross product, but set in the Lorentzian three-space

$\mathbb{R}^{1,2}$ . Given two vectors  $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$  and  $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$  in  $\mathbb{R}^{1,2}$ , their Lorentzian cross-product is defined as: [60]

$$\mathbf{a} \times_L \mathbf{b} := \det \begin{pmatrix} -\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{1,2} \quad (4-34)$$

$$= (a_3 b_2 - a_2 b_3)\hat{\mathbf{i}} + (a_3 b_1 - a_1 b_3)\hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\hat{\mathbf{k}}. \quad (4-35)$$

In addition, we define the notion of *Lorentz orthogonality* based on the Lorentzian inner product defined previously: two vectors are Lorentz-orthogonal if their Lorentzian inner product is equal to zero. In an analogous fashion to the conventional cross product, the Lorentz cross product of two vectors is Lorentz orthogonal to both of those vectors.



**Figure 4-13:** Vector generated by the Lorentzian cross product by  $\mathbf{a}$  on the vectors in the plane that is Lorentz-orthogonal to  $\mathbf{a}$ . The positive sheet of the hyperboloid is shown as well.

Given that  $\mathbf{a}$  is the vector part of the split-quaternion representation of the state-transition matrix  $A$ , we wish to relate the vector field generated by  $A$  to the vector field generated by  $\mathbf{a}$  as ‘operator’:  $\mathbf{a}_{\times_L}(\mathbf{b}) := \mathbf{a} \times_L \mathbf{b}$ . The operator  $\mathbf{a}_{\times_L}$  has an invariant linear subspace: the plane that is Lorentz-orthogonal to the vector  $\mathbf{a}$ . This is visualized in Figure 4-13. Expressing  $\mathbf{a}_{\times_L}$  as a matrix, we get:

$$[\mathbf{a}_{\times_L}](\mathbf{b}) = \begin{pmatrix} 0 & a_3 & -a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \quad (4-36)$$

The eigenvalues of  $[\mathbf{a}_{\times_L}]$  are  $\{0, \pm i\sqrt{a_1^2 - a_2^2 - a_3^2}\}$ , i.e. the imaginary part of the eigenvalues is the same as the imaginary part of the eigenvalues of  $A$  (or the eigenvalues of  $A$  with trace

removed), with the addition of zero. The zero direction is easily seen to be colinear with the vector  $\mathbf{a}$ . Hence, this means that in the Lorentz orthogonal plane, the action of  $[\mathbf{a}_{\times_L}]$  is the same as the action of  $A - \frac{\text{tr}(A)}{2}I$  (the associated matrix with trace removed) on the phase plane, *given the right basis*.

We can find the correct basis by relating the major axes of the elliptic trajectory generated by  $[\mathbf{a}_{\times_L}]$ , denoted by  $(\mathbf{s}_+, \mathbf{s}_-)$  (also shown in Figure 4-13) to the major axes of the elliptic trajectory  $(\mathbf{r}_+, \mathbf{r}_-)$  given by Equation (4-33).

The semi-major axis  $\mathbf{s}_+$  of the elliptic trajectory generated by the Lorentzian cross-product can be obtained by projecting the matrix  $\hat{\mathbf{a}}$  on the plane that is Lorentz orthogonal to it. The semi-minor axis is simply found by applying the Lorentzian cross-product to the semi-major axis. That is,

$$\mathbf{s}_+ = \hat{\mathbf{a}} - \frac{\langle \hat{\mathbf{a}}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} \mathbf{n}, \quad (4-37)$$

$$\mathbf{s}_- = \hat{\mathbf{a}} \times_L \mathbf{s}_+,$$

where  $\mathbf{n} = a_1\hat{\mathbf{i}} - a_2\hat{\mathbf{j}} - a_3\hat{\mathbf{k}}$  is the normal vector to the plane that is Lorentz-orthogonal to  $\mathbf{a}$ . Please note that we are using two types of orthogonality: both Lorentz and conventional. The inner product  $\langle \cdot, \cdot \rangle$  refers to the normal inner product as opposed to the Lorentzian inner product.

*Proof.* A normal vector to the Lorentz-orthogonal subspace is  $\mathbf{n} = a_1\hat{\mathbf{i}} - a_2\hat{\mathbf{j}} - a_3\hat{\mathbf{k}}$ . Then, the basis vectors are

$$\begin{aligned} \mathbf{s}_+ &= \mathbf{a} - \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} \mathbf{n} \\ \mathbf{e}_3 &= \mathbf{a} \times_L \mathbf{s}_+ = -\frac{\langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} (\mathbf{a} \times_L \mathbf{n}), \end{aligned} \quad (4-38)$$

because the Lorentz-cross product distributes over addition and  $\mathbf{a} \times_L \mathbf{u} = \mathbf{0}$ .

If  $(\mathbf{s}_+, \mathbf{s}_-)$  are the major axes of the elliptic trajectory generated by the cross product, then they must be the real and imaginary part of an eigenvector of  $[\mathbf{a}_{\times_L}]$ . Hence, it must be the case that

$$[\mathbf{a}_{\times_L}](\mathbf{e}_2 + i\mathbf{e}_3) = \lambda(\mathbf{e}_2 + i\mathbf{e}_3),$$

where  $\lambda$  is then an eigenvalue of the matrix [64]. This can be verified by replacing the action of  $[\mathbf{a}_{\times_L}]$  with the cross product.

Plugging in the definition and exploiting the linearity of the Lorentz cross-product, we obtain:

$$\begin{aligned} \mathbf{a} \times_L (\mathbf{s}_+ + i\mathbf{s}_-) &= \mathbf{a} \times_L \mathbf{s}_+ + i(\mathbf{a} \times_L \mathbf{s}_-) \\ &= \mathbf{s}_- + (\mathbf{a} \times_L \mathbf{s}_-)i \\ &= \mathbf{s}_- + (\mathbf{a} \times_L (\mathbf{a} \times_L \mathbf{s}_+))i \\ &= \mathbf{s}_- - \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} (\mathbf{a} \times_L (\mathbf{a} \times_L \mathbf{n}))i. \end{aligned}$$

The triple cross-product expansion, or ‘Lagrange formula’, relates the regular cross product to the corresponding dot product:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \langle \mathbf{z}, \mathbf{x} \rangle - \mathbf{z} \langle \mathbf{x}, \mathbf{y} \rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\mathbf{x} \times_L (\mathbf{y} \times_L \mathbf{z}) = \mathbf{y} \langle \mathbf{z}, \mathbf{x} \rangle_L - \mathbf{z} \langle \mathbf{x}, \mathbf{y} \rangle_L.$$

Using the Lagrange formula, the above expression becomes

$$\begin{aligned} \mathbf{a} \times_L (\mathbf{s}_+ + i\mathbf{s}_-) &= \mathbf{s}_- - \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} (\mathbf{a} \langle \mathbf{a}, \mathbf{n} \rangle_L - \mathbf{n} \langle \mathbf{a}, \mathbf{a} \rangle_L)i \\ &= \mathbf{s}_- - \left( \mathbf{a} \frac{\langle \mathbf{a}, \mathbf{n} \rangle_L \langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} - \mathbf{n} \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} \right)i \\ &= \mathbf{s}_- - \left( \mathbf{a} - \mathbf{n} \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle} \right)i \\ &= \mathbf{s}_- - \mathbf{s}_+ i. \end{aligned}$$

The latter is the scalar multiple of the vector  $\mathbf{s}_+ + \mathbf{s}_-$  by  $-i$  - hence, this is indeed an eigenvector of the corresponding matrix. ■

Knowing the eigenvectors of both matrices, the state transition matrix  $A$  and matrix generated by the Lorentzian cross-product  $[\mathbf{a}_{\times_L}]$  can be related through the following similarity transformation:

$$A = T^{-1} [\mathbf{a}_{\times_L}] T,$$

with

$$T = (\mathbf{r}_+ \quad \mathbf{r}_-) (\mathbf{s}_+ \quad \mathbf{s}_-)^{-1}.$$



---

## Chapter 5

---

# **Conclusion and Recommendations**



---

## Appendix A

---

# Geometry of Lagrangian Mechanics

Just like the cotangent bundle, the tangent bundle admits a canonical structure, which is called the *vertical endomorphism*. Its construction is slightly more convoluted than the canonical symplectic structure of the cotangent bundle, but nevertheless essential for a proper geometric interpretation of Lagrangian mechanics.

### A-1 The double tangent bundle

The *double tangent bundle* is the tangent bundle to  $TQ$ , denoted by  $T(TQ)$ . This space has not one but two canonical vector bundle structures, defined by projection maps from  $T(TQ) \rightarrow TQ$ . First, there is the trivial projection  $\pi_{TQ}$  that ‘forgets’ about the tangent elements to  $TQ$ . Secondly, there is  $(\pi_Q)_*$  the pushforward (tangent map) of the projection map  $\pi_Q : TQ \rightarrow Q$ . [11]

$$\begin{array}{ccc} & T(TQ) & \\ (\pi_Q)_* \swarrow & & \searrow \pi_{TQ} \\ TQ & & TQ \\ \pi_Q \searrow & & \swarrow \pi_Q \\ Q & & \end{array}$$

Vectors on the tangent bundle  $TQ$  (they live in  $T(TQ)$ ) are called vertical if they vanish under the action of  $(\pi_Q)_*$ . These vectors point entirely in the ‘direction’ of the fiber: in the Lagrangian formalism, they reflect a pure change in velocity, and no change in the generalized position. The *vertical lift*  $\Psi$  maps a vector on  $Q$  to a vertical vector on  $TQ$ . [9]

$$\Psi_v : T_q Q \rightarrow T_v(T_q Q) :$$

$$\Psi_v(\mathbf{w}) f = \frac{d}{dt} f(\mathbf{v} + t\mathbf{w}) \Big|_{t=0} \quad q \in Q, \quad \mathbf{v}, \mathbf{w} \in T_q Q, \quad f \in C^\infty(TQ). \tag{A-1}$$

In components, the effect of the vertical lift is as follows:

$$\Psi_v : \quad \mathbf{w} = w^i \frac{\partial}{\partial q^i} \Big|_q \quad \mapsto \quad \Psi_v(\mathbf{w}) = w^i \frac{\partial}{\partial v^i} \Big|_{(q,v)}.$$

The vertical lift can also lift entire sections of  $TQ$  by simply applying the vertical lift pointwise.

Using the concept of the vertical lift, we can define the *vertical isomorphism*  $S$  from the double tangent bundle to itself, first by projecting with  $(\pi_Q)_*$  and then lifting again:

$$S : T(TQ) \rightarrow T(TQ) : \quad S(q, v) u = (\Psi_v \circ (\pi_Q)_*) u \quad u \in T_{(q,v)} TQ.$$

The action of  $S$  can also be stated in the form of the following diagram:

$$\begin{array}{ccc} T(TQ) & \xrightarrow{S} & T(TQ) \\ (\pi_Q)_* \downarrow & & \uparrow \Psi \\ TQ & \xrightarrow{\text{id}_{TQ}} & TQ \end{array}.$$

The action of the vertical endomorphism on the chart-induced basis is:

$$S : \quad \frac{\partial}{\partial q^i} \Big|_{(q,v)} \mapsto \frac{\partial}{\partial v^i} \Big|_{(q,v)} \quad \frac{\partial}{\partial v^i} \Big|_{(q,v)} \mapsto 0.$$

The vertical isomorphism is therefore a tensor of valence  $(1, 1)$  — it takes a vector and produces another. Locally,  $S$  can be expressed as:

$$S = \frac{\partial}{\partial v^i} \otimes dq^i.$$

with  $\otimes$  being the tensor product. [9]

The Lagrangian formalism only applies to second-order vector fields. A second-order vector field is a vector field  $X$  such that  $(\pi_Q \circ X) = \text{id}_{TQ}$ ; i.e. the following diagram commutes: [11]

$$\begin{array}{ccc} & T(TQ) & \\ (\pi_Q)_* \swarrow & & \nwarrow X \\ TQ & \xrightarrow{\text{id}_{TQ}} & TQ \end{array}.$$

The identity on  $TQ$  is  $\text{id}_{TQ} : (q, v) \mapsto (q, v)$ . Therefore, for a vector field  $X$  to be second order, we should have that the component in  $\frac{\partial}{\partial q^i}$  that is picked out by  $(\pi_Q)_*$  should be equal to  $v^i$ ; for example

$$X = \sum_{i=1}^n \left[ v^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial v^i} \right].$$

The corresponding differential equations are

$$\frac{dq^i}{dt} = v^i \quad \frac{dv^i}{dt} = F^i,$$

which means that the second-order vector field coincides with the notion of a 'second-order differential equation' in  $q^i$ .

## A-2 The Euler-Lagrange equations

With the infrastructure set up in the preceding section, we can now define the precise geometric setting of Lagrangian mechanics. Given a Lagrangian function  $L \in C^\infty(TQ)$ , define the *Lagrange 1-form*<sup>1</sup>

$$\vartheta_L \equiv dL \circ S = \sum_{j=1}^n \frac{\partial L}{\partial v^j} dq^j. \quad (\text{A-2})$$

Observe that the Lagrange 1-form is also equal to the pullback of the Liouville form under the Legendre transformation:  $\vartheta_L = (\mathbb{F}L)^*\vartheta$ . [11] Secondly, we define the *Lagrange 2-form* as: [9, 11]

$$\omega_L \equiv -d\vartheta_L = \frac{\partial^2 L}{\partial v^i \partial v^j} dq^j \wedge dv^i + \frac{\partial^2 L}{\partial q^i \partial v^j} dq^j \wedge dq^i. \quad (\text{A-3})$$

Because the exterior derivative and the pullback commute, the Lagrange 2-form is equal to the pullback of the symplectic 2-form under the Legendre transform. If the rank of the Hessian  $\frac{\partial^2 L}{\partial v^i \partial v^j}$  is full (and constant), then  $\omega_L$  is nondegenerate and therefore defines a symplectic structure on  $TQ$ . However, observe that whether  $\omega_L$  is symplectic or not depends on the nature of the Lagrangian, while the symplectic structure in the Hamiltonian setting is canonically derived from the cotangent bundle itself — there is no need for the Hamiltonian to be regular.

The final ingredient for the Euler-Lagrange equations is the *energy function*

$$E \equiv Z(L) - L,$$

where  $Z = \sum v^i \frac{\partial}{\partial v^i}$  is the Liouville vector field on  $TQ$ .

The *Lagrangian vector field*  $X_L$  is then the unique vector field that satisfies the equation: [19]

$$X_L \lrcorner \omega_L = dE, \quad (\text{A-4})$$

In components, the right hand side of this equation is:

$$\begin{aligned} dE &= \sum_{i,j} \left( \frac{\partial^2 L}{\partial v^j \partial q^i} v^j dq^i + \frac{\partial^2 L}{\partial v^j \partial v^i} v^j dv^i + \frac{\partial L}{\partial v^j} dv^j \right) - dL, \\ dE &= \sum_{i,j} \left( \frac{\partial^2 L}{\partial v^j \partial q^i} v^j dq^i + \frac{\partial^2 L}{\partial v^j \partial v^i} v^j dv^i - \frac{\partial L}{\partial q^j} dq^j \right). \end{aligned} \quad (\text{A-5})$$

Let  $X_L = \sum_i \left( A_i \frac{\partial}{\partial q^i} + B_i \frac{\partial}{\partial v^i} \right)$ ; the left hand side can then be written as follows:

$$X_L \lrcorner \omega_L = - \sum_{i,j} A_i \frac{\partial^2 L}{\partial q^i \partial v^j} dq^j + \sum_{i,j} A_j \frac{\partial^2 L}{\partial q^i \partial v^j} dq^i - \sum_{i,j} B_i \frac{\partial^2 L}{\partial v^i \partial v^j} dv^j + \sum_{i,j} A_j \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i. \quad (\text{A-6})$$

Comparing this expression with Equation (A-5), it is immediately clear that

$$A_j \frac{\partial^2 L}{\partial v^i \partial v^j} = v^j \frac{\partial^2 L}{\partial v^i \partial v^j}.$$

<sup>1</sup>Cariñena [9] calls  $\vartheta$  the Euler-Poincaré 1-form.

We therefore have that  $A_j = v^j$ , but *only* if the Hessian of  $L$  with respect to the velocities is nonsingular. If this is indeed the case (i.e.  $L$  is regular), and the condition implies that the vector field  $X_L$  is second-order. We can use this knowledge to obtain a second condition (since the terms in  $dq^i$  cancel):

$$\sum_i B_i \frac{\partial^2 L}{\partial v^i \partial v^j} = \frac{\partial L}{\partial q^j} - \sum_i v^i \frac{\partial^2 L}{\partial q^i \partial v^j}.$$

The Hessian of  $L$  in the velocities  $M_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j}$  is also called the mass matrix of the system. We have already assumed that this matrix is invertible (i.e.  $L$  is regular). As such, we have that

$$\sum_i \frac{\partial^2 L}{\partial v^i \partial v^j} \frac{d^2 q^j}{dt^2} + \sum_i \frac{\partial^2 L}{\partial q^i \partial v^j} \frac{dq^i}{dt} = \frac{\partial L}{\partial q^j},$$

or equivalently

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^j} \right) - \frac{\partial L}{\partial q^j} = 0,$$

which is the traditional form of the Euler-Lagrange equations.

Provided that  $X_L$  is a second-order vector field, the equation Equation (A-4) is equivalent to the following statement:

$$\mathcal{L}_{X_L} \vartheta_L = dL. \quad (\text{A-7})$$

The equivalence is easily shown using the Cartan formula:

$$\mathcal{L}_{X_L} \vartheta_L = dL$$

$$d(X_L \lrcorner \vartheta_L) + X_L \lrcorner d\vartheta_L = dL$$

$$d(X_L \lrcorner \vartheta_L) - X_L \lrcorner \omega_L = dL$$

The fact that  $X_L$  is second-order implies that  $X_L \lrcorner \vartheta_L = Z(L)$ . Therefore

$$d(Z(L)) - X_L \lrcorner \omega_L = dL$$

$$X_L \lrcorner \omega_L = dZ(L) - L$$

$$X_L \lrcorner \omega_L = dE.$$

Lagrangians are not unique: from Equation (A-4) we can deduce that the addition of a closed 1-form (as a map from  $TQ \rightarrow \mathbb{R}$ ) to the Lagrangian will not alter the Euler-Lagrange equations. The closed 1-forms on  $Q$  therefore constitute the *gauge group* of Lagrangian mechanics. An equivalent statement is that the Euler-Lagrange equations remain invariant if a total derivative is added to the Lagrangian function. [11]

---

## Appendix B

---

# Contact Geometry

This appendix provides a short introduction to the basic concepts of contact geometry that are relevant in this thesis, particularly Chapter 3.

### B-1 Contact structures

A *contact element* on a manifold  $M$  is a point  $m \in M$  combined with a tangent hyperplane  $\xi_m \subset T_m M$  (a subspace of the tangent space with codimension 1). The term ‘contact’ refers to the intuitive notion that if two submanifolds ‘touch’, they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [14] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

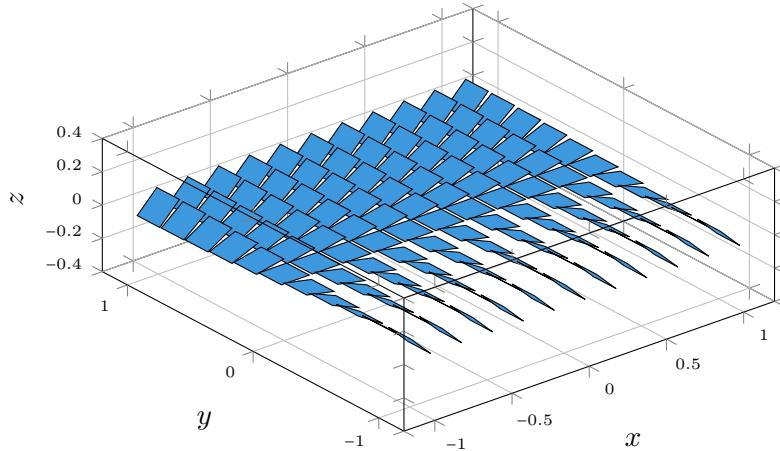
A *contact manifold* is a manifold  $M$  (of dimension  $2n + 1$ ) with a *contact structure*, which is a smooth field (or distribution) of contact elements on  $M$ . Locally, any contact element determines a 1-form  $\alpha$  (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{B-1}$$

This  $\alpha$  is called the (local) *contact form*, and it acts like a ‘normal (co-)vector’ to the hyperplane. For the field hyperplanes to be a constant structure, it must satisfy a nondegeneracy condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [11, 14, 15]

$$\alpha \wedge (d\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of  $M$  such that  $\xi$  is everywhere tangent to it, or (ii) that  $d\alpha|_\xi$  is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient  $TM/\xi$  is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [17].



**Figure B-1:** The standard contact structure on  $\mathbb{R}^3$ , given by the contact form  $dz - y dx$ ; the hyperplanes tilt more in the increasing  $y$ -direction.

The *Darboux theorem* for contact manifolds states that it is always possible to find coordinates  $z, x_i, y_i$  such that locally, the contact form is equal to

$$dz - \sum y_i dx_i,$$

which is also called the standard or natural contact structure. The standard contact structure on  $\mathbb{R}^3$  is illustrated in Figure B-1. Finally, it is clear that the contact form singles out a ‘special direction’ in the tangent space at every point of the manifold. This direction is given by the unique *Reeb vector field*,

$$R_\alpha \in \mathfrak{X}(M) : \quad R_\alpha \lrcorner d\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner \alpha = 1. \quad (\text{B-2})$$

The special direction identified by the Reeb vector field is referred to as the *vertical* direction. Likewise, vector field components in the direction of the Reeb vector field are vertical. A vector field with no vertical component is called *horizontal*.

## B-2 The manifold of contact elements

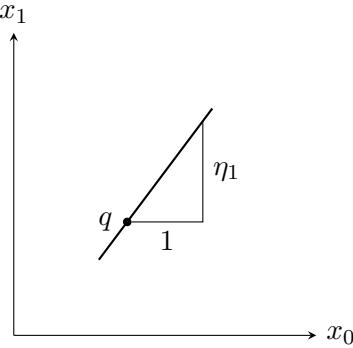
A contact manifold is a manifold with a contact structure. One can, however, associate a *canonical*  $(2n - 1)$ -dimensional contact manifold to *any*  $n$ -dimensional manifold  $Q$ , just like one can always find a canonical symplectic structure on  $T^*Q$ . Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold  $Q$ . As it turns out, this ‘manifold of contact elements’ has a natural contact structure.

The *manifold of contact elements* of an  $n$ -dimensional manifold is [14]

$$CQ = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } T_q Q\}.$$

This manifold  $CQ$  has dimension  $2n - 1$ . It is clear that  $C$  has a natural bundle structure, i.e.  $C \xrightarrow{\pi} Q$  where the bundle projection ‘forgets’ the contact element, that is

$$\pi : CQ \rightarrow Q : (q, \xi_q) \mapsto q.$$



**Figure B-2:** A point in the manifold of contact elements on  $Q = \mathbb{R}^2$ . A coordinate system for  $CQ$  consists of  $(x_0, x_1)$  to indicate a point on  $Q$ , and projective coordinates  $[\eta_0 : \eta_1]$ , which denote the contact element at that point. Without loss of generalization, one can choose  $\eta_0 = 1$ , and the remaining coordinate  $\eta_1$  covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the ‘hyperplane’ and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which  $n - 1 \neq 1$ .

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to  $Q$ , denoted by  $\mathbb{P}T^*Q$ . This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in T_q^*Q \setminus \{\mathbf{0}\} : (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in Equation (B-1), in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure.  $\mathbb{P}T^*Q$  is then the quotient set of  $T^*Q$  (without zero section) with respect to the equivalence relation  $\sim$ . Visually, the projectivization of an  $n$ -dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension  $n - 1$ . It can be shown that this space is bundle-isomorphic to the manifold  $CQ$ . [14]

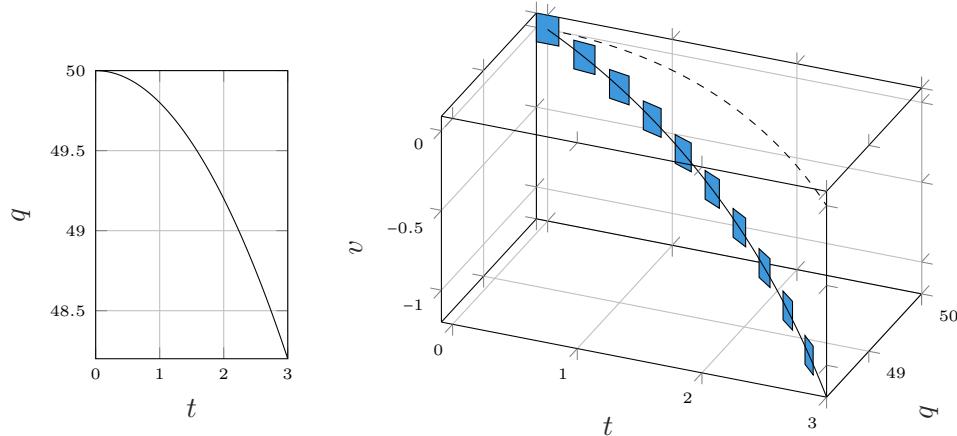
As shown in Figure B-2, coordinates of the equivalence class of 1-forms are ‘projective coordinates’,  $[\eta_0 : \eta_1 : \dots : \eta_{n-1}]$ , where  $\eta_i$  are coordinates for  $T_q^*Q$ . The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes  $\eta_0$  to be nonzero, the tuple  $(1, \eta_1, \dots, \eta_n)$  provides coordinates that cover most of  $\mathbb{P}T^*Q$ .

Now, it remains to be explained why the ‘manifold of contact elements’ is itself a contact manifold. Indeed, there is a canonical field of hyperplanes on  $CQ$ , which lifts the hyperplane tangent to  $Q$  to a hyperplane tangent to  $CQ$  (this is akin to the ‘tautological’ trick played in the symplectic structure of the cotangent bundle). The contact structure distinguishes the curves in  $CQ$  that are lifted versions from curves in  $Q$ . This is illustrated in Figure B-3. [23] Said otherwise, a tangent vector on  $CQ$  lies in the hyperplane defined by the contact structure if its projection down on  $Q$  lies in the hyperplane on  $Q$  defined by the given point on the  $CQ$ .

This contact structure is associated with the 1-form:

$$\alpha = dx_0 + \sum_{i=1}^{n-1} \eta_i dx_i,$$

given that the  $\eta_0$  is the ‘special’ coordinate which is chosen to be 1.



**Figure B-3:** Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let  $(t, q) \in Q$ , and let  $v$  be a coordinate for the contact (line) element. The standard contact form is then  $dq - v dt$ . On the left, the curve corresponding to a falling object is shown in  $Q$ . When this curve is ‘lifted’ to  $CQ$ , the contact structure imposes that it be locally tangent to the contact structure, or that  $v = \frac{dq}{dt}$ . If the vertical direction is projected down onto the  $(q - t)$ -plane ( $C(Q) \rightarrow Q$ ), the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making  $v$  the actual velocity of the curve.

## B-3 Contact Hamiltonian systems

### B-3-1 Contact Hamiltonian vector fields

Just like in the symplectic case, the contact Hamiltonian formalism defines an automorphism between a function on the contact manifold  $K \in C^\infty(M)$ , and an associated ‘Hamiltonian’ vector field  $X_K \in \mathfrak{X}(M)$ . While the isomorphism is rather straightforward for symplectic manifolds, the contact counterpart is not so perspicuous: this is the prime reason behind the computational advantage of symplectification, as opposed to performing the calculations directly on the contact manifold.

#### Coordinate-free derivation

Given a contact manifold  $(M, \xi)$  with contact form  $\alpha$  (i.e.  $\xi \in \ker \alpha$ ), the tangent bundle  $M$  can be decomposed into two subbundles: [14, 16]

$$TM = \ker \alpha \oplus \ker d\alpha,$$

where  $\oplus$  denotes the Whitney sum. The first subbundle is referred to as the *horizontal* bundle, the second as the *vertical* bundle. The vertical subbundle is of rank 1 and its fiber is spanned

by the Reeb vector field (cf. Equation (B-2)). As mentioned to in Appendix B-1, *any* vector field  $X \in \mathfrak{X}(M)$  may be decomposed accordingly. This decomposition is unique and given by

$$X = \underbrace{(X \lrcorner \alpha)R_\alpha}_{X^{\text{ver}}} + \underbrace{[X - (X \lrcorner \alpha)R_\alpha]}_{X^{\text{hor}}}. \quad (\text{B-3})$$

Observe that indeed  $X^{\text{ver}} \in \ker d\alpha$  and  $X^{\text{hor}} \in \ker \alpha$ . [14, 16, 76]

We now wish to find the relation between the contact Hamiltonian  $K \in C^\infty(M)$  and the associated Hamiltonian vector field  $X_K \in \mathfrak{X}(M)$ . This one-to-one relation is uniquely determined by two conditions. Firstly, we impose that<sup>1</sup>

$$K \equiv -X_K \lrcorner \alpha.$$

This condition already provides us with the vertical component of the Hamiltonian vector field, namely

$$X_K^{\text{ver}} = -KR_\alpha.$$

Secondly, the automorphism generated by the Hamiltonian vector field must be a *contact automorphism*: it must preserve the contact structure. This condition is encoded in terms of the Lie derivative:<sup>2</sup>

$$X_K \text{ is an infinitesimal contact automorphism} \Leftrightarrow \mathcal{L}_{X_K} \alpha = s\alpha,$$

where  $s \in C^\infty(M)$ . The function  $s$  is there because  $s\alpha$  and  $\alpha$  give rise to the same hyperplane distribution. Using Cartan's 'magic' formula, the Lie derivative can be expanded as follows:

$$\mathcal{L}_{X_K} \alpha = s\alpha$$

$$d(X_K \lrcorner \alpha) + X_K \lrcorner d\alpha = s\alpha$$

$$-dK + X_K \lrcorner d\alpha = s\alpha$$

Contracting both sides with the Reeb vector field yields:

$$R_\alpha \lrcorner (-dK + X_K \lrcorner d\alpha) = R_\alpha \lrcorner (s\alpha)$$

$$-R_\alpha \lrcorner dK + R_\alpha \lrcorner X_K \lrcorner d\alpha = s R_\alpha \lrcorner \alpha$$

$$-R_\alpha \lrcorner dK - X_K \lrcorner R_\alpha \lrcorner d\alpha = s.$$

Hence, we have  $s = -R_\alpha \lrcorner dK$ . Because the vertical component of  $X_K$  is spanned by the Reeb vector field, its contraction with  $d\alpha$  vanishes. As a result, we can rewrite the previous expression in terms of the *horizontal* component of  $X_K$ :

$$X_K \lrcorner d\alpha = X_K^{\text{hor}} \lrcorner d\alpha = [dK - (R_\alpha \lrcorner dK)\alpha], \quad (\text{B-4})$$

<sup>1</sup>This is the sign convention observed by Bravetti et al. [42] en van der Schaft [34], as opposed to Libermann and Marle [16].

<sup>2</sup>Terminology differs somewhat in literature on this point: some authors, such as de León and Lainz [76] only refer to contactomorphisms as the special case where  $g = 0$ ; while the more general case is called *conformal* contactomorphisms.

We must therefore recover  $X_K^{\text{hor}}$  from the above expression. Define the mapping

$$\alpha^\flat : TM \rightarrow T^*M : X \mapsto X \lrcorner d\alpha,$$

when restricted to the space of horizontal vector fields, this mapping is an isomorphism onto the ‘semi-basic’ forms<sup>3</sup>. Define the inverse mapping of  $\alpha^\flat$  by  $\alpha^\sharp$ , such that

$$X_K^{\text{hor}} = \alpha^\sharp(dK - (R_\alpha \lrcorner dK) \alpha).$$

As such, the Hamiltonian vector field associated to the contact Hamiltonian  $K$  is

$$X_K = KR_\alpha + \alpha^\sharp(dK - (R_\alpha \lrcorner dK) \alpha). \quad (\text{B-5})$$

### Coordinate expression

Given the contact manifold  $(M, \xi)$  with contact form

$$dq_0 - \sum_{i=1}^n p_i dq_i,$$

and define the contact Hamiltonian  $K = K(q_0, q_1, \dots, q_n, p_1, \dots, p_n)$ . The vertical component of the Hamiltonian vector field is straightforward (cf. Equation (B-2)):

$$X_K^{\text{ver}} = -K \frac{\partial}{\partial q_0}.$$

For the horizontal component, first evaluate the right hand side of Equation (B-4) in coordinates:

$$X_K^{\text{hor}} \lrcorner d\alpha = \sum_{i=1}^n \left( \frac{\partial K}{\partial q_i} + p_i \frac{\partial K}{\partial q_0} \right) dq_i + \frac{\partial K}{\partial p_i} dp_i.$$

In terms of the basis vectors, the mapping  $\alpha^\flat$  is

$$\frac{\partial}{\partial q_i} \mapsto dp_i \quad \frac{\partial}{\partial p_i} \mapsto -dq_i \quad \frac{\partial}{\partial q_0} \mapsto 0 \quad i = 1, \dots, n.$$

The inverse transformation is slightly ambiguous at first sight, since any  $\frac{\partial}{\partial q_0}$  cannot be recovered directly from the ‘forward’ mapping. However, we know that  $\alpha^\sharp$  must produce a horizontal vector field. Therefore, first perform the inverse mapping in the  $q_i, p_i$ -components to obtain

$$-\sum_{i=1}^n \left( \frac{\partial K}{\partial q_i} + p_i \frac{\partial K}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial K}{\partial p_i} \frac{\partial}{\partial q_i}.$$

Contracting this expression with  $\alpha$  produces  $-\sum_{i=1}^n p_i \frac{\partial K}{\partial p_i}$ . Hence, we can use this knowledge to find the actual horizontal component:

$$X_K^{\text{hor}} = \sum_{i=1}^n p_i \frac{\partial K}{\partial p_i} \frac{\partial}{\partial q_0} - \sum_{i=1}^n \left( \frac{\partial K}{\partial q_i} + p_i \frac{\partial K}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial K}{\partial p_i} \frac{\partial}{\partial q_i}.$$

---

<sup>3</sup>Semi-basic forms are forms that vanish when contracted with a vertical vector field. [16]

As such, the coordinate expression of Equation (B-5) is

$$X_K = \left( \sum_{i=1}^n p_i \frac{\partial K}{\partial p_i} - K \right) \frac{\partial}{\partial q_0} - \sum_{i=1}^n \left( \frac{\partial K}{\partial q_i} + p_i \frac{\partial K}{\partial q_0} \right) \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial K}{\partial p_i} \frac{\partial}{\partial q_i} \quad (\text{B-6})$$

Furthermore, we have

$$\mathcal{L}_{X_K} \alpha = -\frac{\partial K}{\partial q_0} \alpha,$$

and

$$\mathcal{L}_{X_K} K = -K \frac{\partial K}{\partial q_0}.$$

### B-3-2 Jacobi brackets

Just like the Poisson brackets define a Poisson algebra of the smooth functions on a symplectic manifold, there is a bracket operation on contact manifolds that serves (about) the same purpose. These brackets do not define a Poisson structure, but rather a *Jacobi structure*, which is a more general notion that includes the Poisson structure as a particular instance. In this treatment we will only focus on the associated *Jacobi bracket* for contact Hamiltonian systems. For more details regarding Jacobi manifolds, the reader is referred to [16, chap. V] and [76].

For two smooth functions  $f, g \in C^\infty(M)$ , and  $M$  a contact manifold with contact form  $\alpha$ , the *Jacobi bracket* is defined as

$$\{ , \} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) : \{f, g\} = -[X_f, X_g] \lrcorner \alpha, \quad (\text{B-7})$$

where  $X_f, X_g \in \mathfrak{X}(M)$  are the contact Hamiltonian vector fields of  $f$  and  $g$  respectively, and  $[\cdot, \cdot]$  is the Lie bracket (i.e. the commutator of vector fields). Equivalent expressions for the Jacobi bracket are: [16]

$$\begin{aligned} \{f, g\} &= -X_f \lrcorner dg + g(R_\alpha \lrcorner df) \\ &= X_g \lrcorner df - f(R_\alpha \lrcorner dg) \\ &= -d\alpha(X_f, X_g) - f(R_\alpha \lrcorner dg) + g(R_\alpha \lrcorner df). \end{aligned} \quad (\text{B-8})$$

From these expressions, it is also clear that the Jacobi bracket is antisymmetric, i.e.  $\{f, g\} = -\{g, f\}$  and  $\{f, f\} = 0$ . As a time evolution operator (with respect to the Hamiltonian  $K$ ), we have

$$\frac{df}{dt} = \{f, K\} + f(R_\alpha \lrcorner dK) = \{f, K\} - fs.$$

Using the same coordinates as in Appendix B-3-1, the Jacobi bracket is equal to:

$$\{f, g\} = \left( \sum_{i=1}^n p_i \frac{\partial g}{\partial p_i} - g \right) \frac{\partial f}{\partial q_0} - \left( \sum_{i=1}^n p_i \frac{\partial f}{\partial p_i} - f \right) \frac{\partial g}{\partial q_0} + \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right).$$



---

# Bibliography

- [1] B. N. M. Krabbenborg, “Duration Matching in the Frequency Domain - An Economic Engineering Approach to Asset and Liability Management,” Ph.D. dissertation, Delft University of Technology, 2021.
- [2] N. L. Balazs and A. Voros, “Chaos on the Pseudosphere,” *Physics Reports*, vol. 143, no. 3, pp. 109–240, 1986.
- [3] R. H. Rapp, “Geometric Geodesy, pt. 1,” Ohio State University - Department of Geodetic Science and Surveying, Columbus, OH, Tech. Rep., 1991.
- [4] G. J. L. Kruimer, “An Engineering Approach to Macroeconomic Scenario Modelling,” Ph.D. dissertation, Delft University of Technology, 2021.
- [5] M. Mendel, “Economic Engineering - Lecture Notes,” Delft, 2019.
- [6] T. Russell, “Symplectic geometry: The natural geometry of economics?” *Economics Letters*, vol. 112, no. 3, pp. 236–238, 2011.
- [7] R. Swierstra, “Foundations of Economic Geometry,” *SSRN Electronic Journal*, pp. 1–8, 2014.
- [8] D. Jeltsema and J. M. Scherpen, “Multidomain Modeling of Nonlinear Networks and Systems: Energy- and power-based perspectives,” *IEEE Control Systems*, vol. 29, no. 4, pp. 28–59, 2009.
- [9] J. F. Cariñena, “Theory of Singular Lagrangians,” *Fortschritte der Physik*, vol. 38, no. 9, p. 693, 1990.
- [10] Varian, Hal R, *Microeconomic Analysis*, 3rd ed. New York, NY: W. W. Norton & Company, 1992.
- [11] R. Abraham and J. E. Marsden, *Foundations of Mechanics*. Addison-Wesley Publishing Company, 1978.

- [12] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*. New York, NY: Springer, 1998.
- [13] L. E. Blume, “Duality,” London, UK, 2020.
- [14] A. Cannas da Silva, *Lectures on Symplectic Geometry*, 1, Ed. Berlin, Heidelberg: Springer, 2001.
- [15] V. Arnol'd, *Mathematical Methods of Classical Mechanics*, J. Ewing, F. Gehring, and P. Halmos, Eds. New York: Springer-Verlag, 1989.
- [16] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*. Dordrecht, Holland: D. Reidel Publishing Company, 1987.
- [17] H. Geiges, *An Introduction to Contact Topology*. Cambridge, UK: Cambridge University Press, 2008.
- [18] V. I. Arnol'd, “Contact Geometry and Wave Propagation,” Geneva, 1989.
- [19] C. Godbillon, *Géométrie Différentielle et Mécanique Analytique*. Paris: Collection Méthodes Hermann, 1969.
- [20] V. Arnol'd, *Singularities of Caustics and Wave Fronts*. Dordrecht: Springer Science+Business Media, 1990, vol. 62.
- [21] ——, “Contact Geometry: The Geometrical Method of Gibbs’s Thermodynamics,” in *Proceedings of the Gibbs Symposium: Yale University, May 15-17*, New Haven, CT, 1989.
- [22] P. Bamberg and S. Sternberg, *A Course in Mathematics for Students of Physics*. Cambridge, UK: Cambridge University Press, 1990, vol. 2.
- [23] W. L. Burke, *Applied Differential Geometry*. Cambridge: Cambridge University Press, 1985.
- [24] R. Hermann, *Geometry, Physics and Systems*. New York, NY: Marcel Dekker, 1973.
- [25] J. W. Gibbs, “Graphical Methods in the Thermodynamics of Fluids,” *Transactions of the Connecticut Academy*, vol. II, pp. 309–342, 1873.
- [26] A. S. Wightman and R. B. Israel, “Convexity and The Notion of Equilibrium State in Thermodynamics and Statistical Mechanics,” in *Convexity in the Theory of Lattice Gases*. Princeton, NJ: Princeton University Press, 1979.
- [27] R. Mrugała, J. D. Nulton, J. Christian Schön, and P. Salamon, “Contact structure in thermodynamic theory,” *Reports on Mathematical Physics*, vol. 29, no. 1, pp. 109–121, 1991.
- [28] R. Mrugała, “On contact and metric structures on thermodynamic spaces,” *RIMS Kokyuroku*, vol. 1142, pp. 167–181, 2000.
- [29] ——, “On equivalence of two metrics in classical thermodynamics,” *Physica A: Statistical Mechanics and its Applications*, vol. 125, no. 2-3, pp. 631–639, 1984.

- [30] ——, “Submanifolds in the Thermodynamic Phase Space,” *Reports on Mathematical Physics*, vol. 21, no. 2, pp. 197–203, 1985.
- [31] ——, “Continuous contact transformations in thermodynamics,” *Reports on Mathematical Physics*, vol. 33, no. 1-2, pp. 149–154, 1993.
- [32] ——, “On a Riemannian metric on contact thermodynamic spaces,” *Reports on Mathematical Physics*, vol. 38, no. 3, pp. 339–348, 1996.
- [33] R. Balian and P. Valentin, “Hamiltonian structure of thermodynamics with gauge,” *European Physical Journal B*, vol. 21, no. 2, pp. 269–282, 2001.
- [34] A. van der Schaft, “Liouville geometry of classical thermodynamics,” *Journal of Geometry and Physics*, vol. 170, p. 104365, 2021.
- [35] A. van der Schaft and B. Maschke, “Homogeneous Hamiltonian Control Systems Part I: Geometric Formulation,” *IFAC-PapersOnLine*, vol. 51, no. 3, pp. 1–6, 2018.
- [36] B. Maschke and A. van der Schaft, “Homogeneous Hamiltonian Control Systems Part II: Application to thermodynamic systems,” *IFAC-PapersOnLine*, vol. 51, no. 3, pp. 7–12, 2018.
- [37] A. Bravetti, C. S. Lopez-Monsalvo, and F. Nettel, “Contact symmetries and Hamiltonian thermodynamics,” *Annals of Physics*, vol. 361, pp. 377–400, 2015.
- [38] A. A. Simoes, M. De León, M. L. Valcázar, and D. M. De Diego, “Contact geometry for simple thermodynamical systems with friction: Contact geometry for thermodynamics,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 476, no. 2241, pp. 1–21, 2020.
- [39] T. Frankel, *The Geometry of Physics*. Cambridge, UK: Cambridge University Press, 2012.
- [40] P. A. Dirac, “Generalized Hamiltonian Dynamics,” *Canadian Journal of Mathematics*, vol. 2, pp. 129–148, 1950.
- [41] E. Fermi, *Thermodynamics*. New York, NY: Dover Publications, 1956.
- [42] A. Bravetti, H. Cruz, and D. Tapias, “Contact Hamiltonian Mechanics,” *Annals of Physics*, vol. 376, pp. 17–39, 2017.
- [43] F. P. Schuller, “The Geometric Anatomy of Theoretical Physics,” Friedrich-Alexander Universität Erlangen-Nürnberg - Institut für Theoretische Physik II, Tech. Rep., 2014.
- [44] R. Hermann, *Topics in the Geometric Theory of Linear Systems*. Brookline, MA: Math Sci Press, 1984.
- [45] M. Verhaegen and V. Verdult, *Filtering and System Identification: A Least-Squares Approach*. Cambridge, UK: Cambridge University Press, 2007, vol. 9780521875.
- [46] P. Caldirola, “Forze Non Conservative Nella Meccanica Quantistica,” *Il Nuovo Cimento*, vol. 18, no. 9, pp. 393–400, 1941.

- [47] E. Kanai, “On the Quantization of the Dissipative Systems,” *Progress of Theoretical Physics*, vol. 3, no. 4, pp. 440–442, 1948.
- [48] M. Tokieda and S. Endo, “Equivalence of Dissipative and Dissipationless Dynamics of Interacting Quantum Systems With Its Application to the Unitary Fermi Gas,” *Frontiers in Physics*, vol. 9, no. September, pp. 1–9, 2021.
- [49] D. Schuch, “Nonunitary connection between explicitly time-dependent and nonlinear approaches for the description of dissipative quantum systems,” *Physical Review A - Atomic, Molecular, and Optical Physics*, vol. 55, no. 2, pp. 935–940, 1997.
- [50] W. Borutzky, *Bond Graph Methodology: Development and Analysis of Multidisciplinary Dynamic System Models*. Springer, 2010, vol. 44.
- [51] C.-M. Marle, “On Jacobi Manifolds and Jacobi Bundles,” in *Symplectic Geometry, Groupoids, and Integrable Systems*, ser. Mathematical Sciences Research Institute Publications, P. Dazord and A. Weinstein, Eds. New York, NY: Springer US, 1991, pp. 227–246.
- [52] A. Einstein and V. Bargmann, “Bivector Fields,” *Annals of Mathematics*, vol. 45, no. 1, pp. 1–14, 1944.
- [53] A. Lichnerowicz, “Les variétés de Poisson et leurs algèbres de Lie associées,” *Journal of Differential Geometry*, vol. 12, no. 2, pp. 253–300, Jan. 1977.
- [54] P. Dazord, A. Lichnerowicz, and C. Marle, “Structure locale des variétés de Jacobi,” *undefined*, 1991.
- [55] F. M. Ciaglia, H. Cruz, and G. Marmo, “Contact manifolds and dissipation, classical and quantum,” *Annals of Physics*, vol. 398, pp. 159–179, Nov. 2018.
- [56] F. Mahmood, “Jacobi Structures And Differential Forms On Contact Quotients,” Ph.D. dissertation, Cornell University, Aug. 2012.
- [57] A. Van Der Schaft, “Port-Hamiltonian systems: An introductory survey,” *International Congress of Mathematicians, ICM 2006*, vol. 3, pp. 1339–1365, 2006.
- [58] J. Stillwell, *Naive Lie Theory*. New York: Springer Science+Business Media, 2008.
- [59] D. S. Dummit and R. M. Foote, *Abstract Algebra*. Hoboken, N.J.: John Wiley and Sons, 2004.
- [60] M. Jafari and Y. Yayli, “Matrix Theory over the Split Quaternions,” *International Journal of Geometry*, vol. 3, no. 2, pp. 57–69, 2014.
- [61] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. San Francisco, CA: W. H. Freeman and Company, 1970.
- [62] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*. Oxford: Pergamon Press, 1971, vol. 2.
- [63] S. Lang, *Algebra*. New York, NY: Springer-Verlag, 2002.

- [64] C. H. Edwards, D. E. Penney, and D. T. Calvis, *Differential Equation & Linear Algebra*. Pearson Education, 2018.
- [65] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*. Cham, Switzerland: Springer International Publishing, 2013.
- [66] A. E. Motter and M. A. F. Rosa, “Hyperbolic Calculus,” *Advances in Applied Clifford Algebras*, vol. 8, no. 1, pp. 109–128, 1998.
- [67] A. A. Harkin and J. B. Harkin, “Geometry of Generalized Complex Numbers,” *Mathematics Magazine*, vol. 77, no. 2, pp. 118–129, 2004.
- [68] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics - A Foundation for Computer Science*. Reading, MA: Addison-Wesley Publishing Company, 1994.
- [69] V. Arnold, *Ordinary Differential Equations*. Berlin: Springer-Verlag, 1984.
- [70] C. Moler and C. Van Loan, “Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later,” *SIAM Review*, vol. 45, no. 1, pp. 3–49, 2003.
- [71] H. Dekker, “Classical and Quantum Mechanics of the Damped Harmonic Oscillator,” Physics Laboratory, TNO, Den Haag, Tech. Rep., 1981.
- [72] G. Dedene, “Oscillators and Complex Hamiltonian Calculus,” *Physica*, vol. 371, no. 103A, pp. 371–378, 1980.
- [73] W. P. Thurston, *Three-Dimensional Geometry and Topology*, S. Levy, Ed. Princeton, NJ: Princeton University Press, 1997, vol. 1.
- [74] T. Needham, *Visual Complex Analysis*. New York: Oxford University Press, 1997.
- [75] ——, *Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts*. Princeton, NJ: Princeton University Press, 2021.
- [76] M. de León and M. Lainz, “A Review on Contact Hamiltonian and Lagrangian Systems,” Instituto de Ciencias Matemáticas, Madrid, Spain, Tech. Rep., 2020.



---

# Glossary

## Economic engineering

$\dot{q}$	Quantity demanded; quantity supplied; flow of goods
$p$	Price
$q$	Quantity; amount of goods

## Geometric structures

$\alpha$	General contact 1-form
$\gamma$	Damping coefficient
$\mathbb{R}^n$	Real coordinate space of dimension $n$
$E \xrightarrow{\pi} B$	Bundle with total space $E$ , projection map $\pi$ and base space $B$
$T_x M$	Tangent space to the manifold $M$ at the point $x$
$T_x^* M$	Cotangent space of the manifold $M$ at the point $x$
$TM$	Tangent bundle of the manifold $M$
$T^* M$	Cotangent bundle of the manifold $M$
$\mathfrak{X}(M)$	Set of vector fields (smooth sections of $TM$ ) on the manifold $M$
$C^\infty(M)$	Set of smooth functions on the manifold $M$
$\Omega^n(M)$	Set of $n$ -forms on the manifold $M$

## Split-quaternions

$1, \hat{i}, \hat{j}, \hat{k}$	Split-quaternion basis elements
$\hat{a}$	Normalized (unit) vector part of the split-quaternion $a$
$a$	Vector part of the split-quaternion $a$
$a$	Split-quaternion

