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E. B. Legrand

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Delft University of Technology
Department of
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The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

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by

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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis...

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology
May 24, 2022

E. B. Legrand

Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator
- (Optimal) control of the distributions using the Brockett approach
- (Stochastic) inputs, link with Langevin equations
- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.
- Bayesian inversion of chaotic systems; guess the initial state by sampling after a certain time
- Define as streamtube, continuity equation asserts that streamlines cannot cross; i.e. streamtubes are conserves. To reduce computational complexity, define level sets (curves in 2-D) and check how they deform through the evolution of the phase space fluid; should always contain the same amount of probability troughout the evolution of the system.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\mathit{\Gamma\Delta\Upsilon\Omega\Theta}$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathit{\Gamma\Delta\Upsilon\Omega\Theta}}$
Tensor	$\boldsymbol{\mathsf{abcde}}$	$\boldsymbol{\mathsf{ABCDE}}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathsf{\Gamma\Delta\Upsilon\Omega\Theta}}$

Table 1-1: Caption

Math constants: $\mathrm{i}\mathrm{e}\pi$
Variation: δS

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner\, \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$
$\Gamma(TM)$
$\mathfrak{X}(TM)$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by Abraham and Marsden [1] and Cannas da Silva [2] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.
- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

Symplectic and Contact Geometry in Economic Engineering

Chapter 3

Symplectified Contact Mechanics for Dissipative Systems

The traditional view is that the methods of analytical mechanics, such as the Lagrangian and Hamiltonian formalisms, are only suited for conservative systems. However, several attempts, especially in the previous century, have been made to extend these principles to dissipative systems as well.

3-1 The damped harmonic oscillator

This chapter (and the application in the following chapter) is primarily concerned with the prototypical dissipative mechanical system: the linearly damped harmonic oscillator depicted in fig. 3-1, with the governing second-order differential equation being

$$m\ddot{q} + b\dot{q} + kq = 0. \tag{3-1}$$

The choice for this system is rather perspicuous, since it is arguably the 'easiest' dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [3, 4]. However, the method described in this section can be generalized directly to a general (possibly time-dependent) potential function $V = V(q, t)$. (... `Discuss parameter notation ...`)

3-2 Historical perspectives

A traditional, engineering-inclined method to incorporate damping in the framework is to include a Rayleigh damping term in the Lagrangian to emulate linear damping forces, and this works 'mathematically' to derive the correct equations of motion [5]. Although frequently

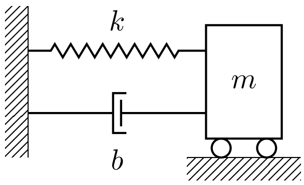

Figure 3-1: Schematic of the mass-spring-damper system.

Table 3-1: Parameter conventions of the damped harmonic oscillator. To avoid confusion with the symplectic form ω, angular frequencies are denoted by Ω instead of the conventional lower case Greek letter.

Name	**Symbol**	**Value**	**Units**
Damping coefficient	γ	b/m	s^{-1}
Undamped frequency	Ω_o	$\sqrt{k/m}$	s^{-1}
Damped frequency	Ω_d	$\sqrt{\Omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$	s^{-1}
Damping ratio	ζ	$\frac{b}{2\sqrt{mk}}$	–

used for practical problems, this damping term is not really part of the *actual* Lagrangian — rather, it simply makes use of the notion of a generalized force that is not inherently part of the system. As such, this method only 'works' on a superficial level: the pristine differential geometric foundations of mechanics do not leave room for such ad hoc tricks. There is, as a result, also no Hamiltonian counterpart for this method.

The historical attempts to do better than the Rayleigh method were primarily motivated by the application of the (dissipative) Hamiltonian formalism in quantum mechanics through discretization. For this application, a sound mathematical structure is of the essence, which calls for a more rigorous approach. A celebrated paper by Dekker [3] provides an excellent summary of many attempts up to 1981. Indeed, the well-studied approach developed by Caldirola [6] and Kanai [7] was intended exactly for this purpose. This method features an explicit time-dependence both in the Lagrangian function

$$L_{\mathrm{CK}}(q, \dot{q}, t) = \mathrm{e}^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right), \tag{3-2}$$

and the corresponding Hamiltonian function:

$$H_{\mathrm{CK}}(q, \rho, t) = \frac{\rho^2}{2m}\mathrm{e}^{-\gamma t} + \frac{1}{2}kq^2\mathrm{e}^{\gamma t}. \tag{3-3}$$

In latter equation, ρ refers to a special 'canonical momentum', that is

$$\rho \equiv \frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}},$$

which is related to the 'true' kinematic momentum by the relation $\rho = p\mathrm{e}^{\gamma t} = m\dot{q}\mathrm{e}^{\gamma t}$. As such, it is also clear that the Caldirola-Kanai Lagrangian and Hamiltonian functions are related by the Legendre transform *with respect to the canonical momentum*:[1]

$$H_{\mathrm{CK}} = \rho\dot{q} - L_{\mathrm{CK}}.$$

From either eq. (3-2) or eq. (3-3), the equations of motion are readily derived (for the Hamiltonian case with respect to ρ after which the transformation to p can be effected). Indeed, after taking the appropriate derivatives, one obtains:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}\right) - \frac{\partial L_{\mathrm{CK}}}{\partial q} &= 0\\ \Rightarrow \mathrm{e}^{\gamma t}(m\ddot{q} + m\gamma\dot{q} + kq) &= 0\end{aligned}$$

for the Lagrangian case. Likewise, Hamilton's equations yield: [9]

$$\begin{aligned}\dot{q} &= \frac{\partial H_{\mathrm{CK}}}{\partial \rho} = \frac{\rho}{m}\mathrm{e}^{-\gamma t} = \frac{p}{m},\\ \dot{\rho} &= -\frac{\partial H_{\mathrm{CK}}}{\partial q} = -kq\mathrm{e}^{\gamma t}.\end{aligned}$$

The relation between the time derivatives of the momenta $\dot{p}$ and $\dot{\rho}$ is slightly more involved since one must invoke the product rule as a result of their time-dependent relation:

$$\dot{\rho} = \mathrm{e}^{\gamma t}(\dot{p} + \gamma p).$$

Substition yields the correct equation for p, though the equation is again multiplied by $\mathrm{e}^{\gamma t}$. Because the latter is sufficiently well-behaved (that is, it has no zeros), it can be removed without any problems.

Geometric perspective To put the above derivation in a geometric setting, define the Liouville 1-form as

$$\alpha = \rho\,\mathrm{d}q \quad \Rightarrow \quad \omega = -\,\mathrm{d}\alpha = \mathrm{d}q \wedge \mathrm{d}\rho\,,$$

where the symplectic 2-form will be used to obtain Hamilton's equations. The Hamiltonian eq. (3-3) is explicitly time-dependent. This will give rise to a time-dependent vector field governing the solution curves.[2] The construction of the vector field associated with a time-dependent Hamiltonian follows the same construction rules as a normal Hamiltonian (using

[1] The 'Legendre transform' refers, in the context of fiber bundles, to the so-called fiber derivative. On a manifold M, let $L \in C^\infty(M)$. Then the fiber derivative is defined als

$$\mathbb{F}L : TL \to T^*L : \mathbb{F}L(\boldsymbol{v}) \cdot \boldsymbol{w} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\boldsymbol{v} + s\boldsymbol{w}).$$

Hence, the Legendre transform is in the first place the mapping that associates the generalized velocities with the associated (canonical) generalized momenta. Importantly, this mapping is a diffeomorphism (that is, invertible and onto) if the Hessian of L is nondegenerate - roughly equivalent to the statement that every generalized velocity has an associated 'mass' to it. [8].

[2] A *time-dependent vector field* on a manifold M is a mapping $X : M \times \mathbb{R} \to TM$ such that for each $t \in \mathbb{R}$, the restriction X_t of X to $M \times \{t\}$ is a vector field on M. [10] An additional construction of importance, called the *suspension* of the vector field, is a mapping

$$\tilde{X} : \mathbb{R} \times M \to T(\mathbb{R} \times M) \quad (t, m) \mapsto ((t, 1), (m, X(t, m))),$$

that is to say, it lifts the vector field to the extended space that also includes t and assigns the time coordinate with a trivial velocity of 1. [1]

the isomorphism given by ω), but 'frozen' at each instant of t. Even more bluntly speaking, one simply ignores the t-coordinate during the derivation, only to acknowledge the dependence at the very end. This leads to the following vector field, 'suspended' on the $\mathbb{R} \times Q$ space:

$$\tilde{X}_{H_{\mathrm{CK}}} = -\mathrm{e}^{\gamma t} kq \frac{\partial}{\partial \rho} + \mathrm{e}^{-\gamma t} \frac{\rho}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

The suspension is important to make the final coordinate transform from ρ to p work properly. Indeed, effecting the transformation $(q, \rho, t) \mapsto (q, \mathrm{e}^{-\gamma t}, t)$, one obtains

$$\tilde{X}_{H_{\mathrm{CK}}} = (-kq - \gamma p) \frac{\partial}{\partial p} + \frac{p}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

It is worthwile to ponder on some apparent peculiarities in the Caldirola-Kanai method, for they will be explained elegantly by the contact-Hamiltonian formalism. Firstly, the role of the two-different momenta is not very clear from the get-go, apart from being a consequence of the way the Caldirola-Kanai Lagrangian is formulated. This has also been the reason for considerable confusion in the academic community (see Schuch [11]). Furthermore, there is the

3-3 Contact manifolds

This section may be moved to an appendix.

Contact manifolds are odd-dimensional manifolds with the addition of a contact structure. This contact structure can be considered to be like a symplectic structure (which is necessarily even-dimensional) with the addition of one 'special' dimension. The relation between contact structures and symplectic structures is crucial for the extension of Lagrangian and Hamiltonian mechanics to contact manifolds.

3-3-1 Contact structures

A *contact element* on a manifold M is a point $m \in M$ combined with a tangent hyperplane $\xi_m \subset T_m M$ (a subspace of the tangent space with codimension 1). The term 'contact' refers to the intuitive notion that if two submanifolds 'touch', they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [2] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

A *contact manifold* is a manifold M (of dimension $2n + 1$) with a **contact structure**, which is a smooth field (or distribution) of contact elements on M. Locally, any contact element determines a 1-form α (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{3-4}$$

This α is called the (local) *contact form*, and it acts like a 'normal (co-)vector' to the hyperplane. For the field hyperplanes to be a constact structure, it must satisfy a nondegeneracy

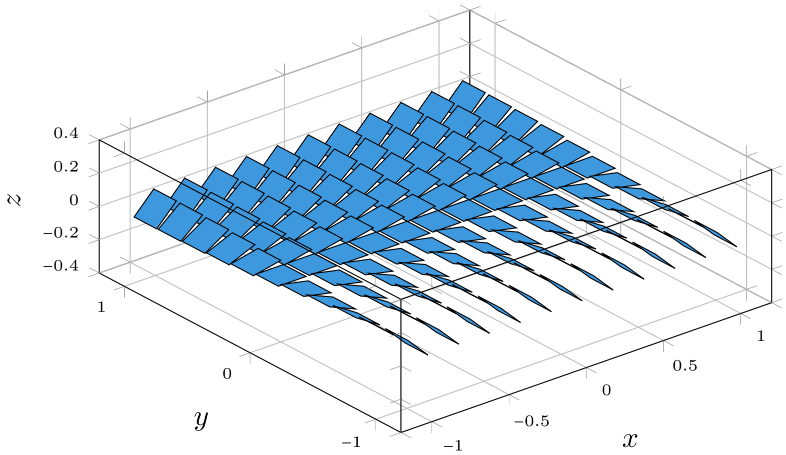

Figure 3-2: The standard contact structure on $\mathbb{R}^3$, given by the contact form $\mathrm{d}z - y\,\mathrm{d}x$; the hyperplanes tilt more in the increasing y-direction.

condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [2, 1, 12]

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of M such that the ξ is everywhere tangent to it, or (ii) that $\mathrm{d}\alpha|_\xi$ is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient TM/ξ is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [13].

The **Darboux theorem** for contact manifolds states that it is always possible to find coordinates z, x_i, y_i such that locally, the contact form is equal to

$$\mathrm{d}z - \sum y_i\,\mathrm{d}x_i\,,$$

which is also called the standard or natural contact structure. The standard contact structure on $\mathbb{R}^3$ is illustrated in fig. 3-2. Finally, it is clear that the contact form singles out a 'special direction' in the tangent space at every point of the manifold. This direction is given by the unique **Reeb vector field**,

$$R_\alpha \in \mathfrak{X}(M): \quad R_\alpha \lrcorner\, \mathrm{d}\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner\, \alpha = 1,$$

that is, it locally points in the 'direction' of the contact form.

3-3-2 The manifold of contact elements

It was prevously noted that a contact manifold is a manifold with a contact structure. There is, associated to any manifold n-dimensional manifold Q a *canonical* contact manifold of dimension $2n - 1$, just like one can always find a canonical symplectic structure on T^*Q of dimension $2n$. Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold Q. As it turns out, this 'manifold of contact elements' also has a natural contact structure.

The *manifold of contact elements* of an n-dimensional manifold is [2]

$$CQ = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } T_qQ\}.$$

This manifold CQ has dimension $2n - 1$. It is clear that C has a natural bundle structure, i.e. $C \xrightarrow{\pi} Q$ where the bundle projection 'forgets' the contact element, that is

$$\pi : CQ \to Q : (q, \xi_q) \mapsto q.$$

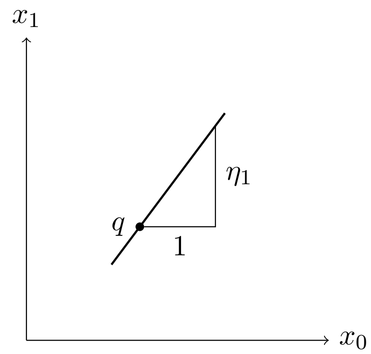

Figure 3-3: A point in the manifold of contact elements on $Q = \mathbb{R}^2$. A coordinate system for CQ consists of (x_0, x_1) to indicate a point on Q, and projective coordinates $[\eta_0 : \eta_1]$, which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$, and the remaining coordinate η_1 covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the 'hyperplane' and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$.

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to Q, denoted by $\mathbb{P}(T^*Q)$. This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in T_q^*Q \setminus \{\mathbf{0}\} : \quad (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in eq. (3-4), in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure. $\mathbb{P}(T^*Q)$ is then the quotient set of T^*Q (without zero section) with respect to the equivalence relation $\sim$. Visually, the projectivization of an n-dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension $n - 1$. It can be shown that this space is bundle-isomorphic to the manifold CQ. [2]

As shown in fig. 3-3, coordinates of the equivalence class of 1-forms are 'projective coordinates', $[\eta_0 : \eta_1 : \ldots : \eta_{n-1}]$, where η_i are coordinates for T_q^*Q. The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes η_0 to be nonzero, the tuple $(1, \eta_1, \ldots, \eta_n)$ provides coordinates that cover most of T_q^*Q.

Now, it remains to be explained why the 'manifold of contact elements' is itself a contact manifold. Indeed, there is a canonical field of hyperplanes *on* CQ, which lifts the hyperplane tangent to Q to a hyperplane tangent to C (this is akin to the 'tautological' trick played in the symplectic structure of the cotangent bundle). Locally, the contact form associated to this contact structure is

$$\alpha = \mathrm{d}x_0 + \sum_{i=1}^{n} \eta_i \,\mathrm{d}x_i + \sum_{i=1}^{n} 0 \,\mathrm{d}\eta_i \,,$$

where any .

3-3-3 Principal bundles

3-3-4 Symplectification and Liouville structures

It was mentioned in **??** that

3-4 Dissipative contact Hamiltonian mechanics

The contact-geometric counterpart of Hamiltonian and Lagrangian mechanics has been the subject of increasing academic interest in recent years, see for example van der Schaft [14], van der Schaft and Maschke [15], Maschke and van der Schaft [16], Bravetti et al. [17], de León and Lainz [18], etc. The conception of the idea arguably traces back to the work of Herglotz [19], who derived it using the variational principle, and the developments in differential geometry, by e.g. Arnol'd [12] and Libermann and Marle [10].

Liouville geometry arises as the result of a Liouville structure, i.e. a homogeneous form.

Introduce projective bundles as principal bundles, cf. Libermann and Marle [10]

- Cosymplectic
- Presymplectic
- Precontact

1. (Briefly) introduce contact Hamiltions á la van der Schaft
2. Briefly explain Bravetti's Hamiltonian
3. From CK to Bravetti
4. Integral invariants + Lagrangian?

Chapter 4

Split-Quaternions as Dynamical Systems

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{\imath}} - u_2\hat{\boldsymbol{\jmath}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned} \boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}\hat{\boldsymbol{n}} \\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}), \end{aligned} \tag{4-1}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, one obtains:

$$\begin{aligned} \hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\ &= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}. \end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{c}, \boldsymbol{a} \rangle - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{c}, \boldsymbol{a} \rangle_{\mathrm{L}} - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$\begin{aligned}
&\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} - \hat{\boldsymbol{n}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle_{\mathrm{L}}) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i} \\
&= \boldsymbol{e}_3 - \boldsymbol{e}_2 \mathrm{i}.
\end{aligned}$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ■

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

Chapter 5

Conclusion

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Glossary

List of Acronyms

Mathematical notation

$\boldsymbol{v}$	A (tangent) vector