Thesis Title

Optional Subtitle

E. B. Legrand

Master of Science Thesis

Delft University of Technology

Delft Center for Systems and Control

Thesis Title

Optional Subtitle

Master of Science Thesis

For the degree of Master of Science in Systems and Control at Delft University of Technology

E. B. Legrand

January 5, 2022

Faculty of Mechanical, Maritime and Materials Engineering · Delft University of Technology

The work in this thesis was supported by Rabobank. Their cooperation is hereby gratefully acknowledged.

Delft University of Technology
Department of
Delft Center for Systems and Control (dcsc)

The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

Thesis Title

by

E. B. Legrand

in partial fulfillment of the requirements for the degree of

Master of Science Systems and Control

Dated: January 5, 2022

Supervisor(s):

prof.em.dr.ir. M. Mendel

Reader(s):

prof.dr.ir. M.Y. First Reader

dr.ir. F.S.T. Reader-two

ir. Th. Reader-three

Abstract

This is an abstract.

Table of Contents

List of Figures

List of Tables

Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis…

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology
January 5, 2022

E. B. Legrand

"To doubt everything, or, to believe everything, are two equally convenient solutions; both dispense with the necessity of reflection."

— *Henri Poincaré*

Chapter 1

Introduction

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\mathit{\Gamma\Delta\Upsilon\Omega\Theta}$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathit{\Gamma\Delta\Upsilon\Omega\Theta}}$
Tensor	$\boldsymbol{\mathsf{abcde}}$	$\boldsymbol{\mathsf{ABCDE}}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathsf{\Gamma\Delta\Upsilon\Omega\Theta}}$

Table 1-1: Caption

Christoffel symbol: Γ
Math constants: $\mathrm{ie}\pi$
Variation: δS
Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$

Chapter 2

Notes

2-1 Mathematical Investigations in the Theory of Value and Prices

Utility as a quantity

The total utility of a given quantity of a commodity at a given time and for a given individual is the integral of the marginal utility times the differential of that commodity:

$$\text{ut. of } (x) = \int_0^x \frac{\mathrm{d}U}{\mathrm{d}x}\,\mathrm{d}x$$

The *gain* or consumer's rent is total utility minus utility value

$$\text{gain} = \underbrace{\int_0^x \frac{\mathrm{d}U}{\mathrm{d}x}\,\mathrm{d}x}_{\text{total utility}} - \underbrace{x\,\frac{\mathrm{d}U}{\mathrm{d}x}}_{\text{utility value}}$$

The latter clearly is a Legendre transform.

Table 2-1: Mechanical analogies as proposed by Fisher [1].

Mechanics	**Economics**
Particle	Individual
Space	Commodity
Force	Marginal utility
Work	Disutility
Energy	Utility

The 'hydraulic' Fisher market

Fisher considers a market with n individuals and m commodities. The commodity quantities are denoted by $A, B, C \ldots$[1], while the individuals are numbered from 1 to m. The market is subject to a few conditions:

- For each of the commodities, there is a total endowment that fixes the total amount of that commodity in the market:
$$\sum_{i=1}^{n} A_i = K_a \quad \sum_{i=1}^{n} B_i = K_b \quad \ldots \quad \sum_{i=1}^{n} M_i = K_m$$
- The total income of any individual is a given as well:
$$\begin{aligned} A_1\, p_a + B_1\, p_b + \ldots + M_1\, p_m &= K_1 \\ A_2\, p_a + B_2\, p_b + \ldots + M_2\, p_m &= K_2 \\ &\vdots \\ A_n\, p_a + B_n\, p_b + \ldots + M_2\, p_m &= K_2 \end{aligned}$$
- Furthermore, the marginal utility associated with the quantity of goods consumed is determined by a certain function that is represented by the the 'cistern shape':
$$\frac{\mathrm{d}U}{\mathrm{d}A_i} = F(A_i) \quad \frac{\mathrm{d}U}{\mathrm{d}B_i} = F(B_i) \quad \ldots \quad \frac{\mathrm{d}U}{\mathrm{d}M_i} = F(M_i)$$
In this case, the cistern shape depends both on the consumer and the commodity, so F is different for all of them; Fisher's notation is somewhat confusing at times. Also, if U is a function that encompasses all consumers and commodities, this derivative should be a partial derivative.
- Finally, there is the *principle of proportion*, which states that the marginal utility of an individual is equal to the product of the marginal utility of money itself with the 'exchange ratio for money and that commodity'; that is, the infinitesimal utility of the product and the exchanged money must be the same every time:
$$\underbrace{\frac{\mathrm{d}U}{\mathrm{d}A}\,\mathrm{d}A}_{\text{inf. utility of the product}} = \underbrace{\frac{\mathrm{d}U}{\mathrm{d}m}\,\mathrm{d}m}_{\text{inf. utility of the money}}$$
Hence,
$$\frac{\mathrm{d}U}{\mathrm{d}A} = \frac{\mathrm{d}U}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}A} = \frac{\mathrm{d}U}{\mathrm{d}m}\, p_a,$$
which basically means that the marginal utility of a product is related to the prices through the personal utility of money of that particular consumer. However, there are two important observations to make here. Firstly, the utility of money is a parameter that is associated with an individual, but it is equal for all the commodities. In contrast, the price of a commodity is the same for all individuals. As such, one can say that
$$p_a : p_b : \ldots : p_m = \frac{\mathrm{d}U}{\mathrm{d}A} : \frac{\mathrm{d}U}{\mathrm{d}B} : \ldots : \frac{\mathrm{d}U}{\mathrm{d}M}$$

[1] Fisher mentions that these quantities are tacitly assumed to be on a yearly basis; in the economic engineering framework, they are $\dot{\boldsymbol{q}}$'s instead of $\boldsymbol{q}$'s.

Chapter 3

Dissipative Classical Mechanics

3-1 The Bateman approach

The approach used by Bateman [2] starts from a simple linear scalar second-order differential equation:

$$\ddot{x} + 2c\dot{x} + kx = 0.$$

This equation can be written as the solution of a variational expression like so:

$$\delta \int \underbrace{y(\ddot{x} + c\dot{x} + kx)}_{\mathcal{L}} \, \mathrm{d}t = 0;$$

where the Lagrangian is the argument of the time integral. To account for the presence of $\ddot{x}$, Euler-Lagrange equation can be readily extended to higher derivatives. The most general expression is, for p functions of m independent variables up to the nth derivative:

$$\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{j=1}^{n} \sum_{\mu_1 \leq \ldots \leq \mu_j} (-1)^j \frac{\partial^j}{\partial t_{\mu_1} \ldots \partial t_{\mu_j}} \left(\frac{\partial \mathcal{L}}{\partial q_{\mu_1 \ldots \mu_j}} \right) = 0,$$

where $i = 1, \ldots p$ and $\mu_j = 1, \ldots, m$. In this case however, there is only one independent variable, $m = 1$, and the highest derivative taken into account is $n = 2$. The variational problem then yields two equations: the original differential equation and a complementary equation in y:

$$\ddot{x} + 2c\dot{x} + kx = 0 \qquad \ddot{y} - 2c\dot{y} + ky = 0$$

However, the presence of the second derivative in the Lagrangian is altoghether undesirable, so one can effect the substitution

$$\ddot{x}y \, \mathrm{d}t = \mathrm{d}(\dot{x}y) - \dot{x}\dot{y} \, \mathrm{d}t \, .$$

Because the solution of the Euler-Lagrange equation is independent from total differentials added to the Lagrangian, the first term can be neglected. As such, the Lagrangian becomes:

$$\mathcal{L} = -\dot{x}\dot{y} + 2cy\dot{x} + kyx.$$

3-1-1 Towards the bicomplex Hamiltonian

From the two resulting differential equations, it is clear that x and y represent the state evolution in opposite directions of time (in case they are initialized properly); because first (odd) derivative carries the minus sign (that is canceled in the second derivative; which is invariant with respect to a time reversal). This symmetry may become more apparent from the Lagrangian by using integration by parts a second time, i.e.

$$\mathrm{d}(xy) = \dot{x}y\,\mathrm{d}t + \dot{y}x\,\mathrm{d}t\,,$$

such that

$$\mathcal{L} = -\dot{x}\dot{y} + c(y\dot{x} + \mathrm{d}(xy) - \dot{y}x) + kyx,$$

where the total differential may again be neglected. This the negative of the Lagrangian considered by Dekker [3]; the latter is assumed in further calculations (of course, multiplying the Lagrangian by -1 does not alter the solutions of the variational problem). Using this Lagrangian, the two two conjugate momenta are, by definition:

$$p_x \triangleq \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{y} - cy \qquad p_y \triangleq \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{x} + cx.$$

A Legendre transform then leads to the associated Hamiltonian

$$\mathcal{H} = p_x\dot{x} + p_y\dot{y} - \mathcal{L} = p_x p_y - c(xp_x - yp_y) + (k - c^2)xy.$$

This expression already reflects the structure of the bicomplex Hamiltonian proposed by Hutters and Mendel [4]. However, it still contains the states of both the system and the antisystem, i.e. x, y, p_y, p_x. As shown by Bopp [5], a complexification of the states allows to rewrite the above Hamiltonian into two separate components corresponding to the system and the antisystem.

TODO write complex transformationf from Bateman to Bopp

Complex state

Now consider the complexified state:

$$z = \frac{1}{\sqrt{2\omega_d}}(p + (\lambda - \mathrm{i}\omega_d)q)$$

with $\omega_d = \sqrt{\omega - \lambda^2}$. The Bopp Hamiltonian then reads

$$\begin{aligned}\mathcal{H}_{\text{Bopp}} =&(\omega_d - \mathrm{i}\lambda)z\bar{z}\\ =&\frac{1}{2}\left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left((p + \lambda q)^2 + \omega_d^2 q^2\right)\\ =&\frac{1}{2}\left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left(p + 2\lambda pq + \lambda^2 q^2 + \omega_d^2 q^2\right)\\ =&\frac{1}{2}\left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left(p + 2\lambda pq + \omega^2 q^2\right)\end{aligned} \tag{3-1}$$

Then, choosing a new state $a = \frac{1}{\sqrt{2\omega}}(\omega q + \mathrm{i}p)$ such that

$$\omega a\bar{a} = \frac{1}{2}\Big(p^2 + \omega^2 q^2\Big)$$

one can substitute

$$\mathcal{H}_{\mathrm{Bopp}} = \left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)(\omega a\bar{a} + \lambda pq) \tag{3-2}$$

Additionally,

$$a^2 = \frac{1}{2\omega}(\omega^2 q^2 - p^2 + 2\mathrm{i}\omega pq)$$

such that $a^2 - \bar{a}^2 = 2\mathrm{i}pq$, which can also be substituted in the Hamiltonian expression:

$$\mathcal{H}_{\mathrm{Bopp}} = \left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left(\omega a\bar{a} + \mathrm{i}\frac{\lambda}{2}\Big(a^2 - \bar{a}^2\Big)\right) \tag{3-3}$$

Bibliography

[1] I. Fisher, "Mathematical Investigations in The Theory of Value and Prices," Ph.D. dissertation, Yale University, 1892.

[2] H. Bateman, "On Dissipative Systems and Related Variational Problems," *Physical Review*, vol. 38, no. 4, pp. 815–819, 1931.

[3] H. Dekker, "Classical and Quantum Mechanics of the Damped Harmonic Oscillator," Physics Laboratory, TNO, Den Haag, Tech. Rep., 1981.

[4] C. Hutters and M. Mendel, "Overcoming the dissipation obstacle with Bicomplex Port-Hamiltonian Mechanics," 2020.

[5] F. Bopp, "Quantisierign des gedämpften harmonischen Oszillators," *Verlag der Bayerischen Akademie der Wissenschaften*, 1974.

Glossary

List of Acronyms