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E. B. Legrand

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E. B. Legrand

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Delft Center for Systems and Control (dcsc)

The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

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E. B. Legrand

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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis…

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology | E. B. Legrand
May 29, 2022

Man must sit in chair with open mouth for very long time before roast duck fly in.

— *Chinese proverb*

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator
- (Optimal) control of the distributions using the Brockett approach
- (Stochastic) inputs, link with Langevin equations
- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.
- Bayesian inversion of chaotic systems; guess the initial state by sampling after a certain time
- Define as streamtube, continuity equation asserts that streamlines cannot cross; i.e. streamtubes are conserves. To reduce computational complexity, define level sets (curves in 2-D) and check how they deform through the evolution of the phase space fluid; should always contain the same amount of probability troughout the evolution of the system.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\mathit{\Gamma\Delta\Upsilon\Omega\Theta}$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathit{\Gamma\Delta\Upsilon\Omega\Theta}}$
Tensor	$\boldsymbol{\mathsf{abcde}}$	$\boldsymbol{\mathsf{ABCDE}}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathsf{\Gamma\Delta\Upsilon\Omega\Theta}}$

Table 1-1: Caption

Math constants: $\mathrm{ie}\pi$
Variation: δS

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner\, \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$
$\Gamma(TM)$
$\mathfrak{X}(TM)$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by Abraham and Marsden [1] and Cannas da Silva [2] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.
- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

Symplectic and Contact Geometry in Economic Engineering

Chapter 3

Symplectified Contact Mechanics for Dissipative Systems

The traditional view is that the methods of analytical mechanics, such as the Lagrangian and Hamiltonian formalisms, are only suited for conservative systems. However, several attempts, especially in the previous century, have been made to extend these principles to dissipative systems as well.

3-1 The damped harmonic oscillator

This chapter (and the application in the following chapter) is primarily concerned with the prototypical dissipative mechanical system: the linearly damped harmonic oscillator depicted in fig. 3-1, with the governing second-order differential equation being

$$m\ddot{q} + b\dot{q} + kq = 0. \tag{3-1}$$

The choice for this system is rather perspicuous, since it is arguably the 'easiest' dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [3, 4]. However, the method described in this section can be generalized directly to a general (possibly time-dependent) potential function $V = V(q, t)$.

3-2 The Caldirola-Kanai method

A traditional, engineering-inclined method to incorporate damping in the framework is to include a Rayleigh damping term in the Lagrangian to emulate linear damping forces, and this works 'mathematically' to derive the correct equations of motion [5]. Although frequently

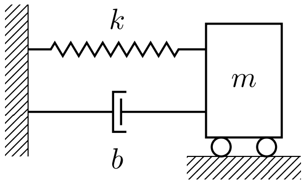

Figure 3-1: Schematic of the mass-spring-damper system.

Table 3-1: Parameter conventions of the damped harmonic oscillator. To avoid confusion with the symplectic form ω, angular frequencies are denoted by $\mathit{\Omega}$ instead of the conventional lower case Greek letter.

Name	**Symbol**	**Value**	**Units**
Damping coefficient	γ	b/m	s^{-1}
Undamped frequency	$\mathit{\Omega}_o$	$\sqrt{k/m}$	s^{-1}
Damped frequency	$\mathit{\Omega}_d$	$\sqrt{\mathit{\Omega}_0^2 - \left(\frac{\gamma}{2}\right)^2}$	s^{-1}
Damping ratio	ζ	$\frac{b}{2\sqrt{mk}}$	–

used for practical problems, this damping term is not really part of the *actual* Lagrangian — rather, it simply makes use of the notion of a generalized force that is not inherently part of the system. As such, this method only works on a superficial level: the pristine differential geometric foundations of mechanics do not leave room for such ad hoc tricks. There is, as a result, also no Hamiltonian counterpart for this method.

The historical attempts to do better than the Rayleigh method were primarily motivated by the application of the (dissipative) Hamiltonian formalism in quantum mechanics through discretization. For this application, a sound mathematical structure is of the essence, which calls for a more rigorous approach. A celebrated paper by Dekker [3] provides an excellent summary of many attempts up to 1981. Indeed, the well-studied approach developed by Caldirola [6] and Kanai [7] was intended exactly for this purpose. This method features an explicit time-dependence both in the Lagrangian function

$$L_{\mathrm{CK}}(q, \dot{q}, t) = \mathrm{e}^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right), \tag{3-2}$$

and the corresponding Hamiltonian function:

$$H_{\mathrm{CK}}(q, \rho, t) = \frac{\rho^2}{2m}\mathrm{e}^{-\gamma t} + \frac{1}{2}kq^2\mathrm{e}^{\gamma t}. \tag{3-3}$$

In latter equation, ρ refers to a special ‘canonical momentum’, that is

$$\rho \equiv \frac{\partial L_{\mathrm{CK}}}{\partial \dot{q}}, \tag{3-4}$$

which is related to the 'true' kinematic momentum by the relation $\rho = pe^{\gamma t} = m\dot{q}e^{\gamma t}$. As such, it is also clear that the Caldirola-Kanai Lagrangian and Hamiltonian functions are related by the Legendre transform *with respect to the canonical momentum*:[1]

$$H_{\text{CK}} = \rho\dot{q} - L_{\text{CK}}.$$

From either eq. (3-2) or eq. (3-3), the equations of motion are readily derived (for the Hamiltonian case with respect to ρ after which the transformation to p can be effected). Indeed, after taking the appropriate derivatives, one obtains:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\text{CK}}}{\partial \dot{q}}\right) - \frac{\partial L_{\text{CK}}}{\partial q} &= 0 \\ \Rightarrow \mathrm{e}^{\gamma t}(m\ddot{q} + m\gamma\dot{q} + kq) &= 0 \end{aligned}$$

for the Lagrangian case. Likewise, Hamilton's equations yield: [9]

$$\begin{aligned} \dot{q} &= \frac{\partial H_{\text{CK}}}{\partial \rho} = \frac{\rho}{m}\mathrm{e}^{-\gamma t} = \frac{p}{m}, \\ \dot{\rho} &= -\frac{\partial H_{\text{CK}}}{\partial q} = -kq\mathrm{e}^{\gamma t}. \end{aligned}$$

The relation between the time derivatives of the momenta $\dot{p}$ and $\dot{\rho}$ is slightly more involved since one must invoke the product rule as a result of their time-dependent relation:

$$\dot{\rho} = \mathrm{e}^{\gamma t}(\dot{p} + \gamma p).$$

Substition yields the correct equation for p, though the equation is again multiplied by $\mathrm{e}^{\gamma t}$. Because the latter is sufficiently well-behaved (that is, it has no zeros), it can be removed without any problems.

Geometric perspective To put the above derivation in a geometric setting, define the Liouville 1-form as

$$\alpha = \rho\,\mathrm{d}q \quad \Rightarrow \quad \omega = -\,\mathrm{d}\alpha = \mathrm{d}q \wedge \mathrm{d}\rho\,,$$

where the symplectic 2-form will be used to obtain Hamilton's equations. The Hamiltonian eq. (3-3) is explicitly time-dependent. This will give rise to a time-dependent vector field governing the solution curves.[2] The construction of the vector field associated with a time-dependent Hamiltonian follows the same construction rules as a normal Hamiltonian (using

[1] The 'Legendre transform' refers, in the context of fiber bundles, to the so-called fiber derivative. On a manifold M, let $L \in C^\infty(M)$. Then the fiber derivative is defined als

$$\mathbb{F}L : TL \to T^*L : \mathbb{F}L(\boldsymbol{v}) \cdot \boldsymbol{w} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\boldsymbol{v} + s\boldsymbol{w}).$$

Hence, the Legendre transform is in the first place the mapping that associates the generalized velocities with the associated (canonical) generalized momenta. Importantly, this mapping is a diffeomorphism (that is, invertible and onto) if the Hessian of L is nondegenerate - roughly equivalent to the statement that every generalized velocity has an associated 'mass' to it. [8]

[2] A *time-dependent vector field* on a manifold M is a mapping $X : M \times \mathbb{R} \to TM$ such that for each $t \in \mathbb{R}$, the restriction X_t of X to $M \times \{t\}$ is a vector field on M. [10] An additional construction of importance, called the *suspension* of the vector field, is a mapping

$$\tilde{X} : \mathbb{R} \times M \to T(\mathbb{R} \times M) \quad (t, m) \mapsto ((t, 1), (m, X(t, m))),$$

that is to say, it lifts the vector field to the extended space that also includes t and assigns the time coordinate with a trivial velocity of 1. [1]

the isomorphism given by ω), but 'frozen' at each instant of t. Even more bluntly speaking, one simply ignores the t-coordinate during the derivation, only to acknowledge the dependence at the very end. This leads to the following vector field, 'suspended' on the $\mathbb{R} \times Q$ space:

$$\tilde{X}_{H_{\text{CK}}} = -\mathrm{e}^{\gamma t} kq \frac{\partial}{\partial \rho} + \mathrm{e}^{-\gamma t} \frac{\rho}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

The suspension is important to make the final coordinate transform from ρ to p work properly. Indeed, effecting the transformation $(q, \rho, t) \mapsto (q, \mathrm{e}^{-\gamma t}, t)$, one obtains

$$\tilde{X}_{H_{\text{CK}}} = (-kq - \gamma p) \frac{\partial}{\partial p} + \frac{p}{m} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}.$$

It is worthwile to ponder on some apparent peculiarities in the Caldirola-Kanai method, for they will be explained elegantly by the contact-Hamiltonian formalism. Firstly, the role of the two-different momenta is not very clear from the get-go, apart from being a consequence of the way the Caldirola-Kanai Lagrangian is formulated. This has also been the reason for considerable confusion in the academic community (see Schuch [11]). Furthermore, there is the

3-3 Symplectification and Liouville structures

We start with an n-dimensional *base manifold* Q. In the context mechanical systems, this manifold is the configuration manifold of the system, *extended* with an additional, 'special' position coordinate coordinate that will be interpreted later. Let us assume that Q has coordinates $\boldsymbol{q} = (q_0, q_1, \ldots, q_n)$. For the damped harmonic oscillator, this manifold is two-dimensional, for it contains just the special coordinate and the position of the mass. Without loss of generalization, we will denote the special position coordinate by q_0.

Now, introduce the *manifold of contact elements* CQ, to the base manifold Q. This is the manifold of all points in Q, with the space of all possible tangent hyperplanes at every point. This manifold has dimension $2n - 1$.

The manifold of contact elements to Q can be identified with the projectivization of the cotangent bundle T^*Q, denoted by $\mathbb{P}(T^*Q)$. This manifold is of dimension $2n - 1$.

3-4 Dissipative contact Hamiltonian mechanics

The contact-geometric counterpart of Hamiltonian and Lagrangian mechanics has been the subject of increasing academic interest in recent years, see for example van der Schaft [12], van der Schaft and Maschke [13], Maschke and van der Schaft [14], Bravetti et al. [15], de León and Lainz [16], etc. The conception of the idea arguably traces back to the work of Herglotz [17], who derived it using the variational principle, and the developments in differential geometry, by e.g. Arnol'd [18] and Libermann and Marle [10].

In this section, the direct connection is made between the Caldirola-Kanai Hamiltonian given by eq. (3-3) and the contact Hamiltonian described by Bravetti et al. [15], using Liouville geometry[3]. It then proceeds to *contact Lagrangian mechanics*, strongly related to the Herglotz'

[3] It is interesting to note that Bravetti gives the Caldirola-Kanai method as an example of dissipative Hamiltonians in his paper, but fails to make the connection with his own method.

work. Finally, the whole theory can be explained from a thermodynamic perspective. While it was already known for some time (dating back to Arnol'd) that contact geometry is the preferred geometry for thermodynamics, its equivalence to contact geometry in (dissipative) classical mechanics has not been desribed in past literature. This somehow underpins a famous statement by Vladimir Arnol'd that 'Contact geometry is all geometry', in the sense that conservative mechanical systems can be considered as part of a larger class of systems for which energy dissipation *is* allowed. [19]

3-4-1 From Caldirola-Kanai to contact mechanics

Recall the Caldirola-Kanai Hamiltonian from section 3-2,

$$H_{\mathrm{CK}}(q,\rho,t) = \frac{\rho^2}{2m}\mathrm{e}^{-\gamma t} + \frac{1}{2}kq^2\mathrm{e}^{\gamma t}, \tag{3-5}$$

and rewrite it as

$$H_{\mathrm{CK}}(q,\rho,t) = \mathrm{e}^{\gamma t}\left(\frac{1}{2m}\left(\frac{\rho}{\mathrm{e}^{\gamma t}}\right)^2 + \frac{1}{2}kq^2\right). \tag{3-6}$$

The idea is to view this Hamiltonian as a homogeneous function by introducing appropriate coordinates. Homogeneous momenta are denoted by ρ_i. By inspection, choose (with slight abuse of notation)

$$\rho_1 = \rho, \quad \rho_0 = \mathrm{e}^{\gamma t} \quad \text{and } q_1 = q$$

such that the Caldirola-Kanai Hamiltonian becomes

$$H(q,\rho_0,\rho_1) = \rho_0\left(\frac{1}{2m}\left(\frac{\rho_1}{\rho_0}\right)^2 + \frac{1}{2}kq_1^2\right). \tag{3-7}$$

which is homogeneous in fiber dimensions ρ_0 and ρ_1. That is to say,

$$H(q,\lambda\rho_0,\lambda\rho_1) = \lambda H(q,\rho_0,\rho_1) \quad \lambda \in \mathbb{R}_0$$

From the definition in eq. (3-4), one can see that the 'real', kinematic momentum is equal to

$$p \equiv \frac{\rho_1}{\rho_0}.$$

As such, we can make the distinction between two Hamiltonian functions: firstly, there is the *contact Hamiltonian*

$$\hat{H}(p,q_1) = \left(\frac{1}{2m}p^2 + \frac{1}{2}kq^2\right),$$

and secondly, the *homogeneous Hamiltonian*,

$$H(\rho_0,\rho_1,q_1) = \rho_0\hat{H}\left(\frac{\rho_1}{\rho_0},q_1\right).$$

which is the 'symplectified' version of the contact Hamiltonian. This allows us to derive the equations of motion using the symplectic structure

$$\omega = \mathrm{d}q_0 \wedge \mathrm{d}\rho_0 + \mathrm{d}q_1 \wedge \mathrm{d}\rho_1$$

and proceed as usual. The contact equations of motion are then obtained through the relation between p, ρ_0 and ρ_1.

3-5 Dissipative contact Lagrangian mechanics

TODO: fix notation, always include ‘1‘ subscripts? T_0^*Q

Chapter 4

Split-Quaternions as Dynamical Systems

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{i}} - u_2\hat{\boldsymbol{j}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$\begin{aligned} \boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}\hat{\boldsymbol{n}} \\ \boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}), \end{aligned} \tag{4-1}$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \mathbf{0}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, one obtains:

$$\begin{aligned} \hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\ &= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\ &= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}. \end{aligned}$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,\langle \boldsymbol{c}, \boldsymbol{a}\rangle - \boldsymbol{c}\,\langle \boldsymbol{a}, \boldsymbol{b}\rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b}\,\langle \boldsymbol{c}, \boldsymbol{a}\rangle_{\mathrm{L}} - \boldsymbol{c}\,\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$\begin{aligned} & \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}(\hat{\boldsymbol{u}}\,\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle_{\mathrm{L}} - \hat{\boldsymbol{n}}\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}\rangle_{\mathrm{L}})\mathrm{i} \\ = {} & \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}}\,\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle_{\mathrm{L}}\,\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle} - \hat{\boldsymbol{n}}\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}\right)\mathrm{i} \\ = {} & \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}}\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}\rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\rangle}\right)\mathrm{i} \\ = {} & \boldsymbol{e}_3 - \boldsymbol{e}_2\mathrm{i}. \end{aligned}$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ∎

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

Chapter 5

Conclusion

Appendix A

Contact geometry

This appendix provides a short introduction to the basic concepts of contact geometry that are relevant in this thesis, particularly chapter 3.

A-1 Contact structures

A *contact element* on a manifold M is a point $m \in M$ combined with a tangent hyperplane $\xi_m \subset T_m M$ (a subspace of the tangent space with codimension 1). The term 'contact' refers to the intuitive notion that if two submanifolds 'touch', they share a contact element: they are *in contact* (which is a slightly weaker condition than tangency). [2] For example, contact elements to a two-dimensional manifold are simply lines through the origin in the tangent space, contact elements on a three-dimensional manifold are planes through the origin, etc.

A *contact manifold* is a manifold M (of dimension $2n+1$) with a *contact structure*, which is a smooth field (or distribution) of contact elements on M. Locally, any contact element determines a 1-form α (up to multiplication by a nonzero scalar) whose kernel constitutes the tangent hyperplane distribution, i.e.

$$\xi_m = \ker \alpha_m \tag{A-1}$$

This α is called the (local) *contact form*, and it acts like a 'normal (co-)vector' to the hyperplane. For the field hyperplanes to be a constact structure, it must satisfy a nondegeneracy condition: it should be *nonintegrable*. This can be expressed as the the Frobenius condition for nonintegrability: [2, 1, 18]

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0,$$

where integrable distributions would have this expression vanish everywhere. Roughly equivalent statements are that (i) one cannot find foliations of M such that the ξ is everywhere tangent to it, or (ii) that $\mathrm{d}\alpha|_\xi$ is a *symplectic form*. In this treatment, all contact forms are assumed to be global, which is the case if the quotient TM/ξ is a trivial line bundle, i.e. the orientation is preserved across the entire manifold [19].

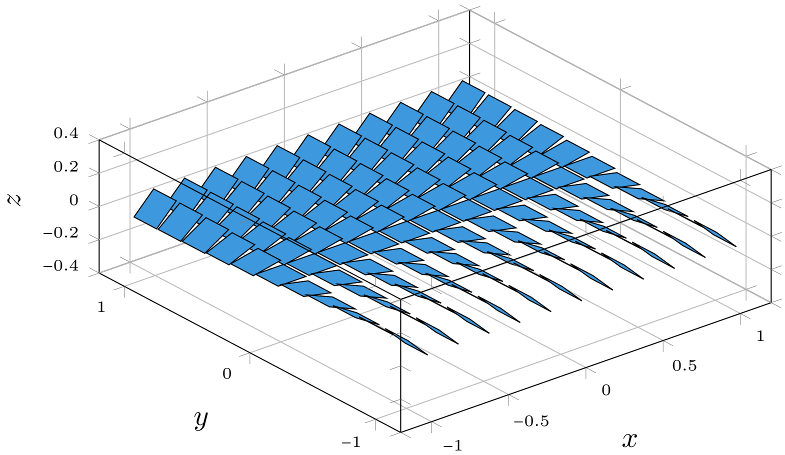

Figure A-1: The standard contact structure on $\mathbb{R}^3$, given by the contact form $\mathrm{d}z - y\,\mathrm{d}x$; the hyperplanes tilt more in the increasing y-direction.

The *Darboux theorem* for contact manifolds states that it is always possible to find coordinates z, x_i, y_i such that locally, the contact form is equal to

$$\mathrm{d}z - \sum y_i \,\mathrm{d}x_i\,,$$

which is also called the standard or natural contact structure. The standard contact structure on $\mathbb{R}^3$ is illustrated in fig. A-1. Finally, it is clear that the contact form singles out a 'special direction' in the tangent space at every point of the manifold. This direction is given by the unique *Reeb vector field*,

$$R_\alpha \in \mathfrak{X}(M): \quad R_\alpha \lrcorner\, \mathrm{d}\alpha = 0 \quad \text{and} \quad R_\alpha \lrcorner\, \alpha = 1,$$

that is, it locally points in the 'direction' of the contact form.

A-2 The manifold of contact elements

A contact manifold is a manifold with a contact structure. One can, however, associate a *canonical* $(2n-1)$-dimensional contact manifold to *any* n-dimensional manifold Q, just like one can always find a canonical symplectic structure on T^*Q. Roughly speaking, this attaches a fiber containing all possible contact elements to every point of the manifold Q. As it turns out, this 'manifold of contact elements' has a natural contact structure.

The *manifold of contact elements* of an n-dimensional manifold is [2]

$$CQ = \{(q, \xi_q) \mid q \in Q \text{ and } \xi_q \text{ a hyperplane on } T_qQ\}.$$

This manifold CQ has dimension $2n-1$. It is clear that C has a natural bundle structure, i.e. $C \xrightarrow{\pi} Q$ where the bundle projection 'forgets' the contact element, that is

$$\pi : CQ \to Q : (q, \xi_q) \mapsto q.$$

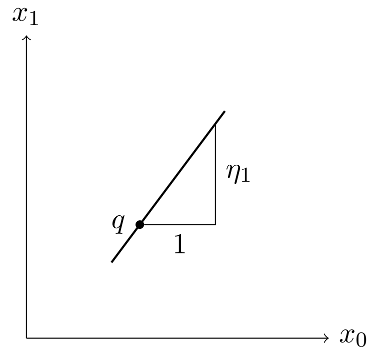

Figure A-2: A point in the manifold of contact elements on $Q = \mathbb{R}^2$. A coordinate system for CQ consists of (x_0, x_1) to indicate a point on Q, and projective coordinates $[\eta_0 : \eta_1]$, which denote the contact element at that point. Without loss of generalization, one can choose $\eta_0 = 1$, and the remaining coordinate η_1 covers all but one points in the projective space. A potential confusion rests in this two-dimensional example, since both the 'hyperplane' and the equivalence class of 1-forms are both lines in the tangent and cotangent space respectively. This is not the case for higher-dimensions, for which $n - 1 \neq 1$.

There is a convenient way to characterize this manifold of contact elements, for it is isomorphic to the *projectivization of the cotangent bundle* to Q, denoted by $\mathbb{P}(T^*Q)$. This projectivization can be defined in terms of an equivalence relation between two nonzero elements in the cotangent bundle at every point in the manifold:

$$\boldsymbol{\eta}, \boldsymbol{\chi} \in T_q^*Q \setminus \{\mathbf{0}\} : \quad (q, \boldsymbol{\eta}) \sim (q, \boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\eta} = \lambda \boldsymbol{\chi}, \quad \lambda \in \mathbb{R}_0, \text{ for all } q \in Q.$$

This equivalence relations identifies all the covectors in the cotangent space that are a nonzero multiple of each other. It is precisely this identification that takes care of the ambiguity in eq. (A-1), in that any nonzero multiple of a 1-form has the same kernel, and therefore gives rise to the same contact structure. $\mathbb{P}(T^*Q)$ is then the quotient set of T^*Q (without zero section) with respect to the equivalence relation $\sim$. Visually, the projectivization of an n-dimensional vector space is the space of all *lines* through the origin in that vector space, which has dimension $n - 1$. It can be shown that this space is bundle-isomorphic to the manifold CQ. [2]

As shown in fig. A-2, coordinates of the equivalence class of 1-forms are 'projective coordinates', $[\eta_0 : \eta_1 : \ldots : \eta_{n-1}]$, where η_i are coordinates for T_q^*Q. The projective coordinates acknowledge the invariance under multiplication by a nonzero number. If one assumes η_0 to be nonzero, the tuple $(1, \eta_1, \ldots, \eta_n)$ provides coordinates that cover most of $\mathbb{P}(T^*Q)$.

Now, it remains to be explained why the 'manifold of contact elements' is itself a contact manifold. Indeed, there is a canonical field of hyperplanes *on* CQ, which lifts the hyperplane tangent to Q to a hyperplane tangent to CQ (this is akin to the 'tautological' trick played in the symplectic structure of the cotangent bundle). The contact structure distinguishes the curves in CQ that are lifted versions from curves in Q. This is illustrated in fig. A-3. [20] Said otherwise, a tangent vector on CQ lies in the hyperplane defined by the contact structure if its projection down on Q lies in the hyperplane on Q defined by the given point on the CQ.

This contact structure is associated with the 1-form:

$$\alpha = \mathrm{d}x_0 + \sum_{i=1}^{n-1} \eta_i \, \mathrm{d}x_i \,,$$

given that the η_0 is the 'special' coordinate wich is chosen to be 1.

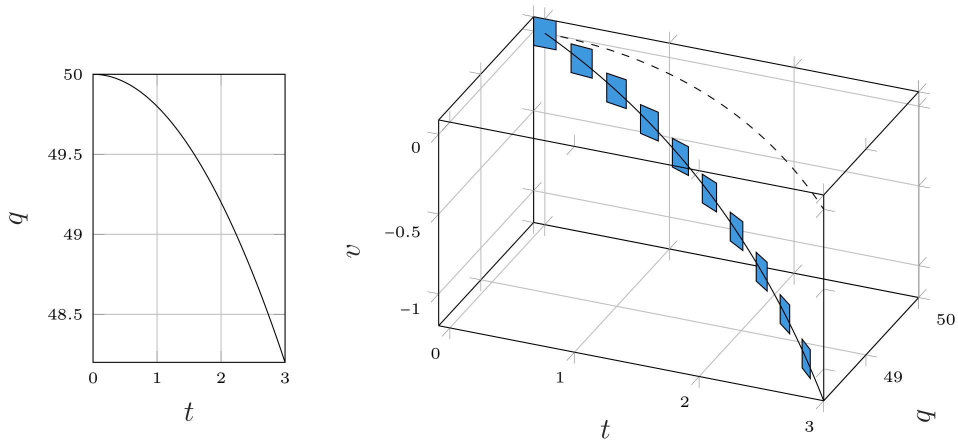

Figure A-3: Intuitive picture of the canonical contact on the manifold of contact elements. In this case, let $(t, q) \in Q$, and let v be a coordinate for the contact (line) element. The standard contact form is then $\mathrm{d}q - v\,\mathrm{d}t$. On the left, the curve corresponding to a falling object is shown in Q. When this curve is 'lifted' to CQ, the contact structure imposes that it be locally tangent to the contact structure, or that $v = \frac{\mathrm{d}q}{\mathrm{d}t}$. If the vertical direction is projected down onto the $(q-t)$-plane ($C(Q) \to Q$), the hyperplanes defined by the contact structure are line elements tangent to the trajectory, making v the actual velocity of the curve.

Zoo of structures in literature

- Cosymplectic = structure on odd-dimensional manifold, closed 2-form ω and closed 1-form η, with $\eta \wedge \omega^n \neq 0$.
- Presymplectic = structure on an even-dimensional manifold, rank of 2-form not maximal but at least constant.
- Precontact system = odd-dimensional extension of presymplectic.

Bibliography

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*. Addison-Wesley Publishing Company, 1978.

[2] A. Cannas da Silva, *Lectures on Symplectic Geometry*, ser. Lecture Notes in Mathematics, 1, Ed. Berlin, Heidelberg: Springer, 2001.

[3] H. Dekker, "Classical and Quantum Mechanics of the Damped Harmonic Oscillator," Physics Laboratory, TNO, Den Haag, Tech. Rep., 1981.

[4] C. Hutters and M. Mendel, "Overcoming the dissipation obstacle with Bicomplex Port-Hamiltonian Mechanics," 2020.

[5] H. Goldstein, C. P. Poole, and J. Safko, *Classical Mechanics*, 3rd ed. Noida, India: Pearson Education, 2011.

[6] P. Caldirola, "Forze Non Conservative Nella Meccanica Quantistica," *Il Nuovo Cimento*, vol. 18, no. 9, pp. 393–400, 1941.

[7] E. Kanai, "On the Quantization of the Dissipative Systems," *Progress of Theoretical Physics*, vol. 3, no. 4, pp. 440–442, 1948.

[8] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd ed., ser. Texts in Applied Mathematics. New York, NY: Springer, 1998. [Online]. Available: http://link.aip.org/link/PHTOAD/v48/i12/p70/s1&Agg=doi

[9] M. Tokieda and S. Endo, "Equivalence of Dissipative and Dissipationless Dynamics of Interacting Quantum Systems With Its Application to the Unitary Fermi Gas," *Frontiers in Physics*, vol. 9, no. September, pp. 1–9, 2021.

[10] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*. Dordrecht, Holland: D. Reidel Publishing Company, 1987.

[11] D. Schuch, "Nonunitary connection between explicitly time-dependent and nonlinear approachesfor the description of dissipative quantum systems," *Physical Review A - Atomic, Molecular, and Optical Physics*, vol. 55, no. 2, pp. 935–940, 1997.

[12] A. van der Schaft, "Liouville geometry of classical thermodynamics," *Journal of Geometry and Physics*, vol. 170, p. 104365, 2021. [Online]. Available: https://doi.org/10.1016/j.geomphys.2021.104365

[13] A. van der Schaft and B. Maschke, "Homogeneous Hamiltonian Control Systems Part I: Geometric Formulation," *IFAC-PapersOnLine*, vol. 51, no. 3, pp. 1–6, 2018. [Online]. Available: https://doi.org/10.1016/j.ifacol.2018.06.001

[14] B. Maschke and A. van der Schaft, "Homogeneous Hamiltonian Control Systems Part II: Application to thermodynamic systems," *IFAC-PapersOnLine*, vol. 51, no. 3, pp. 7–12, 2018. [Online]. Available: https://doi.org/10.1016/j.ifacol.2018.06.002

[15] A. Bravetti, H. Cruz, and D. Tapias, "Contact Hamiltonian mechanics," *Annals of Physics*, vol. 376, pp. 17–39, 2017.

[16] M. de León and M. Lainz, "A Review on Contact Hamiltonian and Lagrangian Systems," Tech. Rep., 2020. [Online]. Available: http://arxiv.org/abs/2011.05579

[17] R. B. Guenther, C. M. Guenther, and J. A. Gottsch, *The Herglotz lectures on contact transformations and Hamiltonian systems*, ser. Lecture Notes in Nonlinear Analysis. Torún: Juliusz Schauder Center for Nonlinear Studies Nicholas - Copernicus University, 1996, vol. 1.

[18] V. Arnol'd, *Mathematical Methods of Classical Mechanics*, 2nd ed., J. Ewing, F. Gehring, and P. Halmos, Eds. New York: Springer-Verlag, 1989.

[19] H. Geiges, *An Introduction to Contact Topology*, ser. Cambridge Studies in Advanced Mathematics. Cambridge, UK: Cambridge University Press, 2008.

[20] W. L. Burke, *Applied Differential Geometry.* Cambridge: Cambridge University Press, 1985. [Online]. Available: https://tudelft.on.worldcat.org/oclc/10949606

Glossary

List of Acronyms

Mathematical notation

$\boldsymbol{v}$	A (tangent) vector