Thesis Title

Optional Subtitle

E. B. Legrand

TUDelft
Delft
University of
Technology

Thesis Title

Optional Subtitle

Master of Science Thesis

For the degree of Master of Science in Systems and Control at Delft
University of Technology

E. B. Legrand

May 3, 2022

DELFT UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF
DELFT CENTER FOR SYSTEMS AND CONTROL (DCSC)

The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

THESIS TITLE

by

E. B. LEGRAND

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE SYSTEMS AND CONTROL

Dated: <u>May 3, 2022</u>

Supervisor(s):

_____
prof.em.dr.ir. M. Mendel

Reader(s):

_____
prof.dr.ir. M.Y. First Reader

_____
dr.ir. F.S.T. Reader-two

_____
ir. Th. Reader-three

Abstract

This is an abstract.

Table of Contents

List of Figures

List of Tables

Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis...

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology   E. B. Legrand
May 3, 2022

Vainly, the author, you picture yourself as your artworks' creator, over the Earth they have hovered forever remaining unnoticed.
Space harbors many invisible shapes and inaudible motives, as it holds many delightful ensembles of colors and letters.

— Aleksey K. Tolstoy

It is incredible how much a well-chosen word can economize thought. Often one only needs to invent a new word, and this word becomes a creator on its own right.

— Henri Poincaré, loosely cited by Vladimir I. Arnol'd

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator

- (Optimal) control of the distributions using the Brockett approach

- (Stochastic) inputs, link with Langevin equations

- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.

- Bayesian inversion of chaotic systems; guess the initial state by sampling after a certain time

- Define as streamtube, continuity equation asserts that streamlines cannot cross; i.e. streamtubes are conserves. To reduce computational complexity, define level sets (curves in 2-D) and check how they deform through the evolution of the phase space fluid; should always contain the same amount of probability troughout the evolution of the system.

Notation check

Object	Roman lower	Roman upper	Greek lower	Greek upper
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\Gamma\Delta\Upsilon\Omega\Theta$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\Gamma\Delta\Upsilon\Omega\Theta}$
Tensor	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\Gamma\Delta\Upsilon\Omega\Theta}$

Table 1-1: Caption

Math constants: $ie\pi$
Variation: δS

Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$
Lie derivative: $\pounds_X H$
Interior product: $X \lrcorner \, \omega$
Lowercase mathcal:
Kinematic momentum: $\mathfrak{p}p$
$E \xrightarrow{\pi} B$

About mathematical notation and sign conventions

For symplectic geometry, the sign convention used by Abraham and Marsden [1] and Cannas da Silva [2] is observed — not the one used by Arnol'd in his *Mathematical methods of classical mechanics*, nonetheless often referred to in this text.

- Matrices, vectors and tensors are bold upper case.

- Differential forms are typically denoted by Greek letters, with their rank as a superscript (cf. Arnol'd).

Chapter 2

A Differential-Geometric Perspective on Economic Engineering

Chapter 3

Symplectified Contact Mechanics for Dissipative Systems

The traditional view is that the methods of analytical mechanics, such as the Lagrangian and Hamiltonian formalisms, are only suited for conservative systems. However, several attempts, especially in the previous century, have been made to extend these principles to dissipative systems as well. This chapter (and the application in the following chapter) will be primarily concerned with the prototypical dissipative mechanical system: the linearly damped harmonic oscillator, with the governing second-order differential equation being

$$m\ddot{q} + b\dot{q} + k\dot{q} = 0. \tag{3-1}$$

The choice for this system is rather perspicuous, it is arguably the 'easiest' dissipative system that also exhibits second-order dynamics and is linear in all terms. Furthermore, as discussed below, it serves as the test case of the overwhelming majority of research into dissipative Lagrangian and Hamiltonian mechanics [3, 4].

The first attempt is arguably made in the work of Bateman [5] in 1931, who used a 'mirror' system in combination with the real system that runs with an opposite time direction — as such, the irreversible dynamics are compensated by the mirror system, which ensures that the overal system can be cast in the necessarily even-dimensional (the Bateman approach automatically doubles the number of dimensions in the system) symplectic setting that underpins the Hamiltonian formalism[1].

From a more practical standpoint, a Rayleigh damping term can be included in the Lagrangian to emulate linear damping forces, and this works 'mathematically' to derive the correct equations of motion [6]. Although frequently used for engineering problems, this damping term is not really part of the *actual* Lagrangian — rather, it simply makes use of the notion of a generalized force that is not inherently part of the system. As such, this method only 'works' on a superficial level: the pristine differential geometric foundations of mechanics do not leave room for such ad hoc tricks.

---

[1]Strictly speaking, Bateman only covered the Lagrangian part of the story in his paper

The historical attempts to do better than the Rayleigh method were primarily motivated by the application of the (dissipative) Hamiltonian formalism in quantum mechanics through discretization. For this application, a sound mathematical structure is of the essence, which calls for a more rigorous approach. A seminal paper by Dekker [3] provides an excellent summary of many attempts up to 1981. A brief summary (with some recent additions) is given below [motivate]. In this and the following chapter, some important connections between these methods that were not mentioned in Dekker's treatment.

- The **mirror system** or **Bateman** approach, doubles the number of system dimensions by including a mirror system that runs opposite in time. Arguably the most flexible of all methods, it is

- Expressing the Hamiltonian in **complex coordinates** has also produced promising results: notable are the attempts of Bopp [7], Dedene [8] and the very recent contribution by Hutters and Mendel [4] in the research group to which the author belongs as well.

- A different approach, related to the contact method, are the

- Contact mechanics

- Mathematical Hamiltonians

[Mention Max as contributor]

Chapter 4

Split-Quaternions as Dynamical Systems

! orthogonal refers to 'regular' orthogonal, Lorentz-orthogonal makes the distinction.

Motivation: $\boldsymbol{u}$ seems to be 'aligned' with major direction of the elliptic trajectory in the Lorentz-orthogonal subspace, generated by the action of its cross-product. Show this formally by making use of the eigenvectors.

The basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, where $\boldsymbol{e}_2$ is the orthogonal projection of the vector $\boldsymbol{e}_1 = \hat{\boldsymbol{u}}$ on its Lorentz-orthogonal subspace, and $\boldsymbol{e}_3 \triangleq \boldsymbol{e}_1 \times_{\mathrm{L}} \boldsymbol{e}_2$, form the real and imaginary parts of two of the eigenvectors of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$.

Because the basis vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the Euclidean sense, the

Proof. Let $\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}$. A normal vector to the Lorentz-orthogonal subspace is $\hat{\boldsymbol{n}} = u_1\hat{\boldsymbol{i}} - u_2\hat{\boldsymbol{j}} - u_3\hat{\boldsymbol{k}}$. Then, the basis vectors are

$$
\begin{aligned}
\boldsymbol{e}_2 &= \hat{\boldsymbol{u}} - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \hat{\boldsymbol{n}} \\
\boldsymbol{e}_3 &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 = -\frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}),
\end{aligned}
\tag{4-1}
$$

because the Lorentz-cross product distributes over addition and $\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{u}} = \boldsymbol{o}$. The proposition above claims that $\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3$ is an eigenvector of the matrix $\boldsymbol{U}_{\times_{\mathrm{L}}}$. Hence, it must be the case that $\boldsymbol{U}_{\times_{\mathrm{L}}}(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) = \lambda(\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3)$, where λ is then an eigenvalue of the matrix. This can be verified by replacing the action of $\boldsymbol{U}_{\times_{\mathrm{L}}}$ with the cross product. Plugging in the definition and exploiting the linearity of the Lorentz cross-product, one obtains:

$$
\begin{aligned}
\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_3) &= \hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2 + \mathrm{i}(\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3) \\
&= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_3)\mathrm{i} \\
&= \boldsymbol{e}_3 + (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \boldsymbol{e}_2))\mathrm{i} \\
&= \boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} (\hat{\boldsymbol{u}} \times_{\mathrm{L}} \hat{\boldsymbol{n}}))\mathrm{i}.
\end{aligned}
$$

The triple cross-product expansion, or 'Lagrange formula', relates the regular cross product to the corresponding dot product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{c}, \boldsymbol{a} \rangle - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle.$$

This well-known identity generalizes (easily verified) to the Lorentzian counterpart of the cross- and inner products:

$$\boldsymbol{a} \times_{\mathrm{L}} (\boldsymbol{b} \times_{\mathrm{L}} \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{c}, \boldsymbol{a} \rangle_{\mathrm{L}} - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathrm{L}}.$$

Using the Lagrange formula, the above expression becomes

$$\boldsymbol{e}_3 - \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} (\hat{\boldsymbol{u}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} - \hat{\boldsymbol{n}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle_{\mathrm{L}}) \mathrm{i}$$

$$= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle_{\mathrm{L}} \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i}$$

$$= \boldsymbol{e}_3 - \left(\hat{\boldsymbol{u}} - \hat{\boldsymbol{n}} \frac{\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{n}} \rangle}{\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} \rangle} \right) \mathrm{i}$$

$$= \boldsymbol{e}_3 - \boldsymbol{e}_2 \mathrm{i}.$$

The latter is the scalar multiple of the vector $\boldsymbol{e}_2 + \boldsymbol{e}_3$ by $-\mathrm{i}$ - hence, this is indeed an eigenvector of the corresponding matrix. ∎

Because $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are also orthogonal in the normal sense, they are aligned with the major axes of the elliptic trajectories generated by the cross product. Hence, they can be used to find a basis of the invariant subspace which makes the trajectories identical to those in the phase plane.

Chapter 5

Conclusion

Bibliography

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*. Addison-Wesley Publishing Company, 1978.

[2] A. Cannas da Silva, "Lectures on Symplectic Geometry," 2001.

[3] H. Dekker, "Classical and Quantum Mechanics of the Damped Harmonic Oscillator," Physics Laboratory, TNO, Den Haag, Tech. Rep., 1981.

[4] C. Hutters and M. Mendel, "Overcoming the dissipation obstacle with Bicomplex Port-Hamiltonian Mechanics," 2020.

[5] H. Bateman, "On Dissipative Systems and Related Variational Problems," *Physical Review*, vol. 38, no. 4, pp. 815–819, 1931.

[6] H. Goldstein, C. P. Poole, and J. Safko, *Classical Mechanics*, 3rd ed. Noida, India: Pearson Education, 2011.

[7] F. Bopp, "Quantisierign des gedämpften harmonischen Oszillators," *Verlag der Bayerischen Akademie der Wissenschaften*, 1974.

[8] G. Dedene, "Oscillators and Complex Hamiltonian Calculus," *Physica*, vol. 371, no. 103A, pp. 371–378, 1980.

Glossary

List of Acronyms

Mathematical notation

$\boldsymbol{v}$ A (tangent) vector