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E. B. Legrand

Master of Science Thesis

Delft Center for Systems and Control

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E. B. Legrand

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Delft University of Technology
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The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled

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Abstract

This is an abstract.

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Preface

Acknowledgements

I would like to thank my supervisor prof.dr.ir. M.Y. First Reader for his assistance during the writing of this thesis...

By the way, it might make sense to combine the Preface and the Acknowledgements. This is just a matter of taste, of course.

Delft, University of Technology — E. B. Legrand
January 8, 2022

"To doubt everything, or, to believe everything, are two equally convenient solutions; both dispense with the necessity of reflection."

— *Henri Poincaré*

Chapter 1

Introduction

Original Liouville ideas:

- Showcase complex behaviour using the van der Pol oscillator
- (Optimal) control of the distributions using the Brockett approach
- (Stochastic) inputs, link with Langevin equations
- Liouville thing (in continuity form, not incompressibility) can be applied to any diff. eq.

Notation check

Object	**Roman lower**	**Roman upper**	**Greek lower**	**Greek upper**
Standard	$abcde$	$ABCDE$	$\alpha\beta\gamma\delta\varepsilon$	$\mathit{\Gamma\Delta\Upsilon\Omega\Theta}$
Vector	$\boldsymbol{abcde}$	$\boldsymbol{ABCDE}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathit{\Gamma\Delta\Upsilon\Omega\Theta}}$
Tensor	$\boldsymbol{\mathsf{abcde}}$	$\boldsymbol{\mathsf{ABCDE}}$	$\boldsymbol{\alpha\beta\gamma\delta\varepsilon}$	$\boldsymbol{\mathsf{\Gamma\Delta\Upsilon\Omega\Theta}}$

Table 1-1: Caption

Christoffel symbol: Γ
Math constants: $\mathrm{i}\mathrm{e}\pi$
Variation: δS
Musical isomorphism
Flat: $X^\flat$
Sharp: $\omega^\sharp$

Chapter 2

Notes

2-1 Mathematical Investigations in the Theory of Value and Prices

Utility as a quantity

The total utility of a given quantity of a commodity at a given time and for a given individual is the integral of the marginal utility times the differential of that commodity:

$$\text{ut. of } (x) = \int_0^x \frac{\mathrm{d}U}{\mathrm{d}x}\,\mathrm{d}x$$

The *gain* or consumer's rent is total utility minus utility value

$$\text{gain} = \underbrace{\int_0^x \frac{\mathrm{d}U}{\mathrm{d}x}\,\mathrm{d}x}_{\text{total utility}} - \underbrace{x\,\frac{\mathrm{d}U}{\mathrm{d}x}}_{\text{utility value}}$$

The latter clearly is a Legendre transform.

Table 2-1: Mechanical analogies as proposed by Fisher [1].

Mechanics	**Economics**
Particle	Individual
Space	Commodity
Force	Marginal utility
Work	Disutility
Energy	Utility

The 'hydraulic' Fisher market

Fisher considers a market with n individuals and m commodities. The commodity quantities are denoted by $A, B, C \ldots$[1], while the individuals are numbered from 1 to m. The market is subject to a few conditions:

- For each of the commodities, there is a total endowment that fixes the total amount of that commodity in the market:
$$\sum_{i=1}^{n} A_i = K_a \quad \sum_{i=1}^{n} B_i = K_b \quad \ldots \quad \sum_{i=1}^{n} M_i = K_m$$
- The total income of any individual is a given as well:
$$\begin{aligned} A_1\,p_a + B_1\,p_b + \ldots + M_1\,p_m &= K_1 \\ A_2\,p_a + B_2\,p_b + \ldots + M_2\,p_m &= K_2 \\ &\vdots \\ A_n\,p_a + B_n\,p_b + \ldots + M_2\,p_m &= K_2 \end{aligned}$$
- Furthermore, the marginal utility associated with the quantity of goods consumed is determined by a certain function that is represented by the the 'cistern shape':
$$\frac{\mathrm{d}U}{\mathrm{d}A_i} = F(A_i) \quad \frac{\mathrm{d}U}{\mathrm{d}B_i} = F(B_i) \quad \ldots \quad \frac{\mathrm{d}U}{\mathrm{d}M_i} = F(M_i)$$
In this case, the cistern shape depends both on the consumer and the commodity, so F is different for all of them; Fisher's notation is somewhat confusing at times. Also, if U is a function that encompasses all consumers and commodities, this derivative should be a partial derivative.

- Finally, there is the *principle of proportion*, which states that the marginal utility of an individual is equal to the product of the marginal utility of money itself with the 'exchange ratio for money and that commodity'; that is, the infinitesimal utility of the product and the exchanged money must be the same every time:
$$\underbrace{\frac{\mathrm{d}U}{\mathrm{d}A}\,\mathrm{d}A}_{\text{inf. utility of the product}} \qquad = \qquad \underbrace{\frac{\mathrm{d}U}{\mathrm{d}m}\,\mathrm{d}m}_{\text{inf. utility of the money}}$$
Hence,
$$\frac{\mathrm{d}U}{\mathrm{d}A} = \frac{\mathrm{d}U}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}A} = \frac{\mathrm{d}U}{\mathrm{d}m}\,p_a,$$
which basically means that the marginal utility of a product is related to the prices through the personal utility of money of that particular consumer. However, there are two important observations to make here. Firstly, the utility of money is a parameter that is associated with an individual, but it is equal for all the commodities. In contrast, the price of a commodity is the same for all individuals. As such, one can say that
$$p_a \,:\, p_b \,:\, \ldots \,:\, p_m = \frac{\mathrm{d}U}{\mathrm{d}A} : \frac{\mathrm{d}U}{\mathrm{d}B} : \ldots : \frac{\mathrm{d}U}{\mathrm{d}M}$$

[1]Fisher mentions that these quantities are tacitly assumed to be on a yearly basis; in the economic engineering framework, they are $\dot{\boldsymbol{q}}$'s instead of $\boldsymbol{q}$'s.

Chapter 3

Dissipative Classical Mechanics

3-1 The Bateman approach

The approach used by Bateman [2] starts from a simple linear scalar second-order differential equation:

$$\ddot{x} + 2c\dot{x} + kx = 0.$$

This equation can be written as the solution of a variational expression like so:

$$\delta \int \underbrace{y(\ddot{x} + c\dot{x} + kx)}_{\mathcal{L}} \,\mathrm{d}t = 0;$$

where the Lagrangian is the argument of the time integral. To account for the presence of $\ddot{x}$, Euler-Lagrange equation can be readily extended to higher derivatives. The most general expression is, for p functions of m independent variables up to the nth derivative:

$$\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{j=1}^{n} \sum_{\mu_1 \leq \ldots \leq \mu_j} (-1)^j \frac{\partial^j}{\partial t_{\mu_1} \ldots \partial t_{\mu_j}} \left(\frac{\partial \mathcal{L}}{\partial q_{\mu_1 \ldots \mu_j}} \right) = 0,$$

where $i = 1, \ldots p$ and $\mu_j = 1, \ldots, m$. In this case however, there is only one independent variable, $m = 1$, and the highest derivative taken into account is $n = 2$. The variational problem then yields two equations: the original differential equation and a complementary equation in y:

$$\ddot{x} + 2c\dot{x} + kx = 0 \qquad \ddot{y} - 2c\dot{y} + ky = 0$$

However, the presence of the second derivative in the Lagrangian is altoghether undesirable, so one can effect the substitution

$$\ddot{x}y \,\mathrm{d}t = \mathrm{d}(\dot{x}y) - \dot{x}\dot{y} \,\mathrm{d}t\,.$$

Because the solution of the Euler-Lagrange equation is independent from total differentials added to the Lagrangian, the first term can be neglected. As such, the Lagrangian becomes:

$$\mathcal{L} = -\dot{x}\dot{y} + 2cy\dot{x} + kyx.$$

3-1-1 Towards the bicomplex Hamiltonian

From the two resulting differential equations, it is clear that x and y represent the state evolution in opposite directions of time (in case they are initialized properly); because first (odd) derivative carries the minus sign (that is canceled in the second derivative; which is invariant with respect to a time reversal). This symmetry may become more apparent from the Lagrangian by using integration by parts a second time, i.e.

$$\mathrm{d}(xy) = \dot{x}y\,\mathrm{d}t + \dot{y}x\,\mathrm{d}t\,,$$

such that

$$\mathcal{L} = -\dot{x}\dot{y} + c(y\dot{x} + \mathrm{d}(xy) - \dot{y}x) + kyx,$$

where the total differential may again be neglected. This the negative of the Lagrangian considered by Dekker [3]; the latter is assumed in further calculations (of course, multiplying the Lagrangian by -1 does not alter the solutions of the variational problem). Using this Lagrangian, the two two conjugate momenta are, by definition:

$$p_x \triangleq \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{y} - cy \qquad p_y \triangleq \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{x} + cx.$$

A Legendre transform then leads to the associated Hamiltonian

$$H = p_x\dot{x} + p_y\dot{y} - \mathcal{L} = p_x p_y - c(xp_x - yp_y) + (k - c^2)xy.$$

This expression already reflects the structure of the bicomplex Hamiltonian proposed by Hutters and Mendel [4]. However, it still contains the states of both the system and the antisystem, i.e. x, y, p_y, p_x. As shown by Bopp [5], a complexification of the states allows to rewrite the above Hamiltonian into two separate components corresponding to the system and the antisystem.

TODO write complex transformation from Bateman to Bopp

Complex state

Now consider the complexified state:

$$z = \frac{1}{\sqrt{2\omega_d}}(p + (\lambda - \mathrm{i}\omega_d)q)$$

with $\omega_d = \sqrt{\omega - \lambda^2}$. The Bopp Hamiltonian then reads

$$\begin{aligned} H_{\text{Bopp}} =& (\omega_d - \mathrm{i}\lambda)z\bar{z} \\ =& \frac{1}{2}\left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left((p + \lambda q)^2 + \omega_d^2 q^2\right) \\ =& \frac{1}{2}\left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left(p + 2\lambda pq + \lambda^2 q^2 + \omega_d^2 q^2\right) \\ =& \frac{1}{2}\left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left(p + 2\lambda pq + \omega^2 q^2\right) \end{aligned} \tag{3-1}$$

Then, choosing a new state $a = \frac{1}{\sqrt{2\omega}}(\omega q + \mathrm{i}p)$ such that

$$\omega a\bar{a} = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

one can substitute

$$H_{\mathrm{Bopp}} = \left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)(\omega a\bar{a} + \lambda pq) \tag{3-2}$$

Additionally,

$$a^2 = \frac{1}{2\omega}(\omega^2 q^2 - p^2 + 2\mathrm{i}\omega pq)$$

such that $a^2 - \bar{a}^2 = 2\mathrm{i}pq$, which can also be substituted in the Hamiltonian expression:

$$H_{\mathrm{Bopp}} = \left(1 - \mathrm{i}\frac{\lambda}{\omega_d}\right)\left(\omega a\bar{a} + \mathrm{i}\frac{\lambda}{2}\left(a^2 - \bar{a}^2\right)\right) \tag{3-3}$$

Chapter 4

The Liouville theorem

4-1 Harmonic oscillator

Although the Liouville theorem is usually expressed directly in terms of Poisson brackets (which, in turn, have a trivial form if expressed in Darboux coordinates), a slightly more insightful approach will be taken here. More specifically, instead of applying the Poisson brackts directly, they are formulated like so:

$$\{f,\, g\} = X_g(f)$$

where X_g is the Hamiltonian vector field associated to g. The defintion of Poisson brackets in terms of Hamiltonian vector fields makes it easy to draw connection between fluid mechanics and the classical mechanics.

For the simple, undamped harmonic oscillator, the configuration manifold M is simply $\mathbb{R}$. As such, the cotangent bundle $T^*M = \mathbb{R}^2$. The Hamiltonian, being a smooth function on T^*M, is simply a 0-form given in Darboux coordinates (p, q) by:

$$H: \quad T^*M \to \mathbb{R}: \quad H(p,q) = \frac{m}{2}p^2 + \frac{k}{2}q^2. \tag{4-1}$$

To apply Liouville's theorem, the Hamiltonian vector field X_H associated with H must be found. By definition, one can do this by virtue of the natural isomorphism induced by the symplectic 2-form:

$$\mathrm{d}H\,(\cdot) = \omega^2(X_H, \cdot),$$

this isomorphism is sometimes called $\omega^\sharp$, or the 'musical isomorphism' [6]. When applied as a simple transformation from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$, this isomorphism can be identified with the transformation matrix [7]

$$\begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

The differential 1-form $\mathrm{d}H$ is

$$\mathrm{d}H = \frac{p}{m}\,\mathrm{d}p + kq\,\mathrm{d}q\,,$$

such that the Hamiltonian vector field becomes (in the chart-induced basis)

$$X_H = kq\frac{\partial}{\partial p} - \frac{p}{m}\frac{\partial}{\partial q}.$$

Having found the Hamiltonian vector field, Liouville's theorem can be applied to to an arbitrary distribution ρ over the phase space:

$$\frac{\partial \rho}{\partial t} = -\{H, \rho\} = \{\rho, H\} = X_H(\rho) = kq\frac{\partial \rho}{\partial p} - \frac{p}{m}\frac{\partial \rho}{\partial q}. \tag{4-2}$$

This is a simple transport equation without diffusion; hence, the initial probability distribution will simply 'drift' along the streamlines of the Hamiltonian flow. As such, this problem is analogous to a flow that is purely characterized by convection. The convection equation may be readily solved using the method of characteristics.

The method of characteristics

Equation (4-2) is part of a larger class of linear first-order PDE's of the form[a] [8, p. 207].

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n, \rho)\frac{\partial \rho}{\partial x_i} = c(x_1, \ldots, x_n, \rho), \tag{4-3}$$

which are traditionally solved using the *method of characteristics*. This method attempts to find characteristic lines along which the solution is constant, as to convert the PDE problem into an ODE problem. More specifically, one whishes to find a parameterization of x_i and ρ such that:

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}s} &= a_i \\ \frac{\mathrm{d}\rho}{\mathrm{d}s} &= c. \end{aligned} \tag{4-4}$$

Given this parameterization, the PDE can be easily rewritten as follows: [8]

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} = \sum_{i=1}^{n} \frac{\partial \rho}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}s}.$$

The solution of the ODE problem then produces the trajectories for the characteristics. The reparameterization in terms of s must be accompagnied by another reparameterization of the initial conditions in terms of the variable(s) r_i; essentially, s provides the parameterization along the characteristic curves while r_i is the parameterization of the initial curves. The expressions for r_i are found by asserting that $x_i(0) = r_i$, and then solving for the integration constants that are still present in the found ODE solutions. Then, finally, one solves the ODE in terms of the characteristic parameterization $(s, r_1, \ldots, r_n)$

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} + c(x_1(s, r), \ldots, x_n(s, r))\rho = 0,$$

after which that solution can be written in terms of the old coordinates to obtain the solution of the PDE.

[a] If the functions a and c depend on ρ, the equation is called *semilinear*. This is, however, never the case for a PDE arising from the Liouville equation.

As it turns out, the method of characteristics takes a particularly simple form for the harmonic oscillator (and Hamiltonian systems in general). The reparameterization in terms of s is

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}s} &= kq \\ \frac{\mathrm{d}q}{\mathrm{d}s} &= -\frac{p}{m} \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= -1.\end{aligned} \tag{4-5}$$

which immediately yields $t = -s + c_1$ (with the immediate choice that c_1 be zero), and the former two equations simply resort to a time-reversed solution of the Hamiltonian problem in terms of p and q. Hence, solving the ODE to obtain the characteristic lines is, rather unsurprisingly, equivalent to finding the phase trajectories. For the harmonic oscillator, these trajectories are

$$\begin{aligned}p(s) &= c_3 \cos(\omega s) + m\omega c_1 \sin(\omega s). \\ q(s) &= c_2 \cos(\omega s) - \frac{c_3}{m\omega} \sin(\omega s)\end{aligned} \tag{4-6}$$

Solving for $q(0) = r_1$ and $p(0) = r_2$, yields $r_1 = c_2$ and $r_2 = c_3$. Now, because the 'forcing term' $c(\cdot)$ is not present in the Liouville equation (for autonomous systems), the solution of the second ODE is trivial:

$$\rho(s) = \rho_0(r_1, \ldots, r_2),$$

where ρ_0 is the initial distribution. It is an encouraging observation that the method of characteristics is easily extended towards non-autonomous systems, leaving the possibility for control action or external disturbances, which may well be of a stochastic nature themselves.

The solution to the Liouville equation is found by writing the initial distribution in terms of p, q and t. Since q and p depend linearly on r_1 and r_2, this is a matter of taking the inverse of the associated matrix.

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \cos(\omega s) & m\omega \sin(\omega s) \\ -\frac{1}{m\omega} \sin(\omega s) & cos(\omega s) \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

This transformation matrix represents a symplectic transformation of the phase plane; symplectic matrices have a unit determinant[1]. Inversion and resubstitution of t then yields:

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos(\omega t) & m\omega \sin(\omega t) \\ -\frac{1}{m\omega} \sin(\omega t) & cos(\omega t) \end{pmatrix}}_{\Phi(t)} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Initial Gaussian distribution The solution of the Liouville equation to any initial distribution is simply found by substituting the (p, q) dependence with transformation stated above. For

[1] Due to the equivalence of $\mathrm{Sp}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{R},)$, having a unit determinant is a necessary and sufficient condition for a 2×2 matrix to be symplectic; this condition is only necessary for higher dimensional vector spaces [7].

example, an initial bivariate Gaussian distribution centered at some initial point (p_0, q_0) with covariance matrix Σ subject to the linear transformation $\Phi(t)$ yields again a Gaussian: [9]

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \quad \sim \quad \mathcal{N}\left(R(t) \begin{pmatrix} p_0 \\ q_0 \end{pmatrix},\, R^\top(t) \Sigma R(t) \right).$$

This result is, after all, not quite a surprise: the Gaussian distribution is transported by the convective stream of the phase space fluid; the mean drifts along its original phase space trajectory as if it where a single particle. The variance changes continuously by the similarity transform given by R. Interestingly, because R has a unit determinant, it does not influence the determinant of the transported distribution; as such, the *entropy* of the Gaussian remains constant throughout, and equal to its initial value

$$\frac{1}{2} \log(\det(2\pi e \sigma)).$$

Averages in time and space The motion of the harmonic oscillator is periodic.

4-2 Damped harmonic oscillator

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Glossary

List of Acronyms