Note: Exactly solvable DMFT

May 21, 2025

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Objective. The objective of this note is to present the calculations involved in dynamical mean-field theory (DMFT) in the context of an entirely solvable model: a linear model with random asymmetric couplings. First, we will exactly solve the model at a microscopic level. Then, we will examine how a single degree of freedom can be treated as a Gaussian process by integrating out the other degrees of freedom. This single degree of freedom becomes the mean-field particle in the DMFT. Finally, we will demonstrate how the same DMFT can be derived without the microscopic solution by using the cavity method.

Setup. Here we'll discuss the following linear model:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{N} A_{ij}x_j - \gamma x_i + h_i, \quad i = 1, \dots, N, \tag{1}$$

where A_{ij} and h_i are random variables with statistics:

$$\begin{gathered}\langle A_{ij}\rangle = 0, \quad \mathrm{cov}[A_{ij}, A_{kl}] = \frac{\sigma^2}{N}[\delta_{ik}\delta_{jl} + \rho\delta_{il}\delta_{jk}(1-\delta_{ij})] \implies \mathrm{var}[A_{ij}] = \frac{\sigma^2}{N}, \quad \mathrm{corr}[A_{ij}, A_{ji}] = \rho \\ \langle h_i\rangle = \langle h\rangle, \quad \mathrm{cov}[h_i, h_j] = \sigma_h^2\delta_{ij},\end{gathered} \tag{2}$$

and γ is a damping constant. The initial conditions have statistics:

$$\langle x_i(0)\rangle = 0, \qquad \mathrm{cov}[x_i(0), x_j(0)] = \sigma_0^2\delta_{ij}, \tag{3}$$

Exact solution of microscopic equations of motion. The solution to the microscopic equations of motion is:

$$\boldsymbol{x}(t) = e^{(A-\gamma 1)t}\big(\boldsymbol{x}(0) + (A-\gamma 1)^{-1}\boldsymbol{h}\big) - (A-\gamma 1)^{-1}\boldsymbol{h}. \tag{4}$$

We have assumed that $A - \gamma 1$ is invertible, which is true with probability 1. This solution can be verified by substituting it back into the equations of motion. Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of A. We can write $A = Q\Lambda Q^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ and Q is a unitary matrix. We then have:

$$x_i(t) = \sum_{j,k=1}^{N} Q_{ij}^{-1}Q_{jk}\left(e^{(\lambda_j-\gamma)t}x_k(0) + \left[\frac{1-e^{(\lambda_j-\gamma)t}}{\lambda_j - \gamma}\right]h_k\right) \tag{5}$$

Integrating $N-1$ degrees of freedom. We can obtain an equation just for x_1 by solving the equations of motion for $x_2, \dots, x_N$. To do this, write:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}_{\setminus 1}}{\mathrm{d}t} &= x_1(t)\boldsymbol{A}_1 + A_{\setminus 1}\boldsymbol{x}_{\setminus 1} - \gamma\boldsymbol{x}_{\setminus 1} + \boldsymbol{h}_{\setminus 1}, \quad i = 2, \dots, N \\ \frac{\mathrm{d}x_1}{\mathrm{d}t} &= A_{11}x_1 + \hat{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{x}_{\setminus 1} - \gamma x_1 + h_1\end{aligned} \tag{6}$$

where $\boldsymbol{A}_1 = (A_{i1})_{i=2}^N$, $\hat{\boldsymbol{A}}_1 = (A_{1j})_{j=2}^N$, $A_{\setminus 1} = (A_{ij})_{i,j=2}^N$, and $\boldsymbol{x}_{\setminus 1} = (x_i)_{i=2}^N$. The solution to the equations of motion for $\boldsymbol{x}_{\setminus 1}$ is:

$$\boldsymbol{x}_{\setminus 1}(t) = e^{(A_{\setminus 1}-\gamma 1)t}\boldsymbol{x}_{\setminus 0}(0) + (A_{\setminus 1} - \gamma 1)^{-1}\Big(e^{(A_{\setminus 1}-\gamma 1)t} - I\Big)\boldsymbol{h}_{\setminus 1} + \int_0^t \mathrm{d}t'\, e^{-(A_{\setminus 1}-\gamma I)(t'-t)}\boldsymbol{A}_1 x_1(t') \tag{7}$$

Substituting this into the equation for x_1 gives:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= A_{11}x_1 + \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{(A_{\setminus 1}-\gamma 1)t}\boldsymbol{x}_{\setminus 0}(0) + \hat{\boldsymbol{A}}_1^{\mathrm{T}}(A_{\setminus 1}-\gamma 1)^{-1}\Big(e^{(A_{\setminus 1}-\gamma 1)t} - I\Big)\boldsymbol{h}_{\setminus 1} \\ &\quad + \int_0^t \mathrm{d}t' \left(\hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{-(A_{\setminus 1}-\gamma I)(t'-t)}\boldsymbol{A}_1\right) x_1(t') - \gamma x_1 + h_1\end{aligned} \tag{8}$$

The spectrum of A. The spectral density of A is given by the elliptic law:

$$\rho_A(x,y)\,\mathrm{d}x\,\mathrm{d}y = \mathrm{d}x\,\mathrm{d}y \begin{cases} \frac{1}{\sigma^2\pi(1-\rho^2)}, & \frac{x^2}{(1+\rho)^2} + \frac{y^2}{(1-\rho)^2} \le \sigma^2, \\ 0, & \text{otherwise.}\end{cases} \tag{9}$$

where $x = \mathrm{Re}[\lambda]$ and $y = \mathrm{Im}[\lambda]$. In the elliptical coordinates (u,ϕ) defined such that $x = u(1+\rho)\cos\phi$, $y = u(1-\rho)\sin\phi$, the spectral density is:

$$\rho_A(u,\phi)\,\mathrm{d}u\,\mathrm{d}\phi = \mathrm{d}u\,\mathrm{d}\phi \begin{cases} \frac{u}{\pi\sigma^2}, & u \le \sigma, \\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

which is much easier for computing integrals.

Obtaining the mean-field theory directly from the microscopic solution. We can analyze each term in the solution to the equations of motion for x_1. This analysis rests on the observation that all the entries in the matrix $A_{\setminus 1}$ are independent of $\boldsymbol{A}_1$ and $\hat{\boldsymbol{A}}_1$ and that we will take the limit $N \to \infty$.

The typical magnitude of the first term is $O(1/\sqrt{N})$, so we can neglect it in the large N limit. Next:

$$\left\langle \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{(A_{\setminus 1}-\gamma 1)t}\boldsymbol{x}_{\setminus 0}(0)\right\rangle = \left\langle \hat{\boldsymbol{A}}_1^{\mathrm{T}}\right\rangle \left\langle e^{(A_{\setminus 1}-\gamma 1)t}\right\rangle \cancel{\left\langle \boldsymbol{x}_{\setminus 0}(0)\right\rangle} = 0, \tag{11}$$

$$\begin{aligned}&\mathrm{cov}[\hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{(A_{\setminus 1}-\gamma 1)t}\boldsymbol{x}_{\setminus 0}(0), \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{(A_{\setminus 1}-\gamma 1)s}\boldsymbol{x}_{\setminus 0}(0)] = \left\langle \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{(A_{\setminus 1}-\gamma 1)t}\boldsymbol{x}_{\setminus 0}(0)\hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{(A_{\setminus 1}-\gamma 1)s}\boldsymbol{x}_{\setminus 0}(0)\right\rangle \\ &\quad= \sum_{i,j,i',j'=2}^{N} \langle A_{1i}A_{1i'}\rangle \left\langle [e^{(A_{\setminus 1}-\gamma 1)t}]_{ij}[e^{(A_{\setminus 1}-\gamma 1)s}]_{i'j'}\right\rangle \langle x_j(0)x_{j'}(0)\rangle \\ &\quad= \frac{\sigma^2\sigma_0^2}{N}\sum_{i,j=2}^{N}\left\langle [e^{(A_{\setminus 1}-\gamma 1)t}]_{ij}[e^{(A_{\setminus 1}-\gamma 1)s}]_{ij}\right\rangle \overset{(*)}{=} \frac{\sigma^2\sigma_0^2}{N}\sum_{i,j=2}^{N}\left\langle e^{(\lambda_i-\gamma)t}e^{(\lambda_j^*-\gamma)s}\delta_{ij}\right\rangle + O(N^{-1}) \\ &\quad= \sigma^2\sigma_0^2\int \mathrm{d}x\,\mathrm{d}y\,\rho_A(x,y)\,e^{(x+iy-\gamma)t+(x-iy-\gamma)s} = \frac{\sigma_0^2}{\pi}\int_0^\sigma \mathrm{d}u\int_0^{2\pi}\mathrm{d}\phi\,u\,e^{-\gamma(t+s)+[e^{i\phi}(t+s\rho)+e^{-i\phi}(s+t\rho)]u} \\ &\quad= \frac{\sigma_0^2\,\sigma}{\sqrt{(t+s\rho)(s+t\rho)}}\,e^{-\gamma(t+s)}\,I_1\Big(2\sigma\sqrt{(t+s\rho)(s+t\rho)}\Big) \\ &\quad\sim \frac{\sigma_0^2\,\sigma}{2(1+\rho)\sqrt{\pi\,\sigma(1+\rho)}}\,t^{-3/2}\,\exp\left(2[\sigma(1+\rho)-\gamma]\,t \;+\; [\sigma(1+\rho)-\gamma]\,\delta t\right), \quad (\delta t = s-t,\ t\to\infty)\end{aligned} \tag{12}$$

In line $(*)$ we used that the eigenvalues of $A_{\setminus 1}^{\mathrm{T}}$ are the complex conjugates of the eigenvalues of $A_{\setminus 1}$ and that the left and right eigenvectors of $A_{\setminus 1}$ become orthogonal in the large-N limit. Here $I_n(x)$ is the modified Bessel function of the first kind. We additionally need to analyze the $\rho \to -1$ case:

$$\lim_{\rho\to -1}\frac{\sigma_0^2\,\sigma}{\sqrt{(t+s\rho)(s+t\rho)}}\,e^{-\gamma(t+s)}\,I_1\Big(2\sigma\sqrt{(t+s\rho)(s+t\rho)}\Big) = \sigma\sigma_0^2\,\frac{e^{-\gamma(t+s)}}{s-t}J_1(2(s-t)\sigma) \tag{13}$$

We will proceed by taking $h=0$ for simplicity. The memory term has statistics:

$$
\begin{aligned}
\left\langle \int_0^t \mathrm{d}t'\, \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{-(A_{\setminus 1}-\gamma I)(t'-t)} \boldsymbol{A}_1 x_1(t') \right\rangle &= \int_0^t \mathrm{d}t' \sum_{i,j=2}^{N} \langle A_{1i} A_{j1} \rangle \left\langle [e^{-(A_{\setminus 1}-\gamma I)(t'-t)}]_{ij} \right\rangle x_1(t') \\
&= \int_0^t \mathrm{d}t' \sum_{i,j=2}^{N} \left(\frac{\rho\sigma^2}{N}\delta_{ij}\right) \left\langle [e^{-(A_{\setminus 1}-\gamma I)(t'-t)}]_{ij} \right\rangle x_1(t') \\
&= \rho\sigma^2 \int_0^t \mathrm{d}t' \left\langle \frac{1}{N}\mathrm{tr}[e^{-(A_{\setminus 1}-\gamma I)(t'-t)}] \right\rangle x_1(t') \\
&= \rho\sigma^2 \int_0^t \mathrm{d}t'\, x_1(t') \int \mathrm{d}x\,\mathrm{d}y\, \rho_A(x,y)\, e^{-(x+iy-\gamma)(t'-t)} \\
&= \frac{\rho}{\pi} \int_0^t \mathrm{d}t'\, x_1(t') \int_0^\sigma \mathrm{d}u \int_0^{2\pi} \mathrm{d}\phi\, u\, e^{(-\gamma+u(e^{i\phi}+e^{-i\phi}\rho))(t'-t)} \\
&= \int_0^t \mathrm{d}t' \left[\sigma\sqrt{\rho}\frac{e^{-\gamma(t'-t)}}{t'-t} I_1\big(2\sigma\sqrt{\rho}(t'-t)\big)\right] x_1(t')
\end{aligned}
\tag{14}
$$

Next, look at the correlator:

$$
\begin{aligned}
&\left\langle \left(\int_0^t \mathrm{d}t'\, \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{-(A_{\setminus 1}-\gamma I)(t'-t)} \boldsymbol{A}_1 x_1(t')\right)\left(\int_0^s \mathrm{d}s'\, \hat{\boldsymbol{A}}_1^{\mathrm{T}} e^{-(A_{\setminus 1}-\gamma I)(s'-s)} \boldsymbol{A}_1 x_1(s')\right) \right\rangle \\
&\quad= \int_0^t \mathrm{d}t' \int_0^s \mathrm{d}s'\, x_1(t') x_1(s') \sum_{i,j,i',j'=2}^{N} \langle A_{1i} A_{j1} A_{1i'} A_{j'1} \rangle \left\langle [e^{-(A_{\setminus 1}-\gamma I)(t'-t)}]_{ij} [e^{-(A_{\setminus 1}-\gamma I)(s'-s)}]_{i'j'} \right\rangle \\
&\quad= \int_0^t \mathrm{d}t' \int_0^s \mathrm{d}s'\, x_1(t') x_1(s') \sum_{i,j,i',j'=2}^{N} \left(\delta_{ij}\tfrac{\rho\sigma^2}{N}\delta_{i'j'}\tfrac{\rho\sigma^2}{N} + 0 + \delta_{ij'}\tfrac{\rho\sigma^2}{N}\delta_{i'j}\tfrac{\rho\sigma^2}{N}\right) \left\langle [e^{-(A_{\setminus 1}-\gamma I)(t'-t)}]_{ij} [e^{-(A_{\setminus 1}-\gamma I)(s'-s)}]_{i'j'} \right\rangle \\
&\quad= (\rho\sigma^2)^2 \int_0^t \mathrm{d}t' \int_0^s \mathrm{d}s'\, x_1(t') x_1(s') \sum_{i,j=2}^{N} \left(\frac{1}{N^2} \left\langle [e^{-(A_{\setminus 1}-\gamma I)(t'-t)}]_{ii} [e^{-(A_{\setminus 1}-\gamma I)(s'-s)}]_{jj} \right\rangle \right. \\
&\qquad\qquad \left. + \frac{1}{N^2} \left\langle [e^{-(A_{\setminus 1}-\gamma I)(t'-t)}]_{ij} [e^{-(A_{\setminus 1}-\gamma I)(s'-s)}]_{ji} \right\rangle \right)
\end{aligned}
\tag{15}
$$

We can see the second term corresponds to the covariance of the two integrals. We can see that it is $O(1/N)$, so we can neglect it in the large N limit.

The mean-field theory is then:

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = \Phi(t) + \rho\sigma^2 \int_0^t \mathrm{d}t'\, \nu(t,t') x(t') - \gamma x(t) \tag{16}
$$

where $\Phi(t)$ is a Gaussian process with statistics:

$$
\begin{aligned}
\langle \Phi(t) \rangle &= 0 \\
\mathrm{cov}[\Phi(t),\Phi(s)] &= \frac{\sigma_0^2\,\sigma}{\sqrt{(t+s\rho)(s+t\rho)}}\, e^{-\gamma(t+s)}\, I_1\Big(2\sigma\sqrt{(t+s\rho)(s+t\rho)}\Big) \equiv \sigma^2\rho\, C(t,s)
\end{aligned}
\tag{17}
$$

and the (deterministic) memory kernel $\nu(t,t')$ is given by:

$$
\nu(t,t') = \frac{1}{\sigma\sqrt{\rho}} \frac{e^{-\gamma(t'-t)}}{t'-t} I_1\big(2\sigma\sqrt{\rho}(t'-t)\big) \tag{18}
$$

The $\rho \to -1$ case needs to be computed explicitly:

$$\begin{aligned}\lim_{\rho\to-1} \mathrm{cov}[\Phi(t), \Phi(s)] &= \sigma\sigma_0^2 \frac{e^{-\gamma(t+s)}}{s-t} J_1(2(s-t)\sigma) \\ \lim_{\rho\to-1} \nu(t,t') &= \frac{1}{\sigma}\frac{e^{-\gamma(t-t')}}{t-t'} J_1(2(t-t')\sigma)\end{aligned} \tag{19}$$

Exactly solving the mean-field theory. Begin by solving:

$$\frac{\mathrm{d}G}{\mathrm{d}t} - \sigma^2\rho\int_0^t \mathrm{d}t'\,\nu(t-t')G(t') + \gamma G(t) = \delta(t-t') \tag{20}$$

The Laplace transform of this equation is:

$$s\tilde{G}(s) - \sigma^2\rho\tilde{\nu}(s)\tilde{G}(s) + \gamma\tilde{G}(s) = e^{-st'}\Theta(t') \tag{21}$$

Using:

$$\tilde{\nu}(s) = \frac{1}{2\sigma^2\rho}\left(s+\gamma-\sqrt{(s+\gamma)^2 - 4\rho\sigma^2}\right) \tag{22}$$

we find:

$$\tilde{G}(s) = \frac{2}{s+\gamma+\sqrt{(s+\gamma)^2-4\rho\sigma^2}}\,e^{-st'}\Theta(t') \tag{23}$$

$$\Downarrow \tag{24}$$

$$G(t-t') = \frac{1}{\sqrt{\rho}\sigma}t^{-1}e^{-\gamma(t-t')}\,I_1(2\sqrt{\rho}\sigma(t-t'))\Theta(t-t') \tag{25}$$

Because the mean-field theory is linear, the solution is:

$$x(t) = G(t)x(0) + \int_0^t \mathrm{d}t'\,G(t-t')\Phi(t') \tag{26}$$

Deriving the mean-field theory using the cavity method. What if we are not able to directly access the microscopic solution to the equations of motion (which is essentially always the case)? We can then use the cavity method to derive the mean-field theory. For the cavity method, we take the microscopic equations of motion and add one extra degree of freedom, x_0. This adds one additional term to the equations of motion for the other degrees of freedom:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^N A_{ij}x_j - \gamma x_i + h_i + A_{i0}x_0, \qquad i = 1,\dots,N \tag{27}$$

This perturbs the auxiliary field as:

$$h_i \mapsto h_i + A_{i0}x_0 \tag{28}$$

As $A_{i0} \sim O(1/\sqrt{N})$, we can apply a linear response argument to approximate how the dynamics of the system changes in response to the addition of x_0:

$$\begin{aligned}x_i(t) &= x_{i\backslash 0}(t) + \int_0^t \mathrm{d}t' \sum_{j=1}^N \nu_{ij}(t,t')A_{j0}x_0(t'), \qquad i = 1,\dots,N \\ \nu_{ij}(t,t') &\equiv \left.\frac{\delta x_i(t)}{\delta h_j(t')}\right|_{h_i(t')=h_i}\end{aligned} \tag{29}$$

Substituting this into the equation of motion for x_0 gives:

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = \sum_{i=1}^{N} A_{0i} x_{i\backslash 0}(t) + \sum_{i,j=1}^{N} \int_0^t \mathrm{d}t'\, \nu_{ij}(t,t') A_{0i} A_{j0} x_0(t') - \gamma x_0 + A_{00} x_0 \tag{30}$$

We can now compute statistics of each of these terms in the large N limit. We do not have to worry about correlations when computing averages because $x_{i\backslash 0}(t)$ is definitionally independent of A_{0i} and x_0. The mean and covariance of the first term is:

$$\begin{aligned}
\left\langle \sum_{i=1}^{N} A_{0i} x_{i\backslash 0}(t) \right\rangle &= \sum_{i=1}^{N} \langle A_{0i} \rangle\, x_{i\backslash 0}(t) = 0 \\
\left\langle \sum_{i=1}^{N} A_{0i} x_{i\backslash 0}(t) \sum_{j=1}^{N} A_{0j} x_{j\backslash 0}(t') \right\rangle &= \sum_{i,j=1}^{N} \langle A_{0i} A_{0j} \rangle\, x_{i\backslash 0}(t) x_{j\backslash 0}(t') = \frac{\sigma^2 \rho}{N} \sum_{i=1}^{N} x_{i\backslash 0}(t) x_{i\backslash 0}(t') = \sigma^2 \rho\, C(t,t')
\end{aligned} \tag{31}$$

with $C(t,t') \equiv N^{-1} \sum_{i=1}^{N} x_{i\backslash 0}(t) x_{i\backslash 0}(t')$. The second term has zero variance in the $N \to \infty$ limit (which can be readily computed) and thus can be replaced with its mean:

$$\left\langle \sum_{i,j=1}^{N} \int_0^t \mathrm{d}t'\, \nu_{ij}(t,t') A_{0i} A_{j0} x_0(t') \right\rangle = \sigma^2 \rho \int_0^t \mathrm{d}t'\, \frac{1}{N} \sum_{i=1}^{N} \nu_{ii}(t,t') x_0(t') = \sigma^2 \rho \int_0^t \mathrm{d}t'\, \nu(t,t') x_0(t') \tag{32}$$

where $\nu(t,t') \equiv N^{-1} \sum_{i=1}^{N} \nu_{ii}(t,t')$ is the trace of the matrix $\nu_{ij}(t,t')$. The term involving A_{00} can be neglected as it is $O(1/\sqrt{N})$. The equation of motion for the cavity degree of freedom is then:

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = \Phi_0(t) + \sigma^2 \rho \int_0^t \mathrm{d}t'\, \nu(t,t') x_0(t') - \gamma x_0(t) \tag{33}$$

where $\Phi_0(t)$ is a zero-mean Gaussian process with statistics $\langle \Phi_0(t) \Phi_0(s) \rangle = \sigma^2 \rho C(t,s)$. To solve this mean-field theory, we must self-consistently solve for ν and C. We start by observing that $\nu(t,t')$ at fixed t' obeys the ordinary differential equation:

$$\frac{\partial}{\partial t} \nu(t,t') = \left\langle \frac{\delta}{\delta h(t')} \frac{\mathrm{d}x(t)}{\mathrm{d}t} \right\rangle = \sigma^2 \rho \int_0^t \mathrm{d}t''\, \nu(t,t'') \nu(t'',t') - \gamma \nu(t,t') + \delta(t-t') \tag{34}$$

This equation implies that $\nu(t,t')$ is a function of $t - t'$. Therefore, we can proceed to solve for $\nu(t)$ by taking the Laplace transform:

$$\begin{aligned}
s\tilde{\nu}(s) = \sigma^2 \rho [\tilde{\nu}(s)]^2 - \gamma \tilde{\nu}(s) + 1 &\implies \tilde{\nu}(s) = \frac{1}{2\rho\sigma^2} \left(s + \gamma - \sqrt{(s+\gamma)^2 - 4\rho\sigma^2} \right) \\
&\implies \nu(t) = \frac{1}{\sqrt{\rho}\sigma} \frac{e^{-\gamma t}}{t} I_1(2\sqrt{\rho}\sigma t)
\end{aligned} \tag{35}$$

This is the same result we obtained from the microscopic solution. We can now compute the covariance of the cavity degree of freedom: Now that we know $\nu(t,t')$, we can write:

$$x_0(t) = G(t) x_0(0) + \int_0^t \mathrm{d}t'\, G(t-t') \Phi_0(t') \tag{36}$$

where $G(t-t')$ is the Green's function of the mean-field theory (which we have already computed). From this, we obtain the self-consistent equation:

$$C(t,t') = G(t)G(t')\sigma_0^2 + \int_0^t \mathrm{d}t'' \int_0^{t'} \mathrm{d}t'''\, G(t-t'') G(t'-t''') C(t'',t''') \tag{37}$$

Taking a double Laplace transform gives:

$$\tilde{C}(s,s') = \sigma_0^2 \, \frac{\tilde{G}(s)\tilde{G}(s')}{1-\tilde{G}(s)\tilde{G}(s')} \tag{38}$$

The two-sided inverse Laplace transform (which is performed by shifting the Laplace variables by γ and then changing time variables $(u,v) = (t + t'\rho, t' + t\rho)$) gives:

$$C(t,t') = \frac{\sigma_0^2}{\sigma\sqrt{\rho}\sqrt{(t+t'\rho)(t'+t\rho)}} \, e^{-\gamma(t+t')} \, I_1\Big(2\sigma\sqrt{(t+t'\rho)(t'+t\rho)}\Big) \tag{39}$$

which is the same result we obtained from the microscopic solution.

The dynamical spectrum. One final interesting result in this context is to see how the spectrum of $\nu_{ij}(t,t')$ evolves. From the microscopic equations of motion, we have:

$$\frac{\partial}{\partial t}\nu_{ij}(t,t') = \frac{\delta}{\delta h_j(t')}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \sum_{k=1}^{N} A_{ik}\nu_{kj}(t,t') - \gamma\nu_{ij}(t,t') + \delta_{ij}\delta(t-t') \tag{40}$$

This equation indicates that $\nu_{ij}(t,t')$ is a function of $t-t'$. We can then take the Laplace transform:

$$\begin{aligned} s\tilde{\nu}_{ij}(s) = \sum_{k=1}^{N} A_{ik}\tilde{\nu}_{kj}(s) - \gamma\tilde{\nu}_{ij}(s) + I \implies \tilde{\nu}_{ij}(s) = \sum_{k=1}^{N} Q_{ik}\frac{1}{s-\lambda_k+\gamma}Q^{-1}_{kj} \\ \implies \nu_{ij}(t,t') = \sum_{k=1}^{N} Q_{ik}e^{(\lambda_k-\gamma)(t-t')}Q^{-1}_{kj} = [e^{(A-\gamma I)(t-t')}]_{ij} \end{aligned} \tag{41}$$

where Q is the matrix of eigenvectors of A and λ_k are the eigenvalues of A. We could have achieved this more directly by just solving $\vec{x}(t)$ as a function of $\vec{h}(t)$ directly. This means the spectrum of $\nu_{ij}(t,t')$ is given by:

$$\rho_\nu(x,y;t) = \int_{\mathbb{C}} \mathrm{d}u\,\mathrm{d}v\,\rho_A(u,v)\,\delta^2(e^{(u+iv-\gamma)t} - (x+iy)) \tag{42}$$

which can be plotted with specific parameter values but is otherwise not analytically tractable.