Camera models and calibration

Read tutorial chapter 2 and 3.1
http://www.cs.unc.edu/~marc/tutorial/

Szeliski’s book pp.29-73

Schedule (tentative)

#	date	topic
1	Sep.16	Introduction and projective geometry
2	Sep.23	Camera models and calibration
3	Sep.30	Invariant features
4	Oct.7	Optical flow & Particle Filters
5	Oct.14	Multi-view geometry & structure from motion
6	Oct.21	Model fitting (RANSAC, EM, …)
7	Oct.28	Stereo matching & multi-view stereo
8	Nov.4	Specific object recognition
9	Nov.11	Tracking
10	Nov.18	Recognition and reconstruction of humans
11	Nov.25	Image segmentation I
12	Dec.2	Image segmentation II & Obj. class recogn. I
13	Dec.9	Object class recognition II
14	Dec.16	Research Overview & Lab tours

Brief geometry reminder

2D line-point coincidence relation: $\mathrm{l}^{\mathsf{T}}\mathrm{x} = 0$

Point from lines: $\mathrm{x} = \mathrm{l} \times \mathrm{l}'$ Line from points: $\mathrm{l} = \mathrm{x} \times \mathrm{x}'$

3D plane-point coincidence relation: $\pi^{\mathsf{T}}\mathrm{X} = 0$

Point from planes: $\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix} \mathrm{X} = 0$ Plane from points: $\begin{bmatrix} \mathrm{X}_1^{\mathsf{T}} \\ \mathrm{X}_2^{\mathsf{T}} \\ \mathrm{X}_3^{\mathsf{T}} \end{bmatrix} \pi = 0$

3D line representation:
(as two planes or two points)

$$\begin{bmatrix} \mathrm{P}^{\mathsf{T}} \\ \mathrm{Q}^{\mathsf{T}} \end{bmatrix} [\mathrm{A}\ \mathrm{B}] = 0_{2\mathrm{x}2}$$

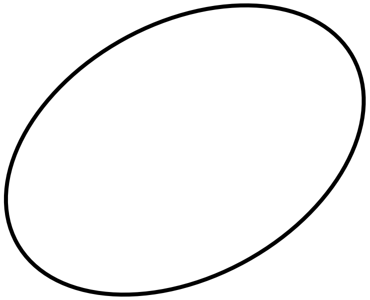
2D Ideal points $(x_1, x_2, 0)^{\mathsf{T}}$ 3D Ideal points $(X_1, X_2, X_3, 0)^{\mathsf{T}}$

2D line at infinity $\mathrm{l}_\infty = (0,0,1)^{\mathsf{T}}$ 3D plane at infinity $\Pi_\infty = (0,0,0,1)^{\mathsf{T}}$

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$



Conics

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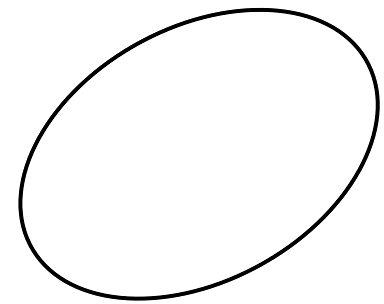

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

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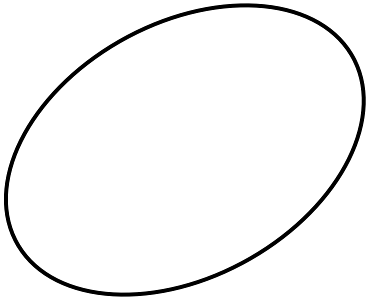

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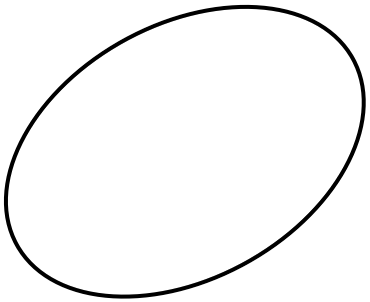
$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

Conics

Curve described by 2nd-degree equation in the plane



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5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$a x_i^2 + b x_i y_i + c y_i^2 + d x_i + e y_i + f = 0$$

or

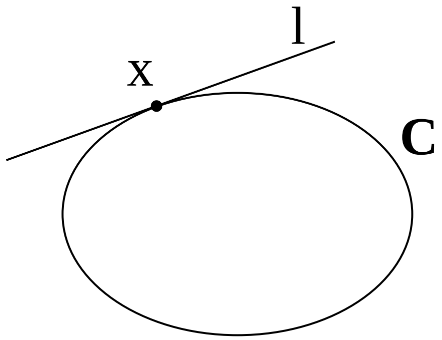
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, 1\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to $\mathbf{C}$ at point x on $\mathbf{C}$ is given by $\mathrm{l}=\mathbf{C}\mathrm{x}$

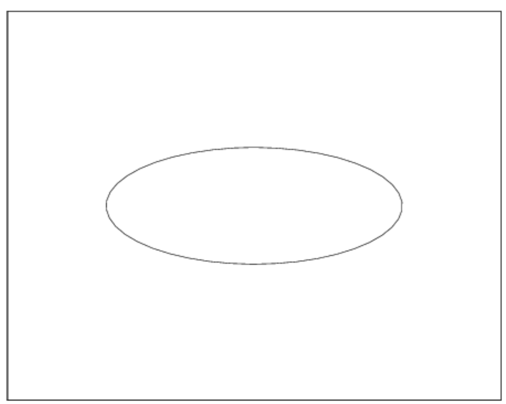


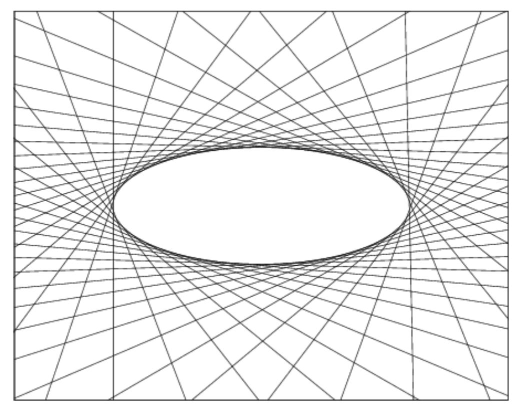
Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*}=\mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes

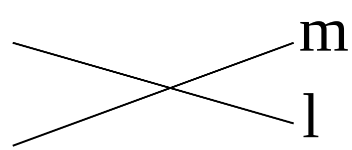




Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

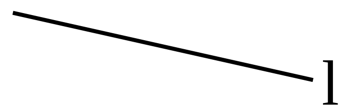
e.g. two lines (rank 2)



$$\mathbf{C} = \mathbf{l}\mathbf{m}^{\mathsf{T}} + \mathbf{m}\mathbf{l}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathbf{l}\mathbf{l}^{\mathsf{T}}$$



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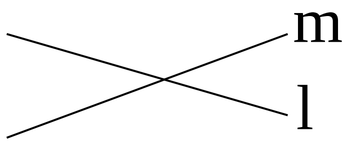

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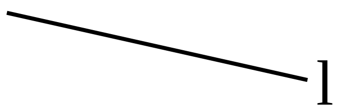
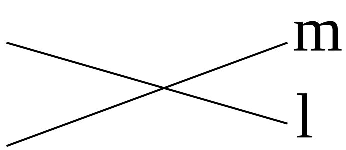


Degenerate line conics: 2 points (rank 2), double point (rank1)

Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

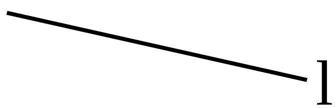
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{lm}^{\mathsf{T}} + \mathrm{ml}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{ll}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Quadrics and dual quadrics

$X^T Q X = 0$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

- 9 d.o.f.
- in general 9 points define quadric
- det Q=0 ↔ degenerate quadric
- tangent plane $\pi = QX$

$\pi^T Q^* \pi = 0$

- relation to quadric $Q^* = Q^{-1}$ (non-degenerate)

2D projective transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points $\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3$ lie on the same line if and only if $h(\mathrm{x}_1),h(\mathrm{x}_2),h(\mathrm{x}_3)$ do.

Theorem:

A mapping $h{:}\mathrm{P}^2 \rightarrow \mathrm{P}^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector x it is true that $h(\mathrm{x})=\mathbf{H}\mathrm{x}$

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathrm{x}' = \mathbf{H}\mathrm{x}$$

8DOF

projectivity=collineation=projective transformation=homography

Transformation of 2D points, lines and conics

For a point transformation

$$x' = \mathbf{H}x$$

Transformation for lines

$$l' = \mathbf{H}^{-\mathsf{T}}\, l$$

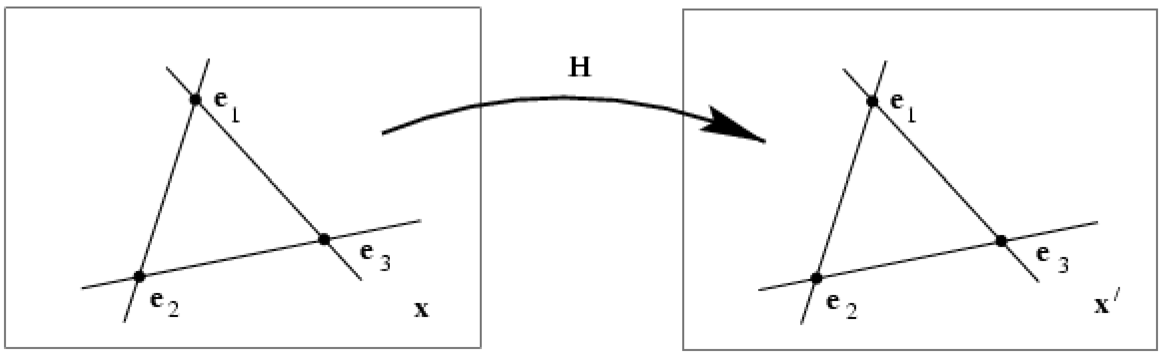
Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

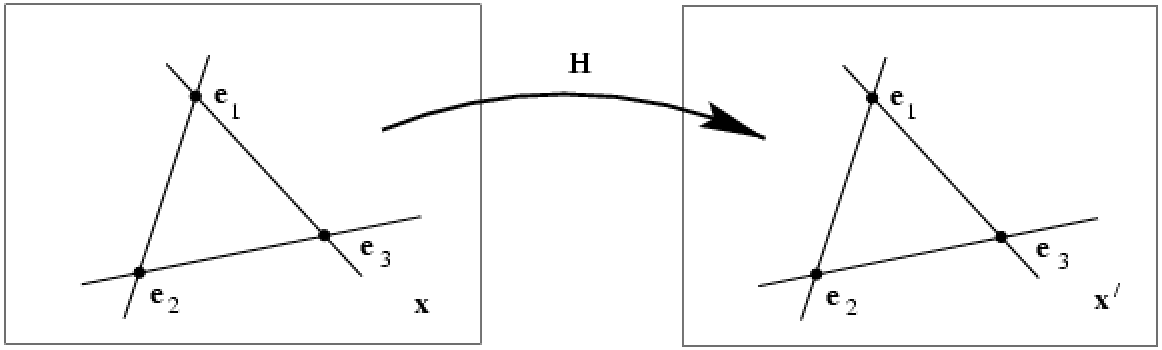
Fixed points and lines



Fixed points and lines

$$\mathbf{H}e = \lambda e$$ (eigenvectors $\mathbf{H}$ =fixed points)

(λ_1=λ_2 $\Rightarrow$ pointwise fixed line)

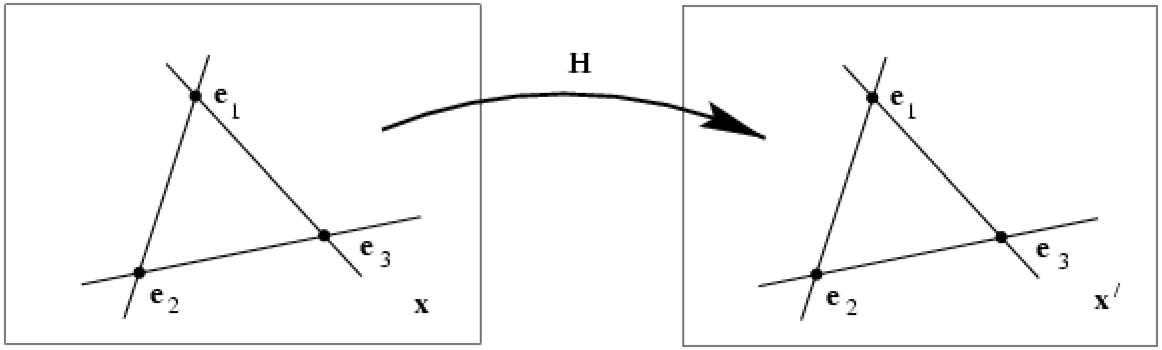


Fixed points and lines

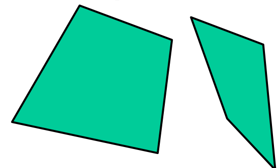
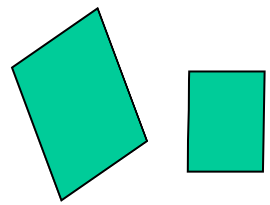
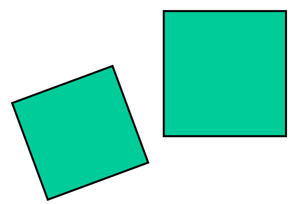
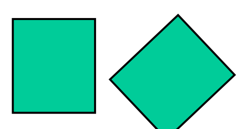
$$\mathbf{H}\mathrm{e} = \lambda\mathrm{e}$$ (eigenvectors $\mathbf{H}$ =fixed points)

(λ_1=λ_2 $\Rightarrow$ pointwise fixed line)

$$\mathbf{H}^{-T}\,\mathrm{l} = \lambda\,\mathrm{l}$$ (eigenvectors $\mathbf{H}^{-T}$ =fixed lines)



Hierarchy of 2D transformations

		transformed squares	invariants
Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity l_∞**
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratios of lengths, angles. **The circular points I,J**
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		lengths, areas.

The line at infinity

$$\mathbf{l}'_\infty = \mathbf{H}_A^{-\mathsf{T}}\,\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{t}^{\mathsf{T}}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity

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The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

The line at infinity

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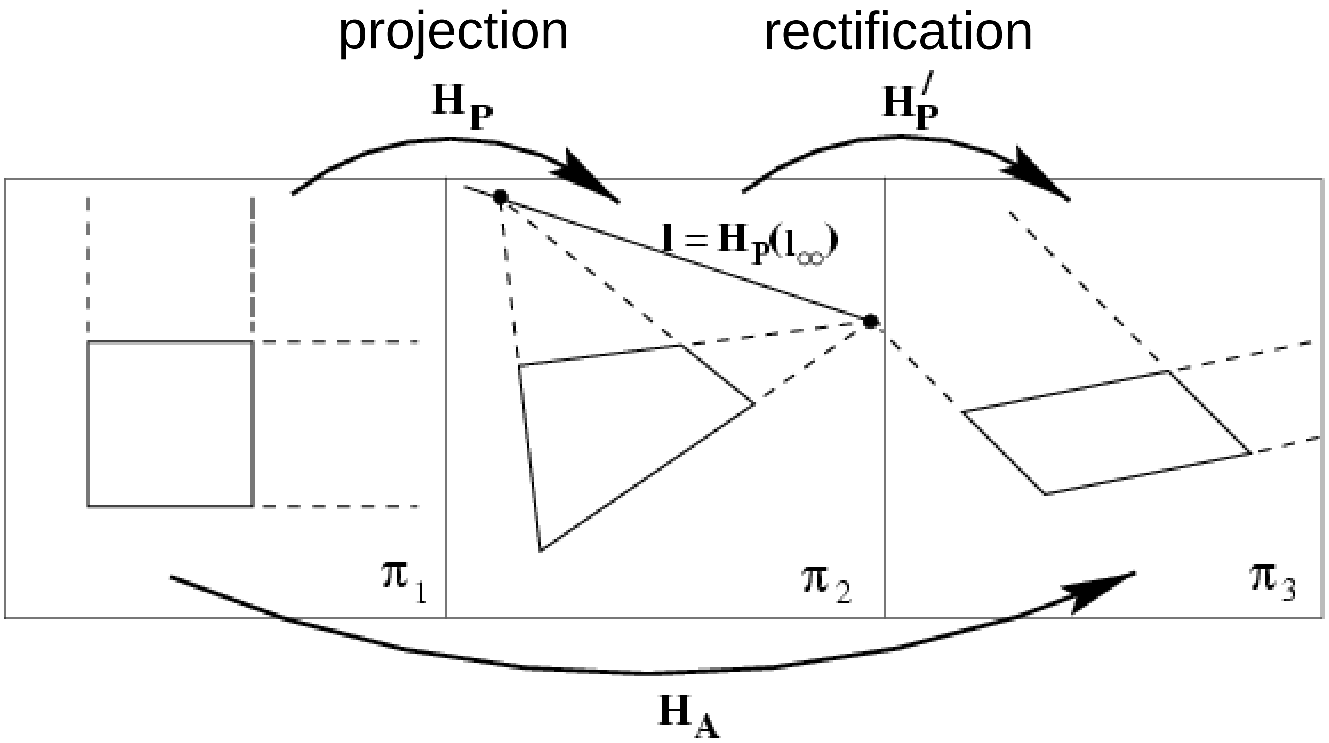
The line at infinity $\mathbf{l}_\infty$ is a fixed line under a projective transformation H if and only if H is an affinity

Note: not fixed pointwise

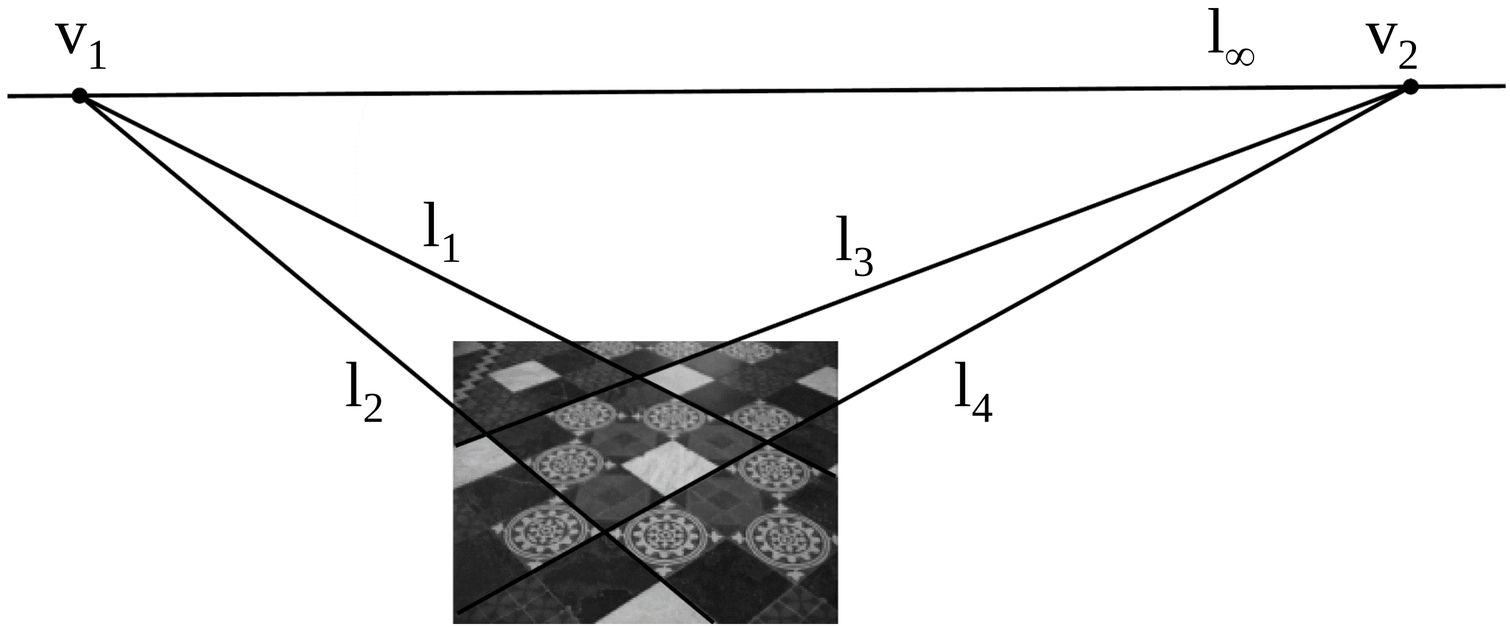
Affine properties from images

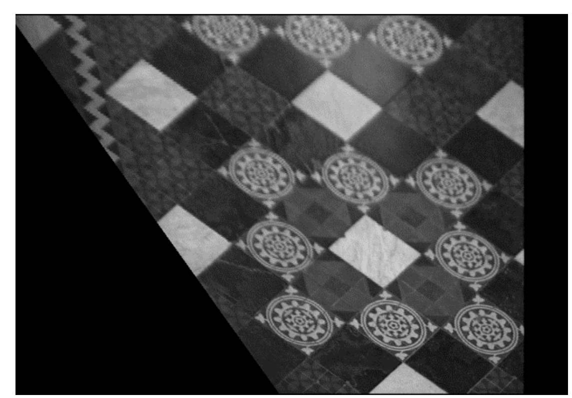


$$\mathbf{H'_P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

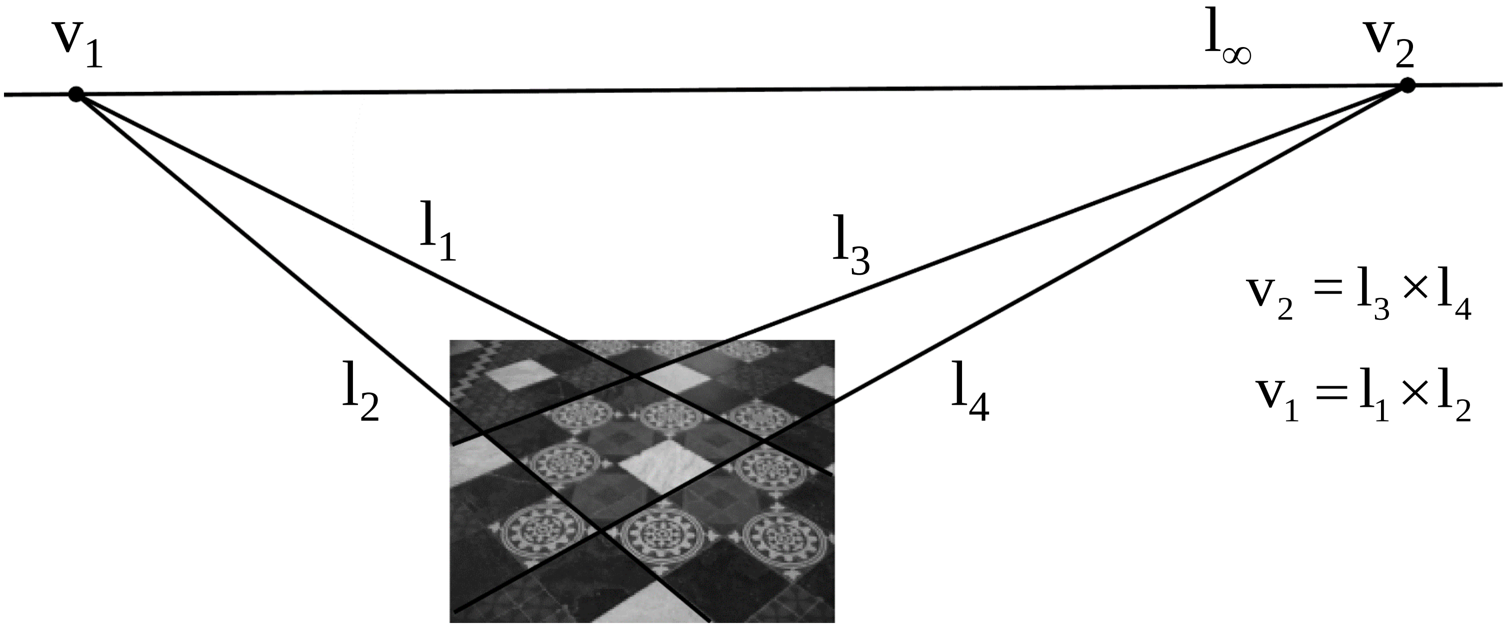
$$\mathbf{l}_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\mathsf{T}, l_3 \neq 0$$

Affine rectification



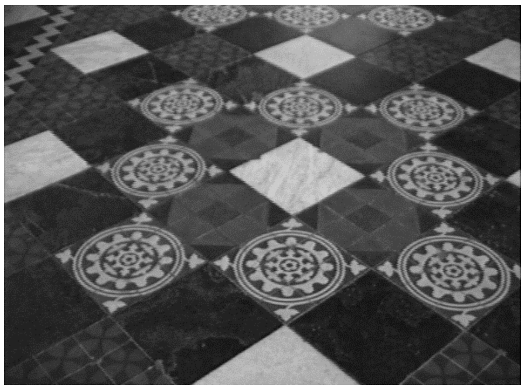


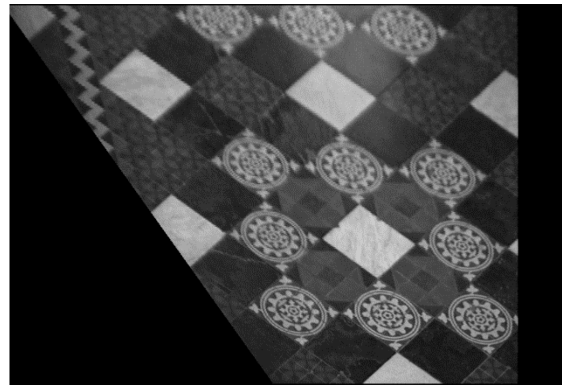
Affine rectification



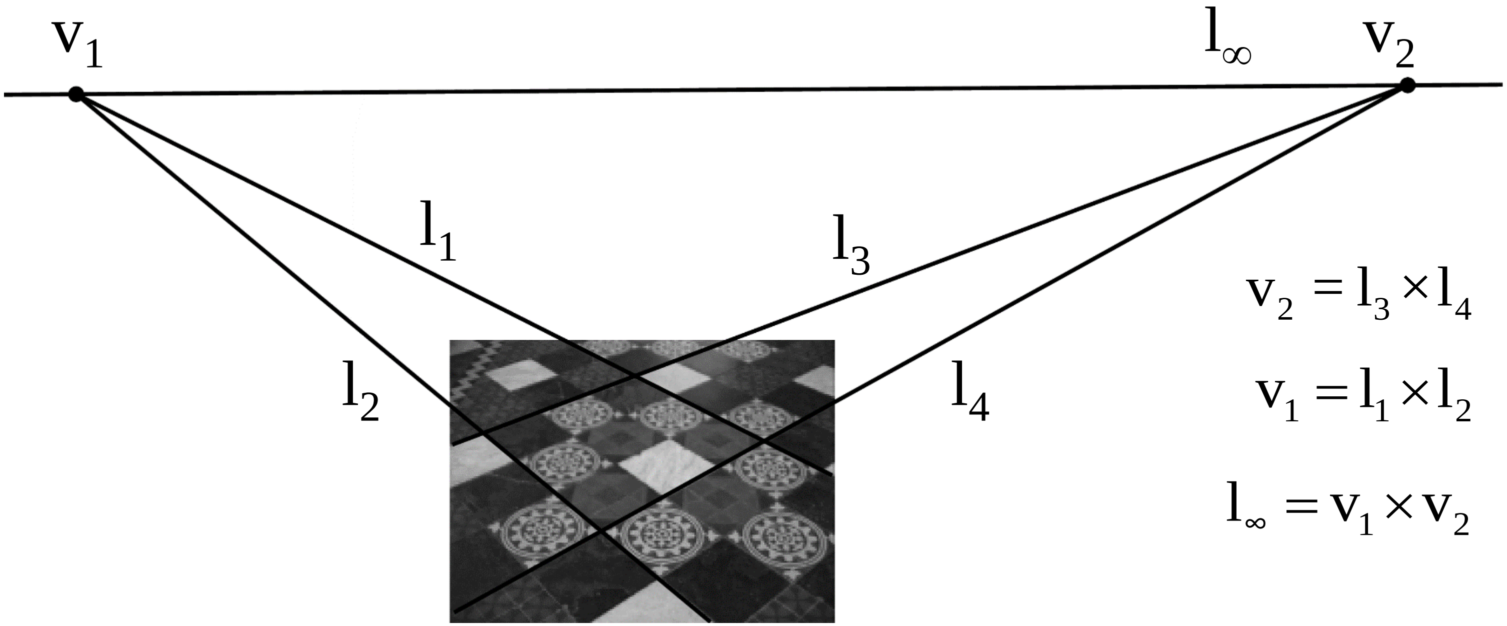
$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$





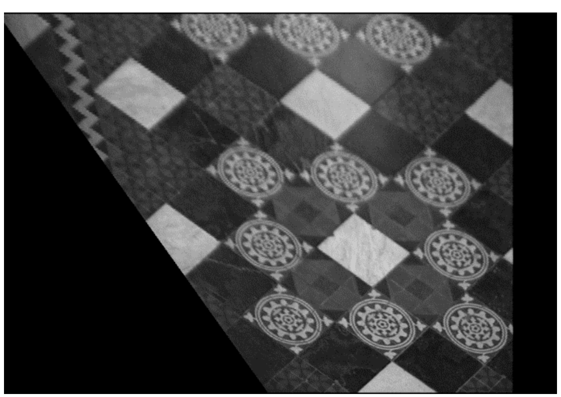
Affine rectification



$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$



The circular points

$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

The circular points

$$\mathrm{I}=\begin{pmatrix}1\\ i\\ 0\end{pmatrix} \qquad \mathrm{J}=\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$

$$\mathrm{I}'=\mathbf{H}_S\,\mathrm{I}=\begin{bmatrix}s\cos\theta & s\sin\theta & t_x\\ -s\sin\theta & s\cos\theta & t_y\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}=se^{i\theta}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}=\mathrm{I}$$

The circular points

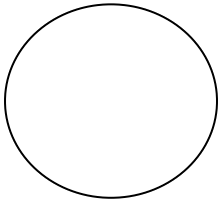
$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & s\sin\theta & t_x \\ -s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity

The circular points

"circular points"

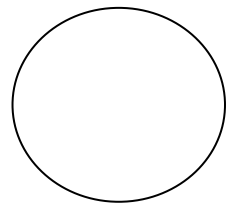


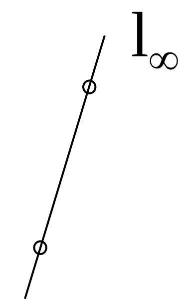
$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

$$x_3 = 0$$

The circular points

"circular points"





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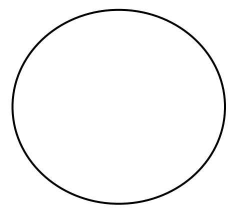
$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

The circular points

"circular points"



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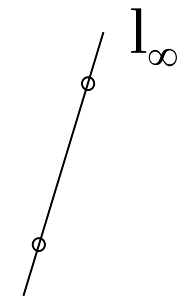
$$x_3 = 0$$



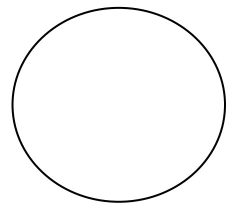
$$x_1^2 + x_2^2 = 0$$

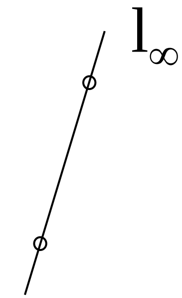
$$\mathrm{I} = (1, i, 0)^\mathsf{T}$$

$$\mathrm{J} = (1, -i, 0)^\mathsf{T}$$

The circular points

"circular points"





$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

$$\mathrm{I} = (1, i, 0)^{\mathsf{T}}$$
$$\mathrm{J} = (1, -i, 0)^{\mathsf{T}}$$

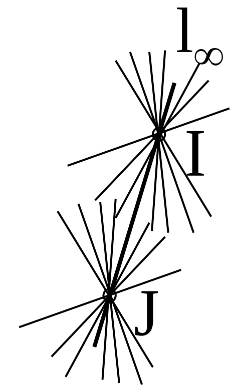
Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^{\mathsf{T}} + i(0,1,0)^{\mathsf{T}}$$

Conic dual to the circular points

$$\mathbf{C}^*_\infty = \mathrm{I}\mathrm{J}^\mathsf{T} + \mathrm{J}\mathrm{I}^\mathsf{T}$$

$$\mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\mathsf{T}$$

Conic dual to the circular points

$$\mathbf{C}^*_\infty = \mathrm{I}\mathrm{J}^\mathsf{T} + \mathrm{J}\mathrm{I}^\mathsf{T}$$

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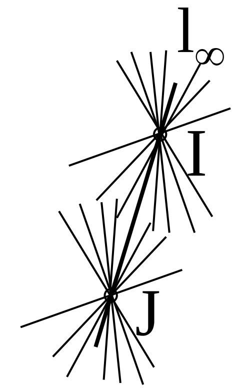

$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}^*_\infty$ is fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Conic dual to the circular points

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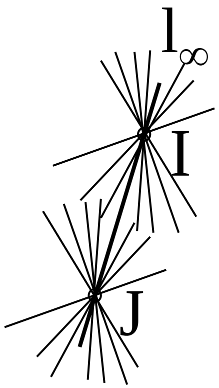


$$\mathbf{C}^{*}_{\infty} = \mathbf{H}_S \mathbf{C}^{*}_{\infty} \mathbf{H}_S^{\mathsf{T}}$$

The dual conic $\mathbf{C}^{*}_{\infty}$ is fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Note: $\mathbf{C}^{*}_{\infty}$ has 4DOF

l_∞ is the nullvector

Angles

Euclidean: $\mathrm{l} = (l_1, l_2, l_3)^{\mathsf{T}} \qquad \mathrm{m} = (m_1, m_2, m_3)^{\mathsf{T}}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Projective: $$\cos\theta = \frac{\mathbf{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^{\mathsf{T}} \mathbf{C}_\infty^* \mathbf{m})}}$$

$\mathrm{l}^{\mathsf{T}} \mathrm{C}_\infty^* \mathrm{m} = 0$ (orthogonal)

Transformation of 3D points, planes and quadrics

For a point transformation (cfr. 2D equivalent)

$$X' = \mathbf{H} X \qquad (x' = \mathbf{H} x)$$

Transformation for planes

$$\pi' = \mathbf{H}^{-T} \pi \qquad (\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l})$$

Transformation for quadrics

$$Q' = H^{-T} Q H^{-1} \qquad (\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1})$$

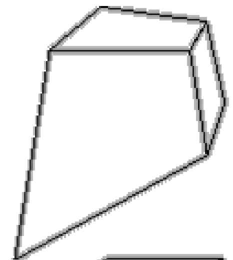
Transformation for dual quadrics

$$Q'^{*} = H Q^{*} H^{T} \qquad (\mathbf{C}'^{*} = \mathbf{H} \mathbf{C}^{*} \mathbf{H}^{T})$$

Hierarchy of 3D transformations

Projective
15dof

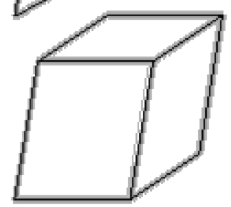
$$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & v \end{bmatrix}$$



Intersection and tangency

Affine
12dof

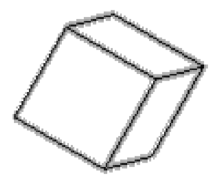
$$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$$



Parallellism of planes,
Volume ratios, centroids,
The plane at infinity π_∞

Similarity
7dof

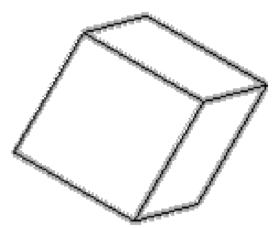
$$\begin{bmatrix} s\mathrm{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$$



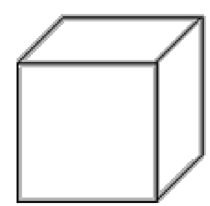
Angles, ratios of length
The absolute conic Ω_∞

Euclidean
6dof

$$\begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$$



Volume



The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}}\pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathrm{t}^{\mathsf{T}}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canonical position $\pi_\infty = (0,0,0,1)^{\mathsf{T}}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^{\mathsf{T}}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $\left(X_1, X_2, X_3\right) \mathrm{I} \left(X_1, X_2, X_3\right)^\mathsf{T}$

(with no real points)

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(with no real points)

> The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

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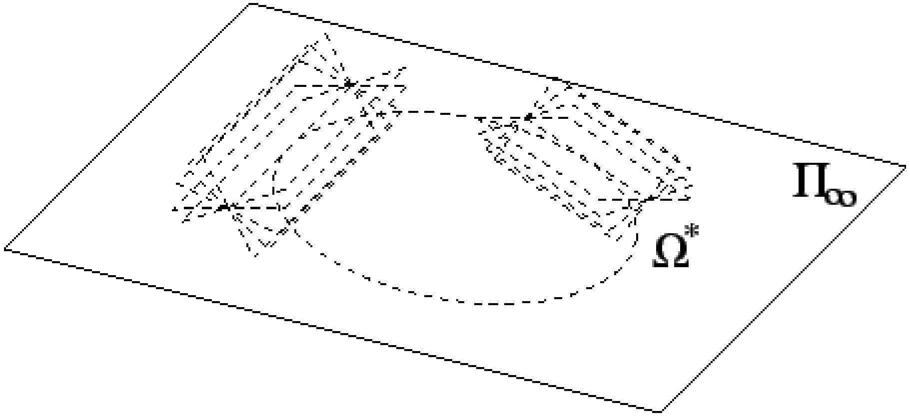
(with no real points)

The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two circular points
3. Spheres intersect π_∞ in Ω_∞

The absolute dual quadric

$$\Omega^*_\infty = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix}$$



The absolute dual quadric

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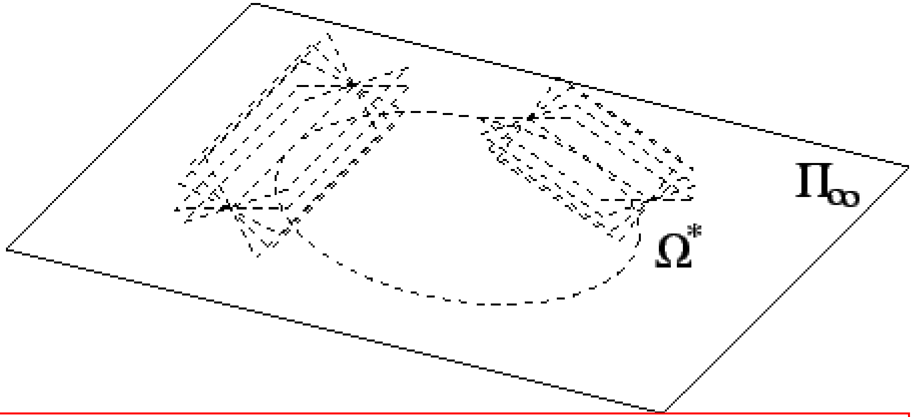
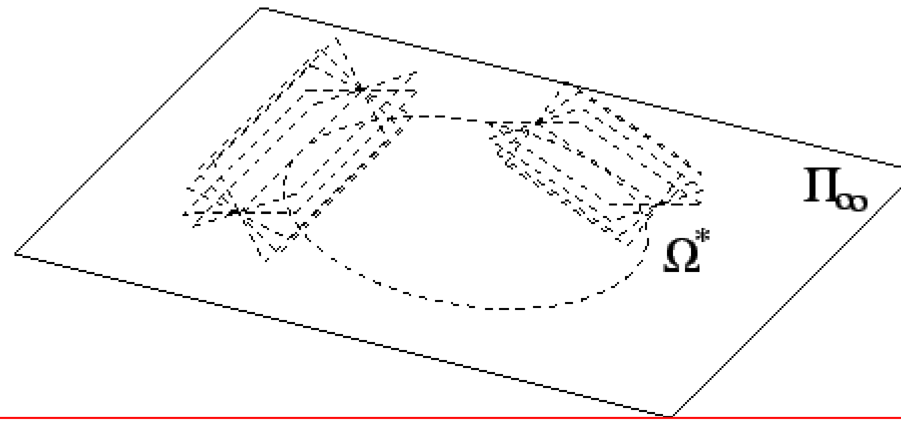


The absolute dual quadric Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

The absolute dual quadric

$$\Omega_\infty^* = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix}$$



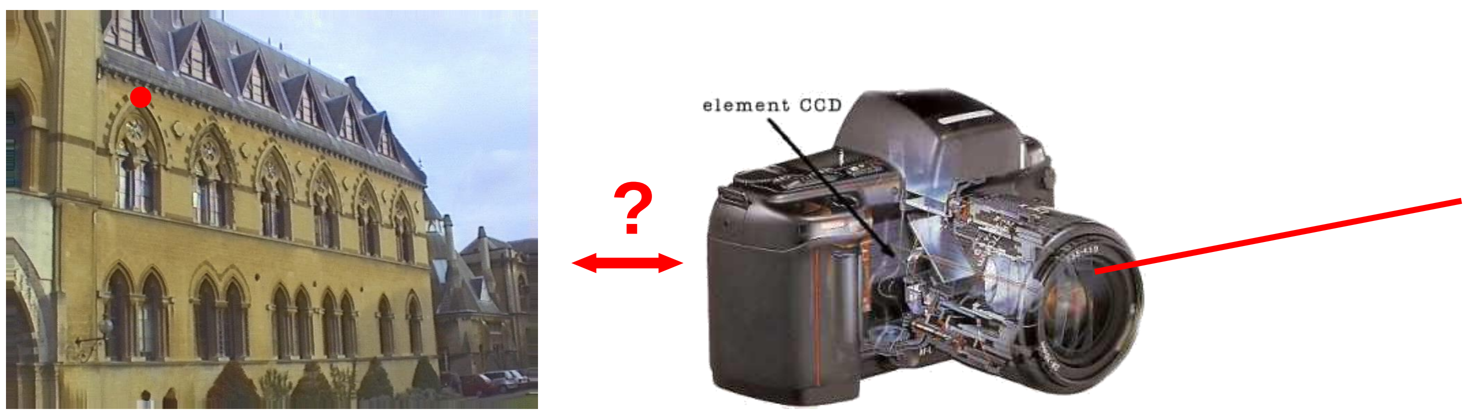
The absolute dual quadric Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the nullvector of Ω_∞
3. Angles:

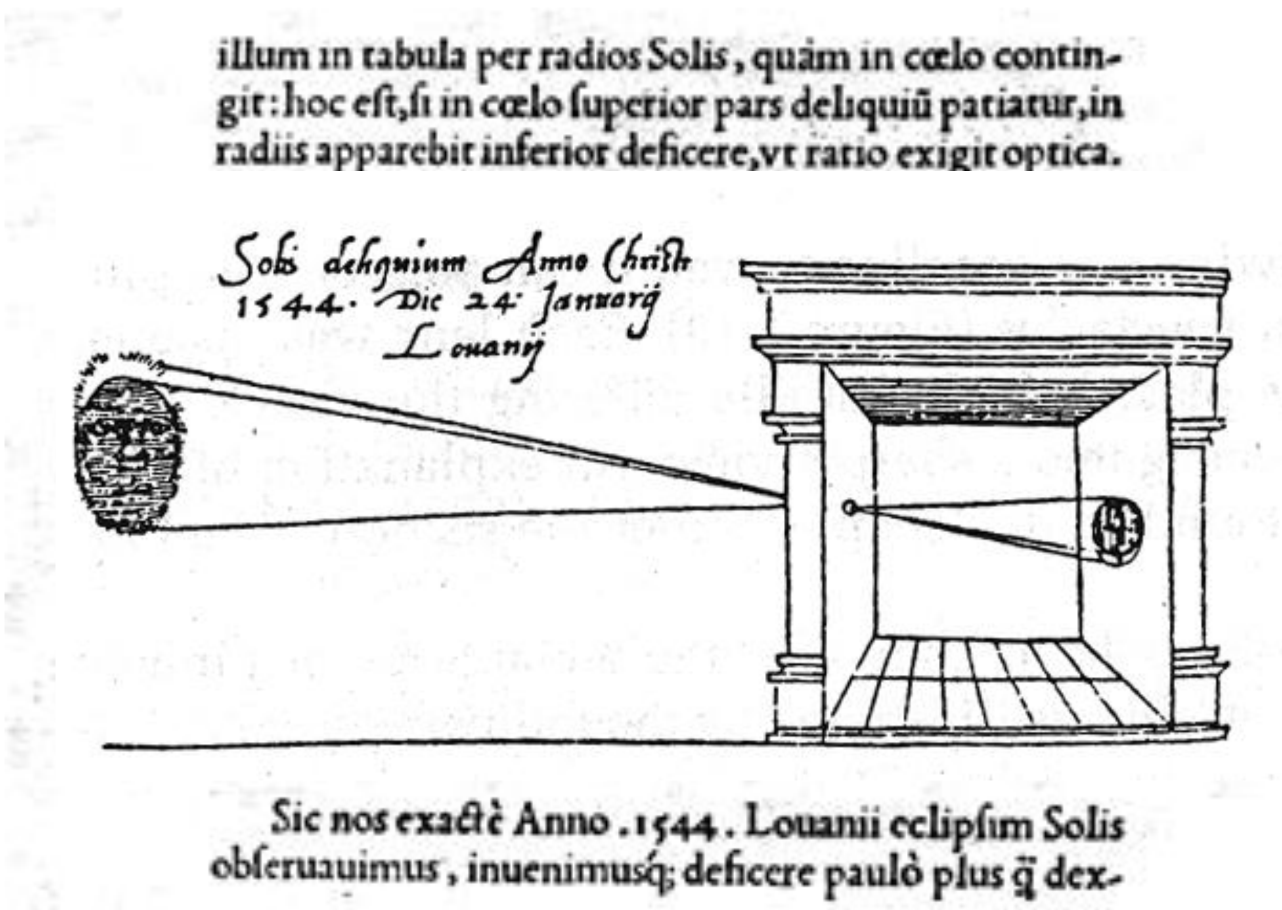
$$\cos\theta = \frac{\pi_1^T \Omega_\infty^* \pi_2}{\sqrt{\left(\pi_1^T \Omega_\infty^* \pi_1\right)\left(\pi_2^T \Omega_\infty^* \pi_2\right)}}$$

Camera model

Relation between pixels and rays in space



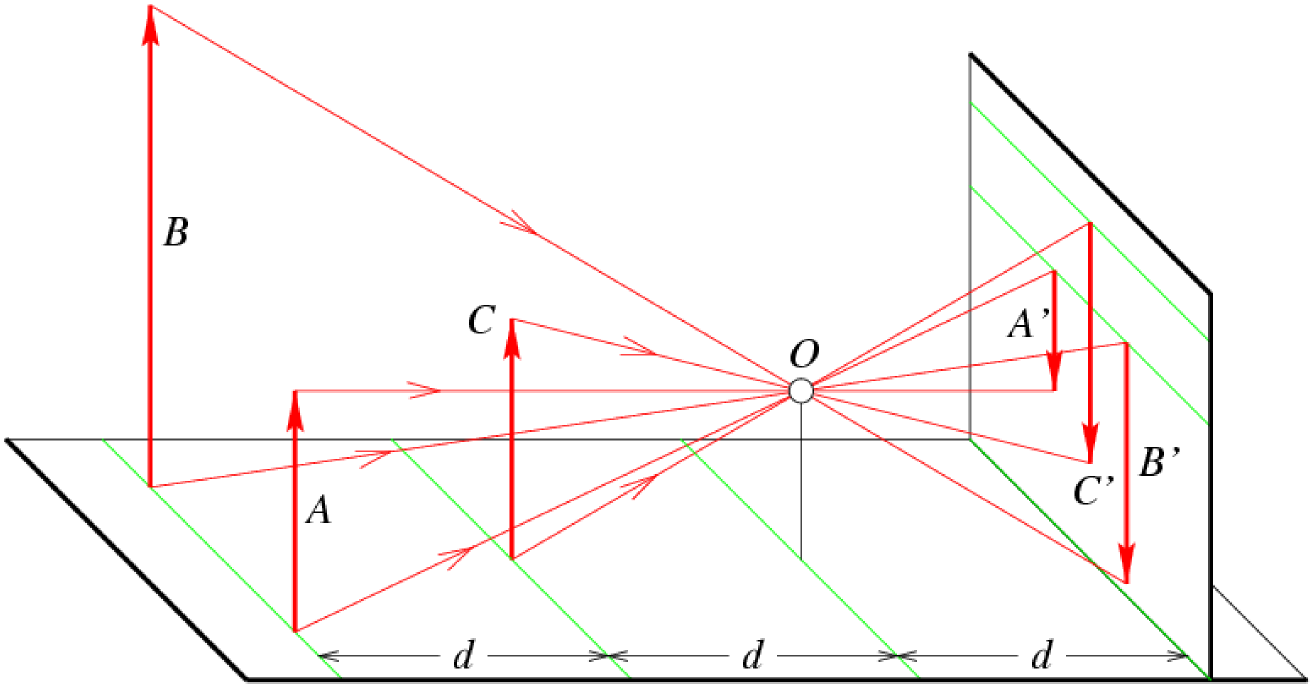
Pinhole camera


illum in tabula per radios Solis, quàm in cœlo contin-
git: hoc eſt, ſi in cœlo ſuperior pars deliquiū patiatur, in
radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipſim Solis
obſeruauimus, inuenimusq; deficere paulò plus q̄ dex-

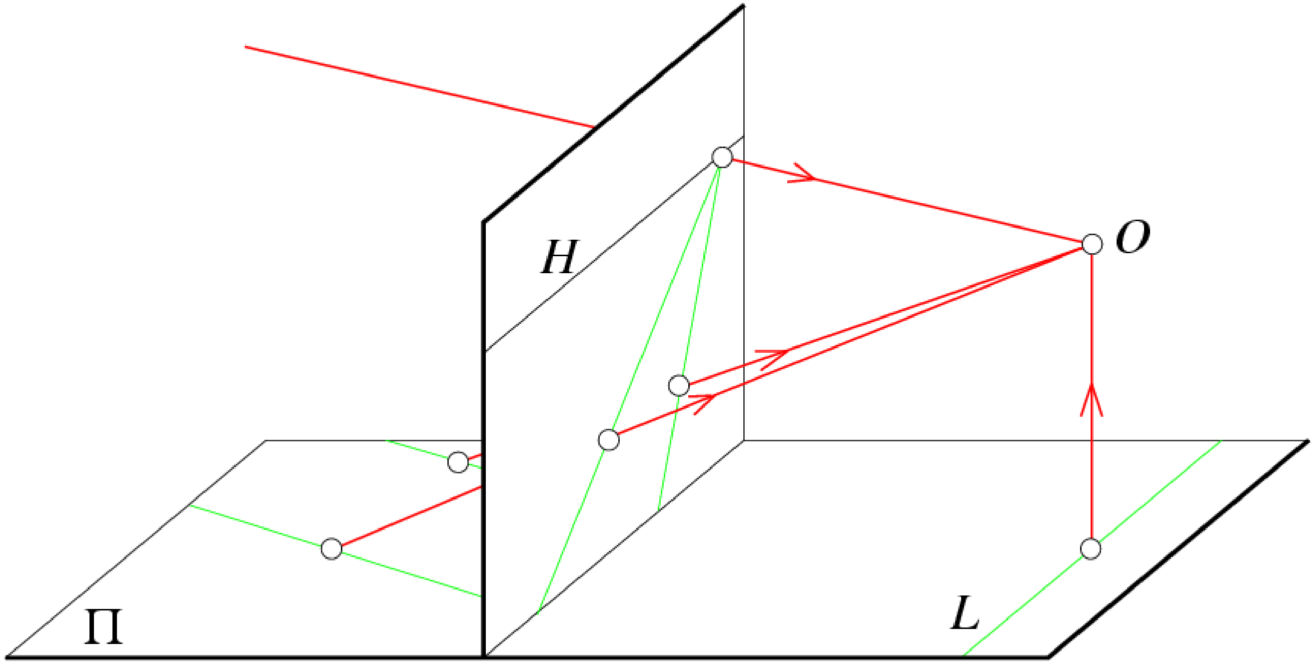
Gemma Frisius, 1544

Distant objects appear smaller



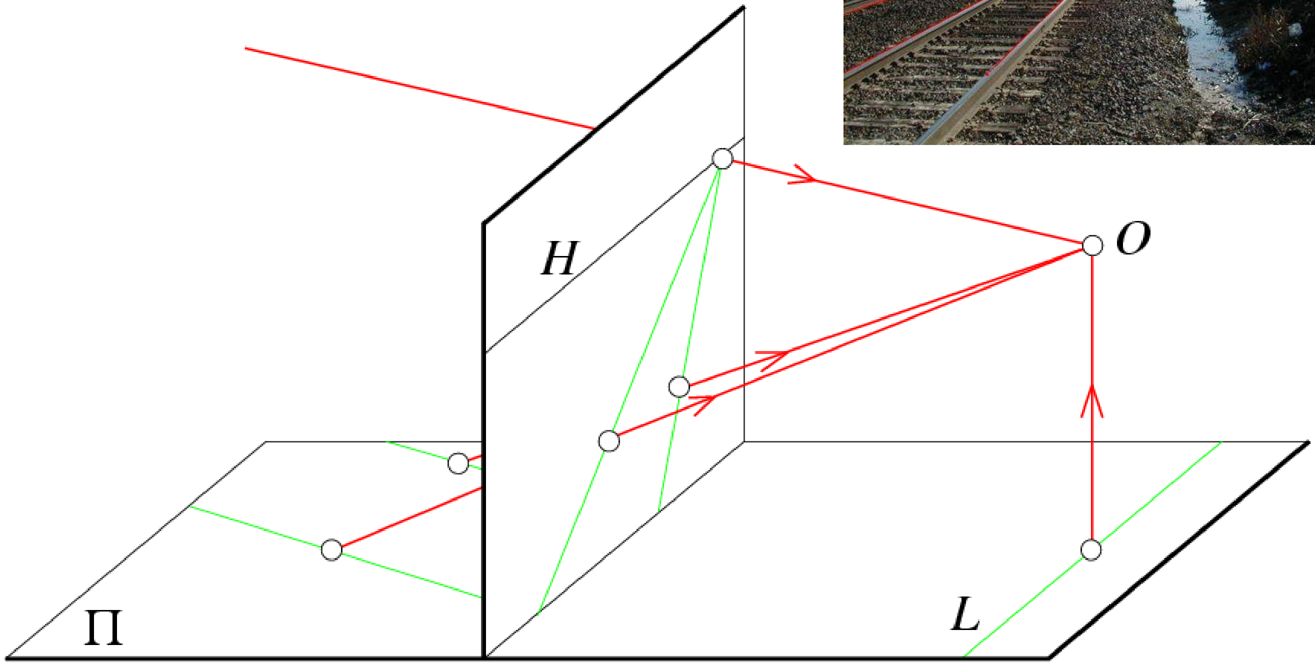
Parallel lines meet

- vanishing point



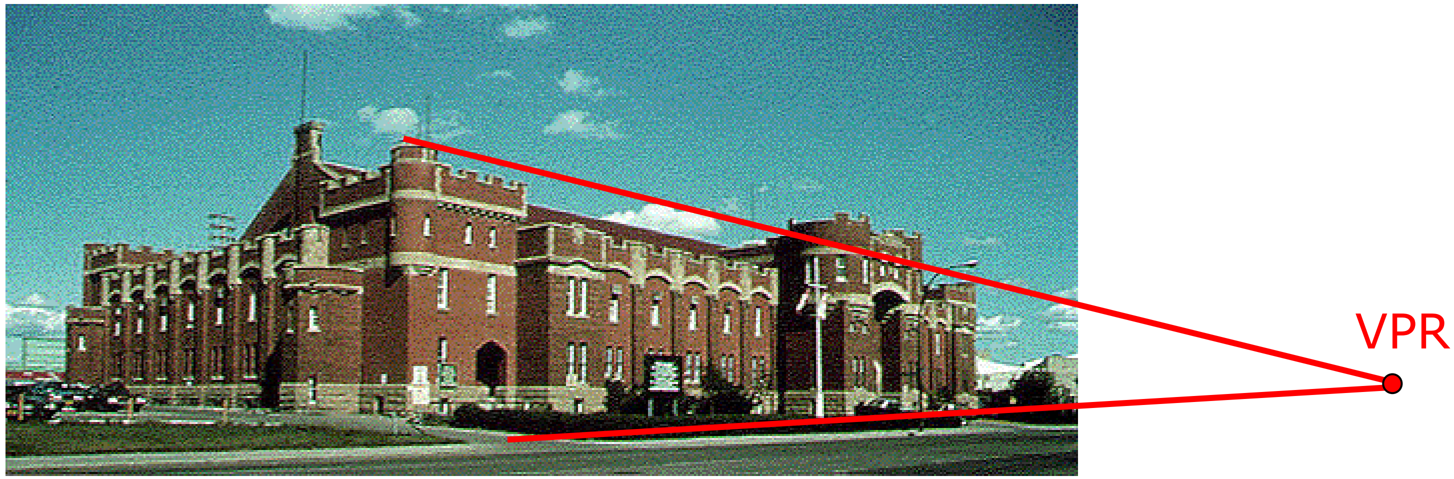
Parallel lines meet

- vanishing point

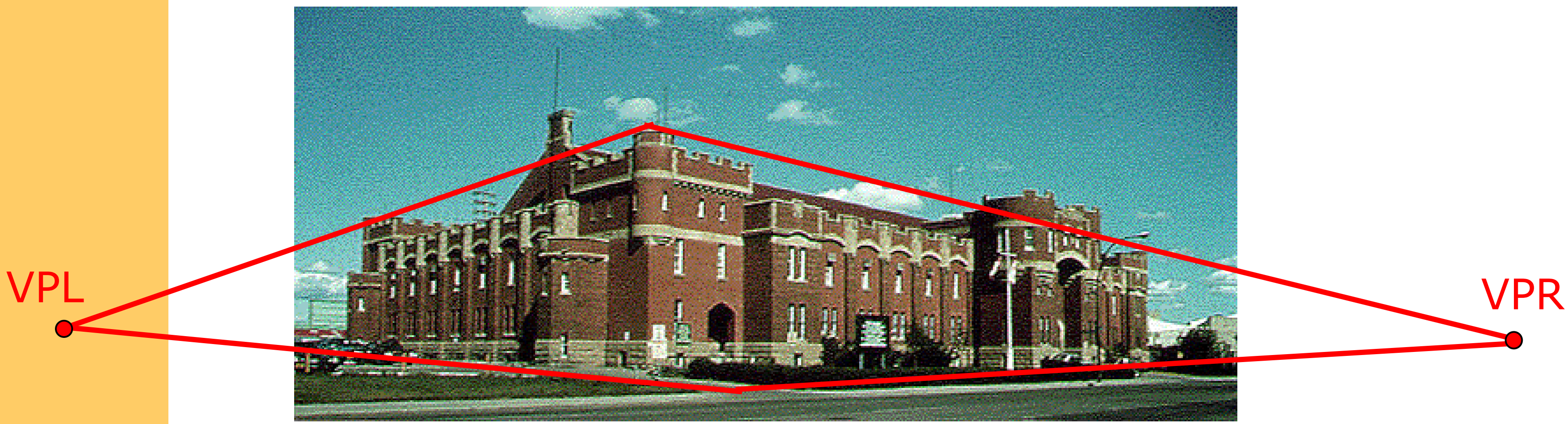


Vanishing points

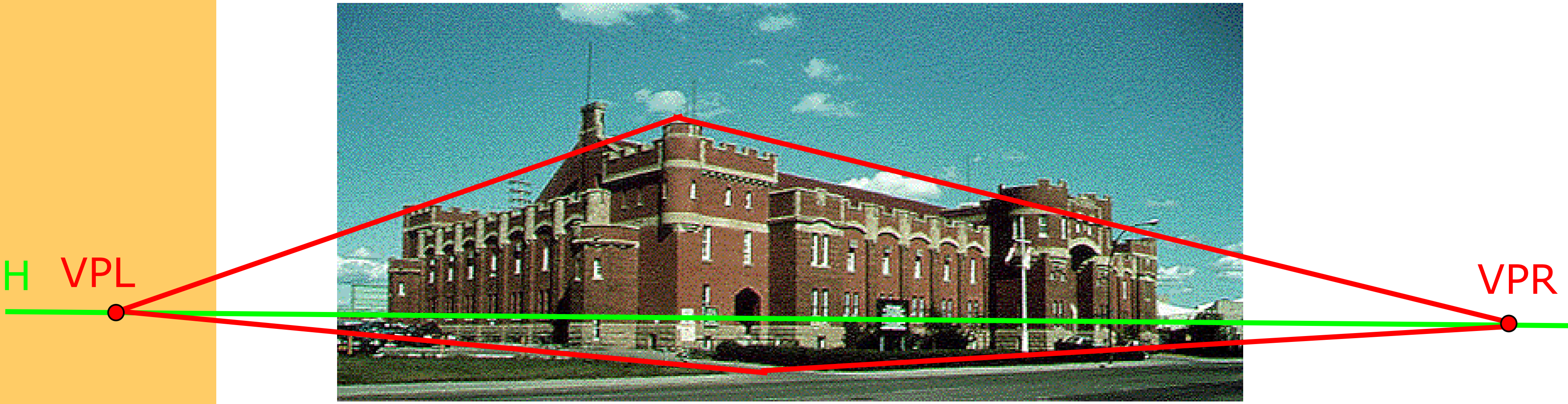
Vanishing points



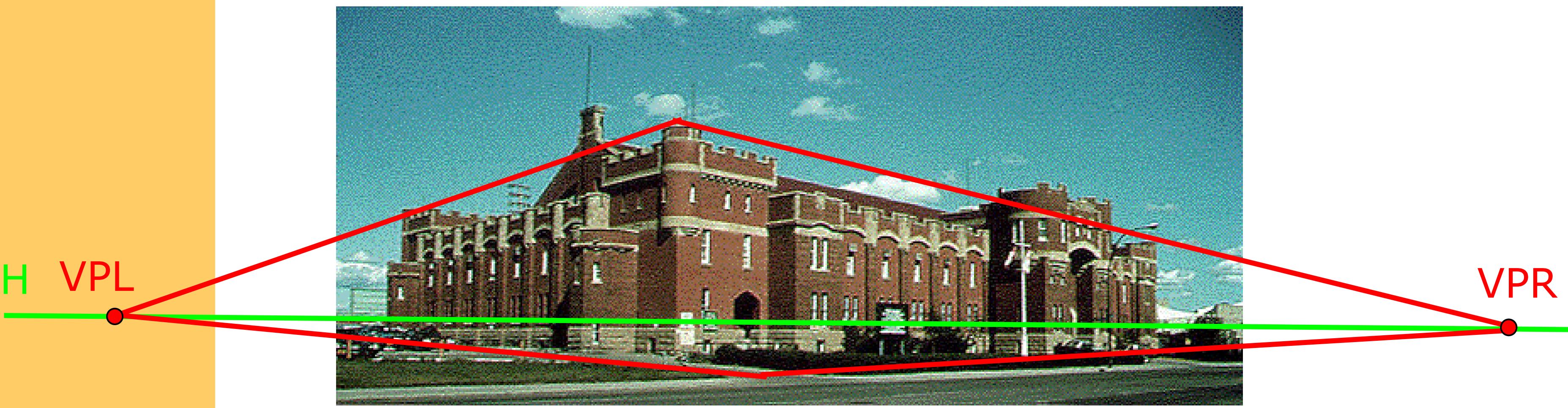
Vanishing points

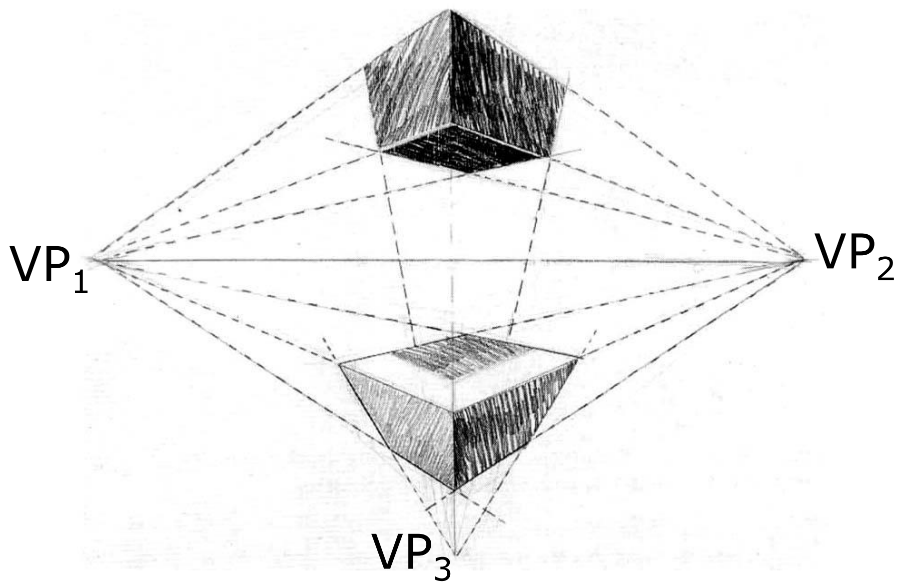


Vanishing points

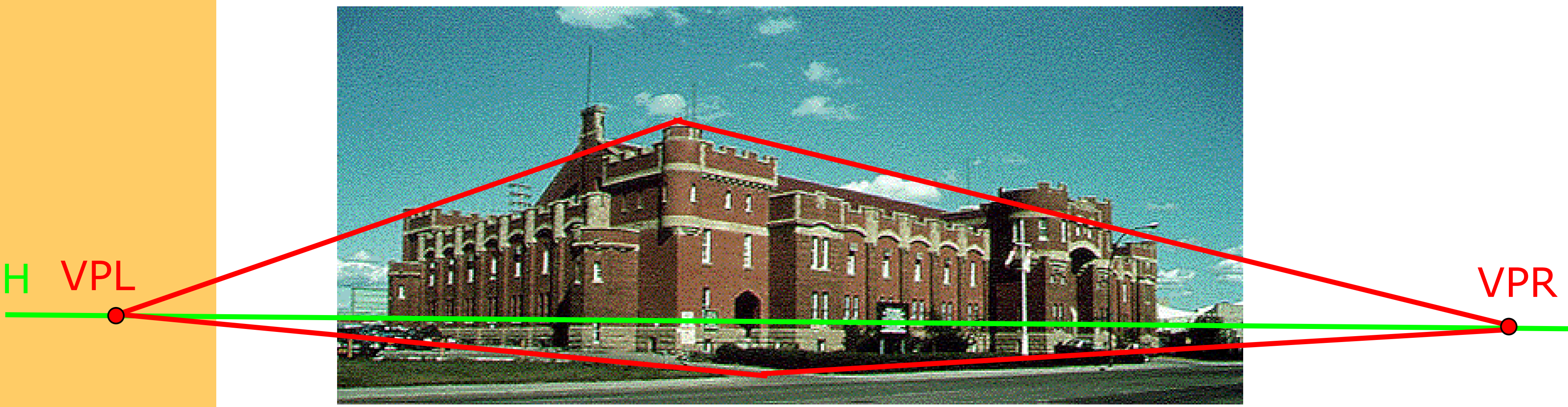


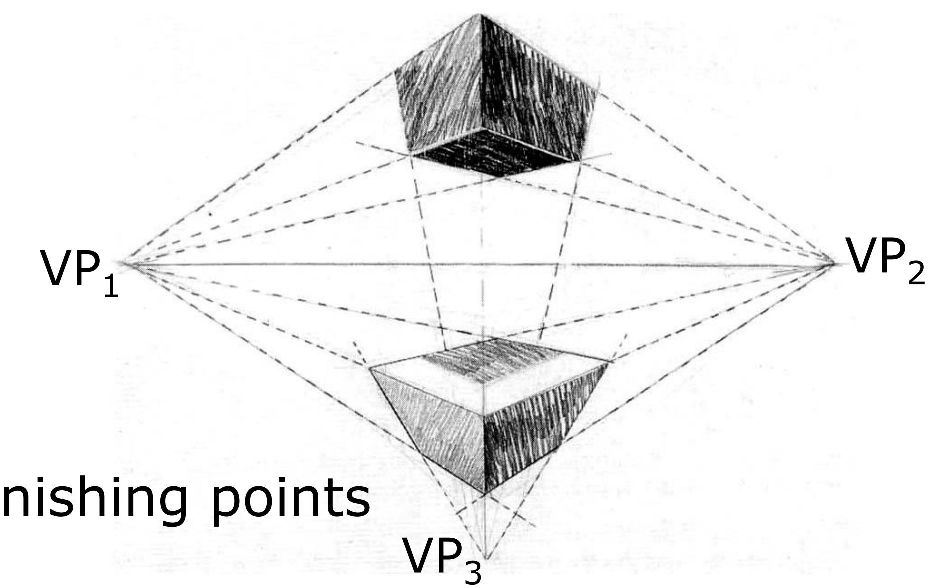
Vanishing points





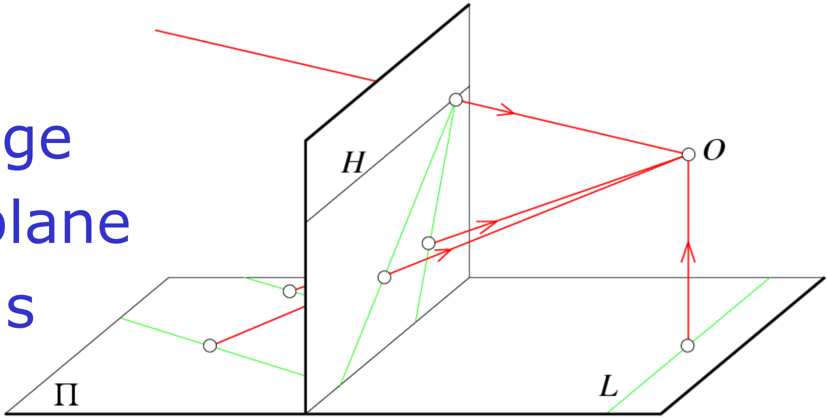
Vanishing points





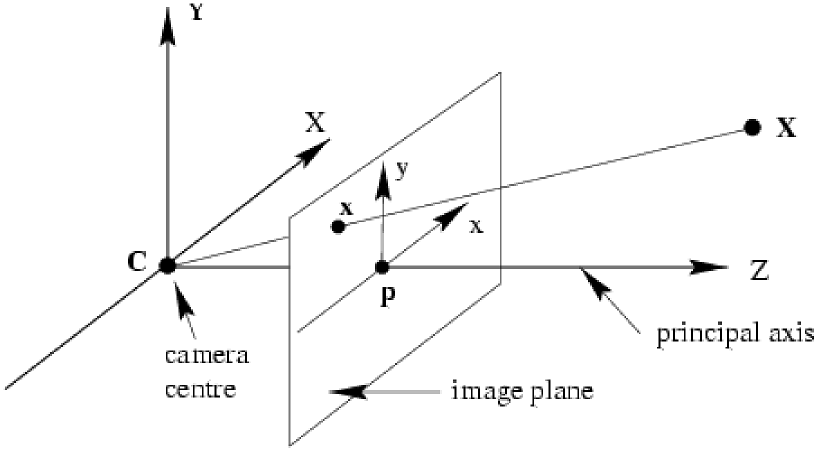
To different directions
correspond different vanishing points

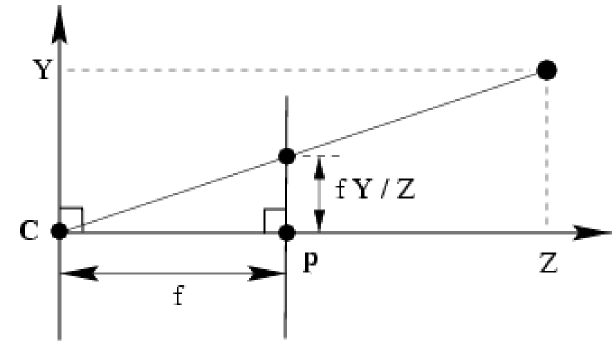
Geometric properties of projection



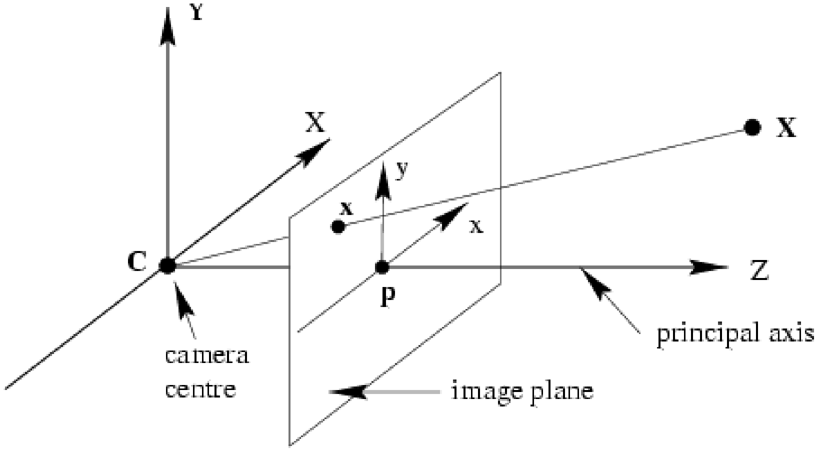
- Points go to points
- Lines go to lines
- Planes go to whole image or half-plane
- Polygons go to polygons
- Degenerate cases:
  - line through focal point yields point
  - plane through focal point yields line

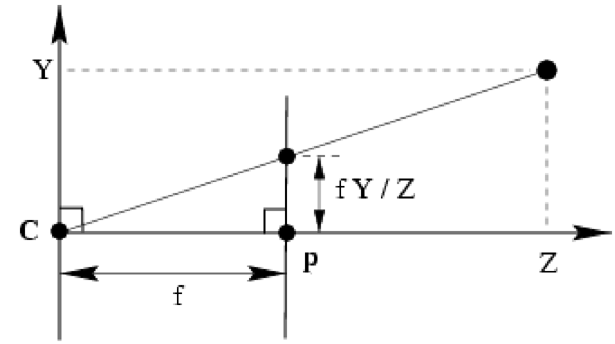
Pinhole camera model





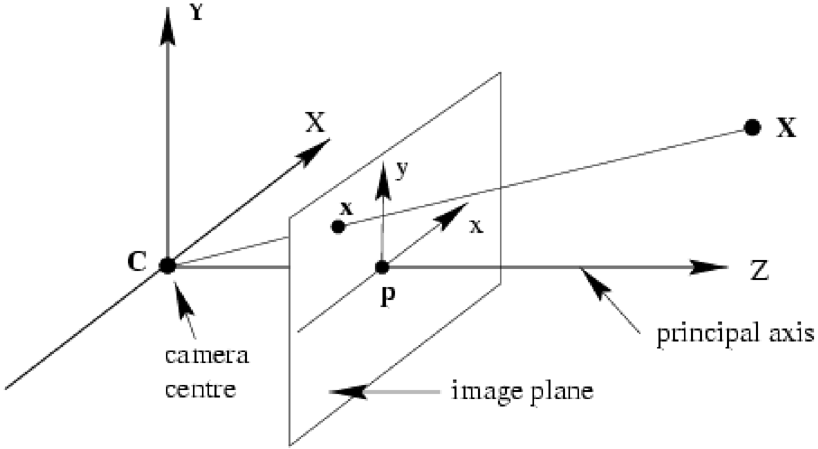
Pinhole camera model

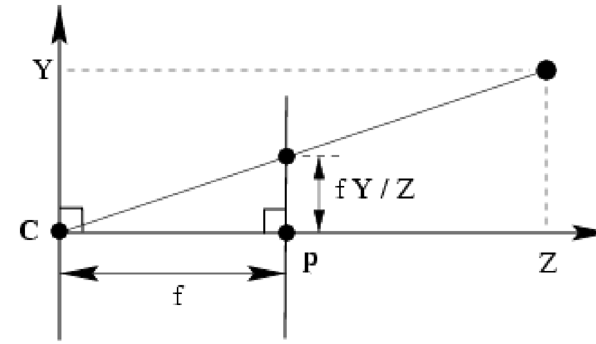




$$(X, Y, Z)^T \mapsto (fX/Z, fY/Z)^T$$

Pinhole camera model

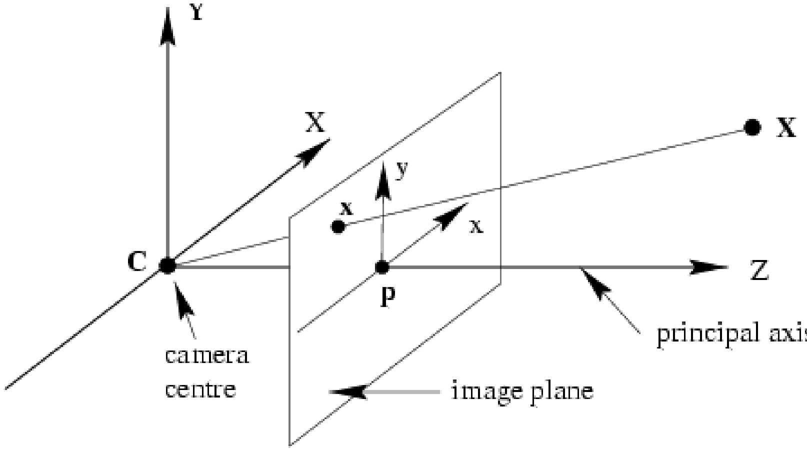


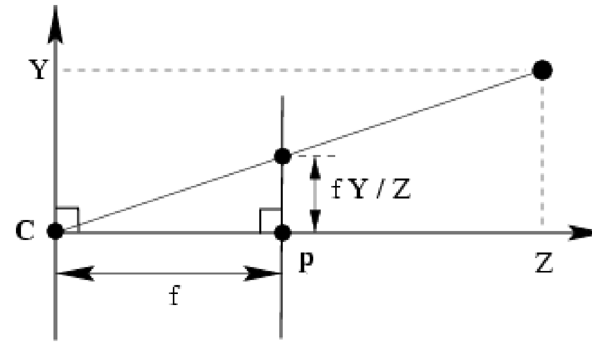


$$(X, Y, Z)^T \mapsto (fX/Z, fY/Z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Pinhole camera model





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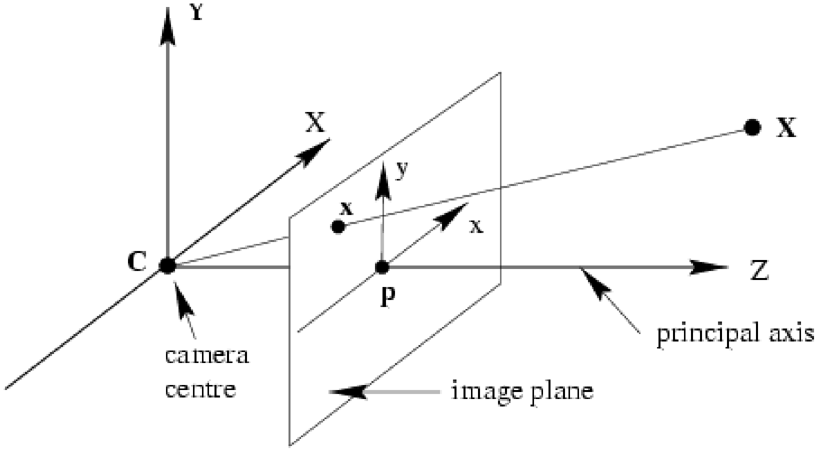
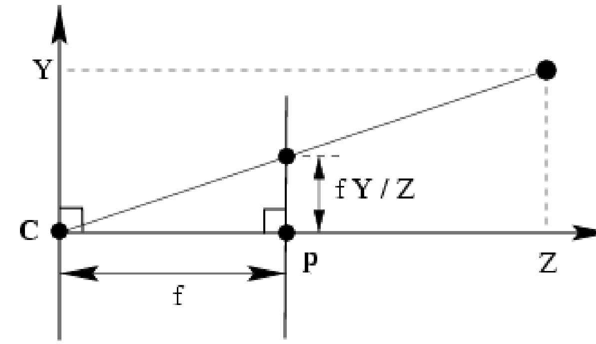
linear projection in homogeneous coordinates!

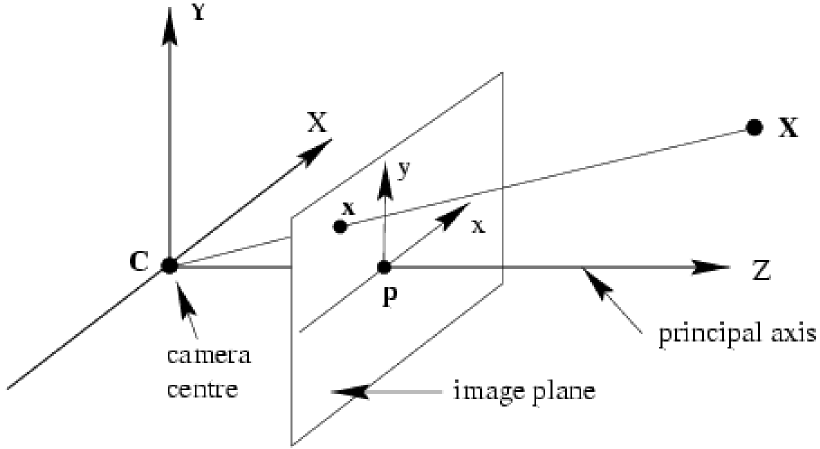
Pinhole camera model

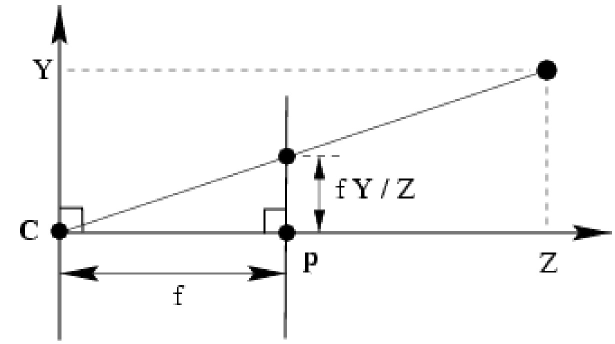




$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

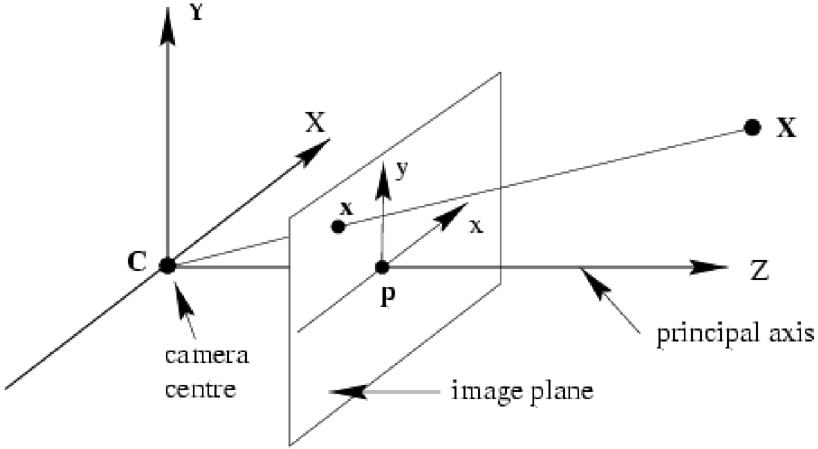
Pinhole camera model

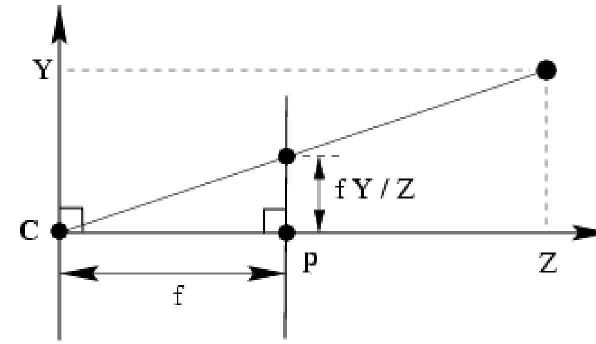




$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Pinhole camera model

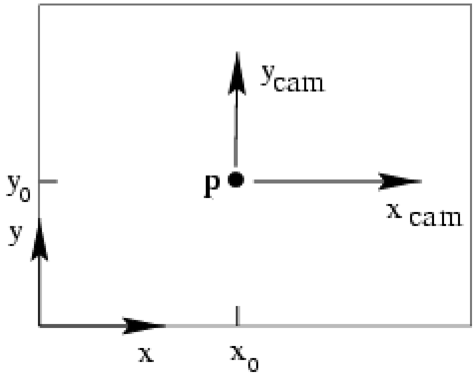




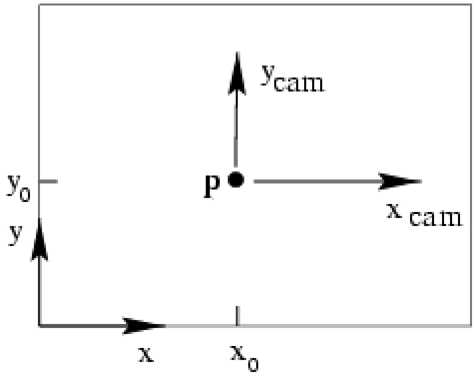
$$x = PX$$

$$P = \mathrm{diag}(f, f, 1)[I \mid 0]$$

Principal point offset



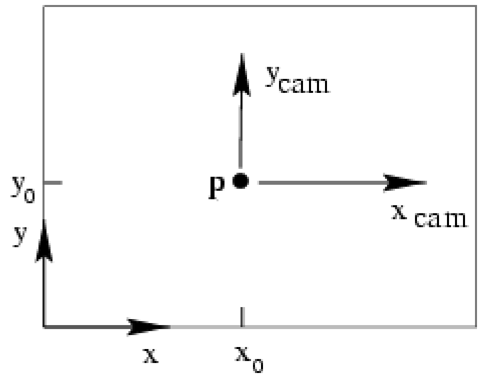
Principal point offset



$$(X,Y,Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

$(p_x, p_y)^T$ principal point

Principal point offset

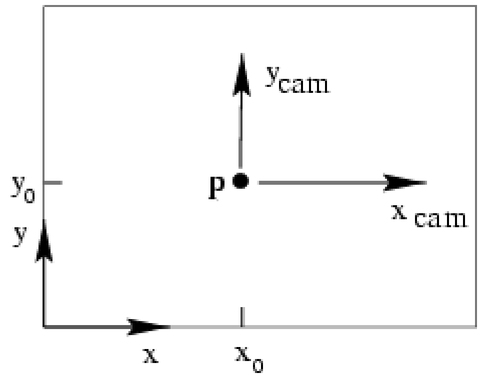


$$(X, Y, Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

$(p_x, p_y)^T$ principal point

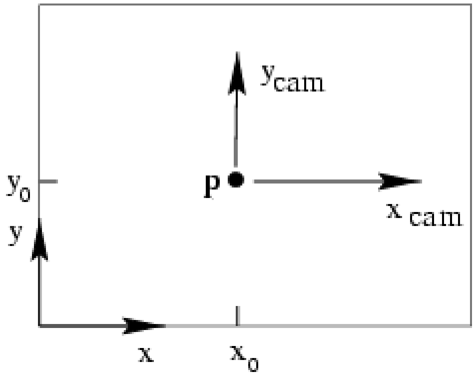
$$\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix} \mapsto \begin{pmatrix} fX+Zp_x\\ fY+Zp_y\\ Z\end{pmatrix} = \begin{bmatrix} f & & p_x & 0\\ & f & p_y & 0\\ & & 1 & 0\end{bmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}$$

Principal point offset



$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

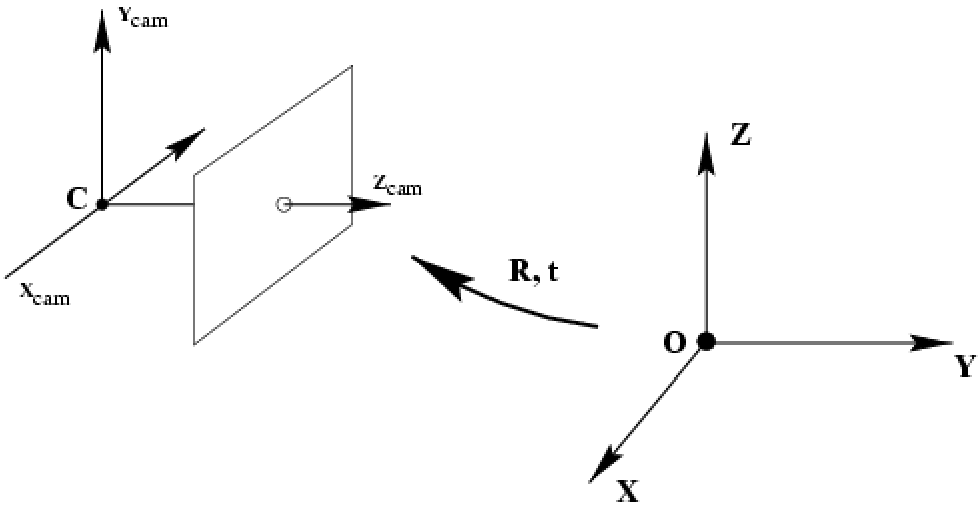
Principal point offset



$$\mathrm{x} = \mathrm{K}\left[\mathrm{I} \mid 0\right]\mathrm{X}_{\mathrm{cam}}$$

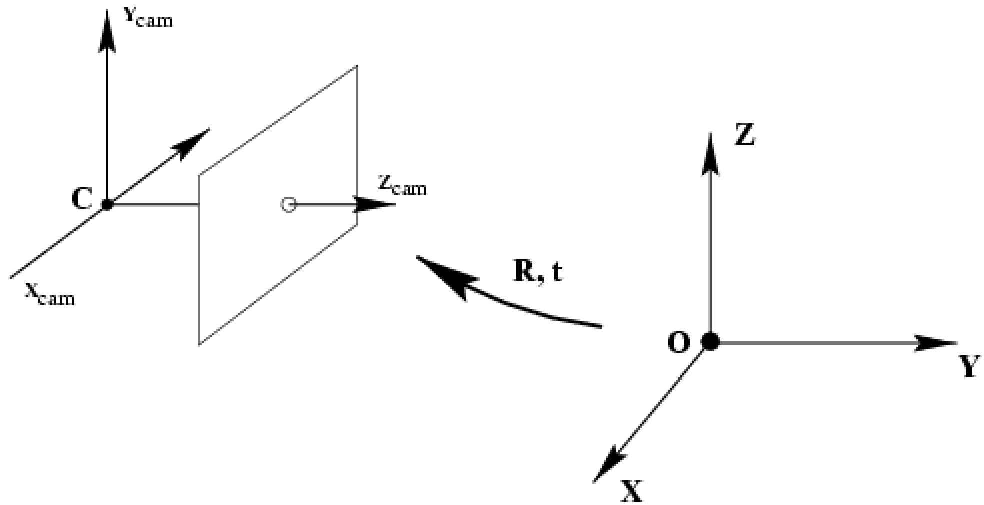
$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

Camera rotation and translation



$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

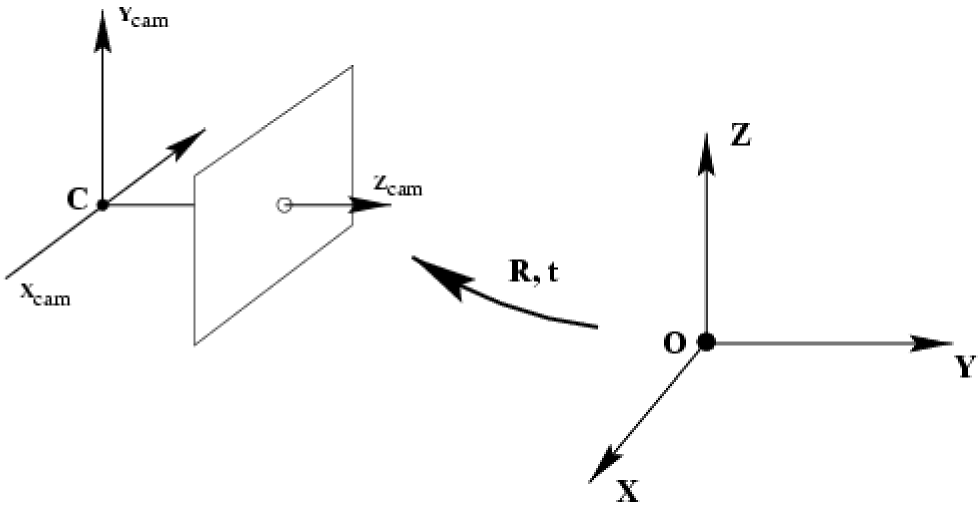
Camera rotation and translation



$$\tilde{\mathrm{X}}_{\mathrm{cam}} = \mathrm{R}\left(\tilde{\mathrm{X}} - \tilde{\mathrm{C}}\right)$$

$$\mathrm{X}_{\mathrm{cam}} = \begin{bmatrix} \mathrm{R} & -\mathrm{R}\tilde{\mathrm{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} \mathrm{R} & -\mathrm{R}\tilde{\mathrm{C}} \\ 0 & 1 \end{bmatrix} \mathrm{X}$$

Camera rotation and translation

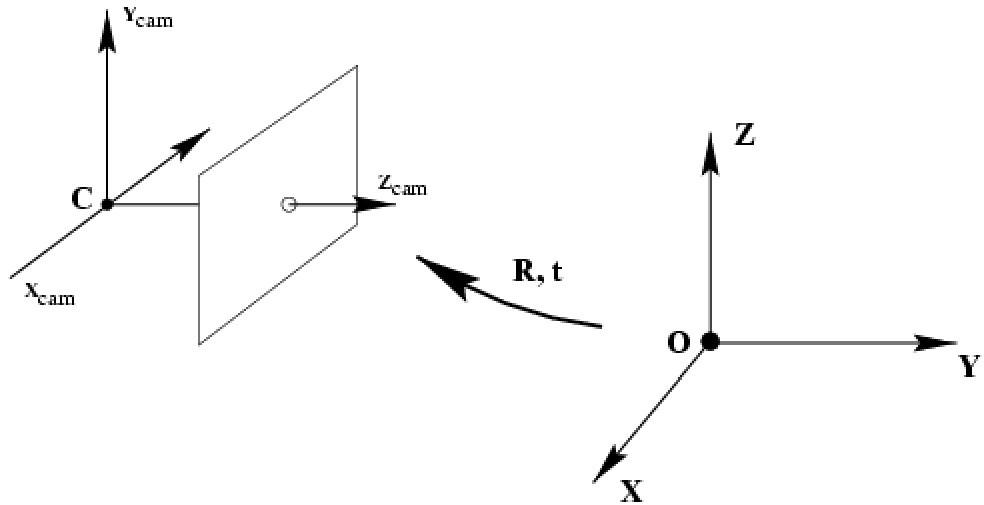


$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = K\left[I \mid 0\right] X_{cam}$$

Camera rotation and translation

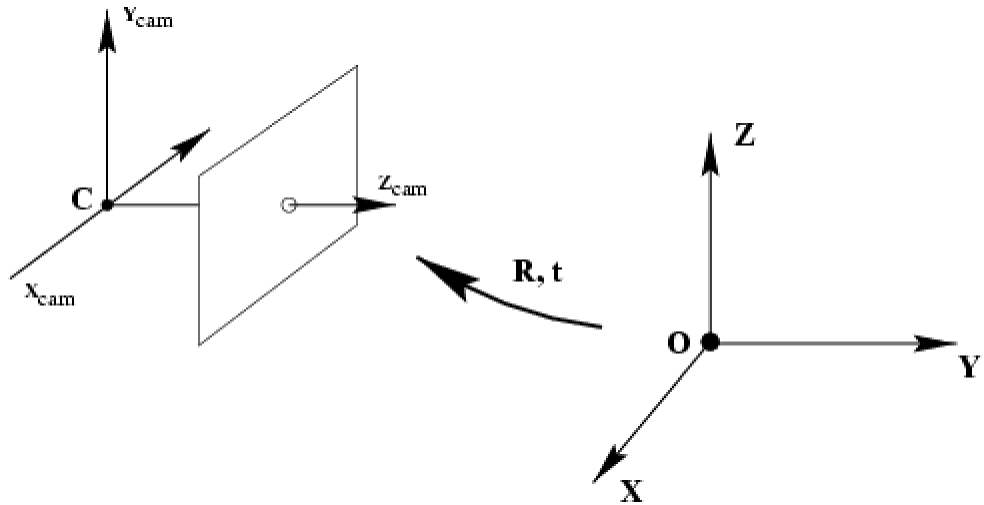


$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = KR\left[I \mid -\tilde{C}\right]X$$

Camera rotation and translation



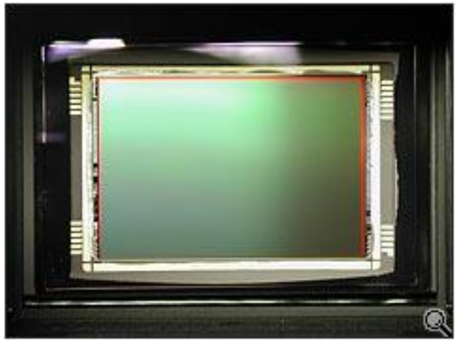
$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = KR\left[I \mid -\tilde{C}\right]X$$

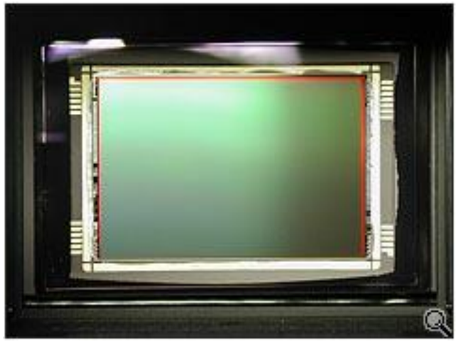
$$x = PX \qquad P = K\left[R \mid t\right] \qquad t = -R\tilde{C}$$

CCD camera



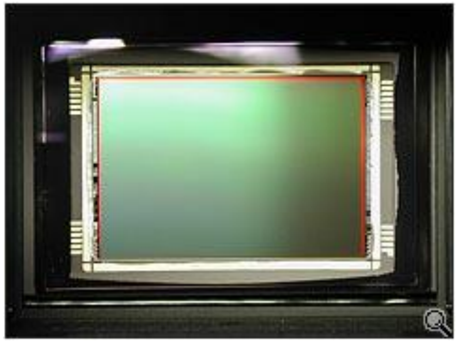
$$K = \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

CCD camera

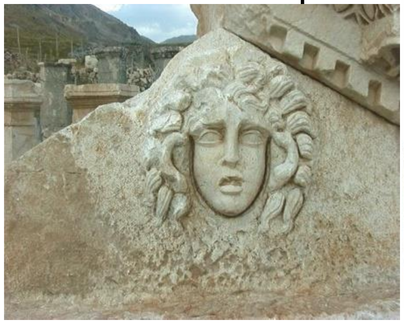


$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

CCD camera



$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$



General projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

$$\mathrm{P} = \mathrm{KR}\left[\mathrm{I} \mid \quad \tilde{\mathrm{C}}\right]$$

General projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

$$\mathrm{P} = \mathrm{KR}\left[\mathrm{I} \mid \quad \tilde{\mathrm{C}}\right]$$ 11 dof (5+3+3)

General projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

$$P = \underbrace{KR}_{\text{non-singular}} \left[I \mid \quad \tilde{C} \right]$$

11 dof (5+3+3)

General projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

$$\mathrm{P} = \underbrace{\mathrm{KR}}_{\text{non-singular}}\left[\mathrm{I} \mid \tilde{\mathrm{C}}\right]$$ 11 dof (5+3+3)

$$\mathrm{P} = \mathrm{K}\left[\mathrm{R} \mid \mathrm{t}\right]$$

intrinsic camera parameters
extrinsic camera parameters

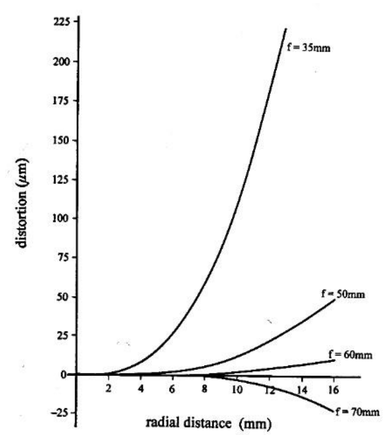
Radial distortion

- Due to spherical lenses (cheap)
- Model:

$$\begin{bmatrix}x\\y\\1\end{bmatrix} \sim \begin{bmatrix} f_x & s & c_x\\ 0 & f_y & c_y \\ 0&0&1\end{bmatrix} R \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}\mathbf{R}^\top & \text{-}\mathbf{R}^\top\mathbf{t}\\ 0_3^\top & 1\end{bmatrix}\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix}\right)$$

$$R \qquad (x,y) = (1+K_1(x^2+y^2)+K_2(x^2+y^2)^2+\ldots)\begin{bmatrix}x\\y\end{bmatrix}$$

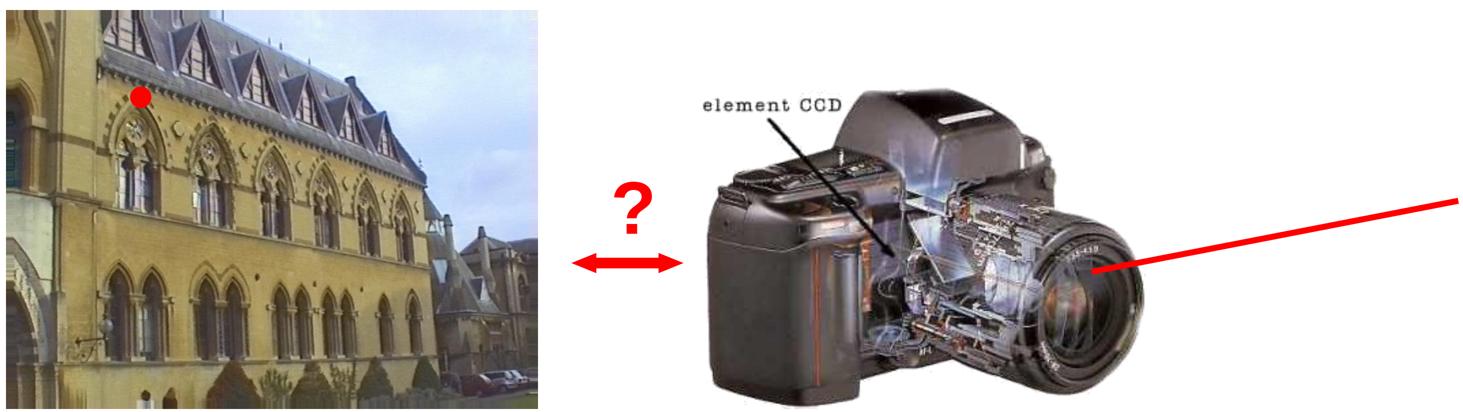
straight lines are not straight anymore



http://foto.hut.fi/opetus/260/luennot/11/atkinson_6-11_radial_distortion_zoom_lenses.jpg

Camera model

Relation between pixels and rays in space



Projector model

Relation between pixels and rays in space

(dual of camera)

(main geometric difference is vertical principal point offset to reduce keystone effect)

Meydenbauer camera

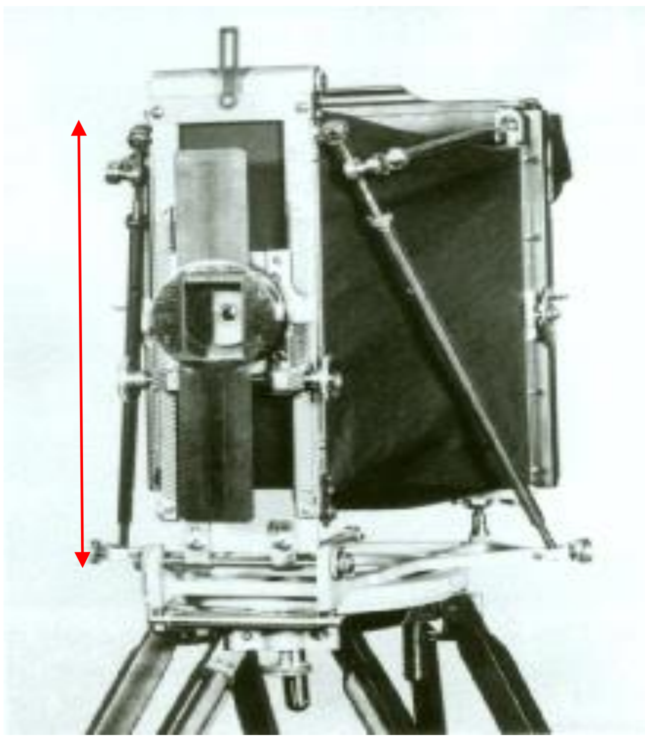


Fig. 5: The principle of »Plane-Table Photogrammetry«
(after an instructional poster of Meydenbauer's institute)

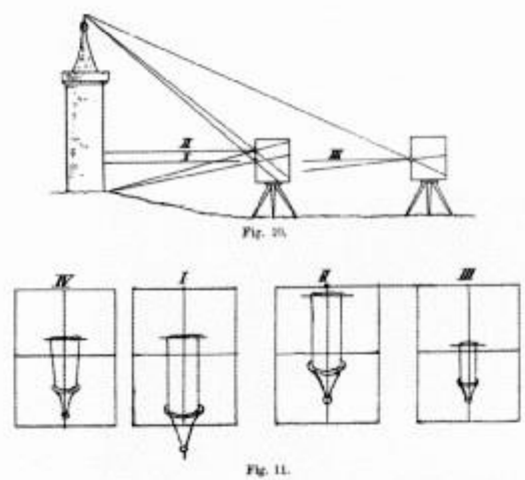
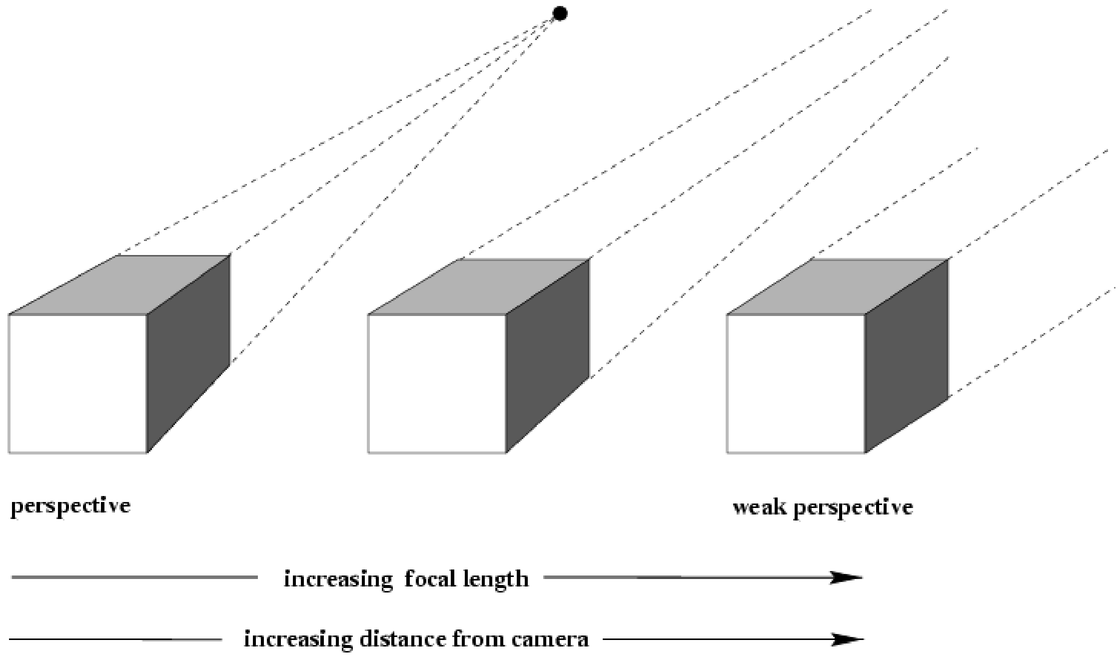


Fig. 6: The effect of a vertical shift of the camera lens;
the position II makes the best use of the image format
(after Meydenbauer's textbook from 1912)

vertical lens shift
to allow direct
ortho-photographs

Affine cameras



Action of projective camera on points and lines

projection of point

$$x = PX$$

forward projection of line

$$X(\mu) = P(A + \mu B) = PA + \mu PB = a + \mu b$$

back-projection of line

$$\Pi = P^T l$$

Action of projective camera on points and lines

projection of point

$$\mathrm{x} = \mathrm{P}\mathrm{X}$$

forward projection of line

$$\mathrm{X}(\mu) = \mathrm{P}(\mathrm{A} + \mu\mathrm{B}) = \mathrm{PA} + \mu\mathrm{PB} = \mathrm{a} + \mu\mathrm{b}$$

back-projection of line

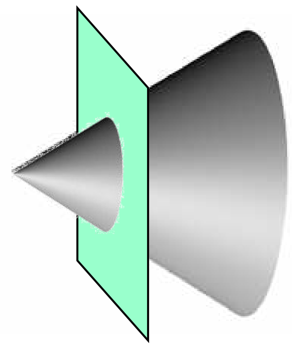
$$\Pi = \mathrm{P}^{\mathsf{T}}\mathrm{l}$$

$$\Pi^{\mathsf{T}}\mathrm{X} = \mathrm{l}^{\mathsf{T}}\mathrm{P}\mathrm{X} \qquad \left(\mathrm{l}^{\mathsf{T}}\mathrm{x} = 0;\ \mathrm{x} = \mathrm{P}\mathrm{X}\right)$$

Action of projective camera on conics and quadrics

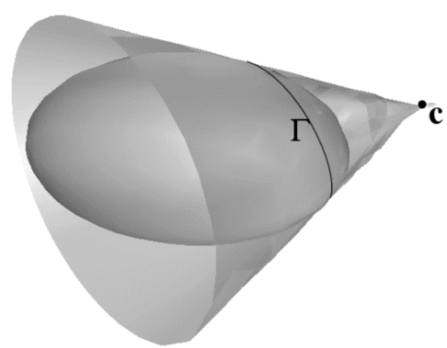
back-projection to cone

$$Q_{co} = P^{T}CP$$



projection of quadric

$$C^{*} = PQ^{*}P^{T}$$



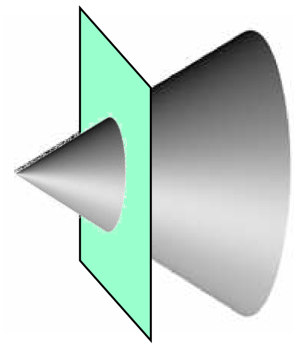
Action of projective camera on conics and quadrics

back-projection to cone

$$Q_{co} = P^T C P$$

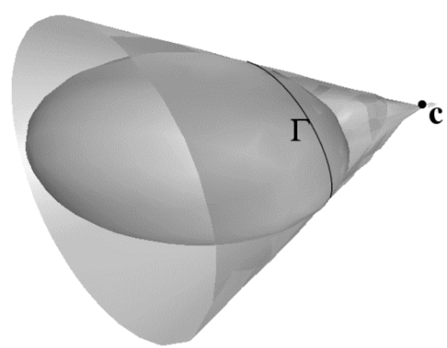
$$x^T C x = X^T P^T C P X = 0$$

$$(x = PX)$$



projection of quadric

$$C^* = P Q^* P^T$$



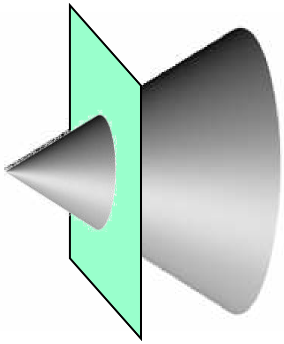
Action of projective camera on conics and quadrics

back-projection to cone

$$Q_{co} = P^T C P$$

$$x^T C x = X^T P^T C P X = 0$$

$$(x = PX)$$

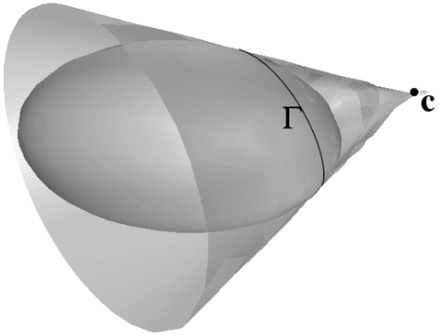


projection of quadric

$$C^* = P Q^* P^T$$

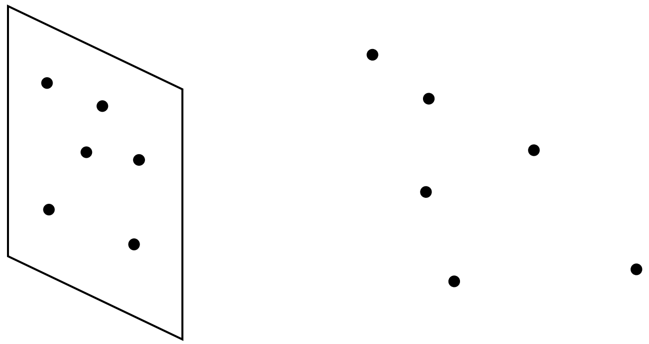
$$\Pi^T Q^* \Pi = l^T P Q^* P^T l = 0$$

$$(\Pi = P^T l)$$



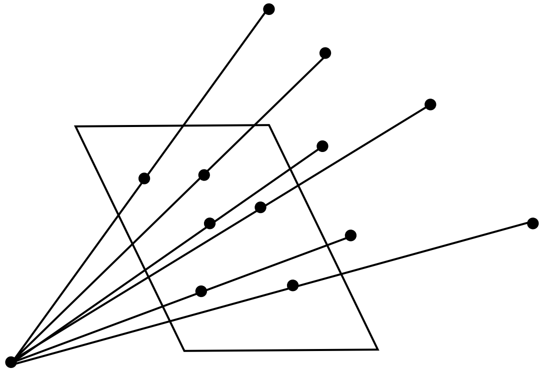
Resectioning

$$X_i \leftrightarrow x_i \qquad P\,?$$



Resectioning

$$X_i \leftrightarrow x_i \qquad P\,?$$



Direct Linear Transform (DLT)

$$\mathrm{x}_i = \mathrm{P}\mathrm{X}_i$$

Direct Linear Transform (DLT)

$$\mathrm{x}_i = \mathrm{P}\mathrm{X}_i \qquad \left[\mathrm{x}_i\right]_{\times}\mathrm{P}\mathrm{X}_i$$

Direct Linear Transform (DLT)

$$\mathrm{x}_i = \mathrm{P}\mathrm{X}_i \qquad [\mathrm{x}_i]_\times \mathrm{P}\mathrm{X}_i \qquad \mathrm{P} = \begin{bmatrix} \mathrm{P}^{1\top} \\ \mathrm{P}^{2\top} \\ \mathrm{P}^{3\top} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \\ -y_i\mathbf{X}_i{}^\top & x_i\mathbf{X}_i{}^\top & \mathbf{0}^\top \end{bmatrix} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

Direct Linear Transform (DLT)

$$x_i = PX_i \qquad [x_i]_\times PX_i \qquad \mathbf{P} = \begin{bmatrix} \mathbf{P}^{1\top} \\ \mathbf{P}^{2\top} \\ \mathbf{P}^{3\top} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \\ -y_i\mathbf{X}_i{}^\top & x_i\mathbf{X}_i{}^\top & \mathbf{0}^\top \end{bmatrix}}_{\text{rank-2 matrix}} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \end{bmatrix} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\mathtt{A}_i\mathtt{p} = 0$$

Direct Linear Transform (DLT)

$$\mathrm{x}_i = \mathrm{P}\mathrm{X}_i \qquad [\mathrm{x}_i]_\times \mathrm{P}\mathrm{X}_i \qquad \mathrm{P} = \begin{bmatrix} \mathrm{P}^{1\top} \\ \mathrm{P}^{2\top} \\ \mathrm{P}^{3\top} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \\ -y_i\mathbf{X}_i{}^\top & x_i\mathbf{X}_i{}^\top & \mathbf{0}^\top \end{bmatrix}}_{\text{rank-2 matrix}} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \end{bmatrix} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\mathtt{A}_i\mathtt{p} = \mathtt{0}$$

$$\mathbf{A}\mathtt{p} = \mathtt{0} \qquad \mathbf{A} = \begin{bmatrix} \mathtt{A}_1 \\ \vdots \\ \mathtt{A}_n \end{bmatrix}$$

Direct Linear Transform (DLT)

$$\mathrm{A}\mathrm{p} = 0$$

Minimal solution

P has 11 dof, 2 independent eq./points

$\Rightarrow$ 5½ correspondences needed (say 6)

Over-determined solution

$n \geq 6$ points

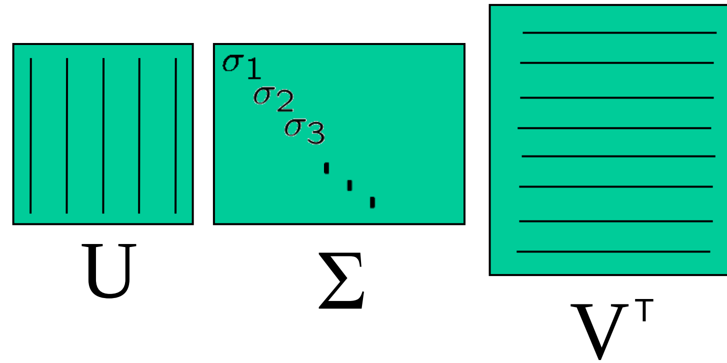
minimize $\|\mathrm{A}\mathrm{p}\|$ subject to constraint

$\|\mathrm{p}\| = 1$

$\longrightarrow$ **use SVD**

Singular Value Decomposition

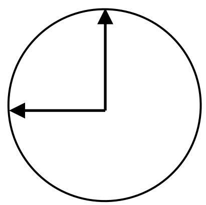
$$A = U\Sigma V^{T}$$

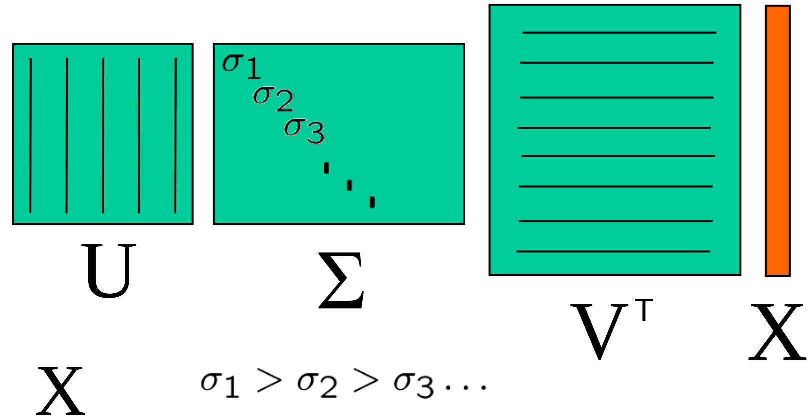


$\sigma_1 > \sigma_2 > \sigma_3 \ldots$

Singular Value Decomposition

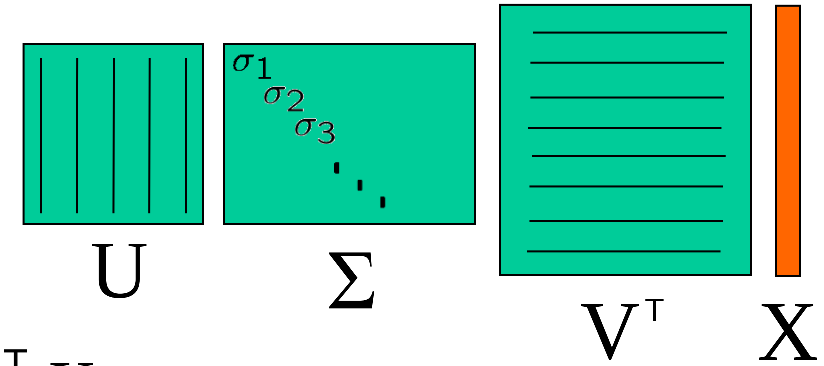
$$A = U \Sigma V^T$$

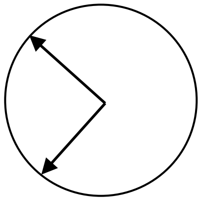




Singular Value Decomposition

$$A = U \Sigma V^T$$



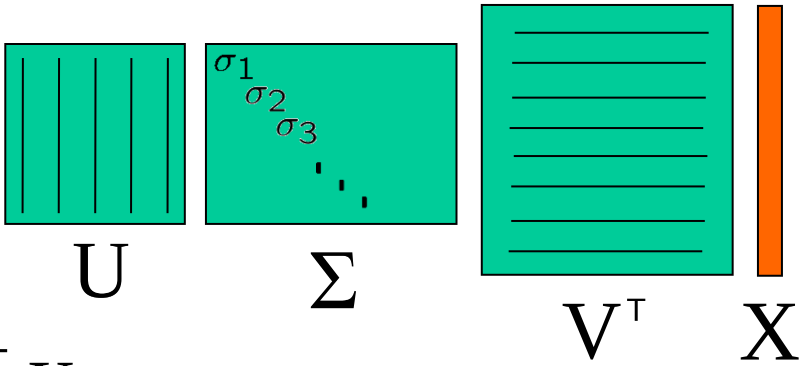


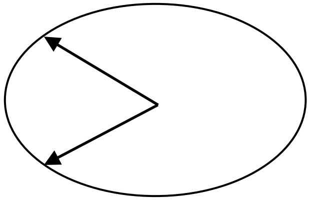
$V^T X$

$\sigma_1 > \sigma_2 > \sigma_3 \ldots$

Singular Value Decomposition

$$A = U\Sigma V^T$$



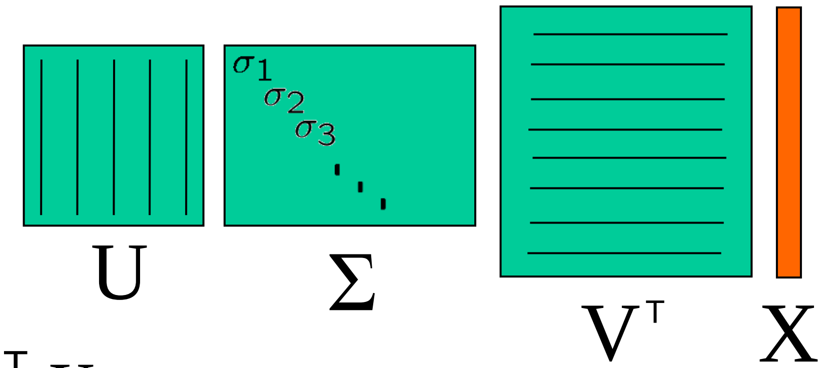


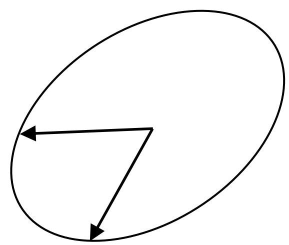
$$\Sigma V^T X$$

$$\sigma_1 > \sigma_2 > \sigma_3 \ldots$$

Singular Value Decomposition

$$A = U\Sigma V^T$$



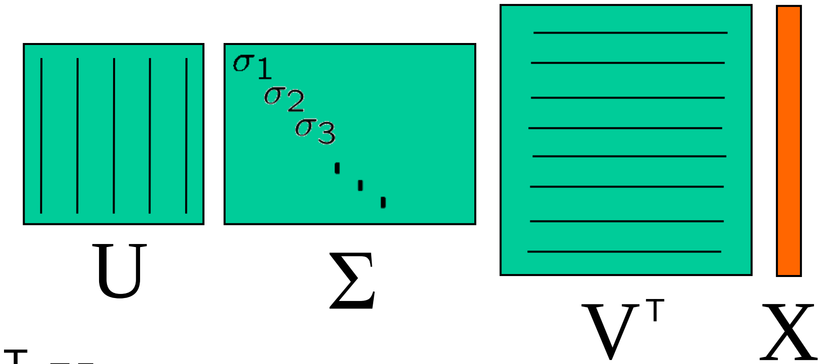


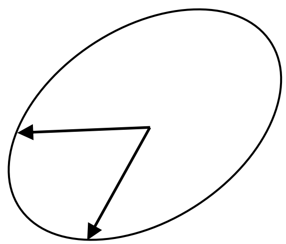
$$U\Sigma V^T X$$

$$\sigma_1 > \sigma_2 > \sigma_3 \ldots$$

Singular Value Decomposition

$$A = U\Sigma V^{T}$$





$$U\Sigma V^{T} X$$

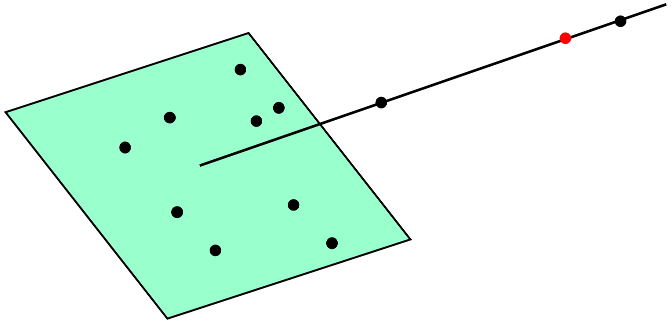
$$\sigma_1 > \sigma_2 > \sigma_3 \ldots$$

Homogeneous least-squares

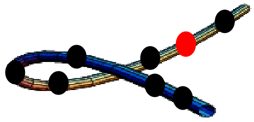
$$\min \|AX\| \text{ subject to } \|X\| = 1 \qquad \text{solution } X = V_n$$

Degenerate configurations

(i) Points lie on plane or single line passing through projection center



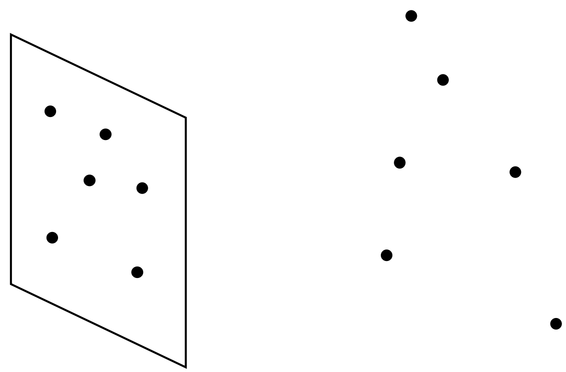
(ii) Camera and points on a twisted cubic



Data normalization

Scale data to values of order 1

1. **move center of mass to origin**
2. **scale to yield order 1 values**



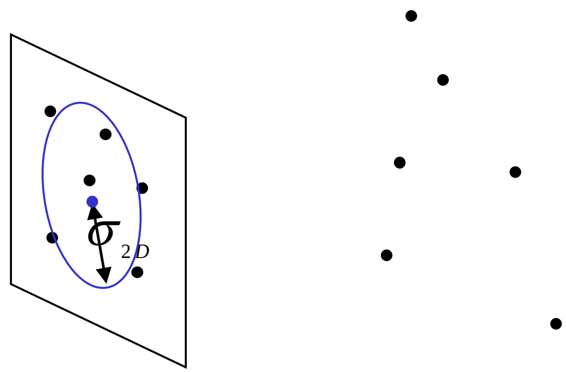
$$\tilde{\mathtt{x}} = \mathtt{T}\mathtt{x} \qquad \tilde{\mathtt{X}} = \mathtt{U}\mathtt{X}$$

$$\mathtt{T} = \begin{bmatrix} \sigma_{2D} & 0 & \bar{x} \\ 0 & \sigma_{2D} & \bar{y} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \qquad \mathtt{U} = \begin{bmatrix} \sigma_{3D} & 0 & 0 & \bar{X} \\ 0 & \sigma_{3D} & 0 & \bar{Y} \\ 0 & 0 & \sigma_{3D} & \bar{Z} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

Data normalization

Scale data to values of order 1

1. **move center of mass to origin**
2. **scale to yield order 1 values**



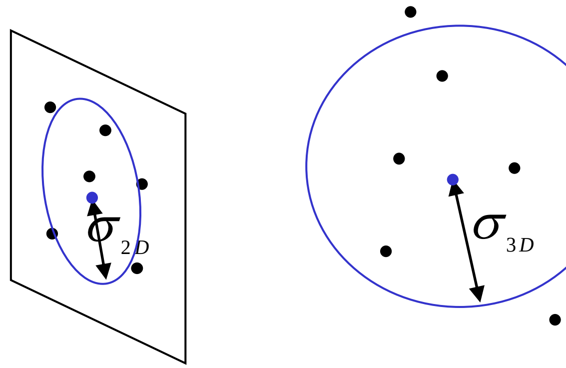
$$\tilde{\mathtt{x}} = \mathtt{T}\mathtt{x} \qquad \tilde{\mathtt{X}} = \mathtt{U}\mathtt{X}$$

$$\mathtt{T} = \begin{bmatrix} \sigma_{2D} & 0 & \bar{x} \\ 0 & \sigma_{2D} & \bar{y} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \qquad \mathtt{U} = \begin{bmatrix} \sigma_{3D} & 0 & 0 & \bar{X} \\ 0 & \sigma_{3D} & 0 & \bar{Y} \\ 0 & 0 & \sigma_{3D} & \bar{Z} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

Data normalization

Scale data to values of order 1

1. **move center of mass to origin**
2. **scale to yield order 1 values**



$$\tilde{\mathtt{x}} = \mathtt{T}\mathtt{x} \qquad \tilde{\mathtt{X}} = \mathtt{U}\mathtt{X}$$

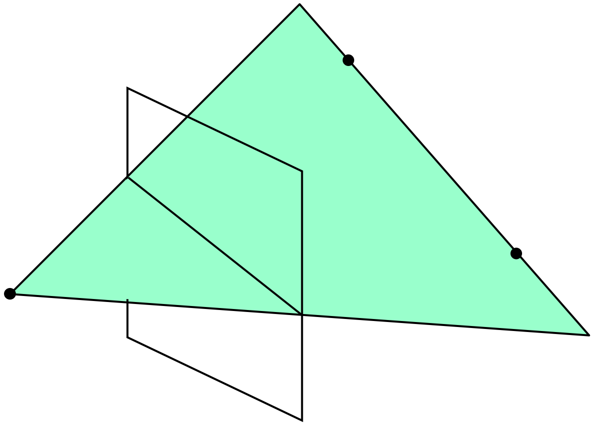
$$\mathtt{T} = \begin{bmatrix} \sigma_{2D} & 0 & \bar{x} \\ 0 & \sigma_{2D} & \bar{y} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \qquad \mathtt{U} = \begin{bmatrix} \sigma_{3D} & 0 & 0 & \bar{X} \\ 0 & \sigma_{3D} & 0 & \bar{Y} \\ 0 & 0 & \sigma_{3D} & \bar{Z} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

Line correspondences

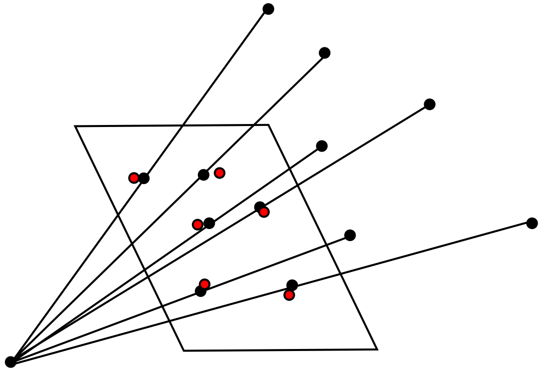
Extend DLT to lines

$$\Pi = \mathrm{P}^{\mathsf{T}}\mathrm{l}_i \qquad \text{(back-project line)}$$

$$\mathrm{l}_i^{\mathsf{T}}\mathrm{P}\mathrm{X}_{1i} \qquad \mathrm{l}_i^{\mathsf{T}}\mathrm{P}\mathrm{X}_{2i} \qquad \text{(2 independent eq.)}$$



Geometric error



$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2$$

$$\min_{\mathrm{P}} \sum_i d(\mathbf{x}_i, \mathrm{P}\mathbf{X}_i)^2$$

Gold Standard algorithm

Objective

Given n≥6 2D to 3D point correspondences $\{X_i \leftrightarrow x_i'\}$, determine the Maximum Likelyhood Estimation of P

Algorithm

(i) Linear solution:

(a) Normalization: $\tilde{X}_i = UX_i \quad \tilde{x}_i = Tx_i$

(b) DLT

(ii) Minimization of geometric error: using the linear estimate as a starting point minimize the geometric error:

$$\min_{\mathrm{P}} \sum_i d(\tilde{\mathbf{x}}_i, \tilde{\mathrm{P}}\tilde{\mathbf{X}}_i)^2$$

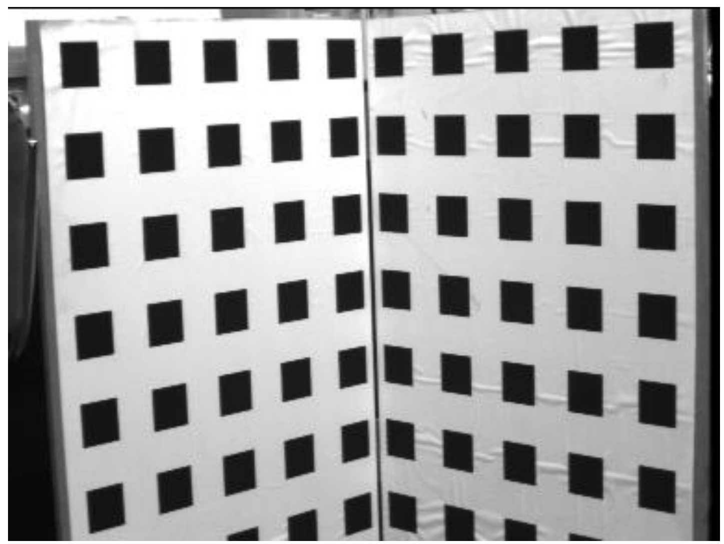
(iii) Denormalization: $P = T^{-1}\tilde{P}U$

Calibration example

(i) Canny edge detection
(ii) Straight line fitting to the detected edges
(iii) Intersecting the lines to obtain the images corners

typically precision <1/10

(H&Z rule of thumb: 5*n* constraints for *n* unknowns)



	f_y	f_x/f_y	skew	x_0	y_0	residual
linear	1673.3	1.0063	1.39	379.96	305.78	0.365
iterative	1675.5	1.0063	1.43	379.79	305.25	0.364

Errors in the image (standard case)

$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2$$

$$\hat{\mathrm{x}}_i = \mathrm{P}\mathrm{X}_\mathrm{i}$$

Errors in the world

$$\sum_i d(\mathbf{X}_i, \widehat{\mathbf{X}}_i)^2$$

$$\mathrm{x}_i = \mathrm{P}\widehat{\mathrm{X}}_i$$

Errors in the image and in the world

$$\sum_{i=1}^{n} d_{\mathrm{Mah}}(\mathbf{x}_i, \mathrm{P}\widehat{\mathbf{X}}_i)^2 + d_{\mathrm{Mah}}(\mathbf{X}_i, \widehat{\mathbf{X}}_i)^2$$

$$\widehat{\mathrm{X}}_i$$

25/09/19

Restricted camera estimation

Find best fit that satisfies

- skew s is zero
- pixels are square
- principal point is known
- complete camera matrix K is known

$$\mathrm{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

Minimize geometric error
$\rightarrow$impose constraint through parametrization

Minimize algebraic error
$\rightarrow$assume map from param q $\rightarrow$ P=K[R|-RC], i.e. p=g(q)
$\rightarrow$minimize ||Ag(q)||

Restricted camera estimation

$$\mathtt{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

Initialization

- Use general DLT
- Clamp values to desired values, e.g. s=0, $\alpha_x = \alpha_y$

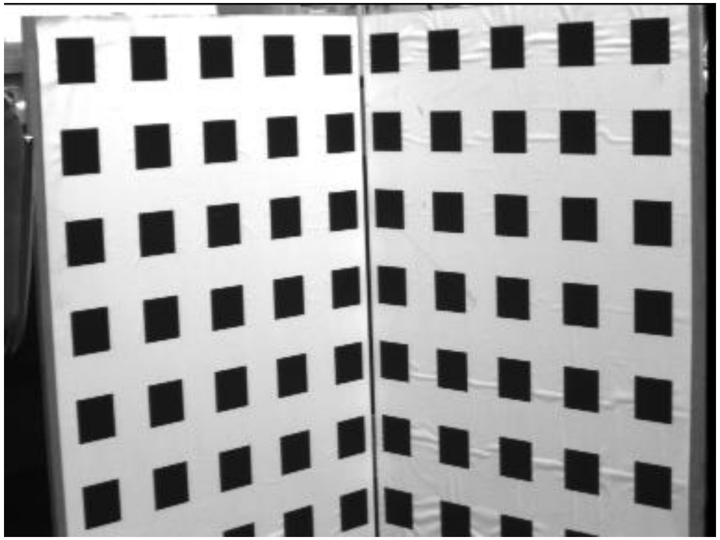
Note: can sometimes cause big jump in error

Alternative initialization

- Use general DLT
- Impose soft constraints

$$\sum_i d(\mathbf{x}_i, \mathrm{P}\mathbf{X}_i)^2 + ws^2 + w(\alpha_x - \alpha_y)^2$$

- gradually increase weights



	f_y	f_x/f_y	skew	x_0	y_0	residual
algebraic	1633.4	1.0	0.0	371.21	293.63	0.601
geometric	1637.2	1.0	0.0	371.32	293.69	0.601

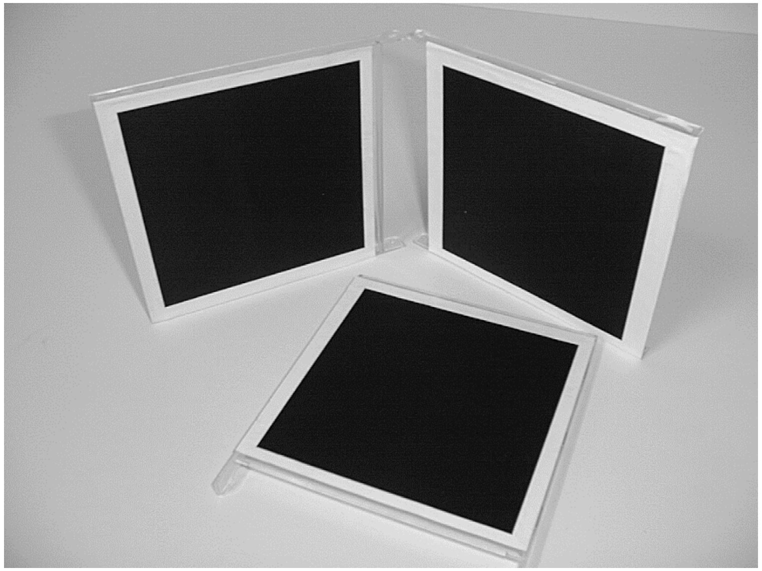
	f_y	f_x/f_y	skew	x_0	y_0	residual
linear	1673.3	1.0063	1.39	379.96	305.78	0.365
iterative	1675.5	1.0063	1.43	379.79	305.25	0.364

Image of absolute conic

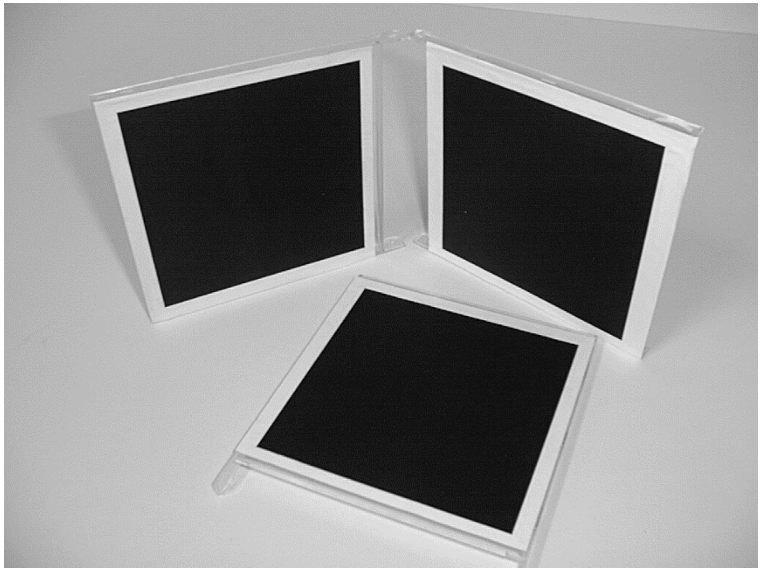
$$
\begin{aligned}
\omega^* &= \mathbf{P}\Omega^*\mathbf{P}^\top \\
&= \mathbf{KR}\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}\mathbf{R}^\top\mathbf{K}^\top \\
&= \mathbf{KK}^\top
\end{aligned}
$$

$$\omega = \mathbf{K}^{-1}\mathbf{K}^{-\top}$$

A simple calibration device

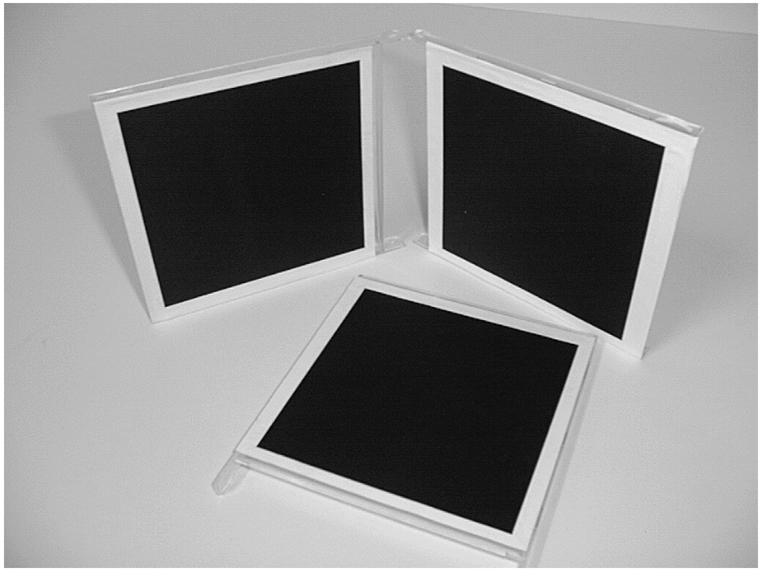


A simple calibration device



(i) compute H for each square
(corners $\rightarrow$ (0,0),(1,0),(0,1),(1,1))
(ii) compute the imaged circular points $H(1,\pm i,0)^T$
(iii) fit a conic to 6 circular points
(iv) compute K from ω through cholesky factorization

A simple calibration device



(i) compute H for each square (corners $\rightarrow$ (0,0),(1,0),(0,1),(1,1))

(ii) compute the imaged circular points $H(1,\pm i,0)^T$

(iii) fit a conic to 6 circular points

(iv) compute K from ω through cholesky factorization

(≈ Zhang's calibration method)

Some typical calibration algorithms

Tsai calibration

Tsai, Roger Y. (1986) “An Efficient and Accurate Camera Calibration Technique for 3D Machine Vision,” *Proceedings of IEEE Conference on Computer Vision and Pattern Recognition*, Miami Beach, FL, 1986, pp. 364–374.

Tsai, Roger Y. (1987) “A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-Shelf TV Cameras and Lenses,” *IEEE Journal of Robotics and Automation*, Vol. RA–3, No. 4, August 1987, pp. 323–344.

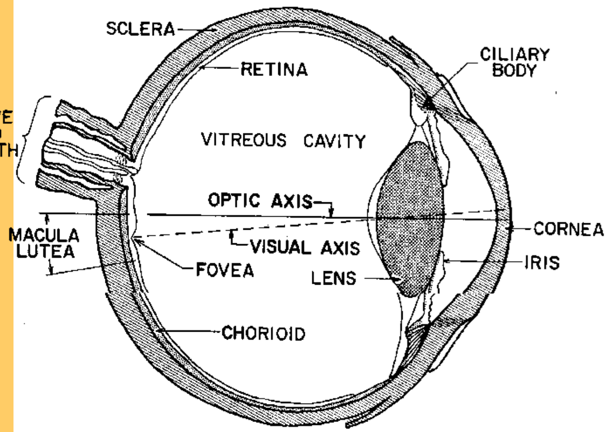
Zhangs calibration

http://research.microsoft.com/~zhang/calib/

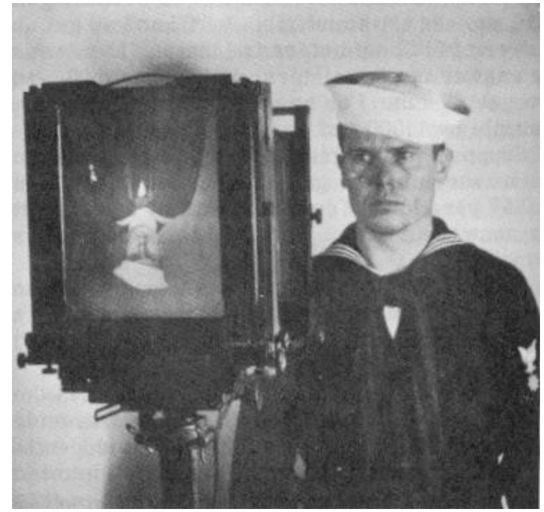
Z. Zhang. A flexible new technique for camera calibration. IEEE Transactions on Pattern Analysis and Machine Intelligence, 22(11):1330-1334, 2000.

Z. Zhang. Flexible Camera Calibration By Viewing a Plane From Unknown Orientations. International Conference on Computer Vision (ICCV'99), Corfu, Greece, pages 666-673, September 1999.

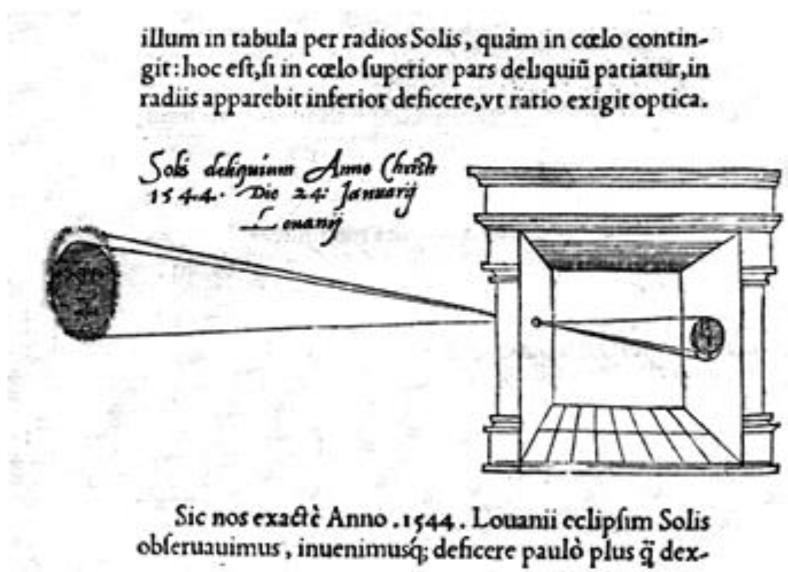
http://www.vision.caltech.edu/bouguetj/calib_doc/



Animal eye:
a looonnng time ago.



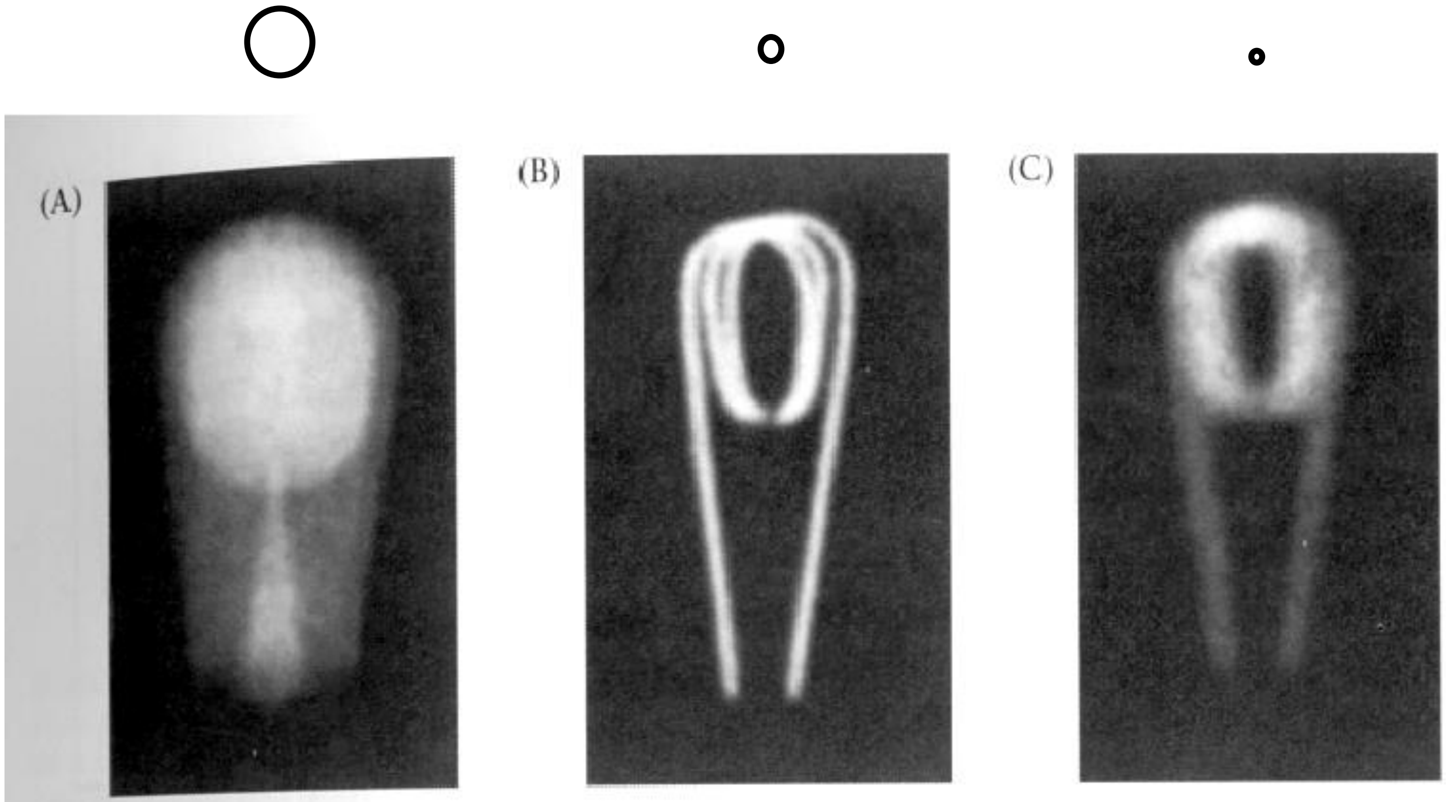
Photographic camera:
Niepce, 1816.


illum in tabula per radios Solis, quàm in cœlo contingit: hoc eſt, ſi in cœlo ſuperior pars deliquiũ patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipſim Solis obſeruauimus, inuenimusq; deficere paulò plus q̃ dex-

Pinhole perspective projection: Brunelleschi, XV^{th} Century.
Camera obscura: XVI^{th} Century.

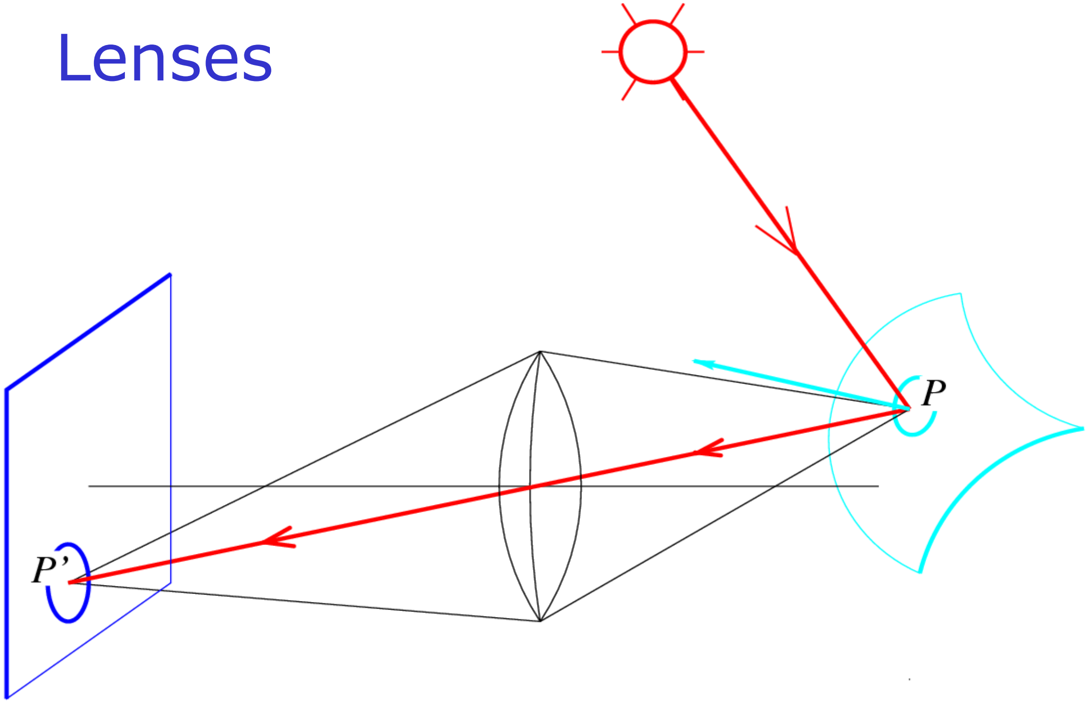
Limits for pinhole cameras



2.18 DIFFRACTION LIMITS THE QUALITY OF PINHOLE OPTICS. These three images of a bulb filament were made using pinholes with decreasing size. (A) When the pinhole is relatively large, the image rays are not properly converged, and the image is blurred. (B) Reducing the size of the pinhole improves the focus. (C) Reducing the size of the pinhole further worsens the focus, due to diffraction. From Ruechardt, 1958.

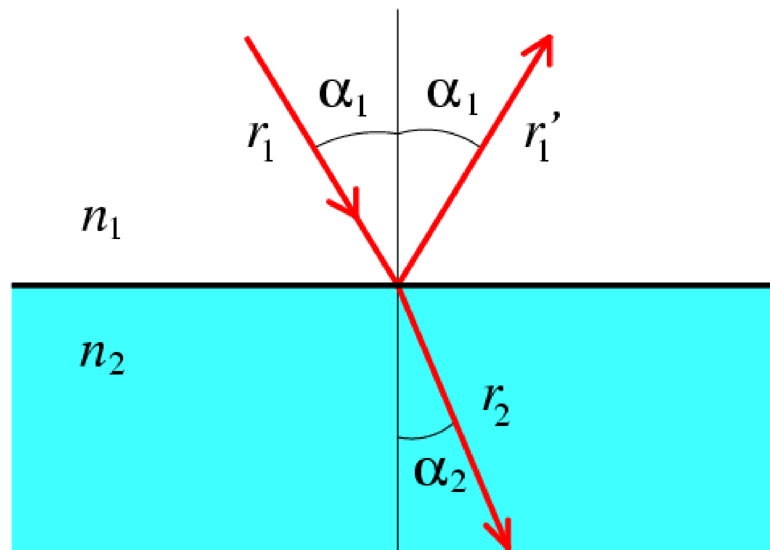
Camera obscura + lens

Lenses

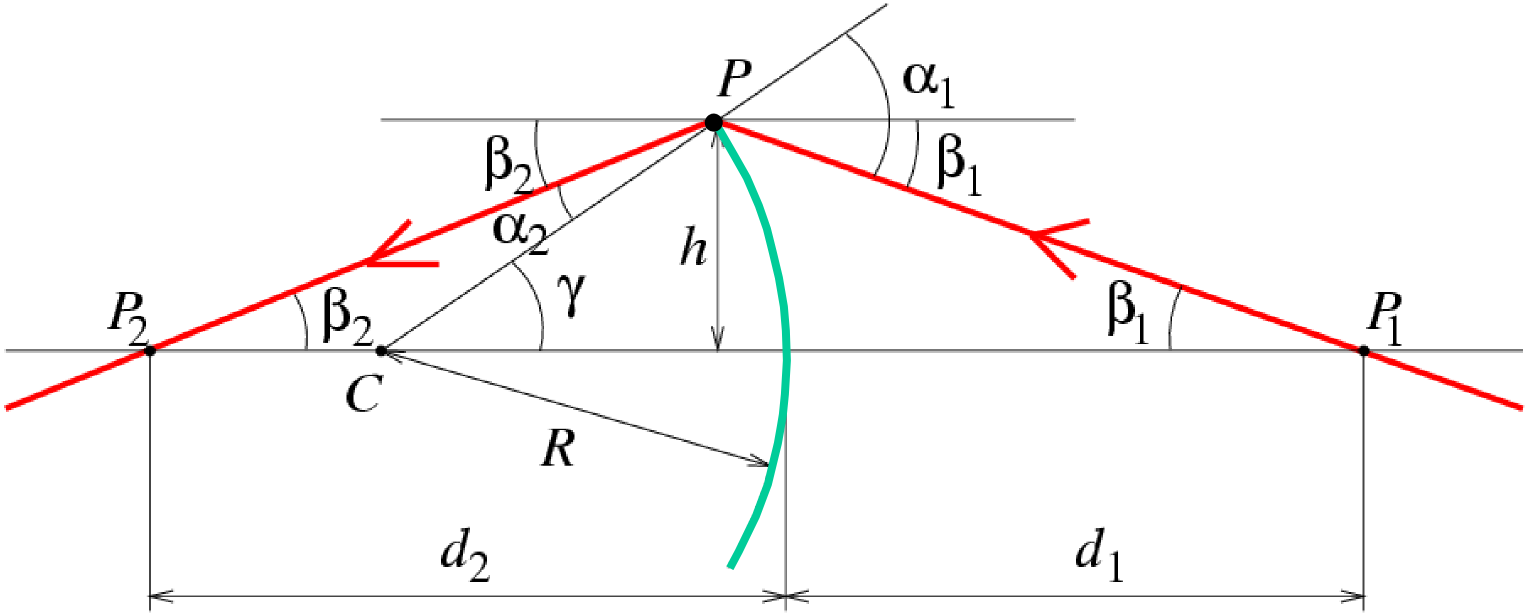


Snell's law

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$



Paraxial (or first-order) optics



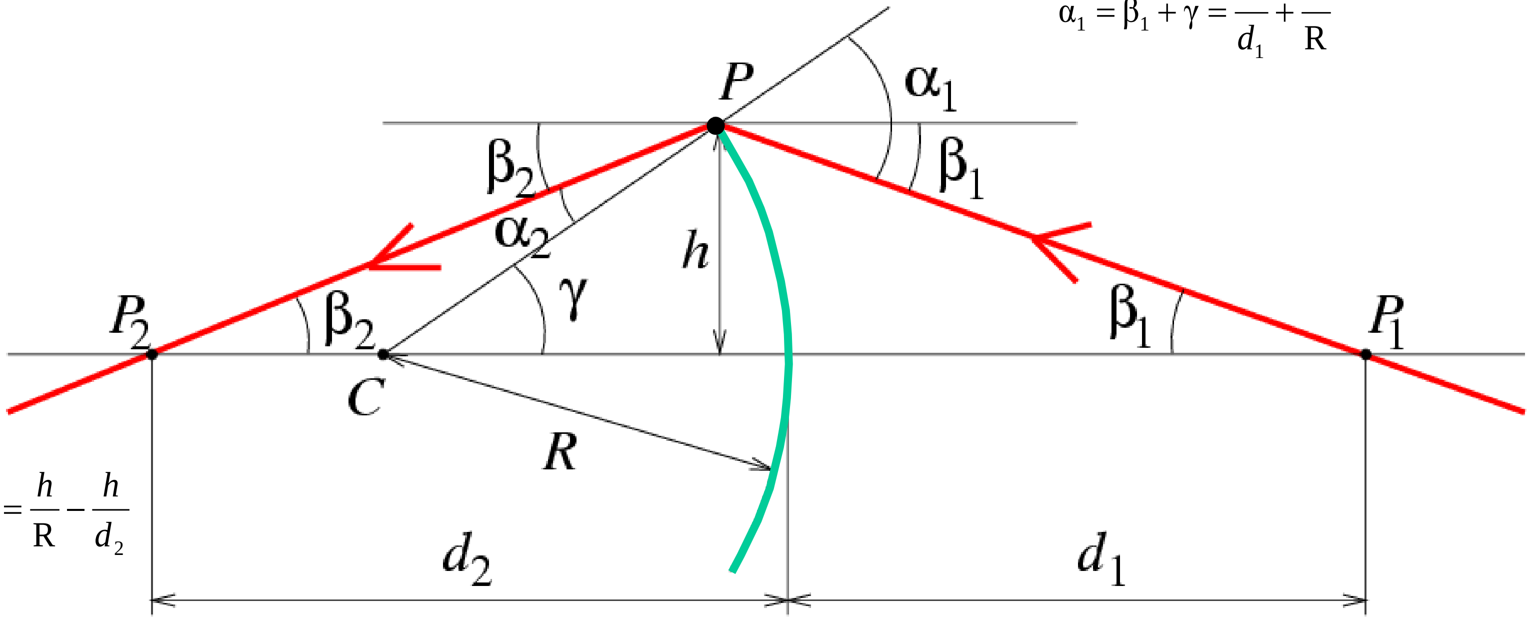
Snell's law:

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

Small angles:

$$n_1 \alpha_1 \approx n_2 \alpha_2$$

Paraxial (or first-order) optics



$$\alpha_1 = \beta_1 + \gamma = \frac{h}{d_1} + \frac{h}{R}$$

$$\alpha_2 = \gamma - \beta_2 = \frac{h}{R} - \frac{h}{d_2}$$

Snell's law:

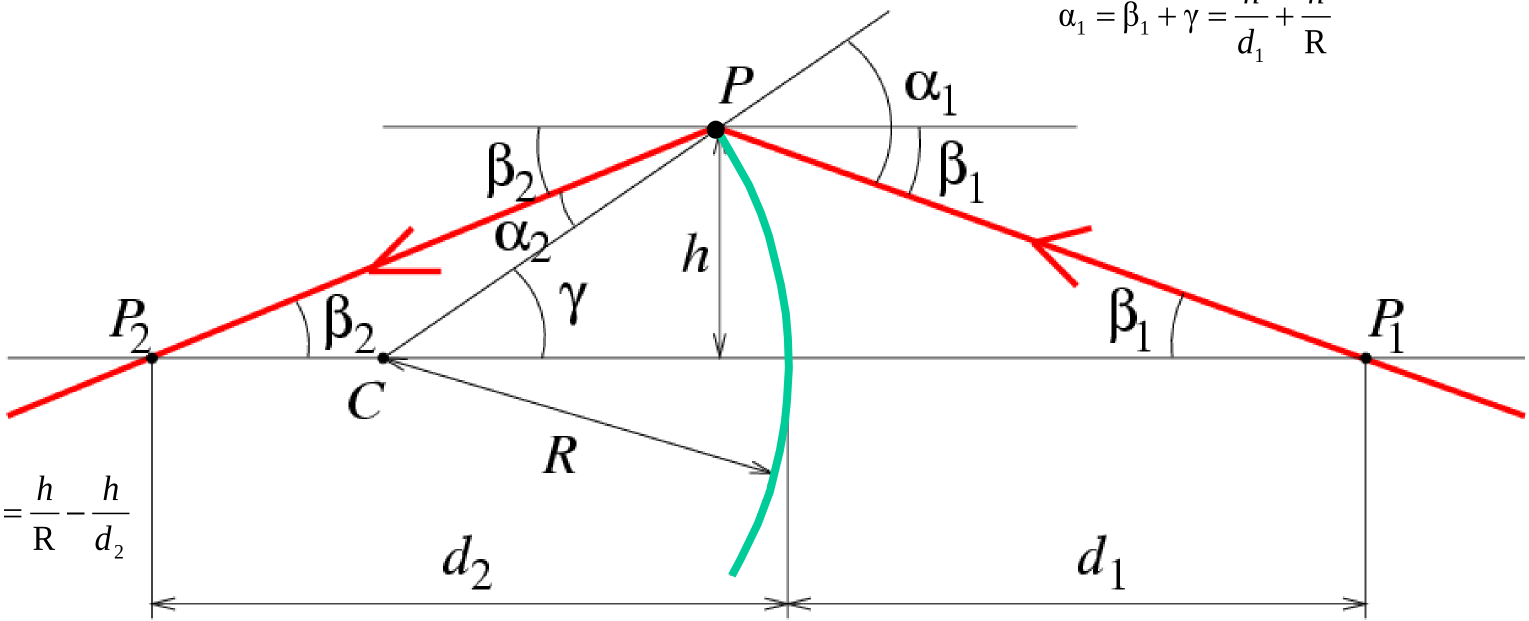
$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

Small angles:

$$n_1 \, \alpha_1 \approx n_2 \alpha_2$$

Paraxial (or first-order) optics

$$\alpha_1 = \beta_1 + \gamma = \frac{h}{d_1} + \frac{h}{R}$$



$$\alpha_2 = \gamma - \beta_2 = \frac{h}{R} - \frac{h}{d_2}$$

Snell's law:

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

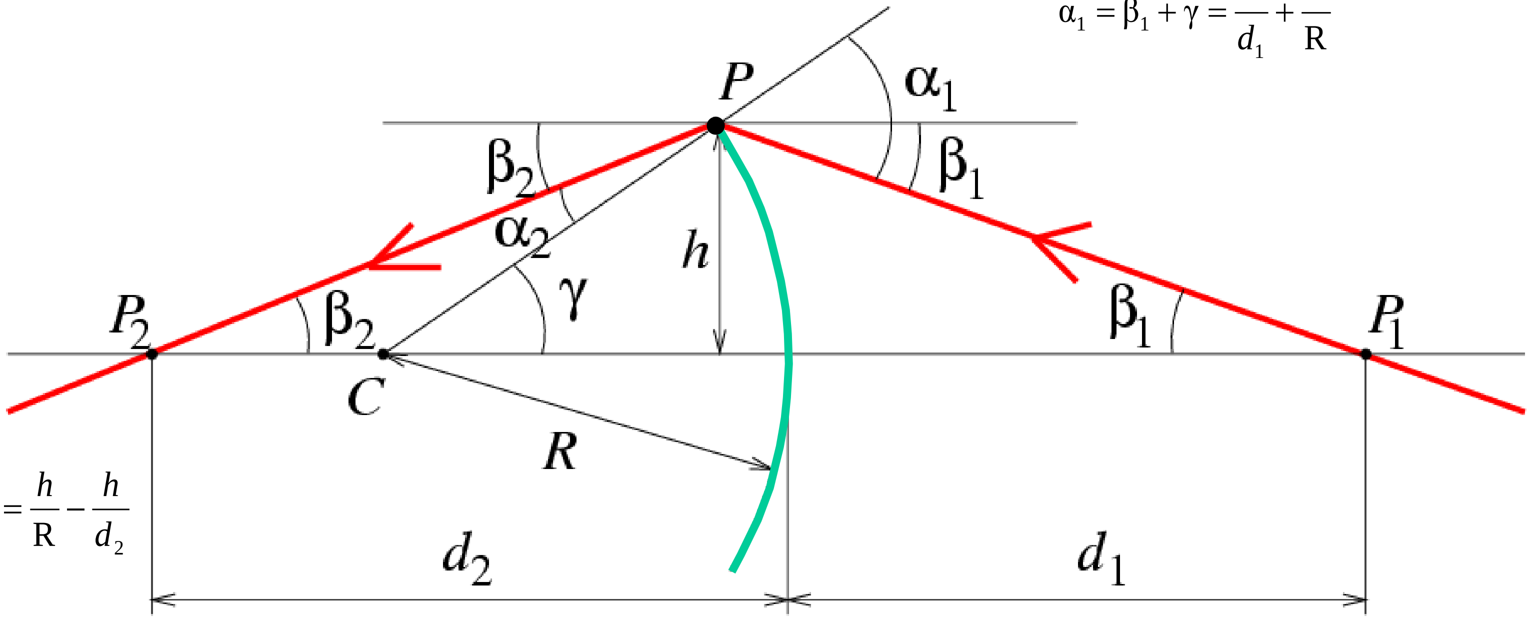
Small angles:

$$n_1\left(\frac{h}{d_1} + \frac{h}{R}\right) = n_2\left(\frac{h}{R} - \frac{h}{d_2}\right)$$

$$n_1 \alpha_1 \approx n_2 \alpha_2$$

Paraxial (or first-order) optics

$$\alpha_1 = \beta_1 + \gamma = \frac{h}{d_1} + \frac{h}{R}$$



$$\alpha_2 = \gamma - \beta_2 = \frac{h}{R} - \frac{h}{d_2}$$

Snell's law:

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

Small angles:

$$n_1 \alpha_1 \approx n_2 \alpha_2$$

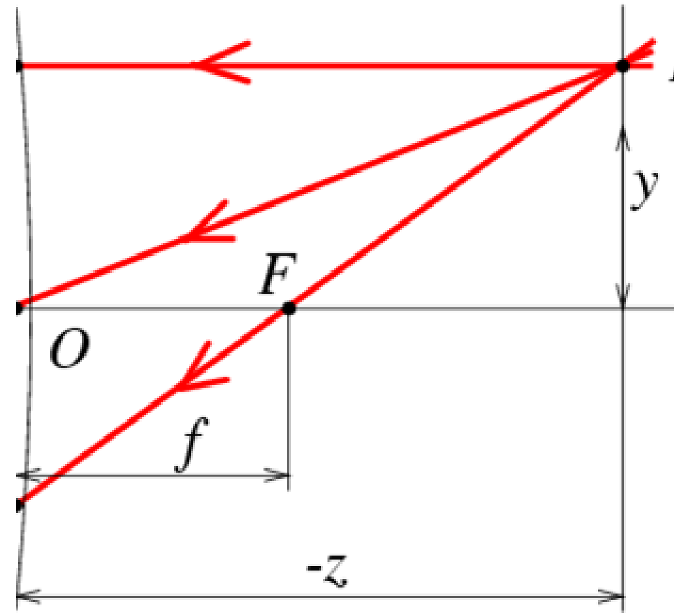
$$n_1\left(\frac{h}{d_1} + \frac{h}{R}\right) = n_2\left(\frac{h}{R} - \frac{h}{d_2}\right)$$

$$\boxed{\frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_2 - n_1}{R}}$$

$$\frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_2 - n_1}{R}$$

Thin Lenses

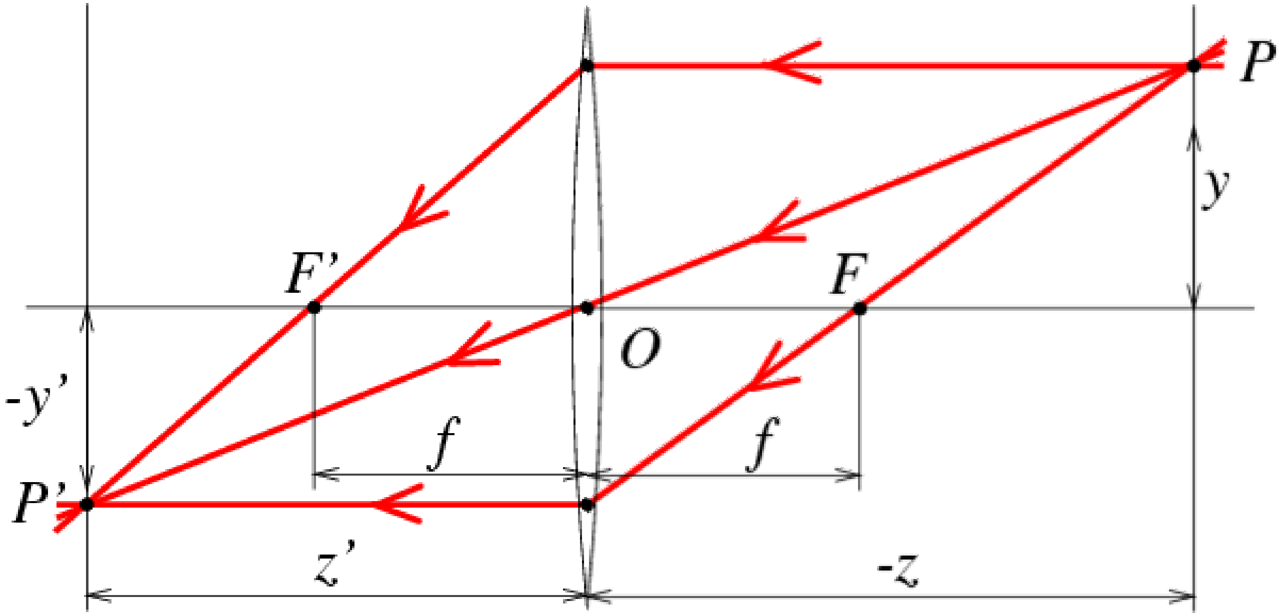
spherical lens surfaces; incoming light ± parallel to axis; thickness << radii; same refractive index on both sides



$$\boxed{\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_2-n_1}{R}}$$

Thin Lenses

spherical lens surfaces; incoming light ± parallel to axis;
thickness << radii; same refractive index on both sides



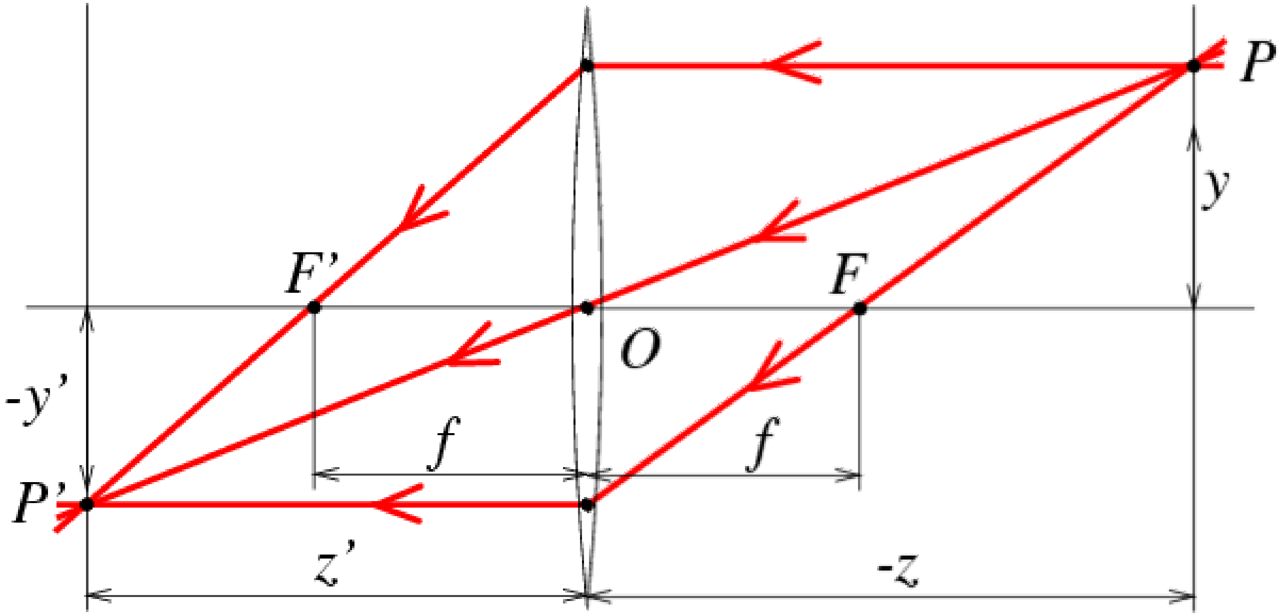
$$\boxed{\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_2-n_1}{R}}$$

$$\frac{1}{Z}+\frac{n}{Z^*}=\frac{n-1}{R}$$

$$\frac{n}{Z^*}+\frac{1}{Z'}=\frac{1-n}{R}$$

Thin Lenses

spherical lens surfaces; incoming light ± parallel to axis; thickness << radii; same refractive index on both sides



$$\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_2-n_1}{R}$$

$$\frac{1}{Z}+\frac{n}{Z^*}=\frac{n-1}{R}$$

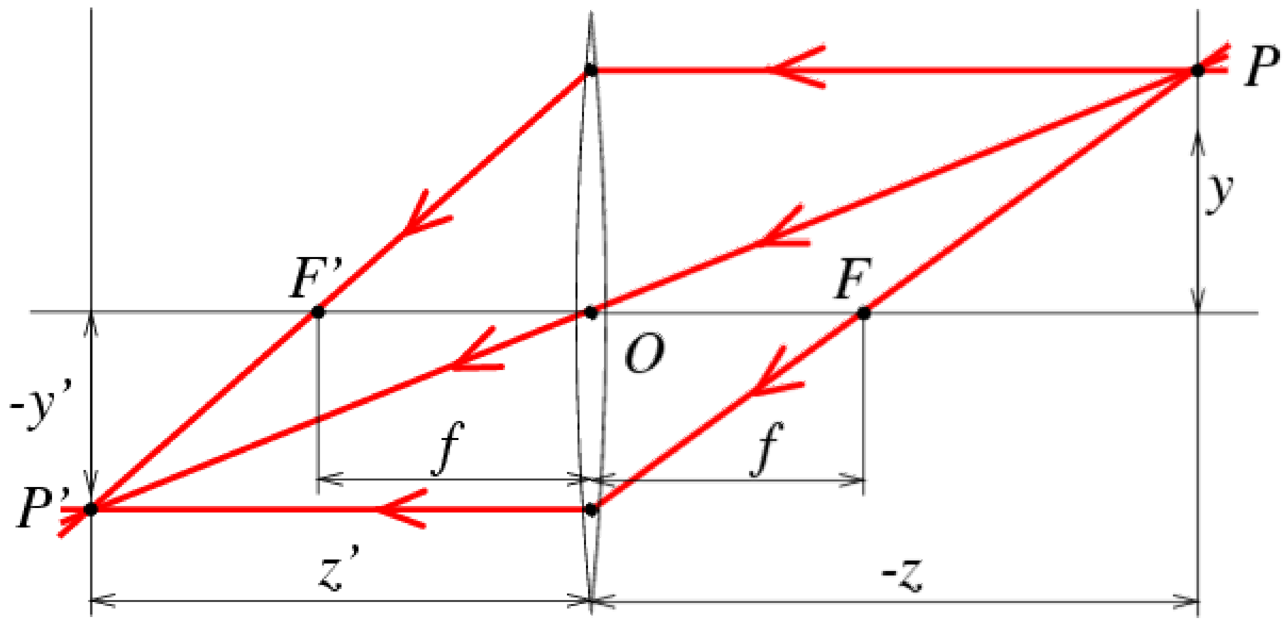
$$\frac{n}{Z^*}+\frac{1}{Z'}=\frac{1-n}{R}$$

$$\frac{n}{Z^*}=\frac{n-1}{R}-\frac{1}{Z}$$

$$\frac{n}{Z^*}=\frac{1-n}{R}-\frac{1}{Z'}$$

Thin Lenses

spherical lens surfaces; incoming light ± parallel to axis;
thickness << radii; same refractive index on both sides



$$\boxed{\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_2-n_1}{R}}$$

$$\frac{1}{Z}+\frac{n}{Z^*}=\frac{n-1}{R}$$

$$\frac{n}{Z^*}+\frac{1}{Z'}=\frac{1-n}{R}$$

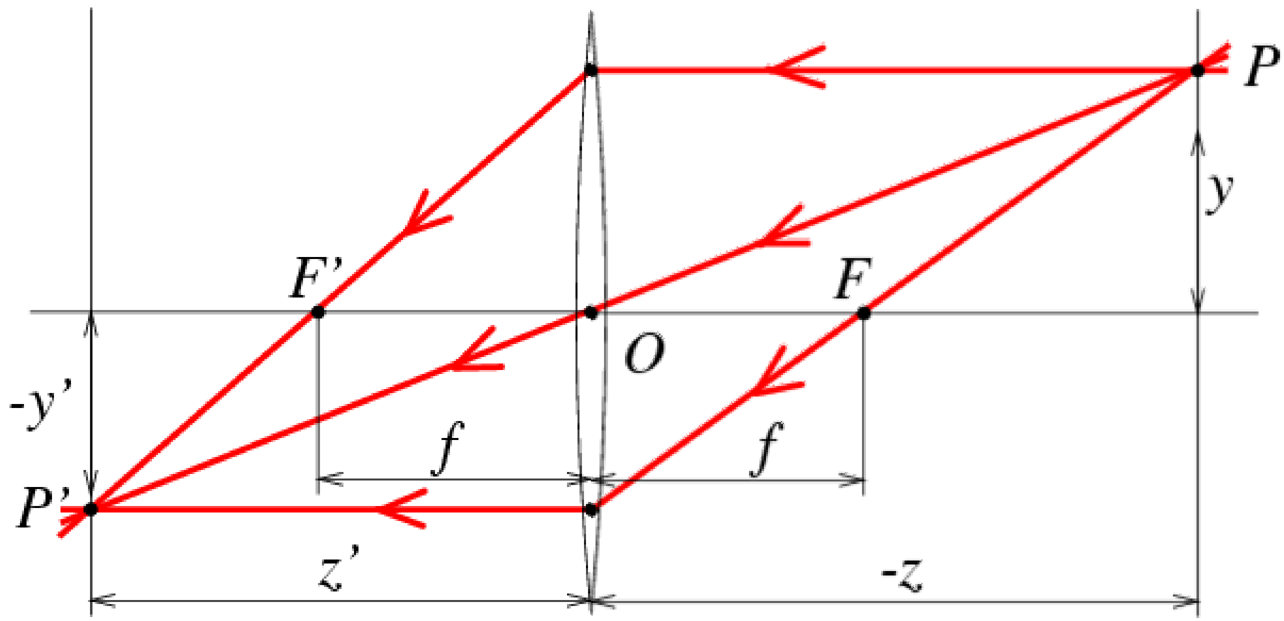
$$\frac{n}{Z^*}=\frac{n-1}{R}-\frac{1}{Z}$$

$$\frac{n}{Z^*}=\frac{1-n}{R}-\frac{1}{Z'}$$

$$\frac{n-1}{R}-\frac{1-n}{R}=\frac{1}{Z}-\frac{1}{Z'}$$

Thin Lenses

spherical lens surfaces; incoming light ± parallel to axis;
thickness << radii; same refractive index on both sides



$$\boxed{\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_2-n_1}{R}}$$

$$\frac{1}{Z}+\frac{n}{Z^*}=\frac{n-1}{R}$$

$$\frac{n}{Z^*}+\frac{1}{Z'}=\frac{1-n}{R}$$

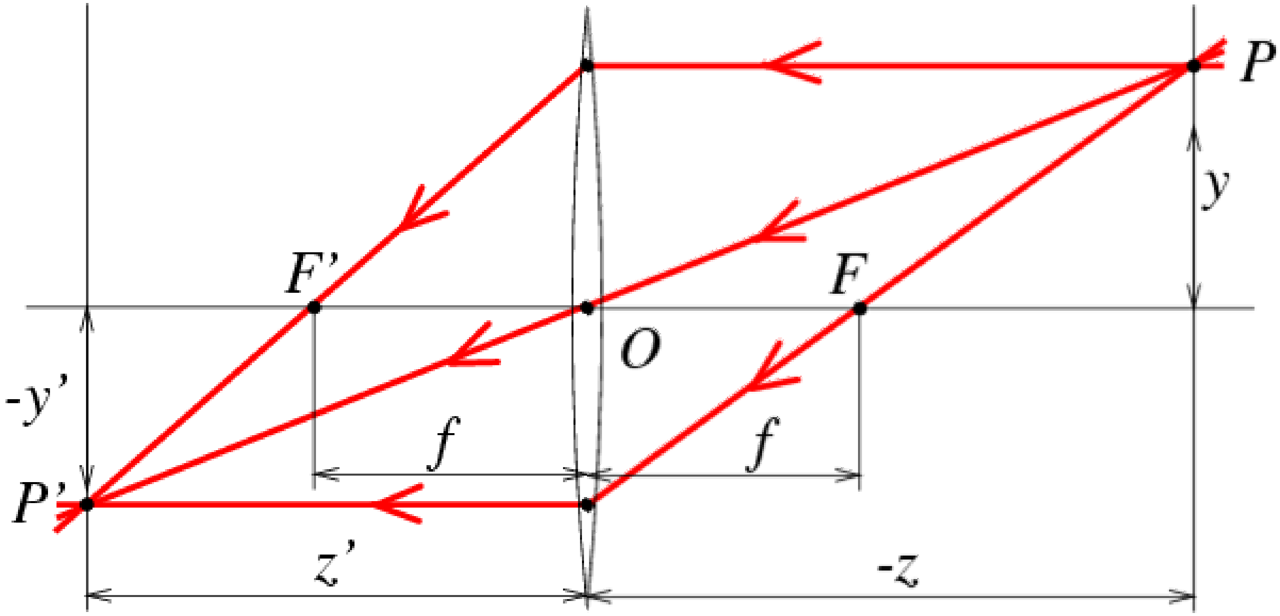
$$\frac{n}{Z^*}=\frac{n-1}{R}-\frac{1}{Z}$$

$$\frac{n}{Z^*}=\frac{1-n}{R}-\frac{1}{Z'}$$

$$\frac{n-1}{R}-\frac{1-n}{R}=\frac{1}{Z}-\frac{1}{Z'}$$

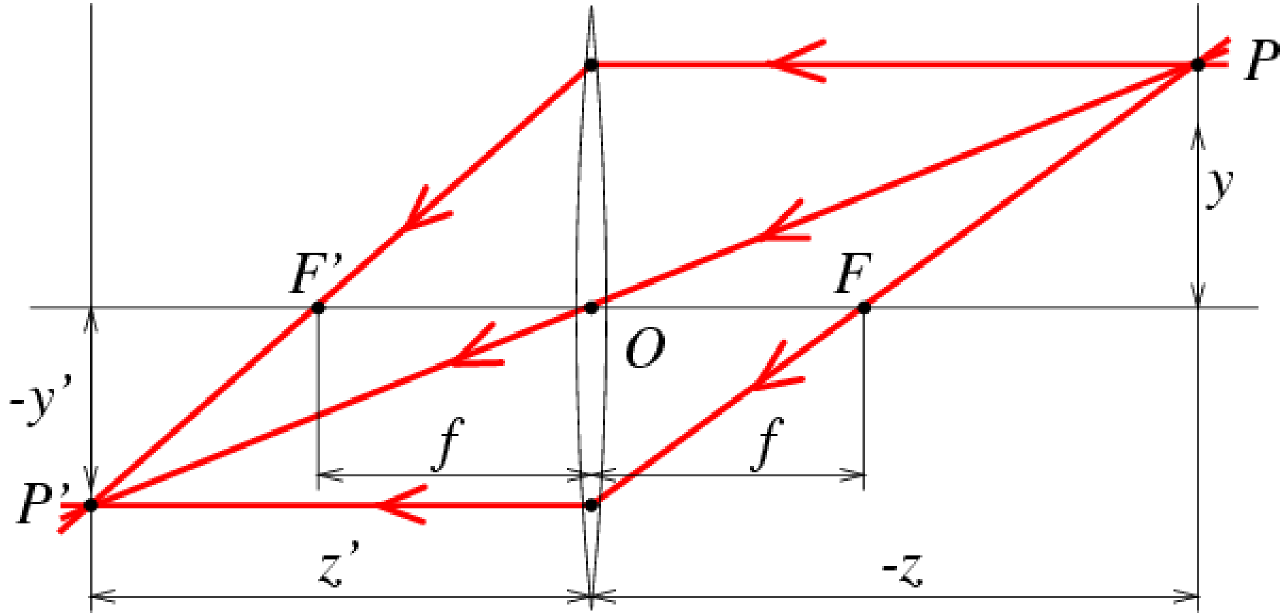
Thin Lenses

spherical lens surfaces; incoming light ± parallel to axis;
thickness << radii; same refractive index on both sides



$$\boxed{\frac{1}{z'}-\frac{1}{z}=\frac{1}{f}}$$ and $$\boxed{f=\frac{R}{2(n-1)}}$$

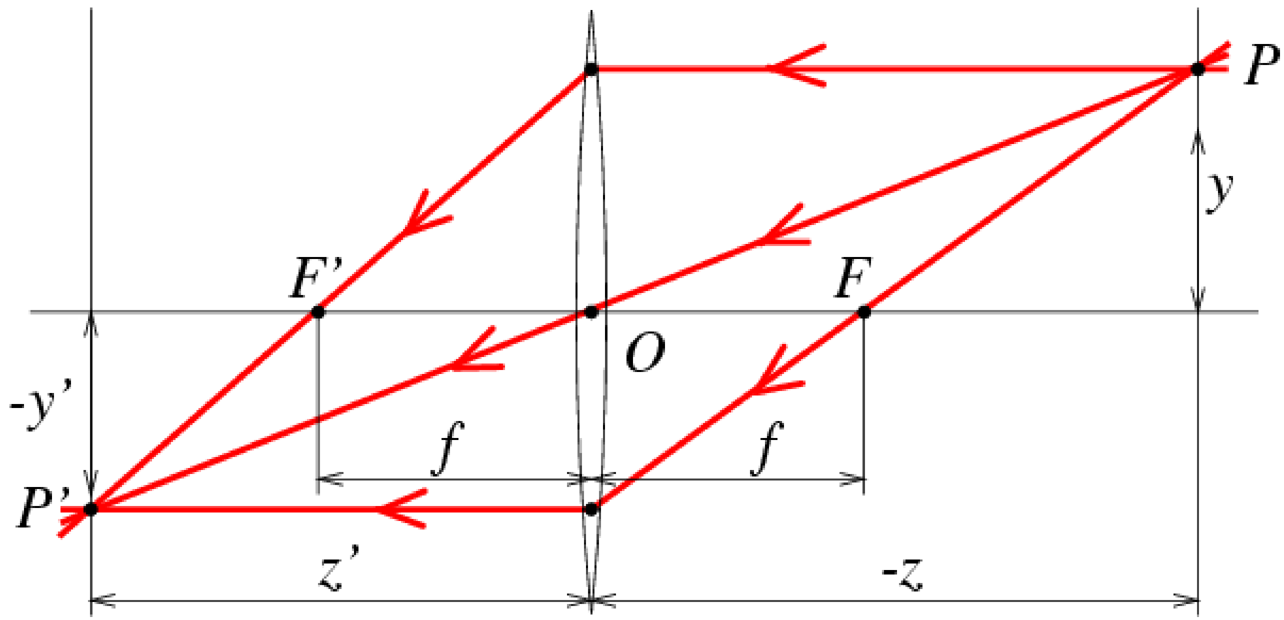
Thin Lenses



$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases} \quad \text{where} \quad \frac{1}{z'} - \frac{1}{z} = \frac{1}{f} \quad \text{and} \quad f = \frac{R}{2(n-1)}$$

http://www.phy.ntnu.edu.tw/java/Lens/lens_e.html

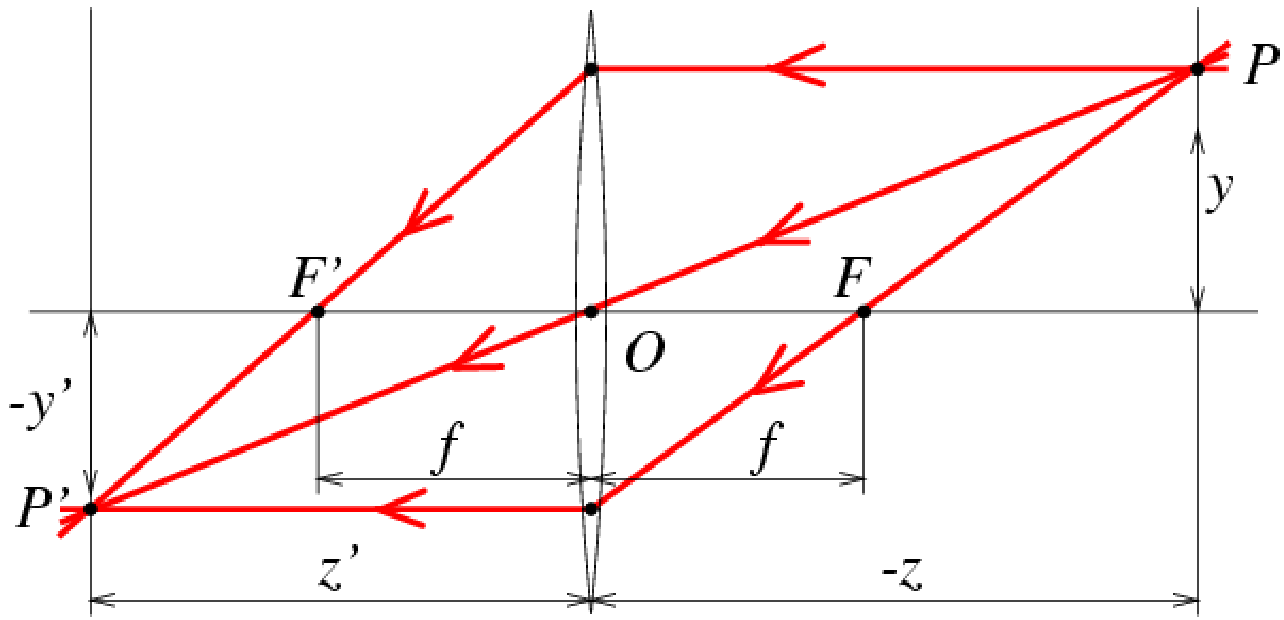
Thin Lenses



$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases}$$ where $$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}$$ and $$f = \frac{R}{2(n-1)}$$

http://www.phy.ntnu.edu.tw/java/Lens/lens_e.html

Thin Lenses

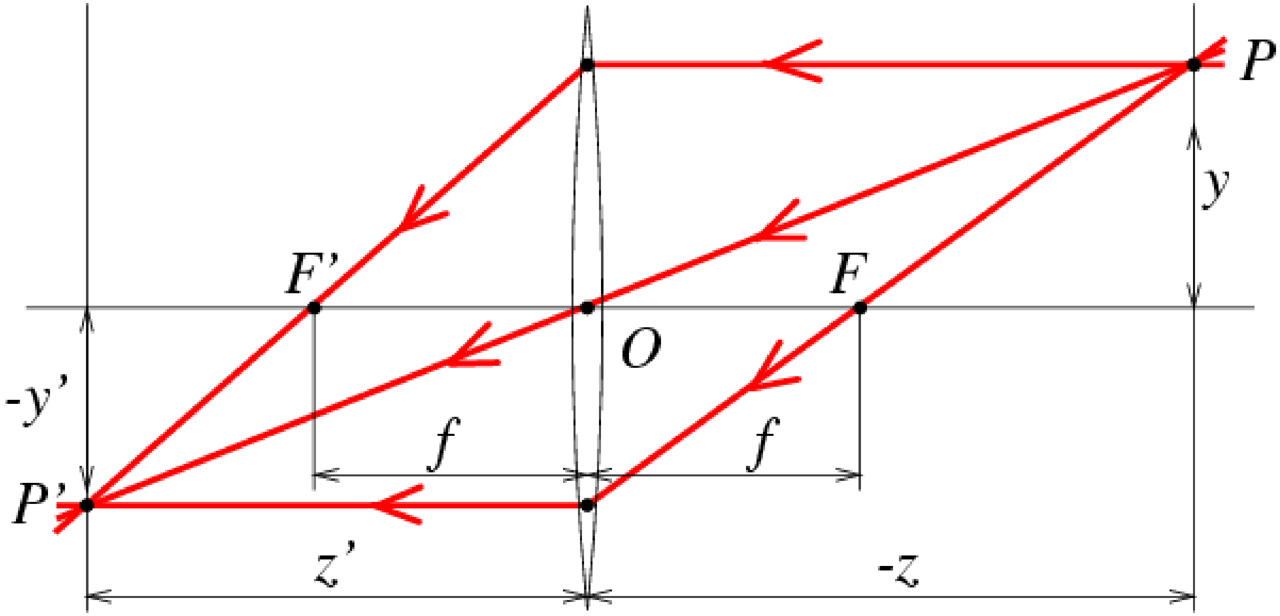


$$\begin{cases} x' = z'\dfrac{x}{z} \\ y' = z'\dfrac{y}{z} \end{cases}$$

where $\boxed{\dfrac{1}{z'} - \dfrac{1}{z} = \dfrac{1}{f}}$ and $f = \dfrac{R}{2(n-1)}$

http://www.phy.ntnu.edu.tw/java/Lens/lens_e.html

Thin Lenses

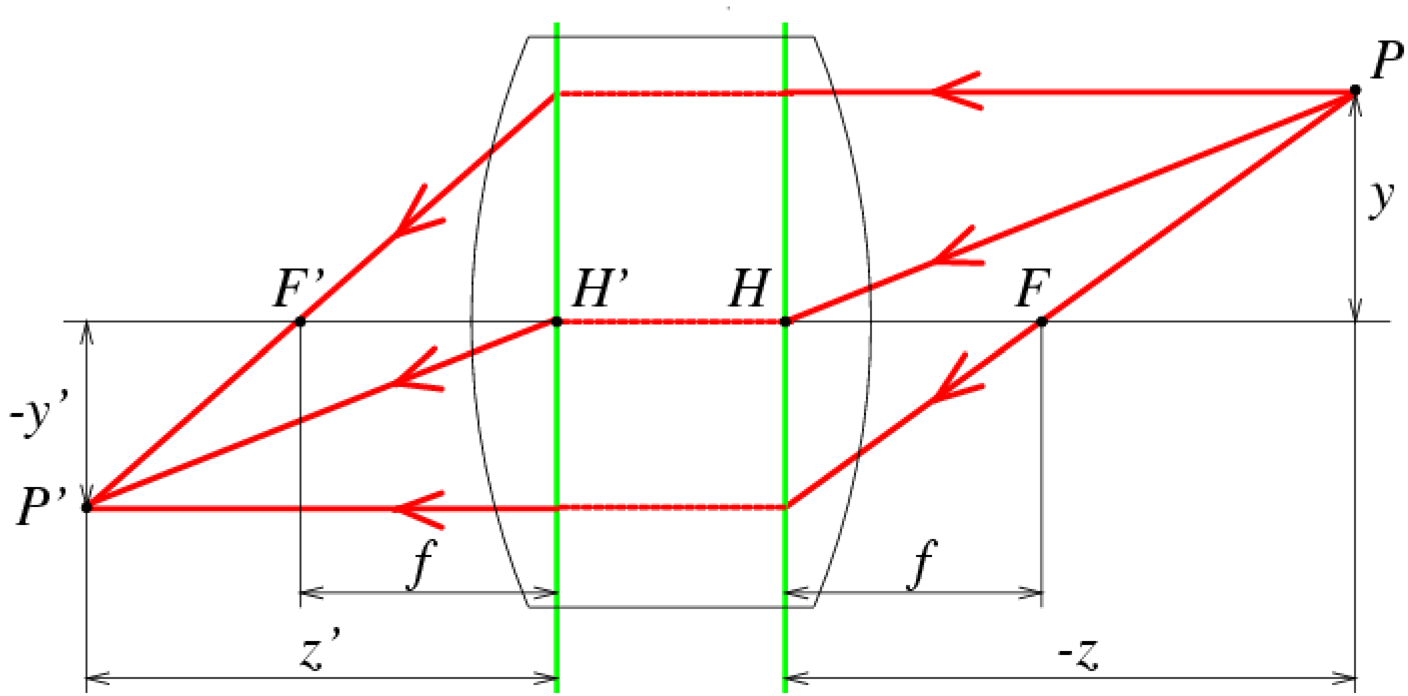


$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases}$$

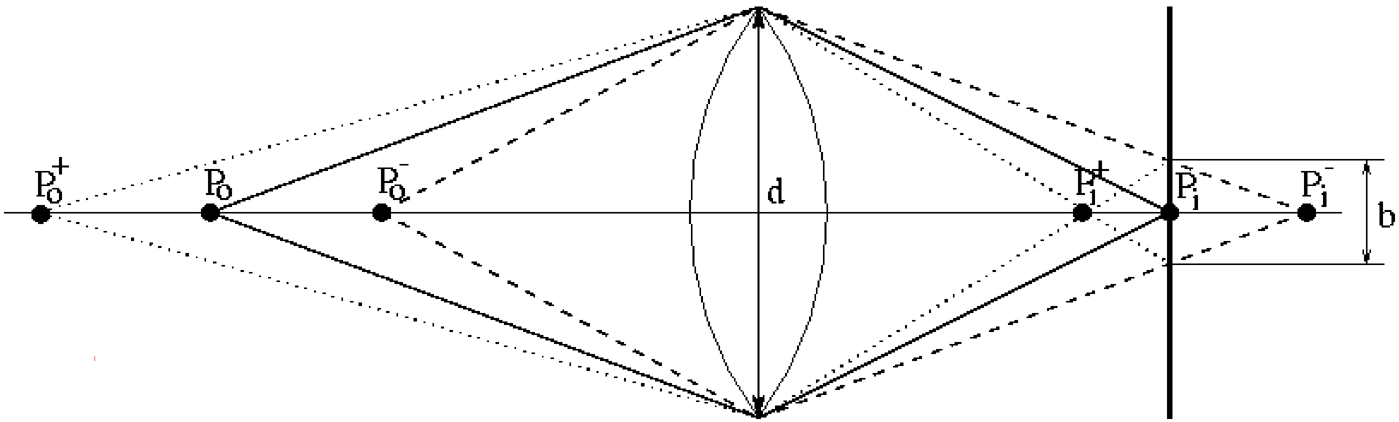
where $\dfrac{1}{z'} - \dfrac{1}{z} = \dfrac{1}{f}$ and $f = \dfrac{R}{2(n-1)}$

http://www.phy.ntnu.edu.tw/java/Lens/lens_e.html

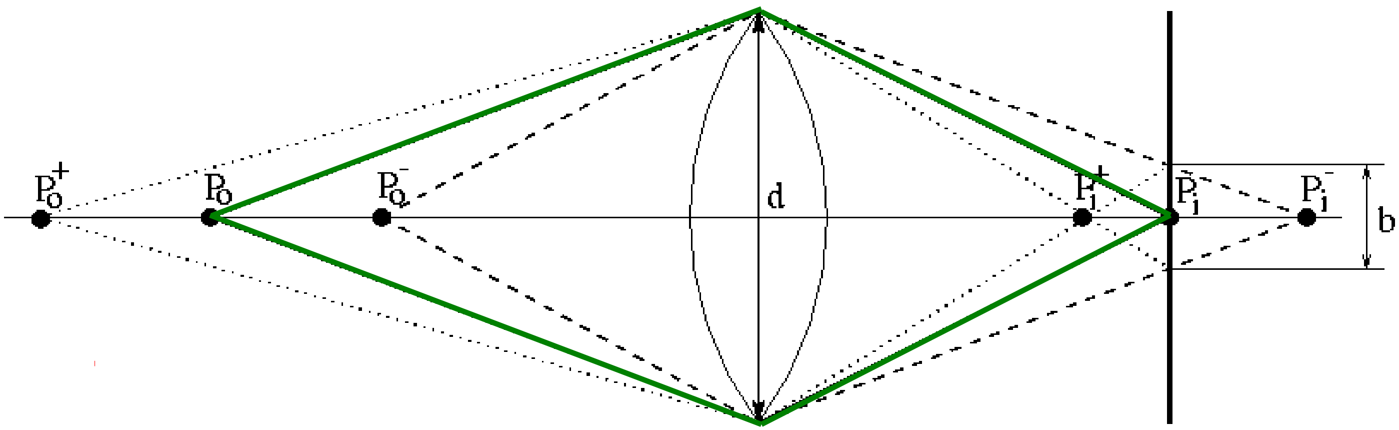
Thick Lens



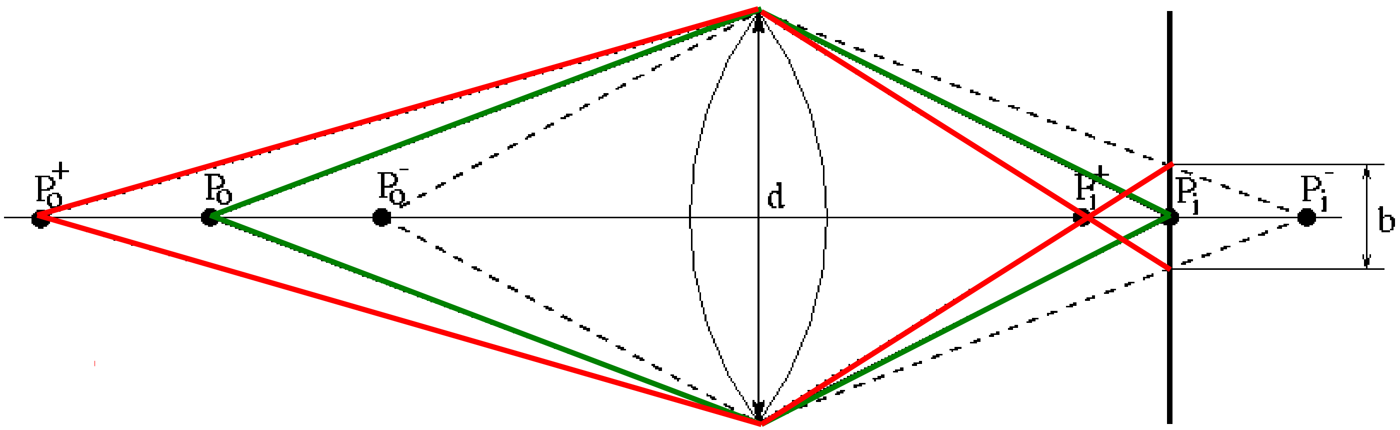
The depth-of-field



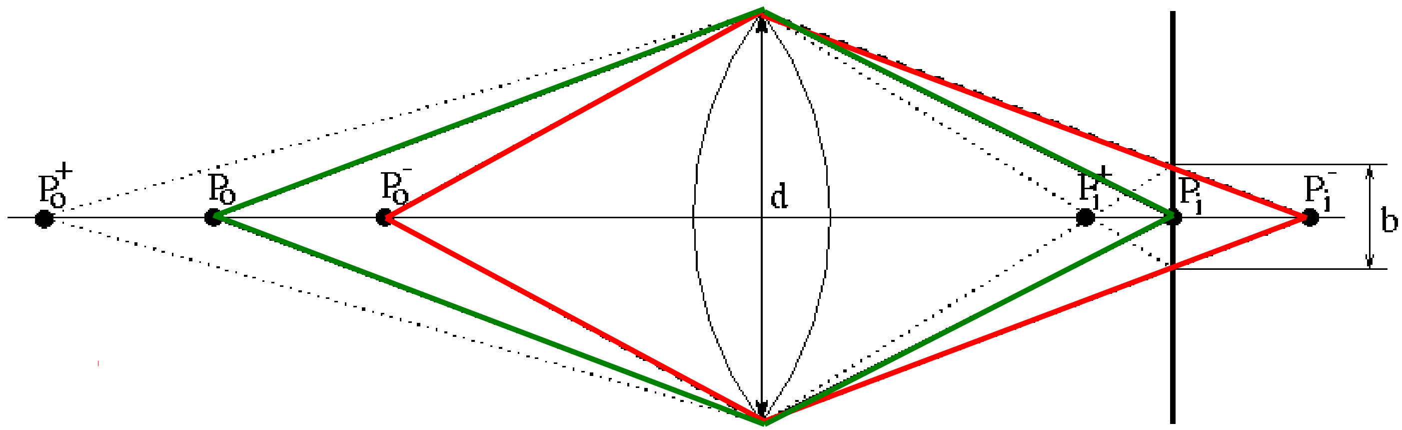
The depth-of-field



The depth-of-field

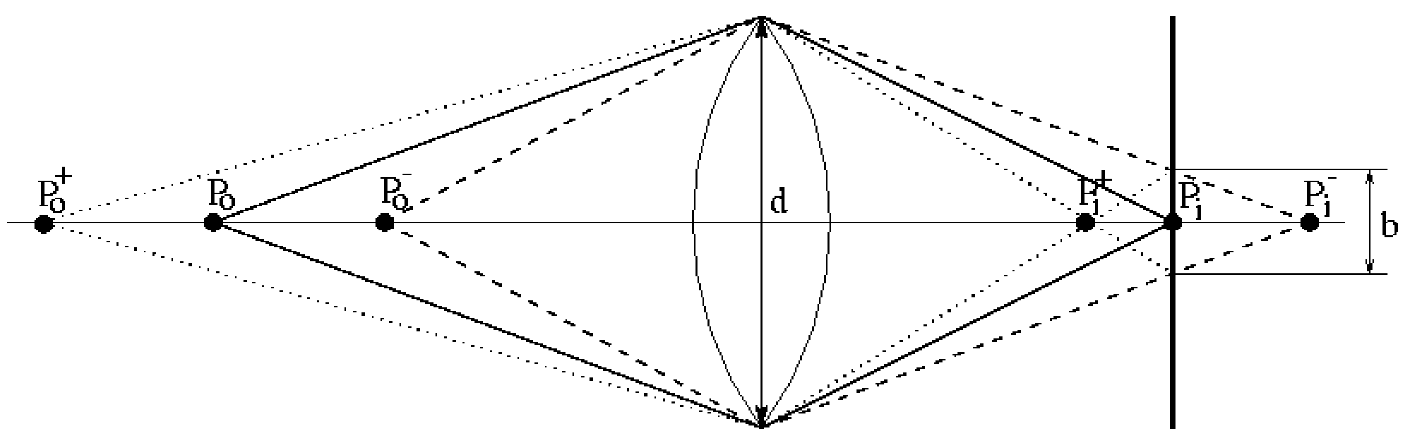


The depth-of-field



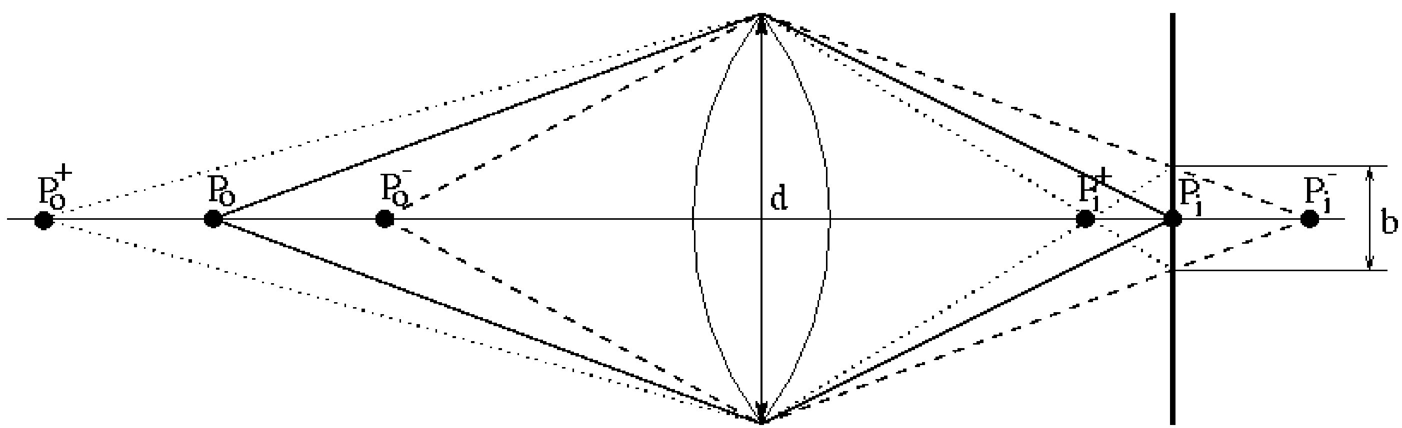
The depth-of-field

$$\frac{1}{Z_o^-} + \frac{1}{\left|Z_i^-\right|} = \frac{1}{f}$$



The depth-of-field

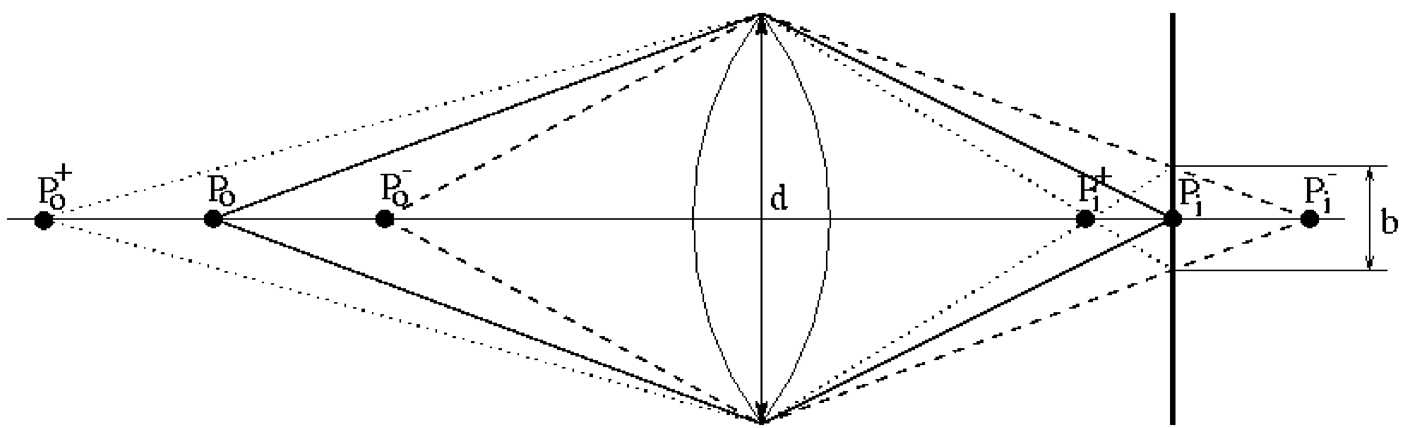
yields $$Z_o^- = f \frac{\left|Z_i^-\right|}{\left|Z_i^-\right| - f}$$



The depth-of-field

yields $$Z_o^- = f \frac{|Z_i^-|}{|Z_i^-| - f}$$

$$\boxed{|Z_i^-| = |Z_i| + \Delta Z_i^-}$$

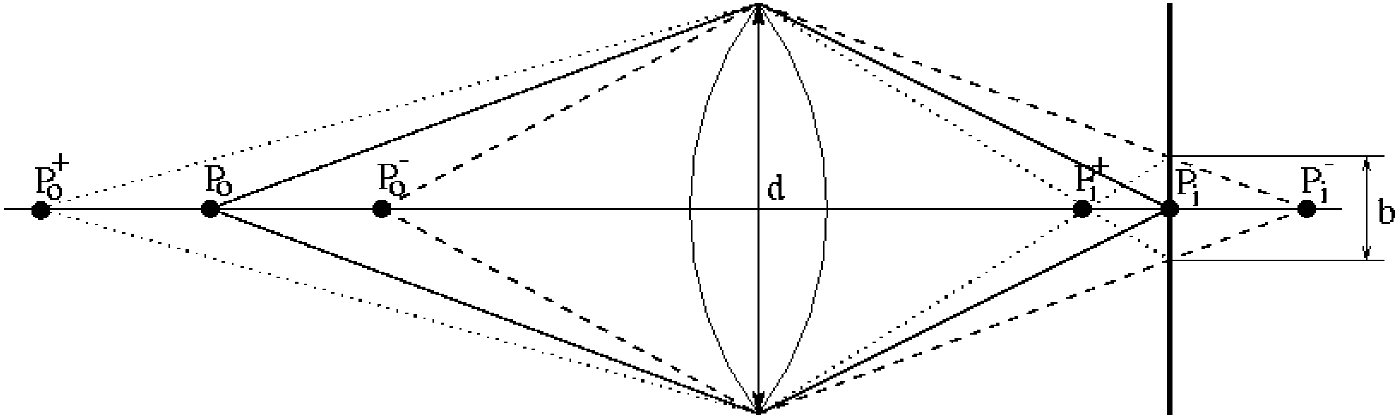


The depth-of-field

yields $$Z_o^- = f \frac{|Z_i^-|}{|Z_i^-| - f}$$

$$\boxed{|Z_i^-| = |Z_i| + \Delta Z_i^-}$$

$$\boxed{\frac{\Delta Z_i^-}{b} = \frac{|Z_i| + \Delta Z_i^-}{d}}$$

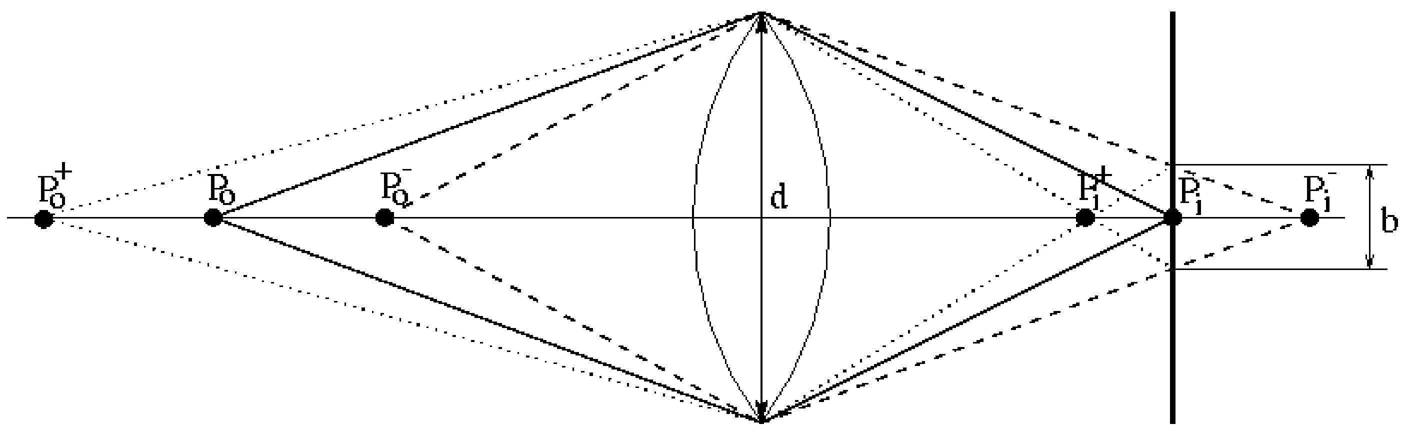


The depth-of-field

yields $$Z_o^- = f\frac{|Z_i^-|}{|Z_i^-| - f}$$

$$\boxed{|Z_i^-| = |Z_i| + \Delta Z_i^-}$$

$$\boxed{\Delta Z_i^- = \frac{b}{d-b}|Z_i|}$$

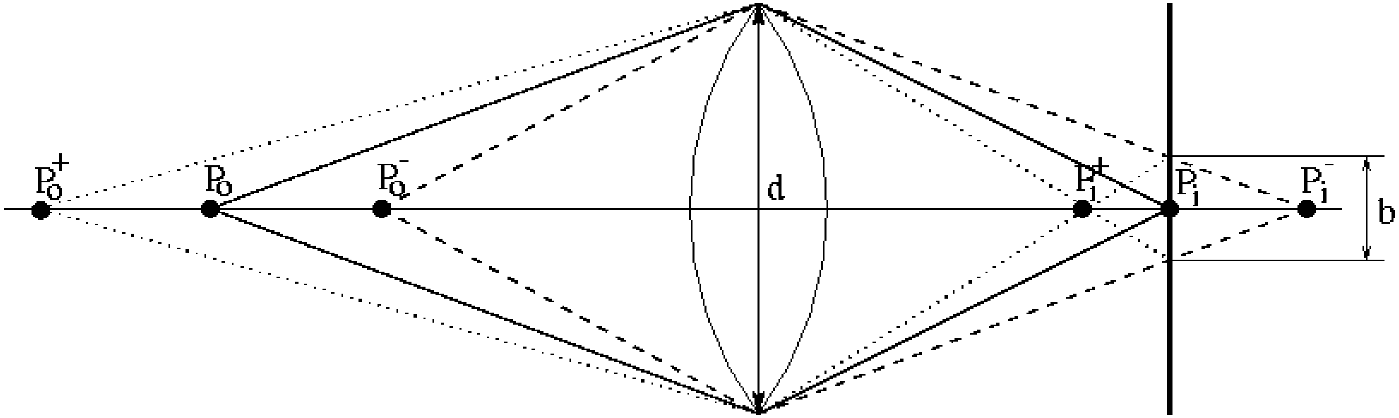


The depth-of-field

yields $$Z_o^- = f \frac{|Z_i^-|}{|Z_i^-| - f}$$

$$\boxed{|Z_i^-| = |Z_i| + \Delta Z_i^-}$$

$$\boxed{\Delta Z_i^- = \frac{b}{d-b} |Z_i|}$$



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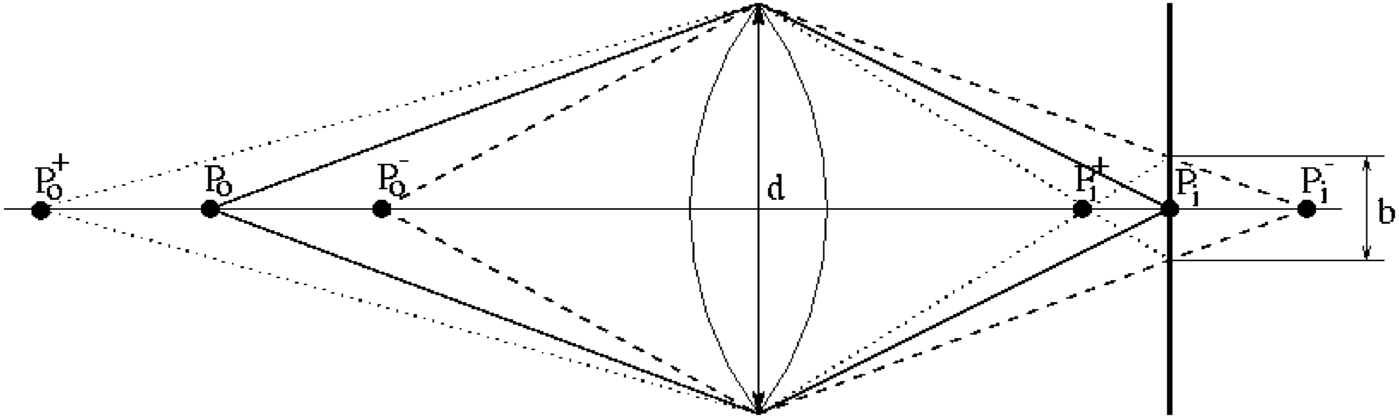

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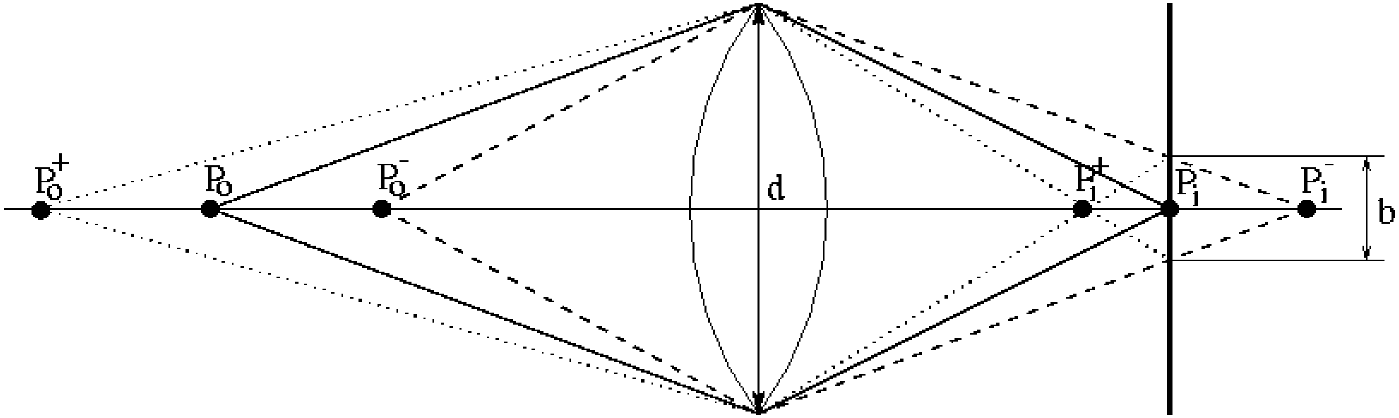
$$\boxed{|Z_i^-| = |Z_i|\, d/(d-b)}$$



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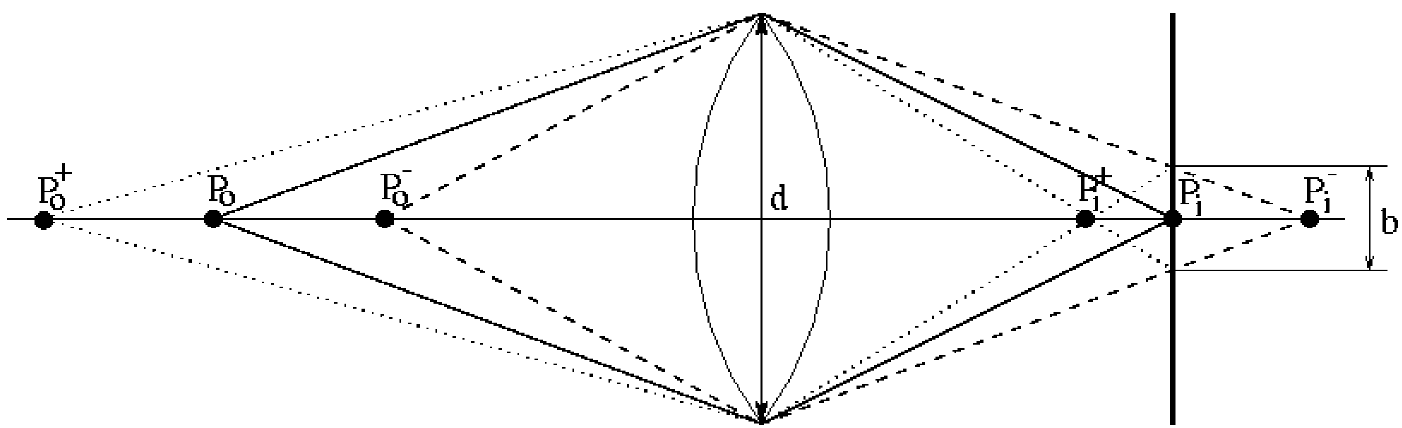

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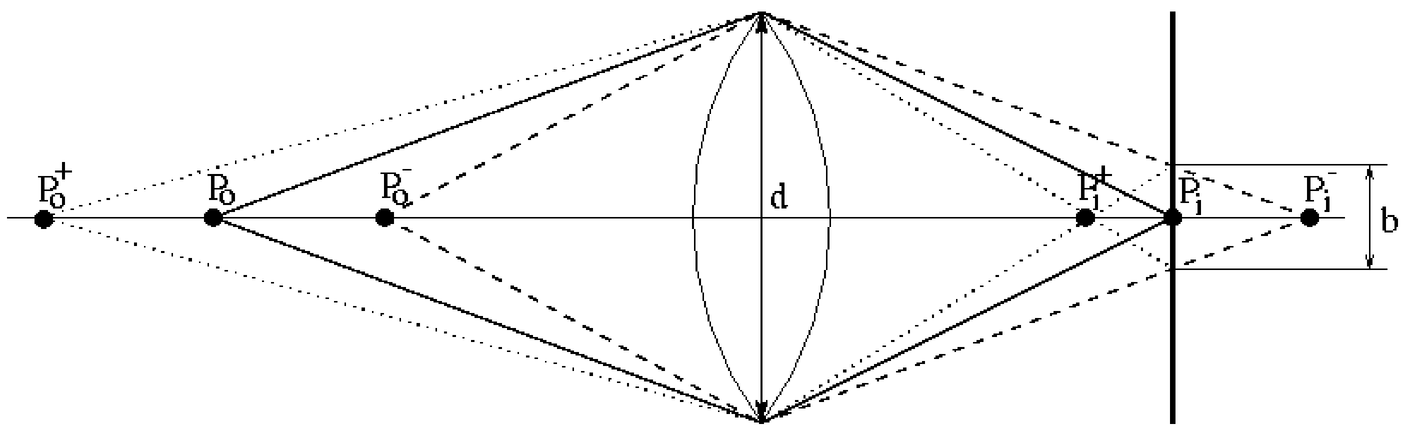

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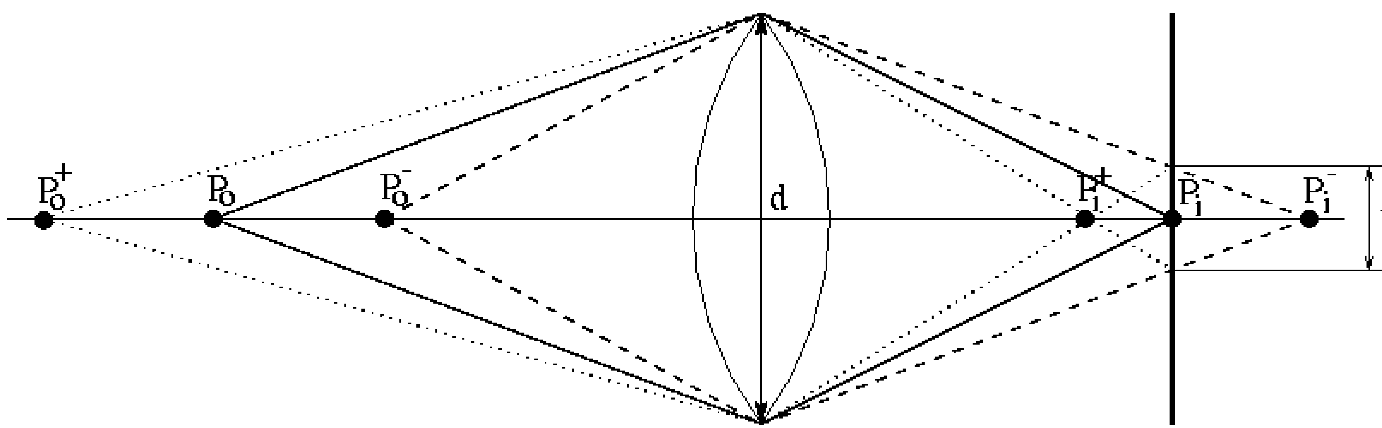
$$Z_o^- = f \frac{d\, Z_o}{b\, Z_0 + f\,(d - b)}$$



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$$\Delta Z_o^- = Z_o - Z_o^- = \frac{Z_o\,(Z_o - f)}{Z_0 + f\,d/b - f}$$

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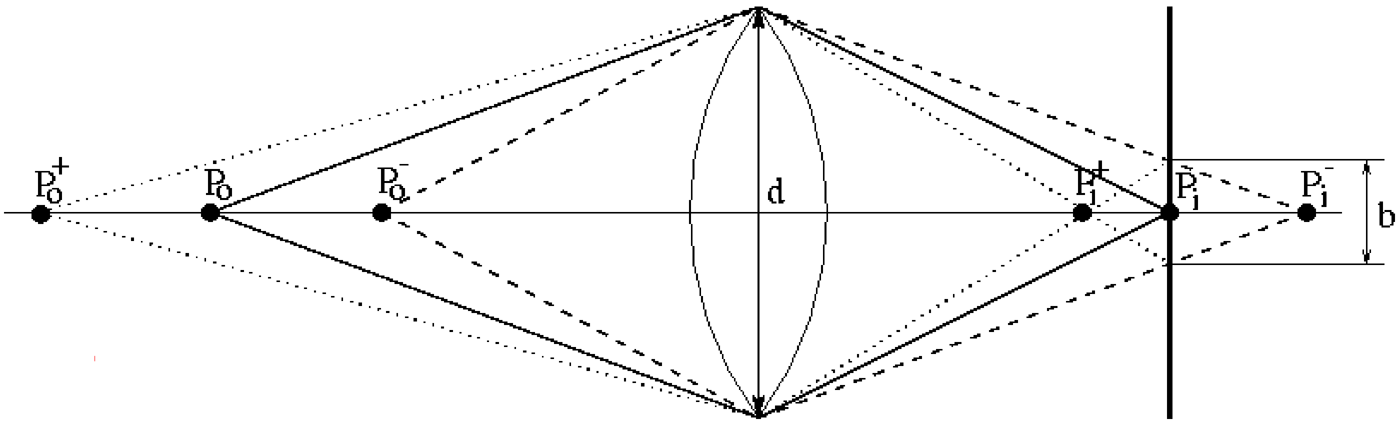
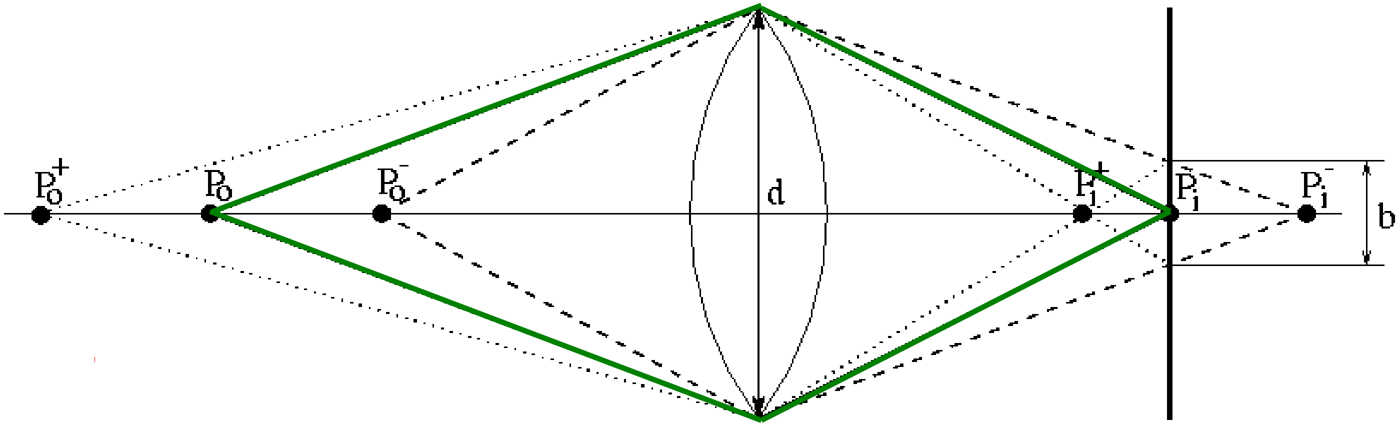
Similar formula for $\Delta Z_o^+ = Z_o^+ - Z_o$

The depth-of-field



The depth-of-field



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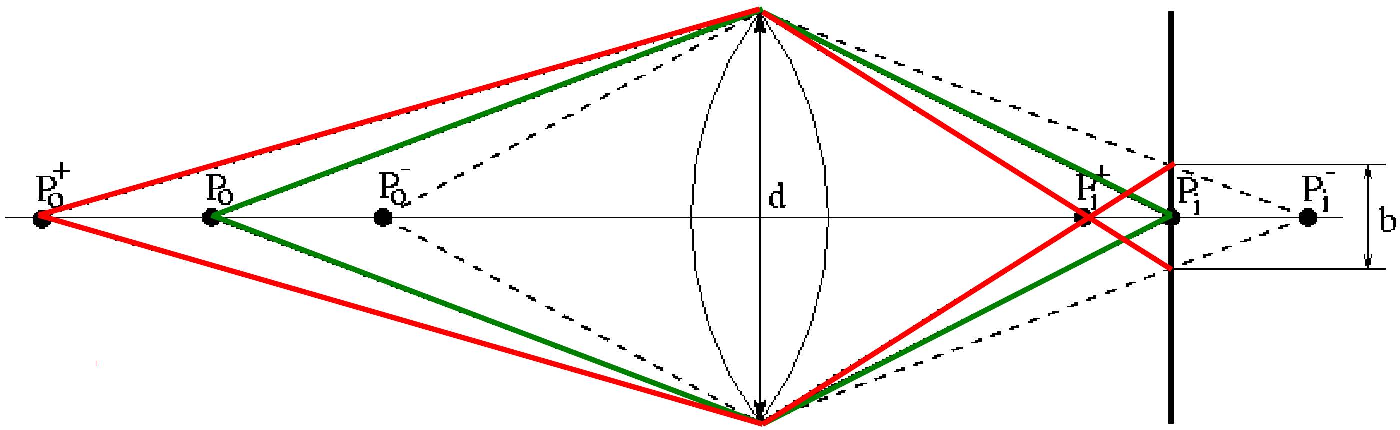

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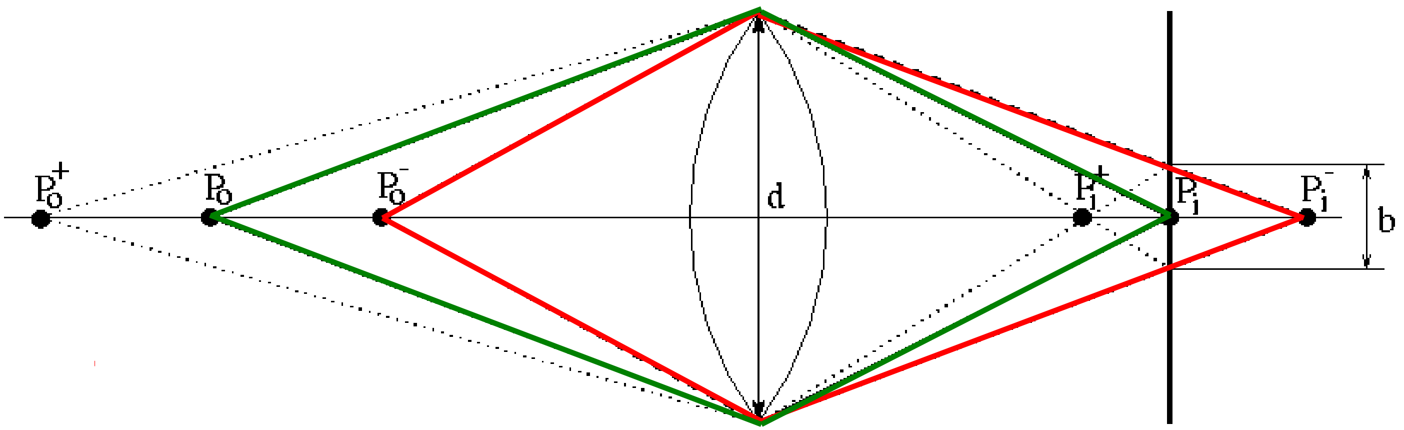

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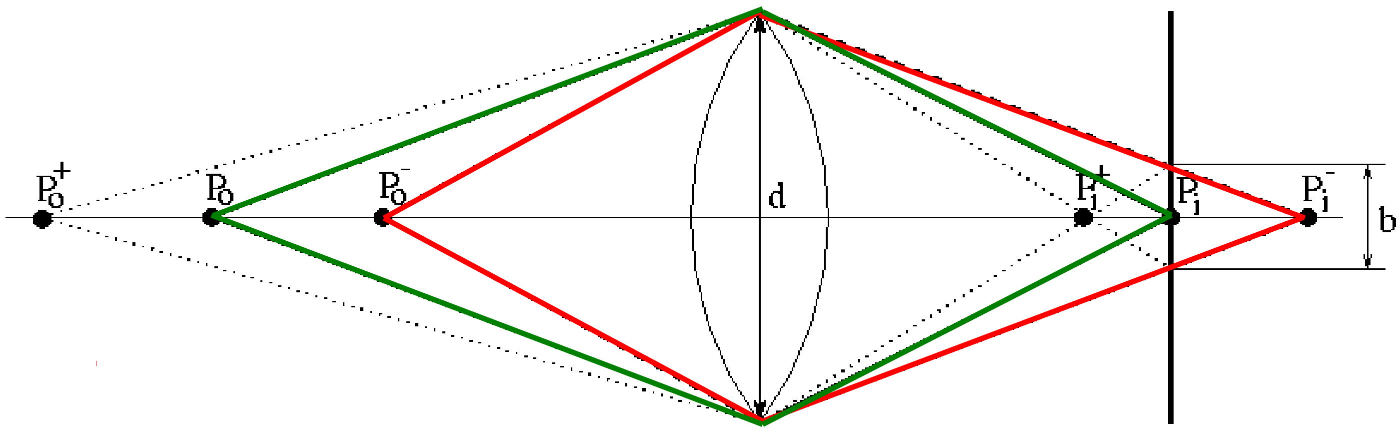
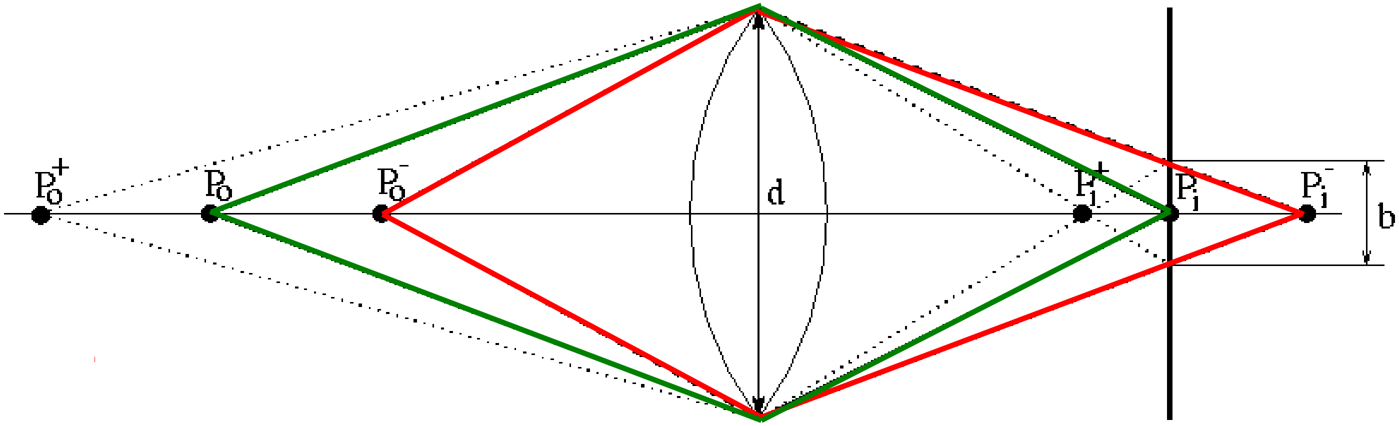


$$\Delta Z_0^- = Z_0 - Z_0^- = \frac{Z_0(Z_0 - f)}{Z_0 + f\,d/b - f}$$

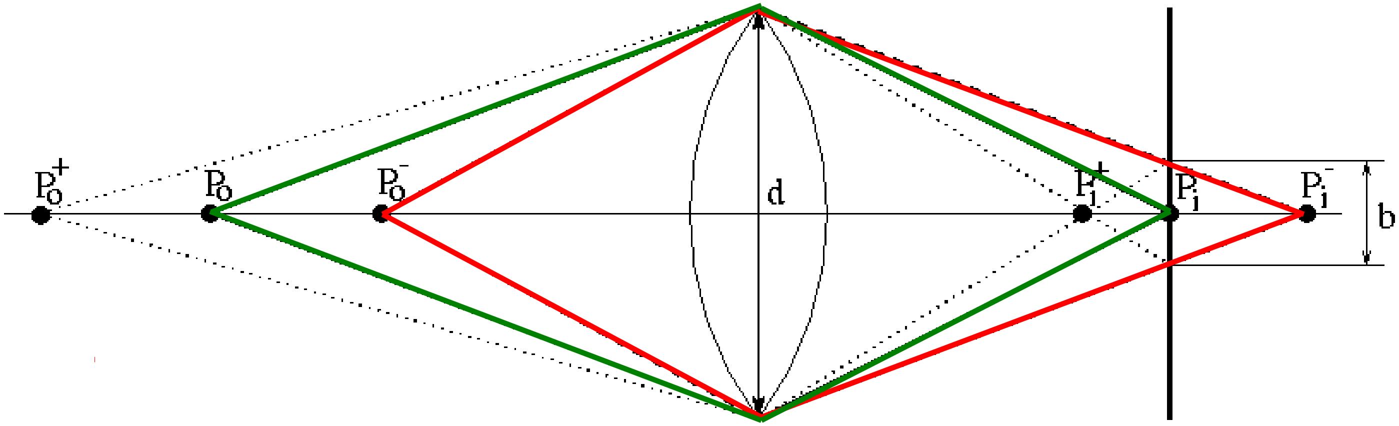
The depth-of-field



$$\Delta Z_0^- = Z_0 - Z_0^- = \frac{Z_0(Z_0 - f)}{Z_0 + f\, d/b - f}$$

decreases with d, increases with Z_0

The depth-of-field



$$\Delta Z_0^- = Z_0 - Z_0^- = \frac{Z_0(Z_0 - f)}{Z_0 + f\, d/b - f}$$

decreases with d, increases with Z_0

strike a balance between incoming light and sharp depth range

Deviations from the lens model

3 assumptions :

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1. all rays from a point are focused onto 1 image point

Deviations from the lens model

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2. all image points in a single plane

Deviations from the lens model

3 assumptions :

1. all rays from a point are focused onto 1 image point

2. all image points in a single plane

3. magnification is constant

deviations from this ideal are *aberrations*

Aberrations

2 types :

Aberrations

2 types :

1. geometrical

geometrical : small for paraxial rays

study through 3rd order optics $\sin(\theta) \approx \theta - \frac{\theta^3}{6}$

Aberrations

2 types :

1. geometrical

2. chromatic

geometrical : small for paraxial rays

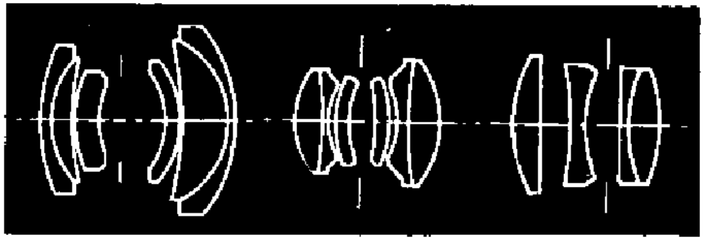
study through 3rd order optics $\sin(\theta) \approx \theta - \frac{\theta^3}{6}$

chromatic : refractive index function of wavelength

Geometrical aberrations

- spherical aberration
- astigmatism
- distortion
- coma

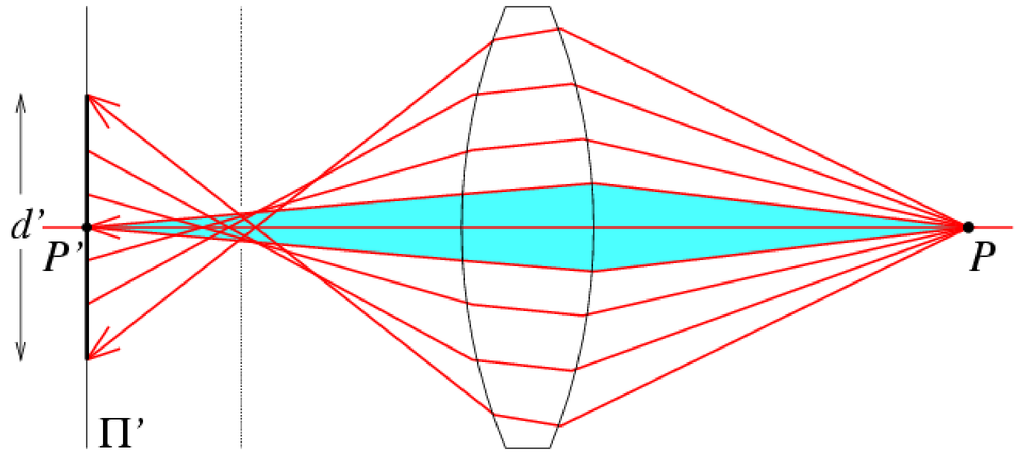
aberrations are reduced by combining lenses



Spherical aberration

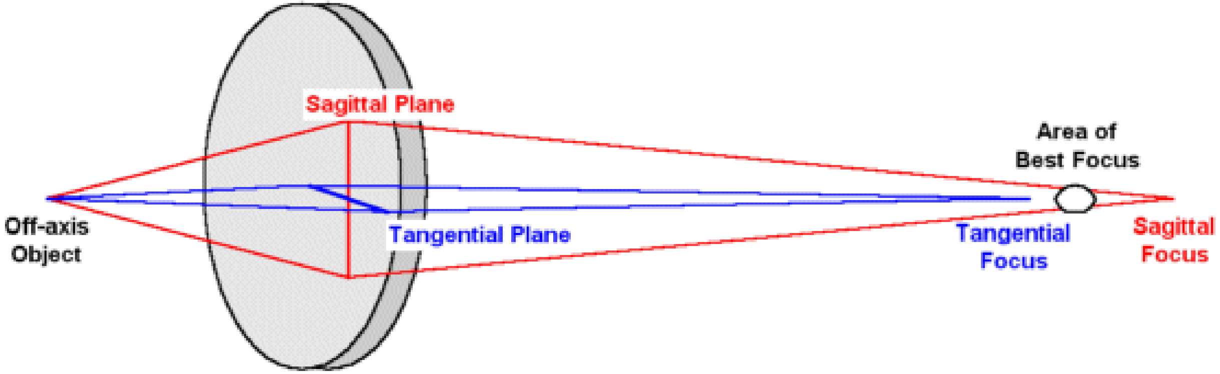
rays parallel to the axis do not converge

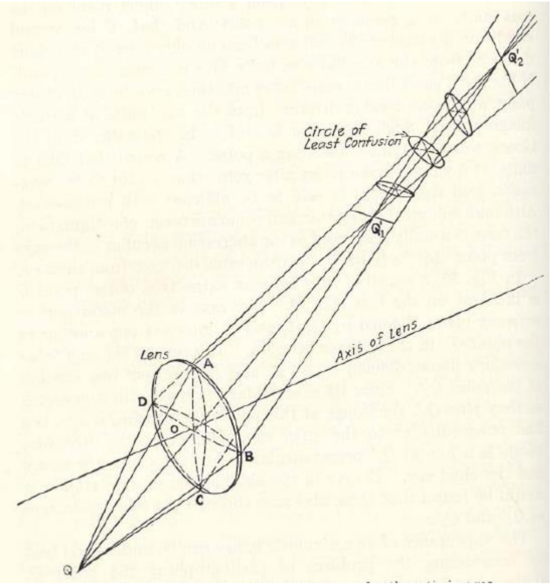
outer portions of the lens yield smaller focal lenghts



Astigmatism

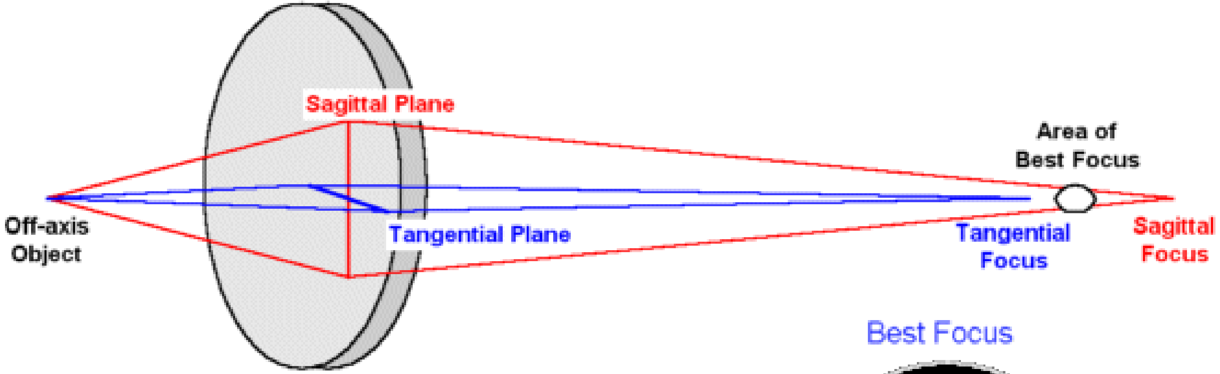
Different focal length for inclined rays

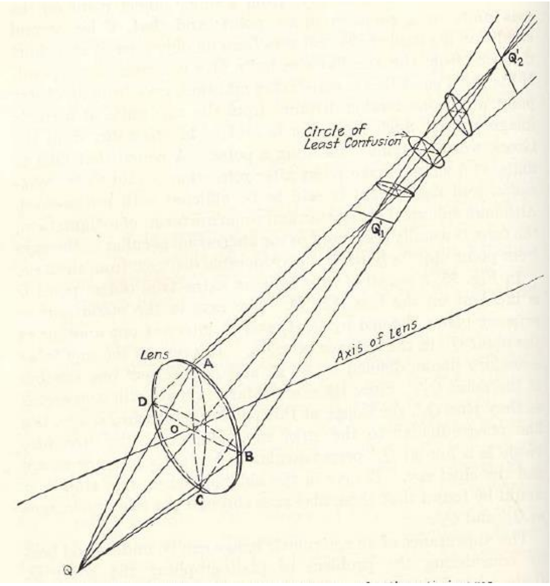


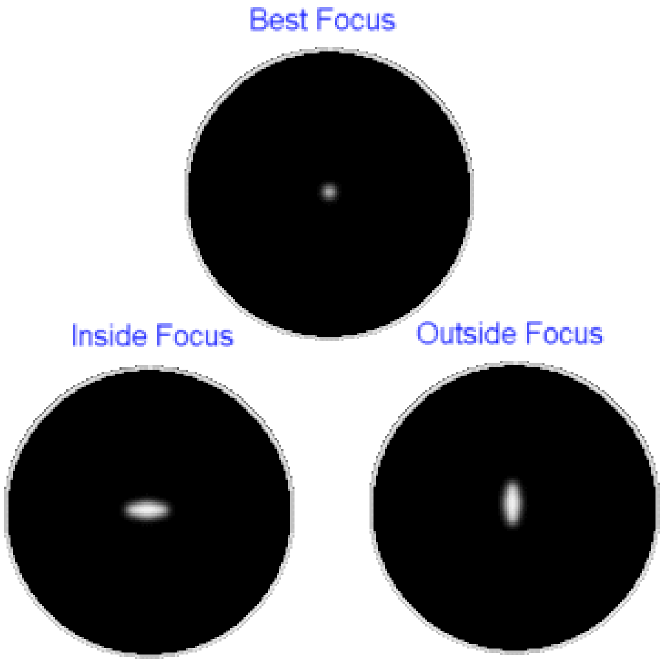


Astigmatism

Different focal length for inclined rays







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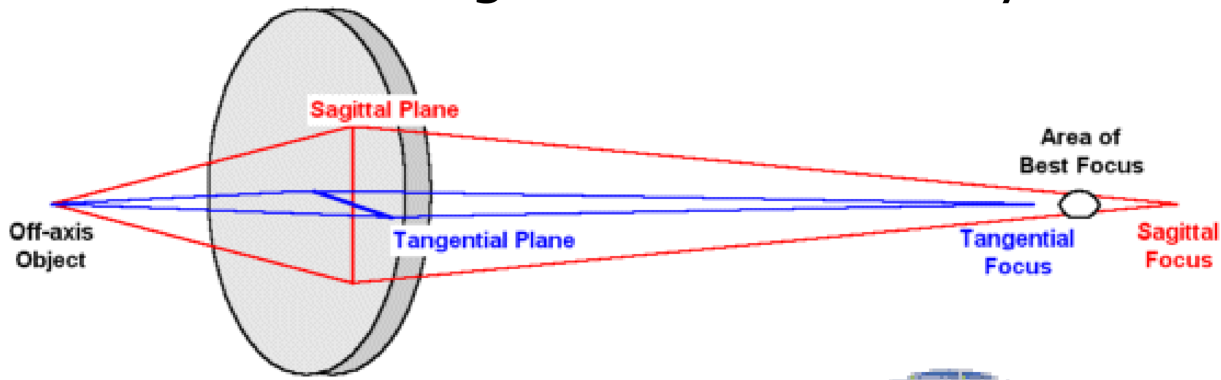
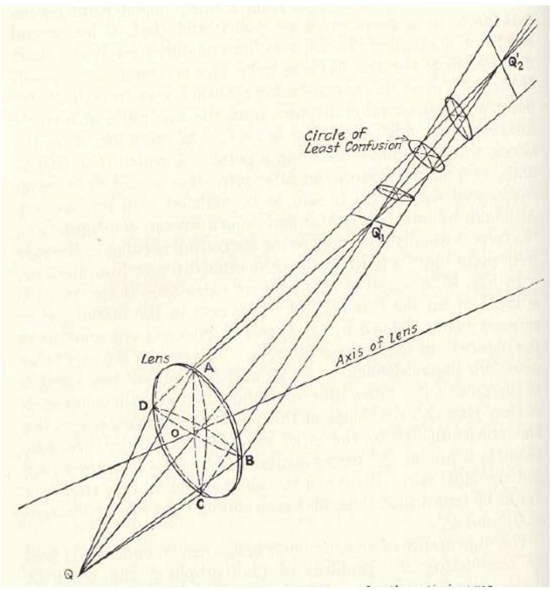
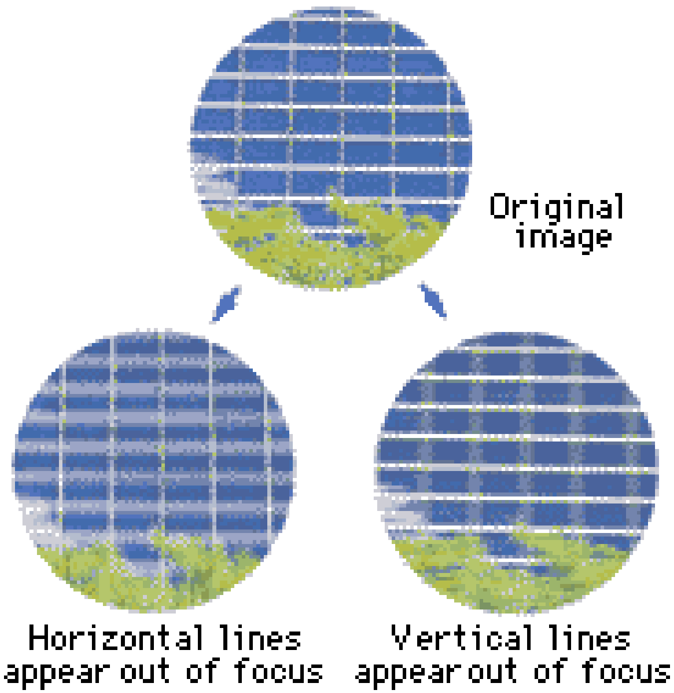
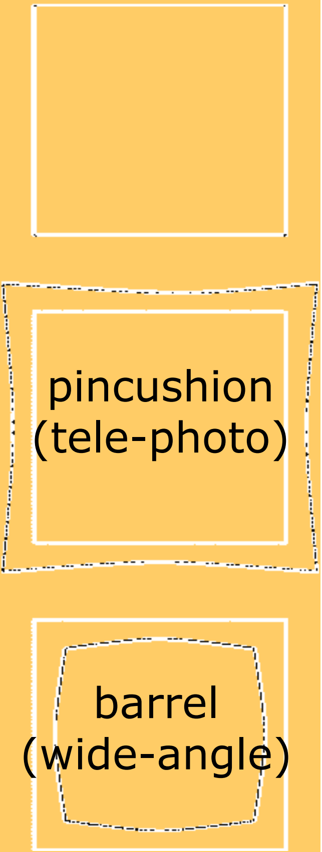


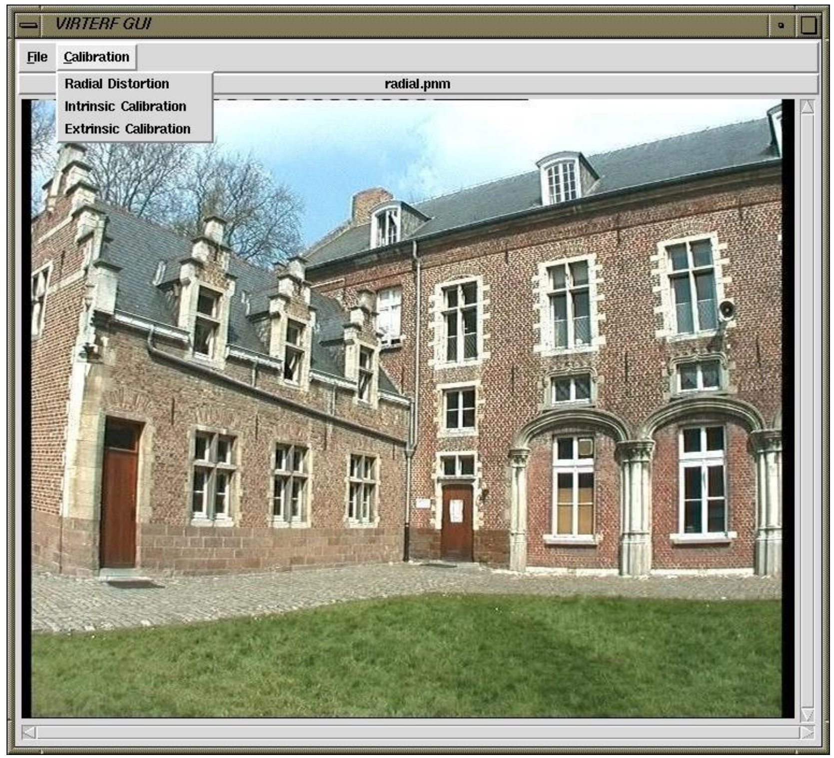




Distortion

magnification/focal length different for different angles of inclination





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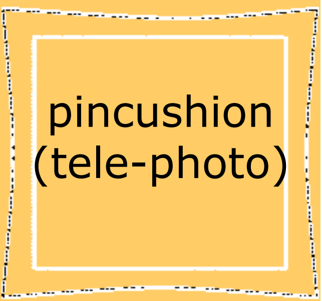
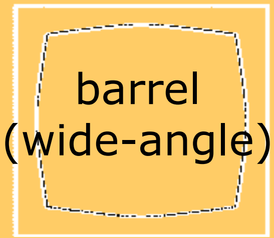
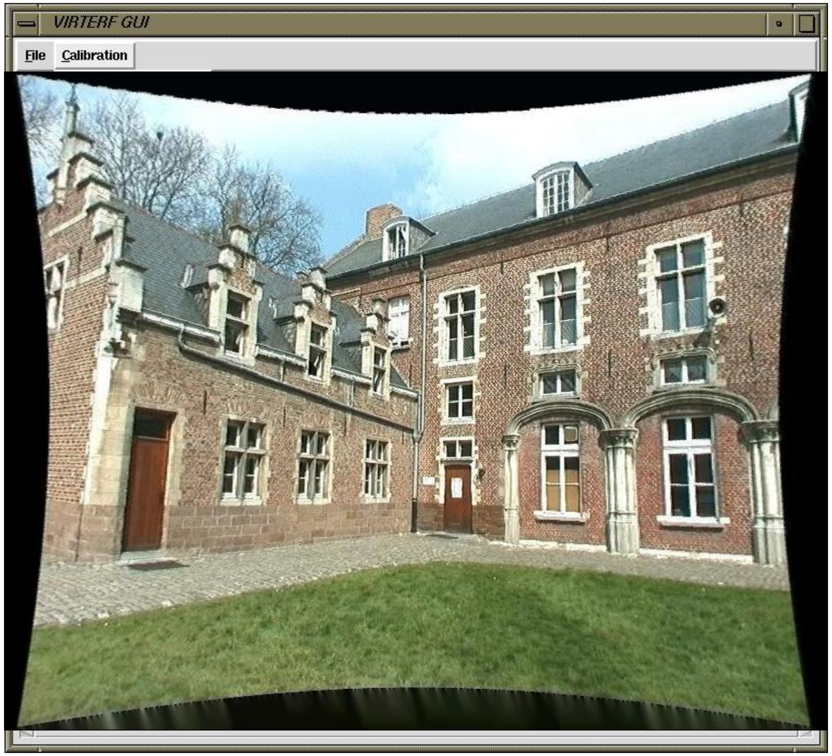






Can be corrected! (if parameters are know)

Ultra wide-angle optics

- Sometimes distortion is what you want

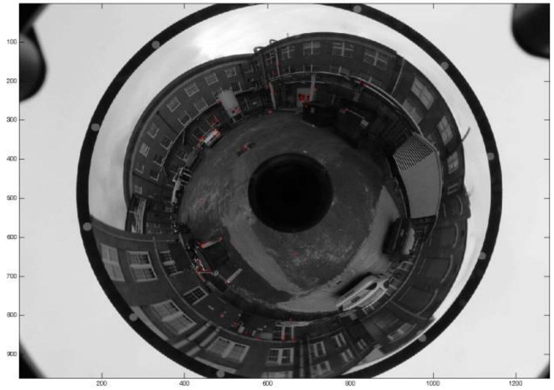
  Fisheye lens

Ultra wide-angle optics

- Sometimes distortion is what you want

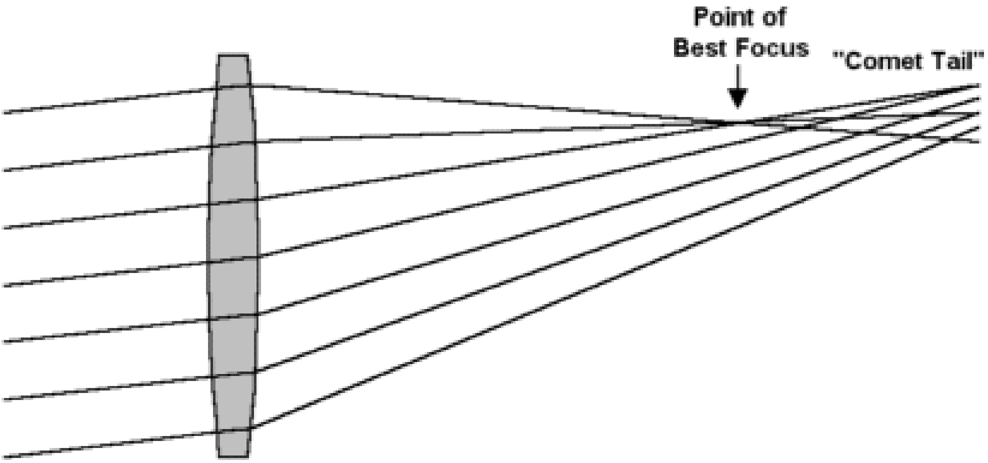
Fisheye lens

Cata-dioptric system (lens + mirror)



Coma

point off the axis depicted as comet shaped blob



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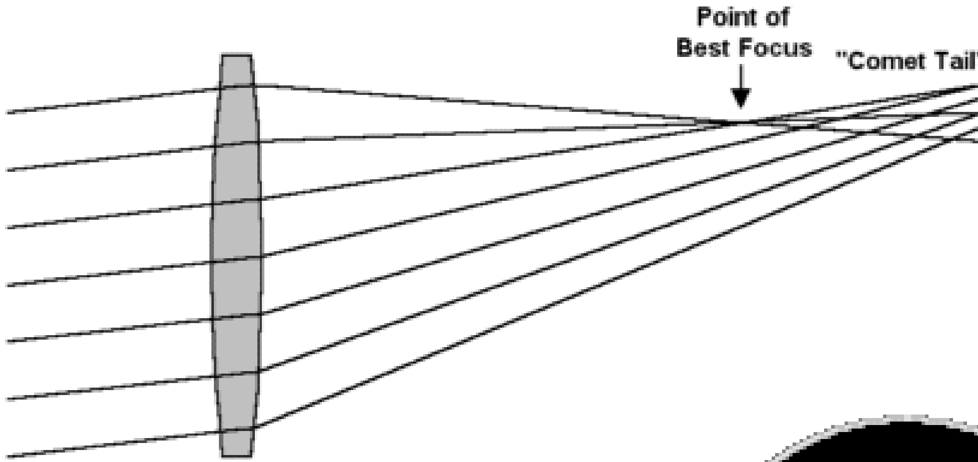
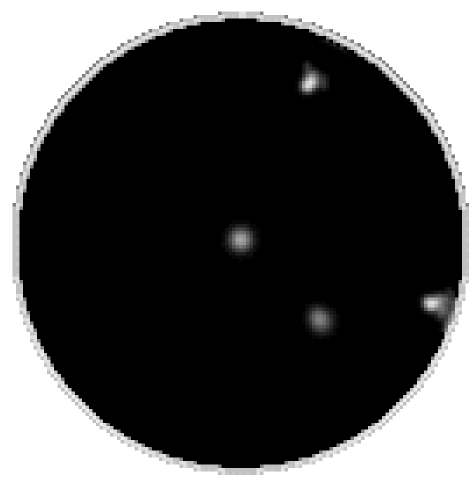
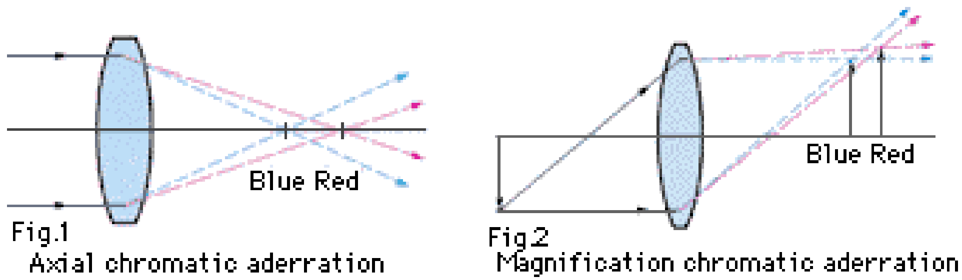




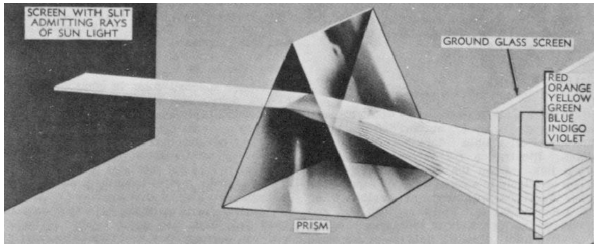
Chromatic aberration

rays of different wavelengths focused in different planes



cannot be removed completely

sometimes *achromatization* is achieved for more than 2 wavelengths



Chromatic aberration

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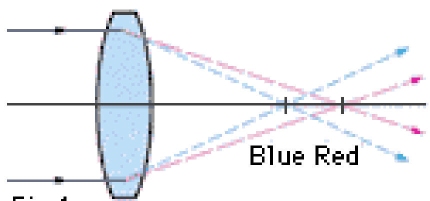


Fig.1
Axial chromatic aderration

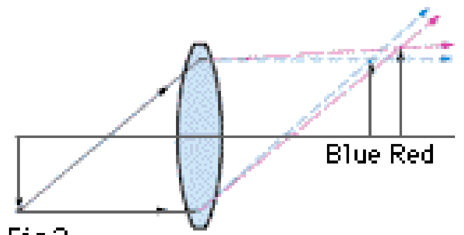
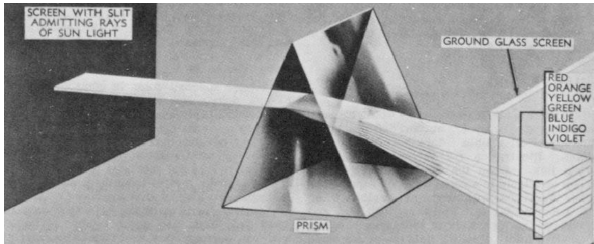


Fig.2
Magnification chromatic aderration

The image is blurred and appears colored at the fringe.

cannot be removed completely

sometimes *achromatization* is achieved for more than 2 wavelengths



Lens materials

reference wavelengths :

$\lambda_F = 486.13nm$
$\lambda_d = 587.56nm$
$\lambda_C = 656.28nm$

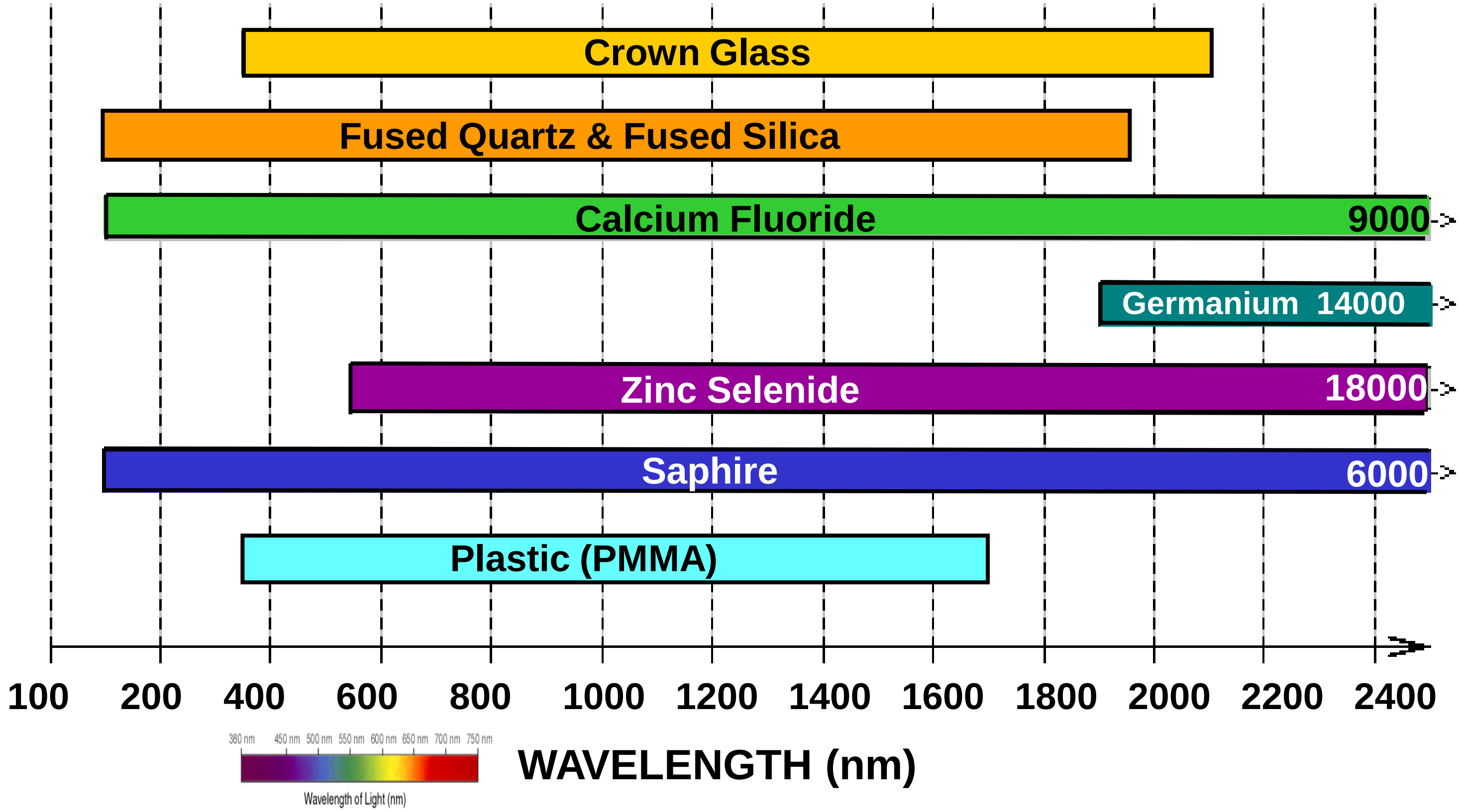
lens characteristics :

1. refractive index n_d
2. Abbe number $V_d = (n_d - 1) / (n_F - n_C)$

typically, both should be high
allows small components with sufficient refraction

notation : e.g. glass BK7(517642)
$n_d = 1.517$ and $V_d = 64.2$

Lens materials



additional considerations :

humidity and temperature resistance, weight, price,...

Vignetting

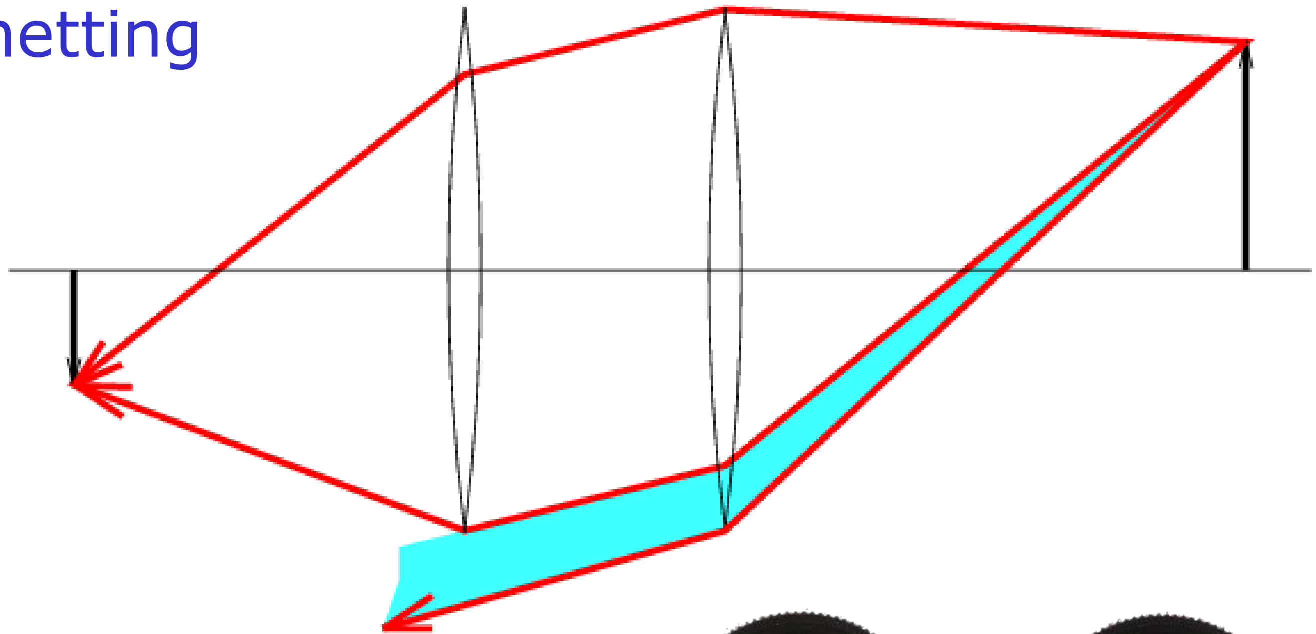
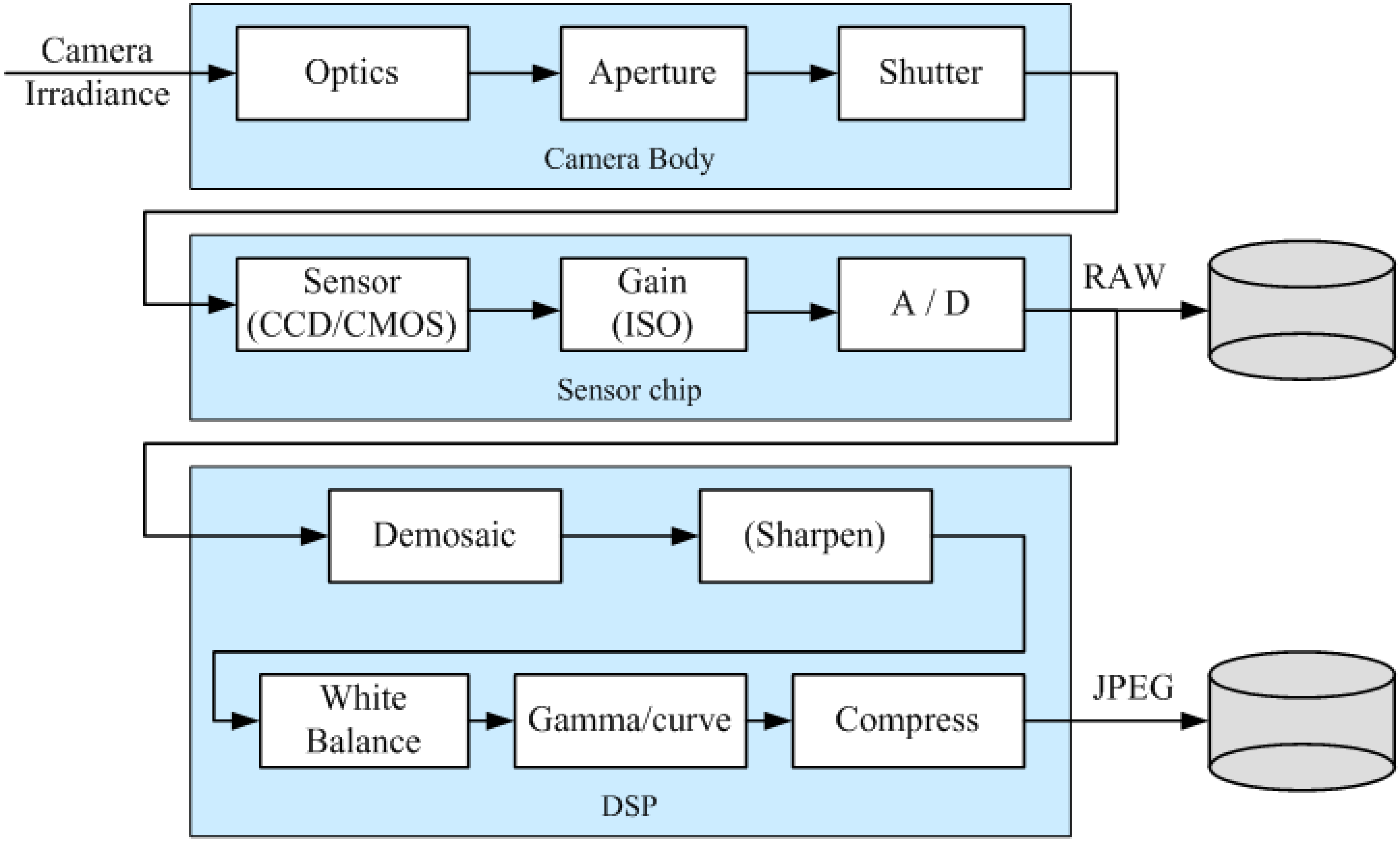


Figure from http://www.vanwalree.com/optics/vignetting.html

http://toothwalker.org/optics/vignetting.html



from Szeliski's book

Next week: Image features