# Logical Relations in Coq

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# Introduction to Logical Relations

**Theorem**

*Normalization of STLC: For all terms $e$, if $\vdash e : \tau$, then there exists a value $v$ s.t. $e \rightarrow^* v$.*

**Proof.**

By induction on the typing derivation?

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Case T-App

$$\frac{\vdash e1 : \tau_2 \rightarrow \tau \quad \vdash e2 : \tau_2}{\vdash e1\, e2 : \tau} \texttt{T-App}$$

By IH,

$$e1\, e2 \rightarrow^* (\lambda x : \tau_2 . e')\, e2 \rightarrow^* e'[v2/x]$$

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IH is too weak!

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$$\begin{aligned} N_{\mathsf{bool}}(e) &\equiv\ \vdash e : \mathsf{bool} \wedge \exists v. e \rightarrow^* v \\ N_{\tau_1 \rightarrow \tau_2}(e) &\equiv\ \vdash e : \tau_1 \rightarrow \tau_2 \wedge \exists v. e \rightarrow^* v \\ &\quad \wedge \forall e'. N_{\tau_1}(e') \Rightarrow N_{\tau_2}(e\, e') \end{aligned}$$

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Now prove:

1. If $\vdash e : \tau$, then $N_\tau(e)$.
2. If $N_\tau(e)$, then $\exists v. e \to^* v$.

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Now prove:

1. If $\Gamma \vdash e : \tau$ and $\gamma \vDash \Gamma$, then $N_\tau(\gamma(e))$.
2. If $N_\tau(e)$, then $\exists v. e \rightarrow^* v$.

where

$$\gamma = \{x_1 \mapsto v1, \ldots, x_n \mapsto v_n\} \vDash \Gamma := \mathrm{dom}(\Gamma) = \mathrm{dom}(\gamma)$$
$$\wedge (\forall x \in \mathrm{dom}(\Gamma). N_{\Gamma(x)}(\gamma(e)))$$

# Formalization in Coq

```
Fixpoint N (T : ty) (e : tm) : Prop :=
  match T with
  | bool => has_type nil e T
    /\ exists e', step e e'
  | arr T1 T2 => has_type nil e T
    /\ exists e', step e e'
    /\ forall e',
      N T1 e' -> N T2 (app e e')
  end.
```