Logical Relations in Coq

Elliot Bobrow

UPenn REPL

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STLC + Bool

Syntax:

$$
\begin{aligned}
e ::= &\ x \\
|\quad &\ \lambda x : \tau . e \\
|\quad &\ e_1\ e_2 \\
|\quad &\ \text{true} \\
|\quad &\ \text{false} \\
|\quad &\ \text{if } e_1 \text{ then } e_2 \text{ else } e_3
\end{aligned}
$$

$$
\begin{aligned}
\tau ::= &\ \mathit{Bool} \\
|\quad &\ \tau_1 \to \tau_2
\end{aligned}
$$

$$
\begin{aligned}
v ::= &\ \lambda x : \tau . e \\
|\quad &\ \text{true} \\
|\quad &\ \text{false}
\end{aligned}
$$

Introduction to Logical Relations

Theorem

Normalization: For all terms e, if $\vdash e : \tau$, then there exists a value v s.t. $e \rightarrow^* v$.

Introduction to Logical Relations

Define a predicate N_τ:

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Define a predicate N_τ:

$$N_{\text{bool}}(e) \equiv\ \vdash e : \text{bool} \ \wedge\ \exists v. e \rightarrow^* v$$

$$N_{\tau_1 \rightarrow \tau_2}(e) \equiv\ \vdash e : \tau_1 \rightarrow \tau_2 \ \wedge\ \exists v. e \rightarrow^* v$$

$$\wedge\ \forall e'. N_{\tau_1}(e') \Rightarrow N_{\tau_2}(e\, e')$$

Introduction to Logical Relations

Now prove:

1. If $\vdash e : \tau$, then $N_\tau(e)$.
2. If $N_\tau(e)$, then $\exists v. e \to^* v$.

Introduction to Logical Relations

Now prove:

1. If $\Gamma \vdash e : \tau$ and $\gamma \vDash \Gamma$, then $N_\tau(\gamma(e))$.
2. If $N_\tau(e)$, then $\exists v. e \to^* v$.

where

$$\begin{aligned} \gamma = \{x_1 \mapsto v_1, \ldots, x_n \mapsto v_n\} \vDash \Gamma := \mathrm{dom}(\Gamma) = \mathrm{dom}(\gamma) \\ \wedge (\forall x \in \mathrm{dom}(\Gamma). N_{\Gamma(x)}(\gamma(e)) \end{aligned}$$

Formalization in Coq

```
Fixpoint N (T : ty) (e : tm) : Prop :=
  match T with
  | bool => has_type nil e T
    /\ exists e', step e e'
  | arr T1 T2 => has_type nil e T
    /\ exists e', step e e'
    /\ forall e',
      N T1 e' -> N T2 (app e e')
  end.
```