Lenguajes de Programación 2022-1
Nota de clase 8: Inferencia de tipos
Funcional (λ)

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En la nota de clase anterior estudiamos como verificar que el tipado de una expresión sea correcto mediante el uso de la semántica estática con la que se definen reglas de tipado para las expresiones. Sin embargo para encontrar el tipo de una expresión la semántica estática no es suficiente, por esta razón en esta nota de clase definiremos un algoritmo de inferencia de tipos que dada una expresión e de MinHs nos regrese un tipo T tal que se cumpla que $\varnothing \vdash e : \mathsf{T}$.

> **Observación.** Para el desarrollo de esta nota eliminaremos las anotaciones de tipo de MinHs , esto con el fin de observar como aún sin anotaciones explicitas se puede inferir el tipo de las expresiones.

1 Estandarización de variables

En MinHs se pueden escribir expresiones como la siguiente:

```
let x = 5 in
    let x = false in
        x
    end
end
```

En donde se asigna la misma variable con dos valores distintos, y más aún estos valores corresponden a tipos diferentes. La expresión anterior define un programa correcto y con sentido que se reduce a `false`.

Sin embargo usando las reglas de tipado definidas en la semántica estática se presenta un problema para la verificación del tipo de la expresión. Ya que el contexto de tipado para las variables debe ser $\{x : \mathsf{Bool}, x : \mathsf{Nat}\}$ pero para la construcción del contexto se tiene la condición de que cada variable debe estar asignada a un único tipo y en este caso x tiene dos tipos en el mismo contexto.

Para corregir esto se pueden usar α-equivalencias para renombrar una de las variables y de esta forma no compartan el mismo nombre y así construir el contexto correctamente. Sabemos que pode-

mos usar α-equivalencias ya que los programas no cuentan con variables libres, de lo contrario sería un programa mal formado sobre el cual no se hace verificación de tipos.

La idea es que para cualquier expresión del lenguaje se encuentre una expresión equivalente en donde cada definición de variable utilice un identificador único. De esta forma el programa anterior quedaría como:

```
let x0  = 5 in
    let x1  = false in
        x1
    end
end
```

Y de esta forma el contexto asociado a la expresión es $\{x_1 : \mathsf{Bool}, x_0 : \mathsf{Nat}\}$ que se trata de un contexto válido.

2 Generación de restricciones

Para la definición de un algoritmo de inferencia de tipos primero se tiene que definir las restricciones de tipado que deben cumplir las expresiones del lenguaje dependiendo de la estructura que éstas tengan.

Estas restricciones definen tipos específicos que son necesarios para que las expresiones tengan sentido y a partir de un conjunto de restricciones se desea encontrar el tipo de la expresión.

A continuación se presenta un algoritmo de generación de restricciones.

Definición 2.1 (Algoritmo de generación de restricciones)**.** A partir de una expresión de MinHs se construye un conjunto de restricciones de tipado que debe cumplir dicha expresión. Se define el algoritmo mediante juicios de la forma:

$$e \mapsto \mathcal{R}$$

que se lee como, la expresión e genera el conjunto $\mathcal{R}$ de restricciones de tipado.
Para definir el algoritmo se extiende la categoría de tipos como sigue:

$$\mathsf{T} ::= \mathsf{X} \mid \mathsf{Nat} \mid \mathsf{Bool} \mid \mathsf{T}_1 \to \mathsf{T}_2 \mid [\![e]\!]$$

en donde X es una variable de tipo. Estas variables nos ayudan en la definición de programas polimórficos, por ejemplo la función general identidad

$$\mathtt{lam}\, x \Rightarrow x$$

tiene el tipo $\mathsf{X} \to \mathsf{X}$ y esta variable de tipo X va a tomar su valor hasta que la función sea utilizada, por ejemplo en la aplicación

$$(\mathtt{lam}\, x \Rightarrow x)\, 5$$

la variable X toma el valor de Nat .

La expresión $[\![e]\!]$ es una construcción sintáctica para definir el tipo de una expresión e del lenguaje y se lee como: el tipo de e. Es importante aclarar que los tipos de la forma $[\![e]\!]$ no pueden figurar en el tipo resultante del algoritmo de inferencia, el tipo de una expresión debe construirse únicamente con el resto de los tipos.
Una restricción es una ecuación de la forma $\mathsf{T}_1 = \mathsf{T}_2$ en donde T_1 y T_2 son tipos. La ecuación indica que T_1 debe ser igual a T_2 bajo unificación.

Variables

$$\frac{}{x_i \mapsto [\![x_i]\!] = \mathsf{X}_i}$$

Para el tipo de las variables se usará una variable de tipo con el mismo nombre de la variable. Todas las apariciones de la misma variable generarán la misma restricción y como los nombres de variables son únicos no habrá dos variables distintas con el mismo tipo.

Valores numéricos

$$\frac{}{\texttt{num}[n] \mapsto [\![\texttt{num}[n]]\!] = \mathsf{Nat}}$$

Valores Booleanos

$$\frac{}{\texttt{bool}[b] \mapsto [\![\texttt{bool}[b]]\!] = \mathsf{Bool}}$$

Operadores

$$\frac{e_1 \mapsto \mathcal{R}_1 \quad e_2 \mapsto \mathcal{R}_2}{\texttt{suma}(e_1, e_2) \mapsto \mathcal{R}_1, \mathcal{R}_2, [\![e_1]\!] = \mathsf{Nat}, [\![e_2]\!] = \mathsf{Nat}, [\![\texttt{suma}(e_1, e_2)]\!] = \mathsf{Nat}}$$

$$\frac{e_1 \mapsto \mathcal{R}_1 \quad e_2 \mapsto \mathcal{R}_2}{\texttt{prod}(e_1, e_2) \mapsto \mathcal{R}_1, \mathcal{R}_2, [\![e_1]\!] = \mathsf{Nat}, [\![e_2]\!] = \mathsf{Nat}, [\![\texttt{prod}(e_1, e_2)]\!] = \mathsf{Nat}}$$

$$\frac{e_1 \mapsto \mathcal{R}_1 \quad e_2 \mapsto \mathcal{R}_2}{\texttt{sub}(e_1, e_2) \mapsto \mathcal{R}_1, \mathcal{R}_2, [\![e_1]\!] = \mathsf{Nat}, [\![e_2]\!] = \mathsf{Nat}, [\![\texttt{sub}(e_1, e_2)]\!] = \mathsf{Nat}}$$

$$\frac{e_1 \mapsto \mathcal{R}_1 \quad e_2 \mapsto \mathcal{R}_2}{\texttt{eq}(e_1, e_2) \mapsto \mathcal{R}_1, \mathcal{R}_2, [\![e_1]\!] = \mathsf{Nat}, [\![e_2]\!] = \mathsf{Nat}, [\![\texttt{eq}(e_1, e_2)]\!] = \mathsf{Bool}}$$

$$\frac{e_1 \mapsto \mathcal{R}_1 \quad e_2 \mapsto \mathcal{R}_2}{\texttt{gt}(e_1, e_2) \mapsto \mathcal{R}_1, \mathcal{R}_2, [\![e_1]\!] = \mathsf{Nat}, [\![e_2]\!] = \mathsf{Nat}, [\![\texttt{gt}(e_1, e_2)]\!] = \mathsf{Bool}}$$

$$\frac{e_1 \mapsto \mathcal{R}_1 \quad e_2 \mapsto \mathcal{R}_2}{\texttt{lt}(e_1, e_2) \mapsto \mathcal{R}_1, \mathcal{R}_2, [\![e_1]\!] = \mathsf{Nat}, [\![e_2]\!] = \mathsf{Nat}, [\![\texttt{lt}(e_1, e_2)]\!] = \mathsf{Bool}}$$

Condicional

$$\frac{c \mapsto \mathcal{R}_1 \quad t \mapsto \mathcal{R}_2 \quad e \mapsto \mathcal{R}_3}{\texttt{if}(c, t, e) \mapsto \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, [\![c]\!] = \mathsf{Bool}, [\![t]\!] = [\![e]\!], [\![\texttt{if}(c, t, e)]\!] = [\![t]\!], [\![\texttt{if}(c, t, e)]\!] = [\![e]\!]}$$

Asignaciones Locales

$$\frac{v \mapsto \mathcal{R}_1 \quad b \mapsto \mathcal{R}_2}{\texttt{let}(v, x_i.b) \mapsto \mathcal{R}_1, \mathcal{R}_2, \mathsf{X}_i = [\![v]\!], [\![\texttt{let}(v, x_i.b)]\!] = [\![b]\!]}$$

Funciones

$$\frac{t \mapsto \mathcal{R}}{\texttt{lam}(x_i.t) \mapsto \mathcal{R}, [\![\texttt{lam}(x_i.t)]\!] = \mathsf{X}_i \to [\![t]\!]}$$

$$\frac{b \mapsto \mathcal{R}_1}{\texttt{recfun}(f.x_i.b) \mapsto \mathcal{R}_1, [\![\texttt{recfun}(f.x_i.b)]\!] = \mathsf{X}_i \to [\![b]\!]}$$

Aplicación de función

$$\frac{f \mapsto \mathcal{R}_1 \qquad p \mapsto \mathcal{R}_2}{\texttt{app}(f,p) \mapsto \mathcal{R}_1.\mathcal{R}_2, [\![f]\!] = [\![p]\!] \to [\![\texttt{app}(f,p)]\!]}$$

3 Algoritmo de unificación

Es importante notar que la relación $\mapsto$ es total, es decir, para cualquier expresión podemos encontrar una lista de restricciones de tipado. Sin embargo, esto no significa que la expresión sea tipable. En otras palabras, la existencia de una lista $\mathcal{R}$ de restricciones no garantiza que exista un tipo para la expresión, como sucede cuando hay restricciones insatisfacibles, por ejemplo cuando una variable tiene como restricción ser natural y booleano simultáneamente.

Definición 3.1 (Unificador)**.** Un unificador μ es una serie de sustituciones de tal forma que convierten una expresión en otra, o unifican una expresión con otra.

Para poder encontrar el tipo asociado a la expresión es necesario que se puedan resolver las restricciones definidas en la lista $\mathcal{R}$, es decir, cuando $\mathcal{R}$ tiene un unificador mas general. Para encontrar este unificador, definimos el algoritmo de unificación $\mathcal{U}$.

Definición 3.2 (Algoritmo de unificación)**.** La entrada del algoritmo es una lista de restricciones y la salida es un unificador μ en caso de que las restricciones se puedan resolver o fail en caso contrario.

$$\begin{array}{rcll}
\mathcal{U}([\,]) & = & [\,] & \\
\mathcal{U}(\mathsf{T} = \mathsf{T} : \mathcal{R}) & = & \mathcal{U}(\mathcal{R}) & \\
\mathcal{U}(\mathsf{X} = \mathsf{T} : \mathcal{R}) & = & \mathcal{U}(\mathcal{R}[\mathsf{X} := \mathsf{T}]) \circ [\mathsf{X} := \mathsf{T}] & \text{si } \mathsf{X} \notin \mathsf{TVar}(\mathsf{T}) \\
\mathcal{U}(\mathsf{X} = \mathsf{T} : \mathcal{R}) & = & \mathsf{fail} & \text{si } \mathsf{X} \in \mathsf{TVar}(\mathsf{T}) \\
\mathcal{U}([\![e]\!] = \mathsf{T} : \mathcal{R}) & = & \mathcal{U}(\mathcal{R}[[\![e]\!] := \mathsf{T}]) \circ [[\![e]\!] := \mathsf{T}] & \\
\mathcal{U}(\mathsf{T} = \mathsf{X} : \mathcal{R}) & = & \mathcal{U}(\mathsf{X} = \mathsf{T} : \mathcal{R}) & \\
\mathcal{U}(\mathsf{T} = [\![e]\!] : \mathcal{R}) & = & \mathcal{U}([\![e]\!] = \mathsf{T} : \mathcal{R}) & \\
\mathcal{U}(\mathsf{S}_1 \to \mathsf{S}_2 = \mathsf{T}_1 \to \mathsf{T}_2 : \mathcal{R}) & = & \mathcal{U}(\mathsf{S}_1 = \mathsf{T}_1 : \mathsf{S}_2 = \mathsf{T}_2 : \mathcal{R}) & \\
\mathcal{U}(\mathcal{R}) & = & \mathsf{fail} &
\end{array}$$

En donde T representa a cualquier tipo y X a una variable de tipo.

4 Algoritmo de inferencia de tipos

Con las definiciones anteriores se presenta un algoritmo de inferencia de tipos sobre el lenguaje MinHs .

Definición 4.1 (Algoritmo de Inferencia de tipos)**.** Se define el algoritmo $\mathcal{T}(e)$ de inferencia de tipos que recibe una expresión e de MinHs y regresa el tipo de esta expresión. El algoritmo se define con los siguientes pasos:

1. Se encuentra la expresión e' con nombres de variables únicas.
2. Se encuentra el conjunto de restricciones $\mathcal{R}$ tal que $e' \mapsto \mathcal{R}$.
3. Utilizando la función $\mathcal{U}$ se calcula el unificador mas general μ del conjunto de restricciones $\mathcal{R}$, tal que $\mathcal{U}(\mathcal{R}) = \mu$.
4. Se busca en μ la ecuación $[\![e']\!] := \mathsf{T}$.
5. T es el tipo mas general de e, es decir, $\mathcal{T}(e) = \mathsf{T}$.

5 Ejemplos

Ejemplo 5.1. Vamos a encontrar el tipo de la expresión:

```
let x = 5 in
    let x = false in
        x
    end
end
```

que corresponde al árbol de sintaxis abstracta:

$$\texttt{let}(5, x.\texttt{let}(\texttt{false}, x.x))$$

Renombramiento de variables

$$\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))$$

Generación de Restricciones .

1. $\texttt{false} \mapsto [\![\texttt{false}]\!] = \mathsf{Bool}$
2. $5 \mapsto [\![5]\!] = \mathsf{Nat}$
3. $x_0 \mapsto [\![x_0]\!] = \mathsf{X}_0$
4. $x_1 \mapsto [\![x_1]\!] = \mathsf{X}_1$
5. $\texttt{let}(\texttt{false}, x_1.x_1) \mapsto \underbrace{\mathsf{X}_1 = [\![\texttt{false}]\!], [\![\texttt{false}]\!] = \mathsf{Bool}, [\![x_1]\!] = \mathsf{X}_1, [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = [\![x_1]\!]}_{\mathcal{R}_1}$
6. $\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))$
   $\mapsto [\![5]\!] = \mathsf{Nat}, \mathcal{R}_1, \mathsf{X}_0 = [\![5]\!], [\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]$

Como resultado se obtiene la lista de restricciones $\mathcal{R}$:

$$\begin{aligned}\mathcal{R} \;=\; & [\![5]\!] = \mathsf{Nat}, \\ & \mathsf{X}_1 = [\![\texttt{false}]\!] \\ & [\![\texttt{false}]\!] = \mathsf{Bool}, \\ & [\![x_1]\!] = \mathsf{X}_1, \\ & [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = [\![x_1]\!], \\ & \mathsf{X}_0 = [\![5]\!] \\ & [\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]\end{aligned}$$

Unificación de $\mathcal{R}$.

Restricciones $\mathcal{R}$	Unificador μ
$[\![5]\!] = \mathsf{Nat}$, $\mathsf{X}_1 = [\![\texttt{false}]\!]$ $[\![\texttt{false}]\!] = \mathsf{Bool}$, $[\![x_1]\!] = \mathsf{X}_1$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = [\![x_1]\!]$, $\mathsf{X}_0 = [\![5]\!]$ $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]$	
$\mathsf{X}_1 = [\![\texttt{false}]\!]$ $[\![\texttt{false}]\!] = \mathsf{Bool}$, $[\![x_1]\!] = \mathsf{X}_1$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = [\![x_1]\!]$, $\mathsf{X}_0 = \mathsf{Nat}$ $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]$	$[\![5]\!] := \mathsf{Nat}$
$[\![\texttt{false}]\!] = \mathsf{Bool}$, $[\![x_1]\!] = [\![\texttt{false}]\!]$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = [\![x_1]\!]$, $\mathsf{X}_0 = \mathsf{Nat}$ $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]$	$[\![5]\!] := \mathsf{Nat}$, $\mathsf{X}_1 := [\![\texttt{false}]\!]$
$[\![x_1]\!] = \mathsf{Bool}$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = [\![x_1]\!]$, $\mathsf{X}_0 = \mathsf{Nat}$ $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]$	$[\![5]\!] := \mathsf{Nat}$, $\mathsf{X}_1 := \mathsf{Bool}$, $[\![\texttt{false}]\!] := \mathsf{Bool}$
$[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] = \mathsf{Bool}$, $\mathsf{X}_0 = \mathsf{Nat}$ $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = [\![\texttt{let}(\texttt{false}, x_1.x_1)]\!]$	$[\![5]\!] := \mathsf{Nat}$, $\mathsf{X}_1 := \mathsf{Bool}$, $[\![\texttt{false}]\!] := \mathsf{Bool}$, $[\![x_1]\!] := \mathsf{Bool}$
$\mathsf{X}_0 = \mathsf{Nat}$ $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = \mathsf{Bool}$	$[\![5]\!] := \mathsf{Nat}$, $\mathsf{X}_1 := \mathsf{Bool}$, $[\![\texttt{false}]\!] := \mathsf{Bool}$, $[\![x_1]\!] := \mathsf{Bool}$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] := \mathsf{Bool}$

$[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = \mathsf{Bool}$	$[\![5]\!] := \mathsf{Nat}$, $\mathsf{X}_1 := \mathsf{Bool}$, $[\![\texttt{false}]\!] := \mathsf{Bool}$, $[\![x_1]\!] := \mathsf{Bool}$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] := \mathsf{Bool}$, $\mathsf{X}_0 := \mathsf{Nat}$
	$[\![5]\!] := \mathsf{Nat}$, $\mathsf{X}_1 := \mathsf{Bool}$, $[\![\texttt{false}]\!] := \mathsf{Bool}$, $[\![x_1]\!] := \mathsf{Bool}$, $[\![\texttt{let}(\texttt{false}, x_1.x_1)]\!] := \mathsf{Bool}$, $\mathsf{X}_0 := \mathsf{Nat}$, $[\![\texttt{let}(5, x_0.\texttt{let}(\texttt{false}, x_1.x_1))]\!] = \mathsf{Bool}$

De el proceso anterior se puede concluir que el tipo de la expresión es Bool

Ejemplo 5.2. Vamos a encontrar el tipo de la expresión:

```
lam x => x x
```

que en sintaxis abstracta es:

$$\texttt{lam}(x.\texttt{app}(x, x))$$

Renombramiento de variables

$$\texttt{lam}(x_0.\texttt{app}(x_0, x_0))$$

Generación de Restricciones .

1. $x_0 \mapsto [\![x_0]\!] = \mathsf{X}_0$
2. $\texttt{app}(x_0, x_0) \mapsto [\![x_0]\!] = \mathsf{X}_0, [\![x_0]\!] = [\![x_0]\!] \to [\![\texttt{app}(x_0, x_0)]\!]$
3. $\texttt{lam}(x_0.\texttt{app}(x_0, x_0)) \mapsto$
   $[\![x_0]\!] = \mathsf{X}_0, [\![x_0]\!] = [\![x_0]\!] \to [\![\texttt{app}(x_0, x_0)]\!], [\![\texttt{lam}(x_0.\texttt{app}(x_0, x_0))]\!] = \mathsf{X}_0 \to [\![\texttt{app}(x_0, x_0)]\!]$

Como resultado se obtiene la lista de restricciones $\mathcal{R}$:

$$\begin{aligned} \mathcal{R} \quad = \quad & [\![x_0]\!] = \mathsf{X}_0, \\ & [\![x_0]\!] = [\![x_0]\!] \to [\![\texttt{app}(x_0, x_0)]\!] \\ & [\![\texttt{lam}(x_0.\texttt{app}(x_0, x_0))]\!] = \mathsf{X}_0 \to [\![\texttt{app}(x_0, x_0)]\!] \end{aligned}$$

Unificación de $\mathcal{R}$.

Restricciones $\mathcal{R}$	Unificador μ

$[\![x_0]\!] = \mathsf{X}_0$, $[\![x_0]\!] = [\![x_0]\!] \to [\![\mathtt{app}(x_0, x_0)]\!]$ $[\![\mathtt{lam}(x_0.\mathtt{app}(x_0, x_0))]\!] = \mathsf{X}_0 \to [\![\mathtt{app}(x_0, x_0)]\!]$	
$\mathsf{X}_0 = \mathsf{X}_0 \to [\![\mathtt{app}(x_0, x_0)]\!]$ $[\![\mathtt{lam}(x_0.\mathtt{app}(x_0, x_0))]\!] = \mathsf{X}_0 \to [\![\mathtt{app}(x_0, x_0)]\!]$	$[\![x_0]\!] := \mathsf{X}_0$
$[\![\mathtt{lam}(x_0.\mathtt{app}(x_0, x_0))]\!] = \mathsf{X}_0 \to [\![\mathtt{app}(x_0, x_0)]\!]$	fail

Como el algoritmo de unificación fallo, entonces la expresión no es tipable.

Ejemplo 5.3. Ahora inferiremos el tipo de la expresión:

```
(recfun fact n =>
    if (n == 0)
    then 1
    else n * fact (n - 1)
) 5
```

en sintaxis abstracta:

$$\mathtt{app}(\mathtt{recfun}(fac.n.\mathtt{if}(\mathtt{eq}(n, 0), 1, \mathtt{prod}(n, \mathtt{app}(fact, \mathtt{sub}(n, 1))))), 5)$$

Renombramiento de variables

$$\mathtt{app}(\mathtt{recfun}(x_0.x_1.\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1))))), 5)$$

Generación de Restricciones .

1. $x_0 \mapsto [\![x_0]\!] = \mathsf{X}_0$
2. $x_1 \mapsto [\![x_1]\!] = \mathsf{X}_1$
3. $0 \mapsto [\![0]\!] = \mathsf{Nat}$
4. $1 \mapsto [\![1]\!] = \mathsf{Nat}$
5. $5 \mapsto [\![5]\!] = \mathsf{Nat}$
6. $\mathtt{eq}(x_1, 0) \mapsto \underbrace{[\![x_1]\!] = \mathsf{X}_1, [\![0]\!] = \mathsf{Nat}, [\![x_1]\!] = \mathsf{Nat}, [\![\mathtt{eq}(x_1, 0)]\!] = \mathsf{Bool}}_{\mathcal{R}_1}$
7. $\mathtt{sub}(x_1, 1) \mapsto \underbrace{[\![x_1]\!] = \mathsf{X}_1, [\![1]\!] = \mathsf{Nat}, [\![x_1]\!] = \mathsf{Nat}, [\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}}_{\mathcal{R}_2}$
8. $\mathtt{app}(x_0, \mathtt{sub}(x_1, 1)) \mapsto \underbrace{[\![x_0]\!] = \mathsf{X}_0, \mathcal{R}_2, [\![x_0]\!] = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]}_{\mathcal{R}_3}$
9. $\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1))) \mapsto \underbrace{\mathcal{R}_3, [\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}}_{\mathcal{R}_4}$

10. $\underbrace{\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1)))) \mapsto \mathcal{R}_1, [\![1]\!] = \mathsf{Nat}, \mathcal{R}_4, [\![\texttt{eq}(x_1,0)]\!] = \mathsf{Bool}, [\![1]\!] = [\![\texttt{prod}(x_1,\texttt{sub}(x_1,1))]\!], [\![\texttt{if}(...)]\!] = [\![1]\!], [\![\texttt{if}(...)]\!] = [\![\texttt{prod}(x_1,\texttt{sub}(x_1,1))]\!]}_{\mathcal{R}_5}$

11. $\underbrace{\texttt{recfun}(x_0.x_1.\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1))))) \mapsto \mathcal{R}_5, [\![\texttt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1))))]\!]}_{\mathcal{R}_6}$

12. $\texttt{app}(\texttt{recfun}(\dots),5) \mapsto \mathcal{R}_6, [\![5]\!] = \mathsf{Nat}, [\![\texttt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\texttt{app}(\texttt{recfun}(\dots),5)]\!]$

Como resultado se obtiene la lista de restricciones $\mathcal{R}$:

$$
\begin{array}{rcl}
\mathcal{R} & = & [\![x_0]\!] = \mathsf{X}_0, \\
 & & [\![0]\!] = \mathsf{Nat}, \\
 & & [\![x_1]\!] = \mathsf{Nat}, \\
 & & [\![\texttt{eq}(x_1,0)]\!] = \mathsf{Bool} \\
 & & [\![1]\!] = \mathsf{Nat} \\
 & & [\![x_1]\!] = \mathsf{X}_1, \\
 & & [\![\texttt{sub}(x_1,1)]\!] = \mathsf{Nat} \\
 & & [\![x_0]\!] = [\![\texttt{sub}(x_1,1)]\!] \to [\![\texttt{app}(x_0,\texttt{sub}(x_1,1))]\!] \\
 & & [\![\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1)))]\!] = \mathsf{Nat} \\
 & & [\![1]\!] = [\![\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1)))]\!], \\
 & & [\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1))))]\!] = [\![1]\!], \\
 & & [\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1))))]\!] = [\![\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1)))]\!] \\
 & & [\![\texttt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1))))]\!] \\
 & & [\![5]\!] = \mathsf{Nat}, \\
 & & [\![\texttt{recfun}(x_0.x_1.\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1)))))]\!] = \\
 & & [\![5]\!] \to [\![\texttt{app}(\texttt{recfun}(x_0.x_1.\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1))))),5)]\!]
\end{array}
$$

Unificación de $\mathcal{R}$.

Restricciones $\mathcal{R}$	Unificador μ
$[\![x_0]\!] = \mathsf{X}_0$, $[\![0]\!] = \mathsf{Nat}$, $[\![x_1]\!] = \mathsf{Nat}$, $[\![\texttt{eq}(x_1,0)]\!] = \mathsf{Bool}$ $[\![1]\!] = \mathsf{Nat}$ $[\![x_1]\!] = \mathsf{X}_1$, $[\![\texttt{sub}(x_1,1)]\!] = \mathsf{Nat}$ $[\![x_0]\!] = [\![\texttt{sub}(x_1,1)]\!] \to [\![\texttt{app}(x_0,\texttt{sub}(x_1,1))]\!]$ $[\![\texttt{prod}(x_1,\texttt{app}(x_0,\texttt{sub}(x_1,1)))]\!] = \mathsf{Nat}$ $[\![1]\!] = [\![\texttt{prod}(\dots)]\!]$, $[\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(\dots))]\!] = [\![1]\!]$, $[\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(\dots))]\!] = [\![\texttt{prod}(\dots)]\!]$ $[\![\texttt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\texttt{if}(\texttt{eq}(x_1,0),1,\texttt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\texttt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\texttt{app}(\texttt{recfun}(\dots),5)]\!]$	

$[\![0]\!] = \mathsf{Nat}$, $[\![x_1]\!] = \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] = \mathsf{Bool}$ $[\![1]\!] = \mathsf{Nat}$ $[\![x_1]\!] = \mathsf{X}_1$, $[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $[\![x_0]\!] = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $[\![1]\!] = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![1]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0$,
$[\![x_1]\!] = \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] = \mathsf{Bool}$ $[\![1]\!] = \mathsf{Nat}$ $[\![x_1]\!] = \mathsf{X}_1$, $[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $[\![1]\!] = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![1]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0$, $[\![0]\!] := \mathsf{Nat}$,
$[\![\mathtt{eq}(x_1, 0)]\!] = \mathsf{Bool}$ $[\![1]\!] = \mathsf{Nat}$ $[\![x_1]\!] = \mathsf{X}_1$, $[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $[\![1]\!] = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![1]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$,

$[\![1]\!] = \mathsf{Nat}$ $\mathsf{Nat} = \mathsf{X}_1,$ $[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $[\![1]\!] = [\![\mathtt{prod}(\dots)]\!],$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![1]\!],$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat},$ $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0,$ $[\![0]\!] := \mathsf{Nat},$ $[\![x_1]\!] := \mathsf{Nat},$ $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$
$\mathsf{Nat} = \mathsf{X}_1,$ $[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $[\![1]\!] = [\![\mathtt{prod}(\dots)]\!],$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![1]\!],$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat},$ $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0,$ $[\![0]\!] := \mathsf{Nat},$ $[\![x_1]\!] := \mathsf{Nat},$ $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$
$\mathsf{X}_1 = \mathsf{Nat},$ $[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $\mathsf{Nat} = [\![\mathtt{prod}(\dots)]\!],$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat},$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{X}_1 \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat},$ $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0,$ $[\![0]\!] := \mathsf{Nat},$ $[\![x_1]\!] := \mathsf{Nat},$ $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$
$[\![\mathtt{sub}(x_1, 1)]\!] = \mathsf{Nat}$ $\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $\mathsf{Nat} = [\![\mathtt{prod}(\dots)]\!],$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat},$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat},$ $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0,$ $[\![0]\!] := \mathsf{Nat},$ $[\![x_1]\!] := \mathsf{Nat},$ $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat},$

$\mathsf{X}_0 = [\![\mathtt{sub}(x_1, 1)]\!] \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $\mathsf{Nat} = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$
$\mathsf{X}_0 = \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $\mathsf{Nat} = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{X}_0$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$
$[\![\mathtt{prod}(x_1, \mathtt{app}(x_0, \mathtt{sub}(x_1, 1)))]\!] = \mathsf{Nat}$ $\mathsf{Nat} = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$
$\mathsf{Nat} = [\![\mathtt{prod}(\dots)]\!]$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = [\![\mathtt{prod}(\dots)]\!]$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$

$\mathsf{Nat} = \mathsf{Nat}$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$, $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$
$[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] = \mathsf{Nat}$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to [\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!]$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$,
$\mathsf{Nat} = \mathsf{Nat}$ $[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$,
$[\![\mathtt{recfun}(...)]\!] = \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$,

$[\![5]\!] = \mathsf{Nat}$, $[\![\mathtt{recfun}(\dots)]\!] = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\mathtt{recfun}(...)]\!] := \mathsf{Nat} \to \mathsf{Nat}$
$[\![5]\!] = \mathsf{Nat}$, $\mathsf{Nat} \to \mathsf{Nat} = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\mathtt{recfun}(...)]\!] := \mathsf{Nat} \to \mathsf{Nat}$
$\mathsf{Nat} \to \mathsf{Nat} = [\![5]\!] \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\mathtt{recfun}(...)]\!] := \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] := \mathsf{Nat}$,

$\mathsf{Nat} \to \mathsf{Nat} = \mathsf{Nat} \to [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\mathtt{recfun}(...)]\!] := \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] := \mathsf{Nat}$,
$\mathsf{Nat} = \mathsf{Nat}$ $\mathsf{Nat} = [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\mathtt{recfun}(...)]\!] := \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] := \mathsf{Nat}$,
$\mathsf{Nat} = [\![\mathtt{app}(\mathtt{recfun}(\dots), 5)]\!]$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\mathtt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\mathtt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\mathtt{app}(x_0, \mathtt{sub}(x_1, 1))]\!]$ $[\![\mathtt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\mathtt{if}(\mathtt{eq}(x_1, 0), 1, \mathtt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\mathtt{recfun}(...)]\!] := \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] := \mathsf{Nat}$,

$[\![\texttt{app}(\texttt{recfun}(\dots),5)]\!] = \mathsf{Nat}$	$[\![x_0]\!] := \mathsf{Nat} \to [\![\texttt{app}(x_0, \texttt{sub}(x_1, 1))]\!]$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\texttt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\texttt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to [\![\texttt{app}(x_0, \texttt{sub}(x_1, 1))]\!]$ $[\![\texttt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\texttt{if}(\texttt{eq}(x_1, 0), 1, \texttt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\texttt{recfun}(\ldots)]\!] := \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] := \mathsf{Nat}$,
	$[\![x_0]\!] := \mathsf{Nat} \to \mathsf{Nat}$, $[\![0]\!] := \mathsf{Nat}$, $[\![x_1]\!] := \mathsf{Nat}$, $[\![\texttt{eq}(x_1, 0)]\!] := \mathsf{Bool}$ $[\![1]\!] := \mathsf{Nat}$ $\mathsf{X}_1 := \mathsf{Nat}$, $[\![\texttt{sub}(x_1, 1)]\!] := \mathsf{Nat}$ $\mathsf{X}_0 := \mathsf{Nat} \to \mathsf{Nat}$ $[\![\texttt{prod}(\dots)]\!] := \mathsf{Nat}$ $[\![\texttt{if}(\texttt{eq}(x_1, 0), 1, \texttt{prod}(\dots))]\!] := \mathsf{Nat}$, $[\![\texttt{recfun}(\ldots)]\!] := \mathsf{Nat} \to \mathsf{Nat}$ $[\![5]\!] := \mathsf{Nat}$, $[\![\texttt{app}(\texttt{recfun}(\dots),5)]\!]:=\mathsf{Nat}$

Por lo que la expresión tiene tipo Nat

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