Lecture #8 : Pure exploration

All previous lectures: maximise cumulative reward
→ exploration / exploitation trade-off

In some applications, there is no price for exploring.

Think for example of a researcher testing drugs on mice / artificial human cells or testing products on some people before commercialisation.

Share similarities with regret minimisation, but good algorithms are actually different.

Setting (simple regret)

At each round $t = 1, \dots, T$:

- pulls an arm $a_t \in [K]$
- observes $X_{a_t}(t) \sim \nu_{a_t} \in \mathcal{D}$

We explore for T rounds and commit to best action at time $T+1$.

Goal: minimise simple regret

$$R_T^{\text{simple}} = \mathbb{E}\left[\mu^* - \mu_{a_{T+1}}\right].$$

Algorithm: Uniform exploration

For $t = 1, \dots, T$:

choose $a_t = 1 + (t \bmod k)$

$A_{T+1} \in \operatorname{argmax}_k \hat{\mu}_k(T)$.

Theorem: Uniform-Exploration satisfies for any $\nu \in \mathcal{P}([0,1])^K$

$$R_T^{\text{simple}} \leq \sum_{k, \Delta_k > 0} \Delta_k \exp\left(-\left\lfloor \frac{T}{K} \right\rfloor \Delta_k^2\right)$$

Proof Let k such that $\Delta_k > 0$.

$$\mathbb{P}\left(\hat{\mu}_{k^*}(T) \leq \hat{\mu}_k(T)\right) = \mathbb{P}\left(\hat{\mu}_{k^*}(T) - \hat{\mu}_k(T) \leq 0\right)$$

The terms where k and k^* are pulled are not random. We can directly apply Hoeffding inequality

$\geq \lfloor \frac{T}{K} \rfloor$

Assum wlog. $N_{k^*}(T) \leq N_k(T)$

$$\mathbb{P}\left(\hat{\mu}_{k^*}(T) \leq \hat{\mu}_k(T)\right) \leq \mathbb{P}\left(\sum_{t=1}^{T} X_{k^*}(t) \mathbb{1}_{\{a_t = k^*\}} \leq \frac{N_{k^*}(T)}{N_k(T)} \sum_{t=1}^{T} X_k(t) \mathbb{1}_{\{a_t = k\}}\right)$$

$$\leq \mathbb{P}\left(\underbrace{\sum_{t=1}^{T} X_{k^*}(t) - \mu_{k^*} \mathbb{1}_{\{a_t = k^*\}}}_{N_{k^*}(T) \text{ variables of range } 1} - \underbrace{\sum_{t=1}^{T} \frac{N_{k^*}(T)}{N_k(T)} (X_k(t) - \mu_k) \mathbb{1}_{\{a_t = k\}}}_{N_k(T) \text{ variables of range } \frac{N_{k^*}(T)}{N_k(T)}} \leq -N_{k^*}(T) \Delta_k\right)$$

$$\leq \exp\left(-\frac{2 N_{k^*}(T)^2 \Delta_k^2}{N_{k^*}(T) + \frac{N_{k^*}(T)^2}{N_k(T)}}\right)$$

$$\leq \exp\left(-N_{k^*}(T) \Delta_k^2\right) \leq \exp\left(-\left\lfloor \frac{T}{K} \right\rfloor \Delta_k^2\right)$$

$$R_T^{\text{simple}} = \sum_{k, \Delta_k > 0} \Delta_k \mathbb{P}(a_{T+1} = k)$$

$$\leq \sum_{k, \Delta_k > 0} \Delta_k \mathbb{P}\left(\hat{\mu}_{k^*}(T) \leq \hat{\mu}_k(T)\right)$$

$$\leq \sum_{k, \Delta_k > 0} \Delta_k \exp\left(-\left\lfloor \frac{T}{K} \right\rfloor \Delta_k^2\right) \quad \square$$

Theorem UE, distribution free bound

for any $\nu \in \mathcal{M}([0,1])^K$, $R_T^{\text{simple}} \leq 2\sqrt{2\frac{K\ln(K)}{T}}$

Proof

Actually, we could have written for any $\tilde{\Delta} \geq 0$

$$R_T^{\text{simple}} \leq \tilde{\Delta} + \sum_{k,\, \Delta_k > \tilde{\Delta}} \Delta_k \, \mathbb{P}(a_{T+1} = k)$$

$$\leq \tilde{\Delta} + K\tilde{\Delta} \exp\left(-\left\lfloor \tfrac{T}{K} \right\rfloor \tilde{\Delta}^2\right)$$

for any $\tilde{\Delta} \geq 0$.

$x \longmapsto x\exp\left(-\left\lfloor \frac{T}{K} \right\rfloor x^2\right)$ is $\downarrow$ on $\left[\sqrt{\frac{1}{\lfloor T/K \rfloor}}, +\infty\right)$

Taking $\tilde{\Delta} = \sqrt{\frac{\ln(K) \vee 1/2}{\lfloor T/K \rfloor}}$, we have:

$$R_T^{\text{simple}} \leq \tilde{\Delta} + \tilde{\Delta} K e^{-(\ln K) \vee 1/2} \leq 2\tilde{\Delta}$$

$$\leq 2\sqrt{\frac{\ln K \vee 1/2}{\lfloor \frac{T}{K} \rfloor}} = 2\sqrt{\frac{\ln K}{\lfloor T/K \rfloor}}$$

if $T \leq K$ $\quad R_T^{\text{simple}} \leq 1$ and the bound holds.

if $T \geq K$, $2\lfloor \frac{T}{K} \rfloor \geq \frac{T}{K}$ so

$$R_T^{\text{simple}} \leq 2\sqrt{2\frac{K(\ln(K) \vee 1)}{T}}$$

□

We can show that the minimax lower bound is larger than $c\sqrt{\frac{K}{T}}$, so Uniform-Exploration is nearly optimal in minimax sense.

Can we do better than UE? i.e get rid of $\sqrt{\ln K}$ term.

Best arm identification

Setting 1: (fixed confidence)

At each round $t = 1, \dots, \infty$:

- agent picks an arm $a_t \in [K]$ (based on previous observations)
- observes $X_{a_t}(t) \sim \nu_{a_t} \in \mathcal{D}$
- decides whether to continue sampling or stop

If stop: return a final choice $\psi \in [K]$

The (random) stopping time is called τ

new

Goal: 1) Have a **sound** strategy: $\mathbb{P}(\tau < \infty \text{ and } \mu_\psi < \mu^*) \leq \delta$ (for any ν) (with confidence level $\delta \in (0,1)$; δ: confidence level)

2) Minimize the exploration time $\mathbb{E}[\tau]$

Our algorithm will be built on the following lower bound

Theorem (lower bound)

Let (π, τ, ψ) be a sound strategy for the bandit model $\mathcal{D}$, with confidence level $\delta \in (0,1)$, and let $\nu \in \mathcal{D}^K$. Then:

$$\mathbb{E}[\tau] \geq c^*(\nu) \ln\left(\frac{1}{4\delta}\right) \quad \text{where}$$

$$c^*(\nu)^{-1} = \sup_{\alpha \in P_K} \left(\inf_{\nu' \in \mathcal{D}_{alt}(\nu)} \sum_{k=1}^{K} \alpha_k \, KL(\nu_k, \nu'_k) \right)$$

where $\mathcal{D}_{alt}(\nu) = \left\{ \nu' \in \mathcal{D}^K \;\middle|\; \operatorname{argmax}_k \mathbb{E}(\nu_k) \cap \operatorname{argmax}_k \mathbb{E}[\nu'_k] = \emptyset \right\}$ (i.e. no arm is optimal for both ν and ν')

Another use of fundamental inequality (with stopping time)

Lemma: (admitted)

For all bandit problems $\nu = (\nu_k)_{k\in[K]}$ and $\nu' = (\nu'_k)_{k\in[K]}$ in $\mathcal{D}^K$ with $\nu_k \ll \nu'_k$ for all k,

for all strategies π, for any stopping time τ with respect to the filtration $(\sigma(H_t))$

and any random variable Z taking values in $[0,1]$, $\sigma(H_\tau)$-measurable,

$$\sigma(H_\tau) = \{ A \in \sigma(H_\infty) \mid A \cap \{\tau \le t\} \in \sigma(H_t) \text{ for all } t \}$$

$$\sum_{k=1}^{K} \mathbb{E}_\nu[N_k(\tau)]\, KL(\nu_k, \nu'_k) \ge KL\big(Ber(\mathbb{E}_\nu[Z]), Ber(\mathbb{E}_{\nu'}[Z])\big)$$

Proof of the Theorem

Assume $\mathbb{E}[\tau] < \infty$ (otherwise the result holds), so that $\mathbb{P}(\tau = \infty) = 0$.

Let $\nu' \in \mathcal{D}_{alt}(\nu)$. We define the $\sigma(H_\tau)$-measurable r.v.

$$Z = \mathbb{1}_{\{\tau < \infty \text{ and } \psi \notin \operatorname{argmax}_k \mathbb{E}(\nu'_k)\}}.$$ then the fundamental inequality (with stopping time)

yields:

$$\sum_{k=1}^{K} \mathbb{E}_\nu[N_k(\tau)]\, KL(\nu_k, \nu'_k) \ge KL\big(Ber(\mathbb{E}_\nu[Z]), Ber(\mathbb{E}_{\nu'}[Z])\big)$$

$\mathbb{E}_\nu[Z] \ge 1-\delta$, $\mathbb{E}_{\nu'}[Z] \le \delta$ as π is sound with confidence level δ

$$\ge (1-\delta)\ln\left(\frac{1-\delta}{\delta}\right) + \delta \ln\left(\frac{\delta}{1-\delta}\right).$$

$$= (1-2\delta) \ln\left(\frac{1-\delta}{\delta}\right) \geq \ln\left(\frac{1}{4\delta}\right)$$

Let $x_k = \frac{\mathbb{E}_\nu[N_a(\tau)]}{\mathbb{E}_\nu[\tau]}$ $\quad x \in \mathcal{P}_K$ and we have shown:

$$\mathbb{E}_\nu[\tau] \sum_{k=1}^{K} x_k \mathrm{KL}(\nu_k, \nu'_k) \geq \ln\left(\frac{1}{4\delta}\right).$$

for any $\nu' \in \mathrm{Alt}(\nu)$ and a specific $x \in \mathcal{P}_K$ that is independent of ν'

so: $\mathbb{E}_\nu[\tau] \sup_{x \in \mathcal{P}_K} \inf_{\nu' \in \mathrm{Alt}(\nu)} \sum_{k=1}^{K} x_k \mathrm{KL}(\nu_k, \nu'_k) \geq \ln\left(\frac{1}{4\delta}\right)$.

$C(\nu)^{-1}$

□

Track-and-stop algorithm

Idea of the algorithm is to track the lower bound and stop when τ is larger than the estimated lower bound.

$x^*_k(\nu)$ corresponds to the proportion of pulls on k.

i.e. we should stop when

$$\inf_{\nu' \in \mathrm{Alt}(\nu)} \sum_{k=1}^{K} N_k(t) \mathrm{KL}(\nu_k, \nu'_k) \gtrsim \ln\left(\frac{1}{\delta}\right).$$

Z_t

Problem ν is unknown, but can be estimated.

Assume in the following $\mathcal{D} = \{\mathcal{N}(\mu, 1) \mid \mu \in \mathbb{R}\}$

In that case, we can approximate Z_t by

$$Z_t := \frac{1}{2} \inf_{\mu' \in \mathrm{Alt}(\hat{\mu}(t))} \sum_{k=1}^{K} N_k(t) \left(\hat{\mu}_k(t) - \mu'\right)^2.$$

$$\mathrm{Alt}(\hat{\mu}) = \left\{\mu' \mid \operatorname{argmax}_k \hat{\mu}_k \cap \operatorname{argmax}_k \mu'_k = \emptyset\right\}$$

Track-and-stop algorithm Input δ and $\beta_t(\delta)$

For $t = 1, \dots, K$:

Pull $a_t = t$.

While $Z_t < \beta_t(\delta)$

| if $\min_k N_k(t) \le \sqrt{t}$ then pull $a_{t+1} \in \operatorname{argmin}_k N_k(t)$ — forced exploration

| else choose $a_{t+1} \in \operatorname{argmax}_k t\hat{\alpha}_k(t) - N_k(t)$ — track

stop and return $\psi \in \operatorname{argmax}_k \hat{\mu}_k(t)$ — stop.

$$\hat{\alpha}(t) \in \operatorname*{argmax}_{\alpha \in \mathcal{P}_K} \inf_{\nu' \in \mathcal{D}_{\mathrm{alt}}(\hat{\nu}(t))} \sum_{k=1}^{K} \alpha_k \, \mathrm{KL}(\hat{\nu}(t), \nu')$$

For our Gaussian setting,

Theorem There exists a choice $\beta_t(\delta)$ such that track-and-stop is sound for the Gaussian setting and for any ν with a unique optimal arm:

$$\lim_{\delta \to 0} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)} = c^*(\nu)$$

Lemma: let $f : [K, +\infty) \to \mathbb{R}$ be given by

$$f(x) = \exp(K - x) \left(\frac{x}{K}\right)^K \quad \text{and} \quad \beta_t(\delta) = K \ln(\alpha t^2 + H)) + f^{-2}(\delta).$$

Then for $\tau = \min\{t \mid Z_t \geq \beta_t(\delta)\}$ it holds $\mathbb{P}(\operatorname{argmax}_k \hat{\mu}_k(t) \neq \operatorname{argmax}_k \mu_k) \leq \delta$.

Proof of the lemma:

If $|\operatorname{argmax}_k \hat{\mu}_k(t)| > 1$, then $Z_t = 0$. So assume in the following $\tau < \infty$ and

$|\operatorname{argmax}_k \hat{\mu}_k(\tau)| = 1$.

denote ψ

By definition of $M_{alt}(\hat{\mu})$, for any $k^* \in \operatorname{argmax}_k \mu_k$

$$\mathbb{P}(k^* \neq \psi \text{ and } \tau < \infty) = \mathbb{P}(\mu \in M_{alt}(\hat{\mu}(\tau)) \text{ and } \tau < \infty)$$

By defn of Z_t, for any $\mu' \in M_{alt}(\hat{\mu}(\tau))$: $\frac{1}{2} \sum_{k=1}^{K} N_k(\tau) (\hat{\mu}_k(\tau) - \mu'_k)^2 \geq \beta_\tau(\delta)$, so

$$\mathbb{P}(k^* \neq \psi \text{ and } \tau < \infty) \leq \mathbb{P}\left(\frac{1}{2} \sum_{k=1}^{K} N_k(\tau) (\hat{\mu}_k(\tau) - \mu_k)^2 \geq \beta_\tau(\delta)\right).$$

$(\beta_t(\delta) = K \ln(\alpha t^2 + H)) + f^{-2}(\delta)$

We can show the two following concentration inequalities, which allow to conclude (see exercises)

$$\mathbb{P}\left(\exists t \geq 1, \ \sum_{k=1}^{K} \frac{N_k(t)}{2} (\hat{\mu}_k(t) - \mu_k)^2 \geq K \ln(\alpha t^2 + H)) + x\right) \leq \left(\frac{x}{K}\right)^K \exp(K - x) = f(\delta)$$

□

Proof of the theorem

Define for any $\nu' \in \mathcal{D}^K$, (where $\mathcal{D} = \{\mathcal{N}(\mu, 1) \mid \mu \in \mathbb{R}\}$.

$$\alpha^*(\nu') = \operatorname*{argmax}_{\alpha \in \mathcal{P}_K} \inf_{\tilde{\nu} \in \mathcal{D}_{alt}(\nu')} \sum_{k=1}^{K} \alpha_k \, KL(\nu'_k, \tilde{\nu}_k).$$

We admit the following lemma:

Lemma

If ν' admits a unique optimal arm:
1) $\alpha^*(\nu')$ is unique
2) α^* is continuous at ν'

Define the distance in $\mathcal{D}^K$: $d(\nu, \nu') = \max_k |\mathbb{E}[\nu'_k] - \mathbb{E}[\nu_k]|$,

and $\Phi(\nu, \alpha) = \frac{1}{2} \min_{k \neq k^*} \frac{\alpha_k + \alpha_{k^*}}{\alpha_k \cdot \alpha_{k^*}} \Delta_k^2$.

Admit that $\alpha^*(\nu) = \operatorname*{argmax}_{\alpha \in \mathcal{P}_K} \Phi(\nu, \alpha)$ and Φ is concave in α.

Let $\varepsilon > 0$ be a small constant and define the random times

$$\tau_\nu(\varepsilon) = 1 + \max\{t \mid d(\hat{\nu}^t, \nu) \geq \varepsilon\}$$
$$\tau_\alpha(\varepsilon) = 1 + \max\{t \mid \|\alpha^*(\nu) - \alpha^*(\hat{\nu}^t)|_\infty \geq \varepsilon\}$$
$$\tau_T(\varepsilon) = 1 + \max\{t \mid \|\alpha^*(\nu) - \frac{N(t)}{t}\|_\infty \geq \varepsilon\}.$$

Note these are not stopping times.

1) We are gonna use the first concentration inequality admitted in the proof of the Lemma that guarantees soudness of Track-and-Stop.

(a) Define the random variable:

$$\Lambda = \min\{\lambda \geq 1 \mid d(\hat{\nu}^t, \nu) \leq \sqrt{\frac{2\ln(2\lambda K t(t+1))}{\min_k N_k(t)}} \text{ for all t}\}.$$

Show that $\mathbb{E}[\ln(\Lambda)^2] < \infty$.

(b) Prove that $\mathbb{E}[\tau_\nu(\varepsilon)] < \infty$ for all $\varepsilon > 0$.

2) Prove that $\mathbb{E}[\tau_\alpha(\varepsilon)] < \infty$ for all $\varepsilon > 0$.

3) Prove that $\mathbb{E}[\tau_T(\varepsilon)] < \infty$ for all $\varepsilon > 0$.

4)

(a) Define for any $\varepsilon > 0$

$$\tau_\beta(\varepsilon, \delta) = 1 + \max\{t \mid t\Phi(\nu, \alpha^*(\nu)) < \beta_t(\delta) + \varepsilon t\}$$
$$\text{and} \quad u(\varepsilon) = \sup_{\nu', \alpha} \{\Phi(\nu, \alpha^*(\nu) - \Phi(\nu', \alpha)) \mid d(\nu', \nu) \leq \varepsilon, \|\alpha - \alpha^*(\nu)\|_\infty \leq \varepsilon\}.$$

Show that $\mathbb{E}[\tau] \leq \mathbb{E}[\tau_\nu(\varepsilon)] + \mathbb{E}[\tau_T(\varepsilon)] + \mathbb{E}[\tau_\beta(u(\varepsilon), \delta)]$.

(b) Conclude that $\lim_{\delta \to 0^+} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)} \leq c^*(\nu)$.

Solution: 1) a) Define the random variable:

$$\Lambda = \min\{\lambda \geq 1 \mid d(\hat{\nu}^t, \nu) \leq \sqrt{\frac{2\ln(2\lambda K t(t+1))}{\min_k N_k(t)}} \text{ for all t}\}.$$

The mentioned concentration inequality (in the hint) implies that $\mathbb{P}(\Lambda \geq x) \leq 1/x$. So that

$$\begin{aligned}\mathbb{E}[\ln(\Lambda)^2] &= \int_0^\infty \mathbb{P}(\Lambda \geq \exp(\sqrt{x}))\mathrm{d}x \\ &\leq \int_0^\infty \exp(-\sqrt{x})\mathrm{d}x \\ = 2\int_0^\infty u e^{-u}\mathrm{d}u = 2.\end{aligned}$$

b) By definition of Λ,

$$\tau_\nu(\varepsilon) \leq 1 + \max\{t \mid \sqrt{\frac{2\ln(\Lambda) + 2\ln(2Kt(t+1))}{\min_k N_k(t)}} > \varepsilon\}.$$

The forced exploration in the algorithm means that $N_k(t) \geq \frac{\sqrt{t}}{2}$ almost surely for t large enough. Hence,

$$\tau_\nu(\varepsilon) = \mathcal{O}\left(\ln(\Lambda)^2\right).$$

So finally $\mathbb{E}[\tau_\nu(\varepsilon)] < \infty$.

2) This is a consequence of 1) and the continuity of α^* at ν.

3) For t large enough ($t = \Omega(\tau_\alpha(\varepsilon/2K))$, the algorithm (tracking part dominates the forced exploration at some point) implies that

$$N_k(t) \leq t(\alpha_k^*(\nu) + \frac{\varepsilon}{2K}) + 1 \qquad \leq t(\alpha_k^*(\nu) + \frac{\varepsilon}{K}).$$

But since $\sum_k N_k(t) = t$, this also implies

$$N_k(t) \geq t(\alpha_k^*(\nu) - \varepsilon).$$

This thus allows to conclude as $\mathbb{E}[\tau_T(\varepsilon)] \leq \mathcal{O}\left(\mathbb{E}[\tau_\nu(\varepsilon/2K)]\right)$.

4) a) When $t \geq \max\{\tau_\nu(\varepsilon), \tau_t(\varepsilon), \tau_\beta(u(\varepsilon), \delta)\}$,

$$\begin{aligned}tZ_t &= t\Phi(\hat{\nu}^t, \frac{N_k(t)}{t}) \\ &\geq t(\Phi(\nu, \alpha^*(\nu)) - u(\varepsilon)) \geq \beta_t(\delta).\end{aligned}$$

So $\tau \leq \max\{\tau_\nu(\varepsilon), \tau_t(\varepsilon), \tau_\beta(u(\varepsilon), \delta)\}$.

b) Taking the limit $\delta \to 0$, there only remains the last term (the pull choices do not depend on δ), i.e.

$$\limsup_{\delta \to 0} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)} \leq \limsup_{\delta \to 0} \frac{\mathbb{E}[\tau_\beta(u(\varepsilon), \delta)]}{\ln(1/\delta)}.$$

Since $\beta_t(\delta) = \frac{\ln(1/\delta)}{t} + o(\frac{\ln(1/\delta)}{t})$, we have $\tau_\beta(u(\varepsilon), \delta) = \frac{\ln(1/\delta)}{\Phi(\nu, \alpha^*(\nu)) - u(\varepsilon)} + o(\ln(1/\delta))$. Continuity of Φ at $(\nu, \alpha^*(\nu))$ ensures that $\lim_{\varepsilon \to 0} u(\varepsilon) = 0$ and the result then follows taking $\varepsilon \to 0$.

Setting 2: (fixed budget)

At each round $t = 1, \dots, T$:

- agent picks an arm $a_t \in [K]$ (based on previous observations)
- observes $X_{a_t}(t) \sim \nu_{a_t} \in \mathcal{D}$

After round T, return $\psi \in [K]$.

Goal: minimize $\mathbb{P}(\mu^* > \mu_\psi)$

→ much harder problem

Complexities: $H_1 = \sum_{a, \Delta_a > 0} \frac{1}{\Delta_a^2}$, $H_2 = \max_{k, \Delta_k > 0} \frac{k}{\Delta_{(k)}^2}$

reordered values: $\Delta_{(1)} \leq \Delta_{(2)} \leq \dots \leq \Delta_{(K)}$

$$H_2 \leq H_1 \leq \ln(2K)\, H_2$$

Lower bound of order $\begin{cases} \exp\left(-\frac{T}{\ln(K) H_1}\right) & \text{if } H_1 \text{ unknown} \\ \exp\left(-\frac{T}{H_1}\right) & \text{if known} \end{cases}$

First approach: uniform exploration of the arms. Good baseline, but not very good when arms have very different means.

Sequential Halving:

Set $L = \lceil \log_2(K) \rceil$ and $A_1 = [K]$

For $\ell = 1, \dots, L$:

> Pull each arm in A_ℓ $T_\ell = \left\lfloor \frac{T}{L |A_\ell|} \right\rfloor$ times
>
> Let $\hat{\mu}_i^\ell$ be the empirical mean of arm i based only on these last T_ℓ samples
>
> Let $A_{\ell+1}$ contain the top $\lceil \frac{|A_\ell|}{2} \rceil$ arms in A_ℓ

Return ψ as the only arm in A_{L+1}

Thm:

If the distributions are 1- sub-Gaussian, then Sequential Halving satisfies

$$P(\mu^* > \mu_\psi) \leq 3 \log_2(K) \exp\left(-\frac{T}{16 H_2 \log_2(K)}\right)$$

Remarks

- close to lower bound
- For uniform exploration, we can bound this probability by

$$\sum_{k:\Delta_k > 0} \exp\left(-\frac{\lfloor T/K \rfloor \Delta_k^2}{4}\right).$$

- UE slightly better than SH when $\Delta_k = \Delta$ for any k $\left(H_2 = \frac{K}{\Delta^2}\right)$

but SH much better than UE when $\Delta_2 = \Delta \ll 1$ $\left(H_2 = \frac{1}{\Delta^2}\right)$

$\Delta_k = 1$ for $k > 2$