Statistical inference - Tutorial 4

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Exercise 4

Let's define X as the number of heads observed in 20 tosses. From the exercise, we know that $X \sim \text{Binomial}(n = 20, p)$.

Part a

We recall the Neyman-Pearson lemma presented on page 5 (Hypothesis testing, weeks 8 and 9).

Lemma. The Neyman-Pearson lemma: Let H_0 and H_a be simple hypothesis and let f_0 and f_a denote the joint pdf or pmf under H_0 and H_a respectively. Then the test that rejects for small values of

$$T(\underline{X}) = \frac{f_0(\underline{x})}{f_a(\underline{x})}$$

is most powerful for testing H_0 against H_a.

Note $T(\underline{X})$ in the above is simply the ratio of the likelihoods under H_0 and H_a.

In our case, we have that

$$H_0 : p = p_0 \text{ versus } H_a : p = p_a, \tag{1}$$

with $p_a > p_0$. From the Neyman-Pearson lemma, we have that the most powerful test for testing H_0 against H_a is given by

$$\begin{aligned} T(\underline{X}) &= \frac{f_0(\underline{x})}{f_a(\underline{x})}, && (2)\\ &= \frac{\prod_{i=1}^{K} \binom{n}{x} p_0^{x_i}(1-p_0)^{n-x_i}}{\prod_{i=1}^{K} \binom{n}{x_i} p_a^{x_i}(1-p_a)^{n-x_i}}, && (3)\\ &= \left(\frac{p_0}{p_a}\right)^{\sum_{i=1}^{K} x_i} \left(\frac{1-p_0}{1-p_a}\right)^{Kn-\sum_{i=1}^{K} x_i}. && (4) \end{aligned}$$

We reject H_0 when $T(\underline{X})$ is small (i.e., when $\sum_{i=1}^{K} x_i$ is large).

part b

We know that $\alpha = 0.05 \Rightarrow z_{(0.05)} = 1.64$, $p_0 = 0.5$ (considering a fair coin) and $n = 20$.

Here, the size of the rejection region is given by $\{X : X \geq C|H_0\}$, where $P(X \geq C|H_0) = 0.05$. In addition, as we're supposed to use the Normal approximation, we know that

$$X \sim \mathrm{N}\left(np_0, np_0(1-p_0)\right), \text{ under } H_0. \quad (5)$$

How do we know that? Let's have a look at page 2 (weeks 8 and 9) and https://en.wikipedia.org/wiki/Binomial_distribution#Normal_approximation. Hence,

$$\begin{aligned} P(X \geq C|H_0) &\approx P\left(\frac{X-\mathbb{E}(X)}{\sqrt{\mathbb{V}(X)}} \geq \frac{C-\mathbb{E}(X)}{\sqrt{\mathbb{V}(X)}}\right) = 0.05 && (6)\\ &= P\left(Z \geq \frac{C-\mathbb{E}(X)}{\sqrt{\mathbb{V}(X)}}\right) = 0.05 && (7)\\ &= P\left(Z \geq \frac{C-np_0}{np_0(1-p_0)}\right) = 0.05 && (8)\\ &= P\left(Z \geq 1.64\right) = 0.05. && (9) \end{aligned}$$

If we solve

$$\frac{C - 20 \times 0.5}{20 \times 0.5(1-0.5)} = 1.64 \quad (10)$$

we'll find that $C = 13.67$. That is, if $X > 13.67$, we have evidence to reject H_0. Given that $X = 14$, we reject H_0 (i.e, we can say that the coin isn't fair).

part c

To calculate the power of the test, we know that $p_a = 0.7$. Again, let's have a look at page 2 (weeks 8 and 9). Here, the power of the test is given by $\{X : X \geq C|H_a\}$, where $P(X \geq C|H_a) =?$.

$$\begin{aligned} P(X \geq C|H_1) &\approx P\left(Z \geq \frac{C - np_a}{np_a(1-p_a)}\right) && (11) \\ &= P\left(Z \geq \frac{13.67 - 20 \times 0.7}{20 \times 0.7(1-0.7)}\right) && (12) \\ &= P(Z \geq -0.163) = 0.564, && (13) \end{aligned}$$

which can be obtained in R via `1 - pnorm(-0.163) = 0.564`.

Exercise 3

part a

We know that $T \sim \mathrm{N}(0,1)$. If a test rejects for large values of $|T|$, then this test rejects when $T < -1.5$ and $T > 1.5$. Hence,

$$\begin{aligned} P(|T| > 1.5) &= P(T < -1.5) + P(T > 1.5), && (14) \\ &= \texttt{pnorm(-1.5,0,1)} + \texttt{(1 - pnorm(1.5,0,1))} && (15) \\ &= 0.1336. && (16) \end{aligned}$$

part b

Answer the same question if the test rejects for large T. That is,

$$\begin{aligned} P(T > 1.5) &= \texttt{1 - pnorm(1.5,0,1)} && (17) \\ &= 0.0668. && (18) \end{aligned}$$

Exercise 2 - hints

part a

1. Find the MLE for θ. That's, find $\hat{\theta} =?$;

2. Find the $\mathbb{E}(\hat{\theta})$ and $\mathbb{V}(\hat{\theta}) =$ CRLB;
3. Go back to page 10 (weeks 8 and 9, Confidence Interval) and that's it.

part b

1. If $Y = n\bar{X}/\theta \sim \chi^2_{2n}$, we can define something like $P(a < Y < b) = 0.95$;
2. Hence, we can try finding something like $P(? < \theta < ??) = 0.95$, if solve it for θ;
3. Finally, at $n = 10$, we can use the `qchisq(*, **)` (or a χ^2 table) to obtain the lower and upper limits of the CI for θ.