UNIVERSITÉ DU QUÉBEC À MONTRÉAL

# THREES PROBLEMS ON CYCLES IN SIMPLE GRAPHS

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# TROIS PROBLÈMES DE CYCLES DANS DES GRAPHES SIMPLES

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# RÉSUMÉ

Le présent mémoire a pour sujet trois concepts différents concernant les cycles dans des graphes simples : indice de décyclage (la taille minimum d'un transversal de cycles), le nombre cyclomatique (la taille d'une base de l'espace des cycles) et la base minimum de l'espace des cycles. Ils ont tous de nombreuses applications en informatique ainsi que dans d'autres sciences appliquées. Nous nous intéressons à ce qui arrive à ces paramètres quand de nouveaux graphes sont construits comme produits divers à partir de graphes donnés. Nous survolons la littérature et nous présentons quelques nouveaux résultats dans ces trois directions. Nous donnons également un contre-exemple à l'algorithme de Kaveh et Mizrai pour trouver une base minimum de l'espace des cycles du produit lexicographique et nous en présentons une version correcte.

**Mots-clés:** Produits de graphes, jointure de graphes, indice de décyclage, nombre cyclomatique, base minimum de cycles

## ABSTRACT

In the present thesis we study three different concepts concerning cycles of graphs: decycling number (feedback vertex number), basis number, and minimum cycle basis (MCB). They all have many applications in computer science and other applied sciences. We are interested in what happens to these parameters when new graphs are constructed from given graphs by various product operations. We review old results and give some new results in these three directions. We also present a counterexample to Kaveh and Mirzaie's algorithm for minimum cycle basis of lexicographic product and present a correct way to construct a minimum cycle basis for this product.

**Keywords:** Graph products, join of graphs, decycling number, basis number, minimum cycle basis.

# INTRODUCTION

Cycles have an important role in graph theory and many applications in other sciences. In this thesis, we are concerned with three parameters connected to cycles in graphs (decycling number, basis number, and minimum cycle basis) and the way they can be obtained for the join of two graphs as well as for various graph products. Our main aim is to find the decycling number, the basis number and the minimum cycle basis of lexicographic and co-normal products of graphs, pointing out an error in [31] and giving a correct construction.

In the first chapter, we review basic concepts, definitions and results relevant to our research.

In the second chapter, we introduce the decycling number of a graph. After reviewing previous results on the decycling number of the Cartesian and strong products of two graphs, we give an upper bound for the lexicographic and co-normal products as well as an exact formula when one factor of the product is a path, a cycle or a complete graph.

In the third chapter we study the basis number of graphs. We review the results on the basis number of the Cartesian, strong and lexicographic products and give a bound for the basis number of the join of two graphs.

Finally, in the fourth chapter, we review definitions, properties and previous results on the minimum cycle basis of the Cartesian, strong and lexicographic products of graphs and give new result on the minimum cycle basis of the join of two graphs. We also give some new results on the minimum cycle basis of the lexicographic and co-normal products of graphs obtained as extensions of the minimum cycle basis of the strong product of the components. In doing so, we point out a mistake in [31] and give a correct construction and a proof.

# CHAPTER I

# BASIC DEFINITIONS

A (finite simple) **graph** , $G$, is an ordered pair of sets $(V, E)$ where $V$ is a finite set of **vertices** and $E$ is a subset of $\binom{V}{2} = \{\{u, v\} \mid u \neq v,\ u, v \in V\}$. The elements of $E$ are called **edges**. We write $G = (V, E)$. When a graph is given only as $G$, we often write $V(G)$ and $E(G)$ for its sets of vertices and edges, respectively. The **order** of the graph $G$ is denoted by $|G|$ and is the number of its vertices, $|V(G)|$. Often we denote $|V(G)|$ and $|E(G)|$ by $p$ and $q$ respectively. The edge $e = \{u, v\}$ is usually denoted by $uv$; $u$ is then said to be **adjacent** to $v$ and vice versa. Also, $u$ and $v$ are said to be **incident** with $e$. For example, Figure 1.1 represents a graph with the set of vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the set of edges $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_5, v_2v_5\}$.
A **path** of length $n$ is an sequence of $n+1$ distinct vertices $v_0 \cdots v_n$ such that $v_iv_{i+1} \in E$ for $i = 0, \ldots, n-1$. Allowing the first and last vertex to be identical we obtain a **cycle**. That is, a cycle of length $n$ is a sequence of $n$ distinct vertices $v_0 \cdots v_{n-1}$ such that $v_iv_{i+1} \in E$ for $i = 0, \ldots, n-1$, addition modulo $n$. If vertices $u$ and $v$ are connected in $G$, the **distance** between $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest path between $u$ and $v$ in $G$; if there is no path connecting $u$ and $v$ we define $d_G(u, v)$ to be infinite. A **chord** in a cycle $C$ is an edge of the graph $G$ that is not part of the cycle but connects two vertices of the cycle. That is, a chord in a cycle $v_1 \ldots v_n$, $n \geq 3$, is an edge $v_iv_j \in V$ such that $1 < |i-j| < n-1$. The edge $v_2v_5$ in the graph of Figure 1.1 is a chord of the cycle $v_1v_2v_3v_4v_5$.

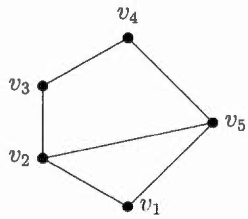



Figure 1.1: A simple graph of 5 vertices and 6 edges.

A **subgraph** $H$ of graph $G$, is a graph whose vertex set is a subset of the vertex set of $G$, and whose edges are a subset of the edge set of $G$ (by the definition of a graph, only edges connecting vertices in $V(H)$ can appear in $E(H)$). We write $H \subseteq G$ to show that $H$ is a subgraph of $G$. A subgraph $H$ of $G$ is **induced** (by the vertex set $V(H)$) if $E(H) = \binom{V(H)}{2} \cap E(G)$. For example, Figure 1.2 shows an induced subgraph on vertices $\{v_1, v_2, v_5\}$, and a non induced subgraph on vertices $\{v_1, v_2, v_3, v_4, v_5\}$ the graph of Figure 1.1. The graph induced by the set $S \subseteq V(G)$ is denoted by $G\langle S\rangle$ or $G_{|S}$ but when the graph $G$ is clear from the context, we write simply $\langle S\rangle$. Similarly, the subgraph induced by $G \setminus S$ is denoted by $G_{|V-S}$ or $G\langle G - S\rangle$, or, more simply, by $G - S$; it is the subgraph obtained from $G$ by deleting the vertices in $S$ together with the edges incident to them. If $S = \{v\}$ we write $G - v$ for $G - \{v\}$.

The **degree** of vertex $u$ of a graph $G$ is the number of edges which are incident with the vertex and is denoted by $d(u)$. If a graph $G$ has $n$ vertices, the **degree sequence** of $G$ is the non-increasing sequence of the degrees of its vertices $d_1, d_2, \cdots, d_n$ with $d_i \geq d_{i+1}$ for $i = 1, \ldots, n-1$.

The **union** of graphs $G$ and $H$ is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G$ and $H$ have no vertex in common, the union is **disjoint** and is often denoted by $G + H$. Note that this allows for $G + G$ where the vertex sets are made disjoint in some standard manner. The intersection $G \cap H$ of $G$ and $H$ is a graph with vertex set $V(G) \cap V(H) \neq \emptyset$ and edge set $V(G) \cap V(H)$.

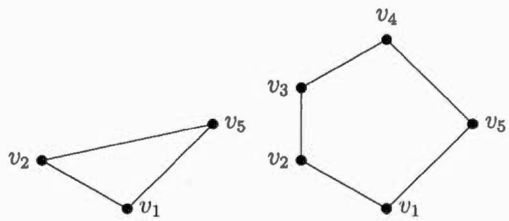



Figure 1.2: An induced subgraph and a non-induced subgraph of the graph of Figure 1.1

A graph is **connected**, if there is a path between each pair of its vertices. Otherwise the graph is **disconnected**. A maximal (vertices) induced connected subgraph of a graph $G$ is called a **connected component** of $G$. The number of connected components of a graph $G$ is denoted by $\omega(G)$. For example, Figure 1.3 shows a graph with three connected components.

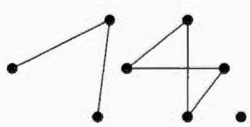


Figure 1.3: A graph with three connected components.

Two graphs $G$ and $H$ are **isomorphic** , written $G \cong H$, if there are bijections $\varphi : V(G) \longrightarrow V(H)$ and $\psi : E(G) \longrightarrow E(H)$ such that $e = ab \in E(G)$ if and only if $\psi(e) = \varphi(a)\varphi(b)$; such a pair $(\varphi, \psi)$ of mappings is called an **isomorphism** between $G$ and $H$.

It is convenient to define some special families of graphs, with appropriate notation.

- A **path graph**, or simply a **path** , denoted by $P_n$, is a graph isomorphic to the graph with vertex set $V(P_n) = \{v_0, ..., v_{n-1}\}$ and edge set

$$E(P_n) = \{\{v_i, v_{i+1}\} \mid 0 \leq i \leq n-1\}.$$

- A **cycle graph**, or a **cycle,** denoted by $C_n$, is a path with an extra edge $\{v_1,\ v_n\}$ between the first and last vertices. Thus a cycle of length $n$ is a graph isomorphic to the graph $C_n$ on the vertex set $\{v_0, ..., v_{n-1}\}$ and edge set $\{\{v_i, v_{i+1}\} \mid 0 \leq i \leq n-1\}$ with addition modulo $n$. Any such cycle is denoted by $C_n = (v_1, v_2, \cdots, v_n)$.

- A **tree** is a connected graph with no cycles, that is, a connected acyclic graph, and is usually denoted by $T$. An acyclic graph is a **forest**; its connected components are trees.

- A **complete graph** is a graph with all possible edges, i.e. $E = \binom{V}{2}$. If a complete graph has $n$ vertices, it is denoted by $K_n$.

- A **bipartite graph** is one whose vertex set can be partitioned into two subsets $X$ and $Y$, so that each edge has one end in $X$ and one end in $Y$; the partition $(X, Y)$ is called a **bipartition** of the graph.

- A **complete bipartite graph** is a bipartite graph with bipartition $(X, Y)$ such that each vertex of $X$ is adjacent to each vertex of $Y$; if $|X| = n$ and $|Y| = m$, such a graph is denoted by $K_{n,m}$.

- A **planar Graph** is one which can be drawn in the plane in such a way that the edges intersect only at their endpoints.

- A **hypercube**, denote by $Q_n$, is the graph whose vertices are the ordered $n$- tuples of 0's and 1's, two vertices being joined if only if they differ in exactly one coordinate.

For a given graph $G = (V, E)$ we define two related graphs.

- A **spanning tree**, denoted by $T_G$, of connected graph $G$ is a subgraph of $G$ that is a tree and contains all the vertices of $G$, i.e. $V(T_G) = V(G)$. Clearly $T_G$ is not unique unless $G$ itself is a tree.

- A **complement** of a graph $G$, denoted by $\overline{G}$, is a graph on the same vertex set as $G$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$.

- A **null graph** is a graph with no edges. A null graph with $n$ vertices is denoted by $\overline{K_n}$.

One other concept will be needed. Given a simple graph G, a minimum distance tree rooted at vertex $v$ is a spanning tree $T$ of $G$, such that the path distance from root $v$ to any other vertex $u$ in $T$ is the shortest path distance from $v$ to $u$ in G. Let $T_G^a$ be a minimum distance tree (breadth-first search tree) rooted at a vertex $a$. This induces a breadth-first search numbering of the vertices of $G$ with $a_0 = a, , a_1, a_2, \cdots, a_{|V(G)|}$ with the obvious property that $d(a_0, a_i) \leq d(a_0, a_j)$ if $i \leq j$ and in particular $d(a_0, a_i) \leq d(a_1, a_{i+1}) = d(a_1, a_i) + 1$.

The removal of an edge from a tree results in a forest with two connected components. Thus each edge of a tree defines a partition of the vertex set. In a connected graph $G$ with a spanning tree $T_G$ we have the same phenomenon - the removal of an edge from $T_G$ defines a partition of the vertex set of $G$, each inducing a connected subgraph. The **fundamental cutset** in a connected graph $G$ with respect to $T_G$ and an edge $e \in E(T_G)$ is defined as the set of edges that must be removed from $G$ to obtain the same partition as that resulting from the removal of $e$ from $T_G$. Thus, each spanning tree defines a set of $V - 1$ fundamental cutsets, one for each edge of the spanning tree.

An **independent (vertex) set** in graph $G$ is a subset of $V$ no two of whose vertices are adjacent in the graph $G$. The **independence number** of a graph $G$, denoted by $\alpha(G)$, is the maximum number of vertices in an independent set in $G$.

A **vertex cover** of a graph $G$ is a subset of $V$ such that every edge of $G$ is incident with a vertex in it. The **covering number**, noted by $\beta(G)$ is the minimum number of vertices in a vertex cover. There is an elementary but useful connection between the independence and covering numbers of a graph. For graph theoretic terms not defined here see [6].

**Theorem 1.1.** *[6] For a graph $G$, $\alpha(G) + \beta(G) = |G|$.*

**Proof.** Suppose $S$ is an independent set in the graph $G$. By definition, there is no edge of $G$ with endpoints in $S$, and so each edge has at least one endpoint in $V \setminus S$, that is $V \setminus S$ is a vertex cover of $G$. Now if $S$ is a maximum independent set of $G$, and $K$ a minimum covering of $G$, then $V \setminus K$ is an independent set and $V \setminus S$ is a covering set of $G$. We have $|G| - \beta(G) = |V \setminus K| \leq \alpha(G)$ and $|G| - \alpha(G) = |V \setminus S| \geq \beta(G)$ and hence $\alpha(G) + \beta(G) = |G|$. $\square$

Recall that the Cartesian product of two non-empty sets $A$ and $B$ is the set of ordered couples $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$.

We can now define the operations on graphs that interest us in this thesis. Suppose $G$ and $H$ are two graphs with vertex sets $V(G)$ and $V(H)$, and edge sets $E(G)$ and $E(H)$. We define the following new graphs.

**Join of two graphs:** The join of two disjoint graphs $G$ and $H$, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{\{u, v\} \mid u \in$

$V(G), v \in V(H)\}$. It follows that $G \vee H$ has $|V(G)| + |V(H)|$ vertices and $|E(G)| + |E(H)| + |V(G)||V(H)|$ edges.

In all the following products of two graphs, the vertex set of the product graph is the set $V(G) \times V(H)$ Cartesian product of the two vertex sets) and so contains $|V(G) \times V(H)| = |V(G)| \times |V(H)|$ vertices. The difference in the various graph products is in the definition of their edge sets. Since the vertices are couples $(a, u)$ we denote the edges by $(a, u)(b, v)$ for readability. For more information on product graphs we refer to [22].

**Cartesian product:** The edge set of the Cartesian product $G \Box H$ of $G$ and $H$ is defined by

$$E(G \Box H) = \{(a, u)(b, v) \mid [a = b \wedge uv \in E(H)] \vee [u = v \wedge ab \in E(G)]\}.$$

By definition, the Cartesian product $G \Box H$ contains $|V(G)|$ copies of the graph $H$ and $|V(H)|$ copies of the graph $G$ and so has $|V(G)||E(H)| + |V(H)||E(G)|$ edges.

**Direct product:** The edge set of the direct product (also called **categorical product**) $G \times H$ of $G$ and $H$ is defined by

$$E(G \times H) = \{(a, u)(b, v) \mid ab \in E(G) \wedge uv \in E(H)\}.$$

Note that $|E(G \times H)| = 2|E(G)||E(H)|$.

**Strong product:** The edge set of the strong product $G \boxtimes H$ of $G$ and $H$ is defined by

$$E(G \boxtimes H) = E(G \Box H) \cup E(G \times H).$$

It is easy to see that $|E(G \boxtimes H)| = |V(G)||E(H)| + |V(H)||E(G)| + 2|E(G)||E(H)|$.

**Lexicographic product:** The edge set of the lexicographic product $G[H]$ of $G$ and $H$ is defined by

$$E(G[H]) = \{(a,u)(b,v) \mid ab \in E(G) \vee [a = b \wedge uv \in E(H)]\}.$$

An easy way to visualise the lexicographic product of graphs $G$ and $H$ is as the graph obtained from graph $G$ by replacing each vertex $a$ by a (disjoint) copy $H_a$ of $H$ and adding all edges between $H_a$ and $H_b$ when $a$ and $b$ are adjacent in $G$. (i.e $H_a \vee H_b$ replaces $ab \in E(G)$).

The edge set of the graph $G[H]$ consists of $|V(G)||E(H)|$ edges for the copies of $H$ and $|V(H)|^2|E(G)|$ edges that join copies of $H$. Thus $|E(G[H])| = |V(G)||E(H)| + |V(H)|^2|E(G)|$.

**Co-normal product** The edge set of the co-normal product $G \bullet H$ of $G$ and $H$ is defined by

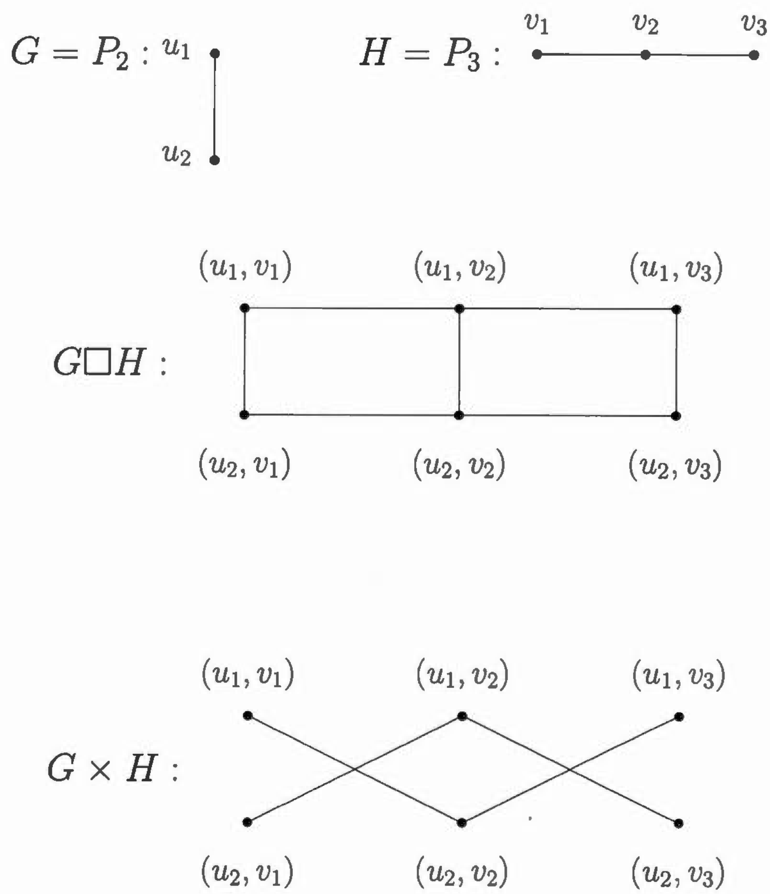
$$E(G \bullet H) = \{(a,u)(b,v) \mid ab \in E(G) \vee uv \in E(H)\}.$$

There is $|G|$ copies of graph $H$ and $|H|$ copies of graph $G$ in $G \bullet H$ such that any two copies of graph $G$ are joined to each other if their corresponding vertices in $H$ are adjacent, as well as any two copies of graph $H$ if their corresponding vertices in $G$ are adjacent, then there is $|V(G)|^2|E(H)| + |V(H)|^2|E(G)|$ edges, but there is $2|E(G)||E(H)|$ common edges in this visualisation then $G \bullet H$ has $|V(G)|^2|E(H)| + |V(H)|^2|E(G)| - 2|E(G)||E(H)|$ edges.

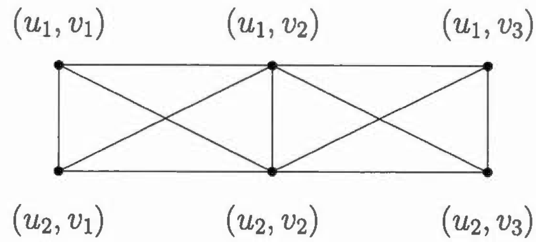
By the definition it is clear that the Cartesian, direct, strong and co-normal products are commutative while the lexicographic product is not. Moreover $G \square H \subseteq G \boxtimes H \subseteq$

$G[H] \subseteq G \bullet H$

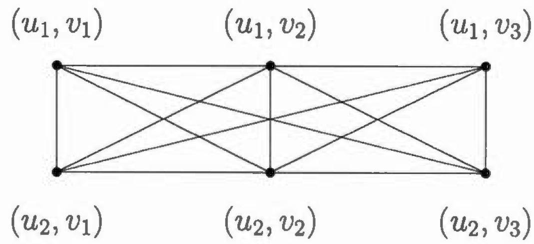
In Figure 1.4 there are the various products of graphs $G = P_2$ and $H = P_3$ :



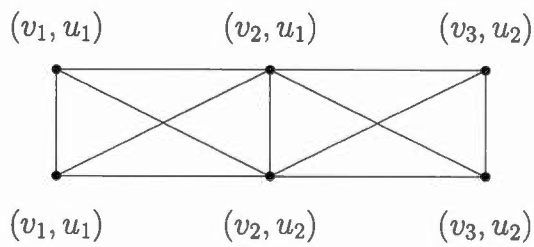
$G \boxtimes H$ :




$G[H]$ :




$H[G]$ :



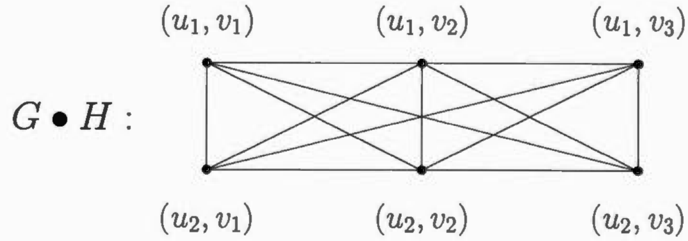



Figure 1.4: Various Products of $P_2$ and $P_3$

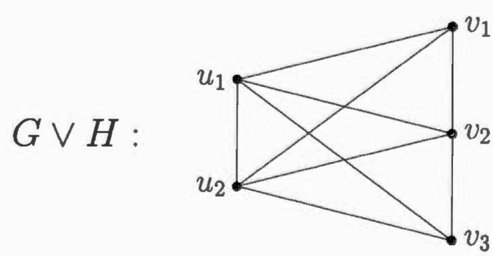



Figure 1.5: The join of $P_2$ and $P_3$.

# CHAPTER II

# THE DECYCLING NUMBER OF GRAPHS

The first subject that we want to consider is the decycling number of a graph. Decycling of graphs has a variety of applications in applied sciences for example in the areas of combinational circuit design [28], deadlock prevention in computer systems [37] and other problems.

Suppose $G(V, E)$ is a simple connected graph. If $S \subseteq V(G)$ and $G - S$ is acyclic, we call $S$ a **decycling (vertex) set** of $G$; it is also known as a **feedback vertex set** . Let us agree that to *decycle* a graph is to remove a set of vertices (and the edges incident to the vertices in the set) so that the remaining graph is acyclic. The smallest size of a decycling vertex set of $G$ is the **decycling number** or **feedback vertex number** of $G$ and is denoted by $\phi(G)$. The problem of finding maximum order of an induced forest is equivalent to this problem since the sum of these two parameters is the size of the vertex set. The analogous concept for edges is the **cycle rank,** the minimum number of edges whose deletion from a graph $G$ leaves an acyclic graph. It is known that the cycle rank of a graph $G$ is $|E| - |V| + \omega(G)$, see [46] for example.

Karp in [29] has shown that the decision problem of finding if $\phi(G) \leq k$ for an arbitrary graph $G$ and $k$ a natural number is $NP$- complete but for some families of graphs ( permutation graphs [35], interval and comparability graphs [36]) there are polynomial

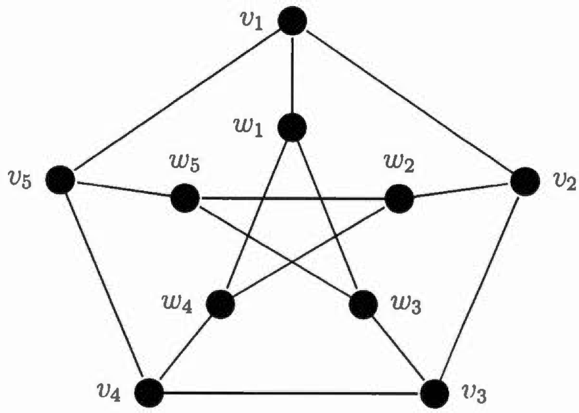



Figure 2.1: Petersen Graph.

time algorithms. Exact values of decycling numbers have been obtained for some families of graphs such as grids and hypercubes [5, 42], generalized Petersen graphs [14]. One list of results on the feedback set problem and the decycling number can be found in the survey by Festa et al. in [12].

It is easy to see that $\phi(G) = 0$ if only if $G$ is an acyclic graph and $\phi(G) = 1$ if only if $G$ has at least one cycle and there is some vertex that is on all the cycles. To remove all cycles from a complete graph $K_n$, we have to remove the vertices one by one to reach $K_3$ and to decycle $K_3$ it suffices to remove just one vertex. Thus $\phi(K_n) = n - 2$. In the case of a complete bipartite graph $K_{n,m}$ with $n \geq m$, we need to remove all but one vertex in one of the sets of the bipartition. Thus $\phi(K_{n,m}) = \min\{n, m\} - 1$. For another example consider the Petersen graph $P$, Figure 2.1. To decycle $P$, we have to remove at least one vertex from external cycle and one vertex from internal cycle. Suppose $v_1$ is the vertex which is deleted from external cycle, if we remove one of the vertices $w_1$, $w_2$ or $w_3$, one cycle will be remain. Then by symmetry of Petersen graph $\phi(P) \geq 3$. But vertex set $\{v_1, v_3, w_4\}$ is a decycling set, so $\phi(P) = 3$.

We now review some of the basic results from [5] by Beineke and Vandell. Then

we review the problem of finding the decycling number of the Cartesian and strong products of graphs and, finally, we will study the decycling number of the lexicographic and co-normal products of graphs.
We begin with a trivial upper bound for an arbitrary graph.

**Lemma 2.1.** *[5] For any non null graph $G, \phi(G) \leq \beta(G) - 1$.*

**Proof.** Suppose a graph $G$ has at least one edge and consider the subgraph induced by a maximum independent set $S$ of $G$ and any vertex of $V(G) \setminus S$. Clearly this subgraph is acyclic and so $\phi(G) \leq |G| - (\alpha(G) + 1) \leq \beta(G) - 1$. □

**Lemma 2.2.** *[5] If neither $G$ nor $H$ is a null graph, then*

$$\phi(G \vee H) \leq \min\{|V(G)| + \phi(H),\ |V(H)| + \phi(G)\}.$$

**Proof.** In $G \vee H$ we have two different kinds of cycles, those in $G$ and in $H$, and the cycles which are produced by the edges between $G$ and $H$. To delete the latter kind of cycles, one can delete all vertices of one of $G$ or $H$. This also trivially destroys all cycles in the graph removed. To destroy the cycles of other graph we need $\phi(H)$ (or $\phi(G)$) vertices. Hence, and by the commutativity of the join $G \vee H$, we have $\phi(G \vee H) \leq \min\{|V(G)| + \phi(H),\ |V(H)| + \phi(G)\}$. □

**Theorem 2.3.** *[5] For any graph $G$,*

$$2\phi(G) \leq \phi(K_2 \square G) \leq \phi(G) + \beta(G).$$

**Proof.** The graph $K_2 \square G$ consists of two disjoint copies of $G$ such that the corresponding vertices are adjacent, see Figure 2.2. To decycle it, we have to remove all cycles

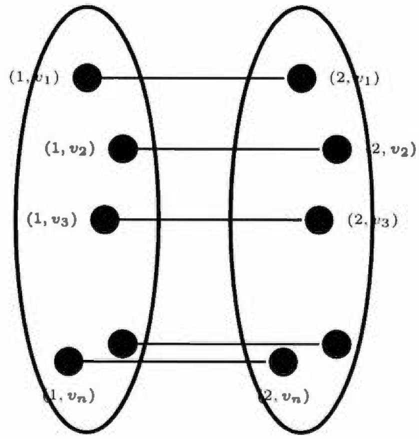



Figure 2.2: Graph $K_2\square G$

of each copy and the first inequality is obvious. Suppose $A = \{u_1, u_2, \cdots u_n\}$ is a maximum independent set of vertices in one copy of $G$ and let $B = \{v_1, v_2, \cdots, v_{|V(G)|-\alpha(G)}\}$ be a vertex set such that $G|_B$ is a maximum induced forest in the other. The subgraph induced by $A \cup B$ is acyclic, so $\phi(K_2\square G) \leq \phi(G) + |G| - \alpha(G)$. □

For some special graphs, the bounds of Theorem 2.3 is sharp. When $G$ is the null graph $\overline{K_n}$ then $\phi(K_2\square G) = 0$ and both bounds hold. For graph $G = K_n$ the lower bound is sharp because $\phi(K_2\square K_n) = 2n - 4 = 2\phi(K_n)$.

It is shown in the next section that $\phi(P_2\square P_n) = \lfloor \frac{n}{2} \rfloor$, which equals the sum of $\phi(P_n)$ and $\beta(P_n)$ since $\phi(P_n) = 0$ and $\beta(P_n) = \lfloor \frac{n}{2} \rfloor$.

**Lemma 2.4.** *[5] Suppose $d_1, d_2, \cdots, d_p$ is the degree sequence of a connected graph $G$ with $p$ vertices and $q$ edges respectively. If $\phi(G) = s$, then*

$$\sum_{i=1}^{s}(d_i - 1) \geq q - p + 1.$$

**Proof.** Suppose $S$ with $|S| = s$ is a decycling set for $G$. Then $G - S$ is a forest containing at most $p - s - 1$ edges. Since by removing $s$ vertices of $G$ at most $d_1 + d_2 + \cdots + d_s$ edges will be removed, we have that $q - \sum_{i=1}^{s}(d_i) \leq p - s - 1$ and so $\sum_{i=1}^{s}(d_i) - s \geq q - p + 1$ and the result follows.□

An edge $uv$ is **subdivided** if it is replaced by the path $uw_{uv}v$ with $w_{uv}$ a new vertex. The graph $G' = (V', E')$ obtained from the graph $G = (V, E)$ by subdividing the edge $uv$ is then defined by $V' = V \cup \{w_{uv}\}$, $w_{uv} \notin V$ and $E' = \{w_{uv}u, w_{uv}v\} \cup (E \setminus \{uv\})$. A subdivision of the graph $G$ is any graph obtained from $G$ by a sequence of subdivisions of edges.

**Definition 2.5.** *Two graphs are said to be homeomorphic if each is a subdivision of the same graph.*

**Theorem 2.6.** *[5] Suppose $G$ and $H$ are homeomorphic graphs, then $\phi(G) = \phi(H)$.*

**Proof.** Suppose that a graph $K'$ is obtained by one subdivision from the graph $K$, that is, an edge $uv$ is replaced by the path $uwv$, $w$ a new vertex. The graph $K'$ does not contains more cycles than $K$ and so every decycling set of $K$ is also a decycling set of $K'$. Thus $\phi(K') \leq \phi(K)$. Conversely, suppose $S$ is a decycling set for $K'$. If $w \notin S$, then $S$ is a decycling set of $K$ and if $w \in S$ then $S - \{w\} \cup \{u\}$ is a decycling set for $K$. Thus $\phi(K) \leq \phi(K')$ .

Suppose $N, N_1, N_2, \cdots, N_i, N_{i+1}, \cdots, N_{n-1}, N_n = G$ and $N, N'_1, N'_2, \cdots, N'_i, N'_{i+1}, \cdots, N'_m = H$ are the two different edges subdividing sequences of the graph $N$ to produce the graphs $H$ and $G$ ( graph $N_{i+1}$ is a obtained by one edge subdividing from the graph $N_i$). By the above, we have $\phi(G = N_n) = \phi(N_{n-1}) = \cdots = \phi(N) = \phi(N'_1) = \cdots =$

$\phi(N'_m = H)$ and the result follows. □

The following useful lower bound for any connected graph $G$ is also given in [5].

**Corollary 2.7.** *[5] If $G$ is a connected graph with maximum degree $\Delta$, then*

$$\phi(G) \geq \frac{q-p+1}{\Delta - 1}.$$

**Proof.** Let $d_1, \ldots, d_p$ be the degree sequence of the graph $G$ and $\phi(G) = s$. We have $\sum_{i=1}^{s}(d_i - 1) = \sum_{i=1}^{s}(d_i) - s \leq s\Delta - s$. By Lemma 2.4, $\sum_{i=1}^{s}(d_i - 1) \geq q - p + 1$ and putting these together gives $q - p + 1 \leq s(\Delta - 1)$, which leads to the result. □

## 2.1 Decycling number of the Cartesian product of graphs

The grid graph, $P_n \square P_m$ is one of the first graphs one studies when new parameters are investigated for the Cartesian product. The decycling number is no exception. Still, we know of no exact formula for the decycling number of $P_n \square P_m$ generally. Some researchers focused on finding a tight bound for the decycling number of grids, while others looked for an exact formula for their subgraphs. In all cases we assume that $m, n \geq 2$ in order to avoid trivialities. We choose a standard labelling for the vertices of $P_n \square P_m$ that corresponds to matrix notation, the $i$th vertex in the $j$th copy of $P_n$ is denoted $v_{ij}$. We sometimes speak of the copies of $P_n$ and $P_m$ as the columns and rows, respectively, of $P_n \square P_m$.

In $P_n \square P_m$, each vertex of a decycling set can break at most 4 cycles of lengths 4, $P_2 \square P_2$. This allows Luccio in [38] to obtain an easy lower bound of $\frac{(n-1)(m-1)+1}{3}$ for the decycling number of grid graphs. For $m = n = 2^r + 1$, [38] gives an algorithm to find a decycling set of the size that agrees with the lower bound.

**Theorem 2.8.** *[5]* $\phi(P_2\square P_m) = \lfloor\frac{m}{2}\rfloor$.

**Proof.** In $P_2\square P_m$ there are $\lfloor\frac{m}{2}\rfloor$ disjoint 4-cycles, so any decycling set must contain at least $\lfloor\frac{m}{2}\rfloor$ vertices. Therefore, it sufficient to find a decycling set with $\lfloor\frac{m}{2}\rfloor$ vertices. Suppose that $V(P_2) = \{1,2\}$, $V(P_m) = \{1,2,\cdots,m\}$ and $k = \lfloor\frac{m}{2}\rfloor$. Then $\{(1,2),(1,4),\cdots,(1,2k)\}$ is such a decycling set. Therefore $\phi(P_2\square P_m) = \lfloor\frac{m}{2}\rfloor$. □

In [5] Beineke and Vandell presented the next lower bound.

**Proposition 2.9.** *[38] If $n, m \geq 3$,*

$$\phi(P_n\square P_m) \geq \lfloor\frac{nm-n-m+2}{3}\rfloor.$$

**Observation 2.10.** *[47] If $G_1 \subseteq G$ , $G_2 \subseteq G$ and $V(G_1)\cap V(G_2) = \emptyset$ then $\phi(G) \geq \phi(G_1)+\phi(G_2)$.*

**Observation 2.11.** *[5] Let $1 < r < m$. Then*

$$\phi(P_n\square P_m) \geq \phi(P_n\square P_r)+\phi(P_n\square P_{m-r}).$$

*Furthermore, suppose $t$ is the number of vertices of a decycling set of $S$ in the $r$th column of $P_n\square P_m$, then*

- $\phi(P_n\square P_m) \geq \phi(P_n\square P_{r-1})+\phi(P_n\square P_{m-r})+t$
- $\phi(P_n\square P_m) \geq \phi(P_n\square P_r)+\phi(P_n\square P_{m-r+1})+t$

Using these results, and applying some standard techniques, Beineke and Vandell in proved the following formulas.

**Theorem 2.12.** *[5] For $m \geq 4$, we have*

- $\phi(P_2 \times P_m) = \lfloor \frac{m}{2} \rfloor$;
- $\phi(P_3 \times P_m) = \lfloor \frac{3m}{4} \rfloor$;
- $\phi(P_4 \times P_m) = m$;
- $\phi(P_5 \times P_m) = \lfloor \frac{3m}{2} \rfloor - \lfloor \frac{m}{8} \rfloor - 1$;
- $\phi(P_6 \times P_m) = \lfloor \frac{5m}{3} \rfloor$;
- $\phi(P_7 \times P_m) = 2m - 1$.

Hartnell and Whitehead in [19] obtained some sharp bounds for the decycling number of the Cartesian product of two graphs when one of the factors is a complete graph. They obtained a result analogous to Theorem 2.3 for any graph $G$ with $p$ vertices,

$$\max\{3\phi(G), p + \phi(G)\} \leq \phi(G \Box K_3) \leq p + 2\phi(G),$$

So when $G$ is a forest $\phi(G \Box K_3) = p$. Also they proved when $G$ is a bipartite graph of order $p \geq 2$ or when $G$ is a graph of order $p \geq 2$ with $\Delta(G) = 3$, then $\phi(G \Box K_3) = p + \phi(G)$.

When one of the graphs is a complete graph with order larger than 4, [19] gives a lower bound.

**Lemma 2.13.** *[19] Let $G$ be a graph of order $p \geq 2$. Then for $r \geq 4$,*

$$\phi(G \square K_r) \geq \max\{p(r-2), r\phi(G)\}.$$

**Proof.** By the definition of Cartesian product of graphs, there are $r$ copies of the graph $G$ and to remove the cycles we need $r\phi(G)$ vertices. Also there are $p$ copies of the graph $K_r$ and $\phi(K_r) = r - 2$, so to decycle the graph $G \square K_r$ we need at least $p(r-2)$ vertices. This gives the result. □

**Theorem 2.14.** *[19] Let $G$ be a bipartite graph of order $p \geq 2$. Then for $r \geq 4$,*

$$\phi(G \square K_r) = p(r-2).$$

For the Cartesian product of two complete graphs, Hartnell and Whitehead have a formula.

**Proposition 2.15.** *[19] Suppose that $n, r$ are integers with $n \geq r \geq 3$. Then*

$$\phi(K_r \square K_n) = \begin{cases} r(n-2) & n > r \\ (r-1)^2 & n = r \end{cases}$$

Decycling sets of the Cartesian product of two cycles were studied by Pike and Zou in [42]. They started with the following lower bound.

**Lemma 2.16.** *[42] For the graph $C_m \square C_n$,*

$$\phi(C_m \square C_n) \geq \frac{mn+1}{3}.$$

**Proof.** $C_m \square C_n$ is a 4-regular graph and so by Corollary 2.7 the result holds.$\square$

The next result is proved in [42] by a series of lemmas:

**Theorem 2.17.** *[42] Let $m \geq 3$ and $n \geq 3$ be integers. Then*

$$\phi(C_m \square C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil & m = 4 \\ \lceil \frac{3m}{2} \rceil & n = 4 \\ \lceil \frac{mn+2}{3} \rceil & otherwise \end{cases}$$

Decycling numbers of the Cartesian product of other classes of graphs are not known. In the next section we study the decycling number of the strong product of graphs.

## 2.2 Decycling number of the strong product of graphs

In the case of strong product, J. Xie in [47] obtained exact formulas. First we review some definitions, simple observations and lemmas.

Suppose the vertex sets of $G$ and $H$ are respectively $V(G)$ and $V(H)$. There are $|V(G)|$ copies of $H$ and $|V(H)|$ copies of $G$ in $G \boxtimes H$. The induced subgraphs $G^y = (G \boxtimes H)_{|\{(x,y)|x \in V(G)\}}$ is a copy of $G$ where $y \in V(H)$ and called a row of $G \boxtimes H$. Also ${}^xH = (G \boxtimes H)_{|\{(x,y)|y \in V(H)\}}$ is a copy of $H$ where $x \in V(G)$, and called a column of $G \boxtimes H$. Also $G^y$ and ${}^xH$ are called the fibers of $G \boxtimes H$.

The strong product is commutative, $G \boxtimes H \cong H \boxtimes G$, so we obtain the following

**Observation 2.18.** *[47] For any graphs $G$ and $H$,*

$$\phi(G \boxtimes H) = \phi(H \boxtimes G).$$

By the definition of the strong product of graphs there are $|G|$ copies of $H$ and $|H|$ copies of $G$ in $G \boxtimes H$. So $\phi(G \boxtimes H) \geq |G|.\phi(H)$ and $\phi(G \boxtimes H) \geq |H|.\phi(G)$. Then the following result follows.

**Lemma 2.19.** *[47] For any graphs $G$ and $H$,*

$$\max\{|G|.\phi(H), |H|.\phi(G)\} \leq \phi(G \boxtimes H).$$

The following lemma is useful when one wants to calculate the decycling number of the strong product of graphs by induction.

**Lemma 2.20.** *[47] Suppose that $G = (V_1, E_1)$, $H = (V_2, E_2)$. For each $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$, we have*

$$\phi(G \boxtimes H) \geq \max\{\phi(G_{|V_1'} \boxtimes H) + \phi(G_{|V_1 - V_1'} \boxtimes H),\ \phi(G \boxtimes H_{|V_2'}) + \phi(G \boxtimes H_{|V_2 - V_2'})\}.$$

**Lemma 2.21.** *[47] For graphs $G$ and $H$,*

$$\phi(G \boxtimes H) \leq \min\{|G||H| - \alpha(G).(|H| - \phi(H)),\ \ |G||H| - \alpha(H).(|G| - \phi(G))\}.$$

**Proof.** Suppose $A = \{u_1, u_2, \cdots, u_{\alpha(G)}\}$ is a maximum independent set in the graph $G$ and $B = \{v_1, v_2 \cdots, v_{|H|-\phi(H)}\}$ is a vertex set such that $H_{|_B}$ is a forest. let $N = G_{|_A} \boxtimes H_{|_B}$ be the graph with vertex set of $V(N) = \{(u_i, v_j) | i = 1, 2, \cdots, \alpha(G); j = 1, 2, \cdots, |H| - \phi(H)\}$. Since $A$ is an independent set in $G$, for any $u_i, u_j \in A, i \neq j, (u_i, u_j) \notin E(G)$, there is no edge whose two ends lie in different rows of $N = G_{|_A} \boxtimes H_{|_B}$. Hence $E(N) = \{((u_i, v_s), (u_i, v_t)) | 1 \leq i \leq \alpha(G), (v_s, v_t) \in H_{|_B}\}$. Since $H_{|_B}$ is a forest and every row in $G_{|_A} \boxtimes H_{|_B}$ is a copy of $H_{|_B}$, $N$ is a forest which has $\alpha(G)$ copies of forest $H_{|_B}$. Then there is no cycle in $N$ and $|N| = \alpha(G).(|H| - \phi(H))$. So $\phi(G \boxtimes H) \leq |G||H| - |N|$. That is, $\phi(G \boxtimes H) \leq |G||H| - \alpha(G).(|H| - \phi(H))$. By symmetry $\phi(G \boxtimes H) \leq |G||H| - \alpha(H).(|G| - \phi(G))$ and by Observation 2.10 the Lemma holds. $\square$

By these two lemmas X. Chen and J. Xie proved the following formulas for the decycling number of the strong product of graphs when one or both of the product factors are complete, path or cycle graphs.

**Theorem 2.22.** *[48]*

- $\phi(P_m \boxtimes P_n) = \min\{m.\lfloor \frac{n}{2} \rfloor, n.\lfloor \frac{m}{2} \rfloor\}$.
- $\phi(P_m \boxtimes C_n) = \min\{m.\lceil \frac{n}{2} \rceil, n.\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil\}$.
- $\phi(C_m \boxtimes C_n) = \min\{n.\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil, m.\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil\}$, *where $m$ and $n$ are even.*
- $n.\lfloor \frac{m}{2} \rfloor \leq \phi(C_m \boxtimes C_n) \leq n.\lfloor \frac{m}{2} \rfloor + 1$, *where $m$ and $n$ are both odd and $m \leq n$.*
- $\phi(K_m \boxtimes C_n) = mn - 2.\lfloor \frac{n}{2} \rfloor$.
- $\phi(K_m \boxtimes P_n) = mn - 2.\lceil \frac{n}{2} \rceil$.

- $\phi(K_m \boxtimes K_n) = mn - 2.$

In the next two sections we investigate the decycling number of the lexicographic and co-normal products of graphs and give our new results.
The co-normal product is the less known of the two. It was introduced by Ore [41] in 1962, but only a few papers (for instance [9], [43]) have appeared about it since. The co-normal product is also known as the disjunctive product [43] and is clearly commutative [18] and [33].

## 2.3 Decycling number of the lexicographic product of graphs

In the case of lexicographic product we prove these results:

**Theorem 2.23.** *For graphs $G$ and $H$,*

$$|G|\phi(H) \leq \phi(G[H]) \leqslant \beta(G)|H| + \alpha(G)\phi(H)$$

.

**Proof.** The left inequality is clear because $G[H]$ contains $|G|$ disjoint copies of the graph $H$ that we may index by the vertices of $G$: $H_u$ is the copy of $H$ induced by the vertex sets $\{u\} \times V(H)$. Two copies $H_u$ and $H_v$ are joined to each other when $uv$ belongs to $E(G)$; "joined" means that all edges between the two copies are present. Therefore there are four kinds of cycles in $G[H]$, the cycles of copies of $H$, the triangles that appear in joins of copies of $H$, the cycles between two copies of $H$ that are not triangles and the cycles induced by those in $G$. To find an acyclic subgraph of $G[H]$, we first delete the set $S \times V(H)$ of vertices such that $S$ is a vertex cover of $G$; there are $\beta(G)|H|$ of them. In the remaining graph there are cycles only within copies of $H$ of which there are $|G| - \beta(G) = \alpha(G)$. So $\phi(G[H]) \leqslant \beta(G)|H| + \alpha(G)\phi(H)$. □

We computed the decycling number of the lexicographic product of various graphs using a small programming written in Sagemath and we conjecture that equality holds for all graphs but we can only prove it when the first factor of the lexicographic product is a path, a cycle, a complete graph or a complete bipartite graph. First we present a lemma that is useful in the sequel.

**Lemma 2.24.** *Suppose that $G = (V_1, E_1)$ and $H = (V_2, E_2)$, for each $V_1' \subseteq V_1$, we have*

$$\phi(G[H]) \geq \phi(G_{|V_1'}[H]) + \phi(G_{|V_1 - V_1'}[H]).$$

**Proof.** For each $V_1' \subseteq V_1$, $G_{|V_1'} \cap G_{|V_1 - V_1'} = \emptyset$. Hence, by the definition of lexicographic product $G_{|V_1}[H] \cap G_{|V_1 - V_1'}[H] = \emptyset$. The result follows by Observation 2.10. □
In particular when the graph $G$ is a path $P_n$ and $|V_1'| = 2$ then $\phi(P_n[G]) \geq \phi(P_2[G]) + \phi(P_{n-2}[G])$.
This allows us to derive an exact expression for this case.

**Theorem 2.25.** $\phi(P_n[G]) = |G|\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil . \phi(G)$.

**Proof.** We have $\beta(P_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. By Theorem 2.23, $\phi(P_n[G]) \leq |G|\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil . \phi(G)$. To prove $\phi(P_n[G]) \geq |G|\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil . \phi(G)$ we use induction on $n$, the length of the path graph. For $n = 2$, $\phi(P_2[G]) = \phi(G \vee G) = \phi(G) + |G|$. Suppose for $n < k$ the inequality holds. For $n = k$, by Lemma 2.24, we have $\phi(P_n[G]) \geq \phi(P_2[G]) + \phi(P_{n-2}[G]) \geq |G| + \phi(G) + |G|\lfloor \frac{n-2}{2} \rfloor + \lceil \frac{n-2}{2} \rceil . \phi(G) = |G| + \phi(G) + |G|(\lfloor \frac{n}{2} \rfloor - 1) + (\lceil \frac{n}{2} \rceil - 1) . \phi(G) = |G|\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil . \phi(G)$, and the result follows. □

Since $\phi(P_m) = 0$ we have the following.

**Corollary 2.26.** $\phi(P_n[P_m]) = m \left\lfloor \frac{n}{2} \right\rfloor$.

When the first graph is a cycle we have the next result.

**Theorem 2.27.** $\phi(C_n[G]) = \lceil \frac{n}{2} \rceil |G| + \lfloor \frac{n}{2} \rfloor \phi(G)$.

**Proof.** For the graph $C_n[G]$ we can use the same argument as in Theorems 2.23 and 2.25, with the difference that $\beta(C_n) = \lceil \frac{n}{2} \rceil$ and $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$. □

**Proposition 2.28.** $\phi(K_n[G]) \leq (n-1)|G| + \phi(G)$.

**Proof.** $K_n[G]$ is a complete graph where vertices are copies of $G$, which are joined to each other. By an argument similar to that in the computation of $\phi(K_n)$, we have $\phi(K_n[G]) = |G|(n-2) + \phi(G \vee G) = |G|(n-2) + |G| + \phi(G) = (n-1)|G| + \phi(G)$. □

**Proposition 2.29.** $\phi(K_{n,m}[G]) \leq m|G| + n\phi(G)$ *where* $m < n$.

**Proof.** The Cartesian product $K_{n,m}[G]$ looks like $K_{n,m}$ with each of its vertices replaced by a copy of $G$ and two copies $G_u$ and $G_v$ of $G$ replacing the vertices $u$ and $v$, respectively are joined (all vertices of $G_u$ are adjacent to all vertices of $G_v$) if $uv \in E(K_{n,m})$. It follows from the structure of complete bipartite graphs that each copy of $G$ in one class of the bipartition of $K_{n,m}$ joined to all vertices of the other class.

We need to remove $\beta(K_{n,m}) = m$ copies of $G$ to delete all cycles that are produced by the joining of the copies. The remaining $n$ disjoint copies of $G$ need $n\phi(G)$ vertices to remove the cycles. □

## 2.4 Decycling number of the co-normal product of graphs

**Lemma 2.30.** *For graphs $G$ and $H$,*

$$\max\{|G|.\phi(H), |H|.\phi(G)\} \leq \phi(G \bullet H).$$

**Proof.** By the definition of co-normal product of graphs, there are $|G|$ copies of $H$ and $|H|$ copies of $G$ in $G \bullet H$. So $\phi(G \bullet H) \geq |G|.\phi(H)$ and $\phi(G \bullet H) \geq |H|.\phi(G)$. □

**Lemma 2.31.** *[49] For any graphs $G$ and $H$ we have $\alpha(G \bullet H) = \alpha(G)\alpha(H)$*

**Proof.** Suppose $S_G$ and $S_H$ are the maximum independent sets in graphs $G$ and $H$ respectively. Clearly $S_G \times S_H$ is an independent set in $G \bullet H$. Therefore $\alpha(G \bullet H) \geq \alpha(G)\alpha(H)$.

Suppose $S$ is a maximum independent set in $G \bullet H$, let $S_G = \{a \in V(G) | \exists u \in V(H), (a, u) \in S\}$ and $S_H = \{u \in V(H) | \exists a \in V(G), (a, u) \in S\}$ be projections of $S$ on $V(G)$ and $V(H)$ respectively. Clearly $S_G$ and $S_H$ are independent set and $|S| \leq |S_G||S_H|$. On the other hand $|S_G||S_H| \leq \alpha(G)\alpha(H)$ so $\alpha(G \bullet H) = |S| \leq \alpha(G)\alpha(H)$ and the result follows.

**Lemma 2.32.** $\phi(G \bullet H) \leq |G||H| - \alpha(G)\alpha(H) - 1$.

**Proof.** Easily comes from Lemma 2.1 and Lemma 2.32. □

Also we have the next result when the graph $G$ is complete.

**Corollary 2.33.** $\phi(K_n \bullet G) = (n-1)|G| + \phi(G)$.

**Proof.** By definition of lexicographic and co-normal product $K_n \bullet G \cong K_n[G]$ and the result follows from Theorem 2.28. □

The bound in 2.32 is tight indeed, if both $G$ and $H$ are complete, we have, by Corollary 2.33
$\phi(K_n \bullet K_m) = (n-1)|K_m| + \phi(K_m) = (n-1)m + (m-2) = nm - 2$ while the upper bound in Lemma 2.32 is $|K_n||K_m| - \alpha(K_n)\alpha(K_m) - 1 = nm - 1 \times 1 - 1 = nm - 2$.

**Lemma 2.34.** *Suppose that $G = (V_1, E_1)$, $H = (V_2, E_2)$. For each $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$, we have*

$$\phi(G \bullet H) \geq \max\{\phi(G_{|V_1'} \bullet H) + \phi(G_{|V_1 - V_1'} \bullet H),\ \phi(G \bullet H_{|V_2'}) + \phi(G \bullet H_{|V_2 - V_2'})\}.$$

**Proof.** For each $V_1' \subseteq V_1$, $G_{|V_1'} \cap G_{|V_1 - V_1'} = \emptyset$. Furthermore $(G_{|V_1'} \bullet H) \cap (G_{|V_1 - V_1'} \bullet H) = \emptyset$. And it is obvious that $G_{|V_1'} \bullet H \subseteq G \bullet H$ and $G_{|V_1 - V_1'} \bullet H \subseteq G \bullet H$. By Observation 2.10 we have $\phi(G \bullet H) \geq \phi(G_{|V_1'} \bullet H) + \phi(G_{|V_1 - V_1'} \bullet H)$. For each $V_2' \subseteq V_2$, by symmetry of co-normal product, $\phi(G \bullet H) \geq \phi(G \bullet H_{|V_2'}) + \phi(G \bullet H_{|V_2 - V_2'})$. This completes the proof the lemma. □

**Observation 2.35.** *Indeed $G \bullet P_2 \cong G \vee G$ so $\phi(G \bullet P_2) = \phi(G \vee G) = |G| + \phi(G)$ and then $\phi(P_n \bullet P_2) = n$. So $\phi(P_3 \bullet P_2) = \phi(P_3 \vee P_3) = 3$ and $\phi(C_3 \bullet P_2) = \phi(C_3 \vee C_3) = 4$.*

**Lemma 2.36.** $\phi(P_3 \bullet P_n) \geq 2n - 2 - \lfloor \frac{n-3}{4} \rfloor$

**Proof.** By induction on $n$ we prove $\phi(P_3 \bullet P_n) \geq 2n - 2 - \lfloor \frac{n-3}{4} \rfloor$. By Observation 2.35, for $n = 2$ the inequality holds. Suppose inequality is correct for each path with length less than $n = k$. We will prove it is satisfied when $n = k + 1$. By Lemma 2.34 we have $\phi(P_3 \bullet P_{k+1}) \geq \phi(P_3 \bullet P_2) + \phi(P_3 \bullet P_{k-1}) \geq 3 + 2(k-1) - 2 - \lfloor \frac{k-1-3}{4} \rfloor \geq 2k - 1 - \lfloor \frac{k}{4} \rfloor + 1 = 2k - \lfloor \frac{k}{4} \rfloor \geq 2k - \lfloor \frac{k-2}{4} \rfloor$. □

We conjecture that equality actually holds.

**Conjecture 2.37.** $\phi(P_3 \bullet P_n) = 2n - 2 - \lfloor \frac{n-3}{4} \rfloor$

Similarly, we have the following.

**Lemma 2.38.** $\phi(C_3 \bullet C_n) \geq 2n + 1$

**Proof.** By induction on $n$ we prove $\phi(C_3 \bullet C_n) \geq 2n + 1$. Since $C_3 \cong K_3$ by Corollary 2.33 $\phi(C_3 \bullet C_3) = 2 \times 3 + 1 = 7$ and the inequality holds. By Lemma 2.34 and Observation 2.35 we have $\phi(C_3 \bullet C_{k+1}) \geq \phi(C_3 \bullet P_2) + \phi(C_3 \bullet C_{k-1}) \geq 4 + 2(k-1) + 1 = 2k + 3 = 2(k+1) + 1$ □

And we think this should be equality.

**Conjecture 2.39.** $\phi(C_3 \bullet C_n) = 2n + 1$

# CHAPTER III

# THE BASIS NUMBER OF GRAPHS

The set all of subsets of the edge set $E$ of the graph $G$ forms an $|E|-$ dimensional vector space over $GF(2)$ with vector addition $X+Y := (X \cup Y) - (X \cap Y)$ and scalar multiplication $1.X = X, 0.X = \emptyset$. It suffices to assign a fixed and arbitrary numbering $\{e_1, \ldots, e_{|E(G)|}\}$ of the edge set of the graph $G$ and allocate a characteristic vector with length of $|E(G)|$ to each subset of the edges. Assign 1 at the $i$-th coordinate of this vector if the edge $e_i$ is in the subset, otherwise put 0.

A (generalized) cycle is a subgraph whose every vertex has even degree. It is denoted by its edge set, $C$. The vertex set of a generalized cycle is denoted by $V(C)$. An elementary cycle is a connected minimal (with respect to number of edges of the cycles) subgraph whose every vertex has degree two. The set $\mathcal{C}$ of all generalized cycles forms a linear subspace of $(P(E), +, .)$ which is called the **cycle space**, $\mathcal{C}(G)$, of the graph $G$, where $P(E)$ is the set of all subsets of $E(G)$. So a set $\mathcal{B} = \{C_1, C_2, C_3, \cdots\}$ is called a **cycle basis** of $\mathcal{C}(G)$ if and only if every element of $\mathcal{C}(G)$ can be written in a unique way as a finite linear combination of cycles from the set $\mathcal{B}$.

**Theorem 3.1.** *[45] The dimension of the cycle space of the graph $G$ is the cyclomatic number (first Betti number) $\nu(G) = \dim(\mathcal{C}) = |E(G)| - |V(G)| + \omega(G)$ (where $\omega(G)$ is the number of connected components of $G$).*

**Proof.** It suffices to prove the claim for $\omega(G) = 1$, when the graph is connected, because the cycle space of a graph is the direct sum of the cycle spaces of its connected

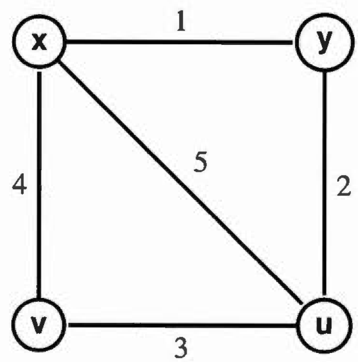



Figure 3.1: Edge Labelling of graphs

components.

Now suppose $T$ is a spanning tree of the graph $G$. For each edge $e \in E(G) - E(T)$, edge set $T \cup \{e\}$ constructs a unique cycle. The set these cycles is independent because each one has an edge that is not in any other cycle. Since $|E(T)| = |V(T)| - 1$, there are $|E(G)| - (|V(G)| - 1)$ such cycles. Therefore $\dim(\mathcal{C}) \geq |E(G)| - |V(G)| + 1$.

On the other hand, suppose $S$ and $\overline{S}$ are vertex sets of the two connected components formed by removing an edge $e \in E(T)$ from $T$. The set $(S, \overline{S})$ is a fundamental cut set in $G$. For each cut set, we define the characteristic vector of a cut to be a vector in $GF[2]^{|E(G)|}$ that has 1's in components corresponding to the edges of G in the cut and 0's in the remaining components. There exist $|V| - 1$ such vectors each defined by an edge of $T$. Each cycle of $G$ must intersect each cut set an even number of times, so the set of characteristic vectors of cut sets are orthogonal to the cycle space of $G$. In addition these characteristic vectors are linearly independent, because each cut is different in at least the tree edge defining the cut. Therefore the dimension of the orthogonal complement to the cycle space is at least $|V(G)| - 1$. Therefore $\dim(\mathcal{C}) \leq |E| - |V| + 1$ and this completes the argument. □

For a simple example, suppose $G$ is the graph in Figure 3.1.

We associate vector [1 1 0 0 1] to the cycle $C_1 = (x, y, u, x)$, vector [0 0 1 1 1] to the cycle $C_2 = (x, u, v, x)$ and vector [1 1 1 1 0] to the cycle $C_3 = (x, y, u, v, x)$. We can write $C_3 = C_1 + C_2$ and consider $\{C_1, C_2\}$ as a cycle basis.

Given a basis $\mathcal{B}$ of the cycle space $\mathcal{C}(G)$, let $f(\mathcal{B}) = \max\{k : \exists e \in E(G) \exists C_1, \ldots, C_i \in \mathcal{B}\}$ such that $e \in C_i$ for $i = \{1, \ldots, k\}$. That is, $f(\mathcal{B})$ is the maximum number of cycles of $\mathcal{B}$ containing any given edge. We call $f(\mathcal{B})$ the **fold** of $\mathcal{B}$. A basis for $\mathcal{C}(G)$ is called a $k-$fold basis if each edge of $G$ occurs in at most $k$ of the cycles in the basis, i.e. $f(\mathcal{B}) = k$. The **basis number** of $G$, denoted by $b(G)$ is the smallest integer $k$ such that $\mathcal{C}(G)$ has a $k-$fold basis. For each edge $b \in E(G)$, $f_{\mathcal{B}}(e)$ is the number of cycles in the basis $\mathcal{B}$ in which $e$ occurs.

The basis number was introduced by Schmeichel [44] in 1981. He proved that for any integer $r$, there is a graph with basis number greater than or equal to $r$. MacLane in 1937 [39] proved the following well-known result.

**Theorem 3.2.** *[39] Graph $G$ is planar if and only if $b(G) \leq 2$.*

Schmeichel [44] proved that $b(K_n) = 3$ whenever $n \geq 5$ and $b(K_{n,m}) \leq 4$ for each $n$ and $m$. In 1982, Banks and Schmeichel [4] proved that $b(Q_n) = 4$ whenever $n \geq 7$. Many papers appeared investigating the basis number of certain graphs, especially the graph products, see [2], [27], [1].

Ali and Marougi in [3] gave the following upper bound for the basis number of the Cartesian product of graphs.

**Theorem 3.3.** *[3] If $G$ and $H$ are two connected disjoint graphs, then $b(G \square H) \leq$*

$\max\{b(G)+\Delta(T_H), b(H)+\Delta(T_G)\}$, *where $T_H$ and $T_G$ are spanning trees of $H$ and $G$, respectively, such that the maximum degrees $\Delta(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of $H$ and $G$.*

In [24] M.M. Jaradat proved some results for the basis number of the strong product.

**Theorem 3.4.** *Let $T_1$ and $T_2$ be two trees. Then $b(T_1 \boxtimes T_2) \le \max\{\lfloor \frac{3\Delta(T_1)+1}{2} \rfloor, \lfloor \frac{3(\Delta(T_2)+1}{2} \rfloor\}$ where $\Delta(T_i)$ is maximum degree of vertices of $T_i$ for $i = 1, 2$.*

**Theorem 3.5.** *If $G$ is a bipartite graph and $H$ is any graph, then $b(G \boxtimes H) \le \max\{b(H)+1,\ 2\Delta(H)+b(G)-1,\ \lfloor \frac{3(\Delta(T_2)+1}{2} \rfloor,\ b(G)+2\}$.*

The basis number of the lexicographic product of graphs was studied by Jaradat and Alzoubi in [26] who proved the following results.

**Lemma 3.6.** *[26] Suppose $M$ is a null graph on vertex set $\{u_1, u_2, \cdots, u_n\}$ and $P_2 = ab$ is a path of order 2. Then*

$$\mathcal{M} = \{(a, u_j)(b, u_l)(a, u_{j+1})(b, u_{l+1}) :\ 1 \le j, l \le n-1\}.$$

*is a 4-fold basis of $C(P_2[M])$*

**Lemma 3.7.** *[26] Let $T$ be a tree and $P_2 = uv$ be a path of order 2. Then $b(P_2[T]) \le 4$. Moreover if $|V(T)| \ge 14$, then equality holds.*

They used a special decomposition of a tree into an edge-disjoint union of subtrees to find a cycle basis for the lexicographic product of trees and then by computation fold of the cycle basis, they found a upper for $b(T_1[T_2])$.

**Lemma 3.8.** *[26] $b(T_1[T_2]) \le \max\ \{2\Delta(T_1), 4\}$.*

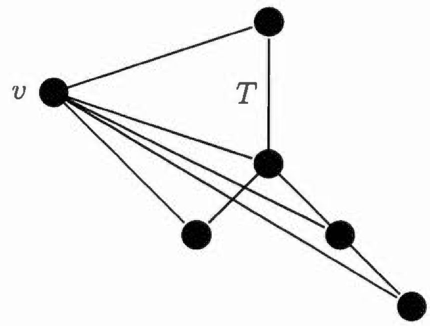



Figure 3.2: $v \vee T$.

The method can be extended to the case when only the second graph is a tree.

**Lemma 3.9.** *[26]* $b(G[T]) \leq \max\{4, 2\Delta(G), 2 + b(G)\}$.

The lemmas allowed Jaradat and Alzoubi to prove a general bound.

**Theorem 3.10.** *[26]* $b(G[H]) \leq \max\{4, 2\Delta(G) + b(H), 2 + b(G)\}$.

## 3.1 Basis number of the join of two graphs

In [50] we gave some results on the basis number of the join of two graphs. Here we review these results and correct the errors have been happened in the proofs.
We will find a cycle basis for the join of two graphs and then by computing its fold, we will have an upper bound for the basis number of the join of two graphs. Suppose a vertex $v$ is not in $V(G)$. We then denote by $v \vee G$ the join of the graphs $G$ and $(\{v\}, \emptyset)$. Similarly, write $v \vee e$ for the join of vertex $v$ and and the single edge $e$. the single edge. More precisely, if $e = ab$ and $v \notin \{a, b\}$ then $v \vee e = (\{v\}, \emptyset) \vee (\{a, b\}, \{ab\})$.

For the following useful lemma we will need Theorem 3.2 and Kuratowski's theorem.

**Theorem 3.11.** *[Kuratowski] A graph is planar if and only if it contains no subdivision of either $K_5$ or $K_{3,3}$.*

**Lemma 3.12.** *Suppose $T$ is a tree with $p \geq 3$ vertices and $u \notin V(T)$. Then the basis number of $u \vee T$ is 2.*

**Proof.** By induction on $|T|$ we prove that $v \vee T$ is a planar graph. Suppose $T$ is a tree with $p$ vertices. If $p = 3$, $T$ is a path of length two and clearly $u \vee T$ is a planar graph. Suppose for $p = k$ graph $v \vee T$ is planar. For $p = k + 1$, assume that $u \vee T$ is a non planar graph. By Theorem 3.11 (Kuratowski's theorem) it contains a subdivision of either $K_{3,3}$ or of $K_5$. The graph $u \vee T_{k+1} - \{u\}$ is a tree but neither a subdivision of $K_{3,3} - u$ nor a subdivision of $K_5 - u$ is a tree, contradicting the non planarity of $u \vee T$. Thus $u \vee T$ is planar graph. Hence $u \vee T$ is a planar graph.
Therefore by Theorem 3.2 we have $b(u \vee T) \leq 2$. If $b(u \vee T) = 1$, then $u \vee T$ has a $1-$fold basis, which implies that $\nu(u \vee T) \leq |E(u \vee T)|/3$ because each cycle contains at least three edges. Since $|E(u \vee T)| = 2p-1$ and $\nu(u \vee T) = 2p-1-(p-1)+1 = p-1$, then $p - 1 \leq \frac{(2p-1)}{3}$ , which implies that $p \leq 2$ and this is a contradiction. Therefore, $b(u \vee T) = 2$. □

**Lemma 3.13.** *[50] Let $H$ be any connected graph and let $v$ be a vertex which is not in $H$. Then $b(u \vee H) \leq b(H) + 2$.*

**Proof.** Let $v_1, v_2, \cdots, v_p$ be the vertices of $H$, $T$ a spanning tree of $H$ and $\mathcal{B}_1$ a $b(H)$-fold basis for $\mathcal{C}(H)$. Let also $\mathcal{B}_2$ be a 2-fold basis for $u \vee T$. In $u \vee T$, any two triangles differ in at least one edge, and since $T$ is acyclic, these cycles are independent. They are also are independent from the cycles in $\mathcal{B}_1$ because they have edges of the form $uv_i$ for some $i \in \{1, 2, \cdots, p\}$ . Then clearly $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is an independent set of cycles with size of $|\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| = q - p + 1 + p - 1 = q$. On the other hand

$\nu(u \vee H) = E(u \vee H) - V(u \vee H) + 1 = q + p - (p+1) + 1 = q$ , hence $\mathcal{B}$ is a basis for $\mathcal{C}(u \vee H)$. If $e = uv_i$, then $f_{\mathcal{B}}(e) \leq f_{\mathcal{B}_2}(e) \leq 2$, and if $e$ is an edge of $H$, then clearly $f_{\mathcal{B}}(e) \leq b(H) + 2$. Thus, $b(u \vee H) \leq b(H) + 2$. □

To find a bound for the basis number of the join of two graphs we first find a cycle basis for it.

**Theorem 3.14.** *Let $G$ and $H$ be two connected graphs with $p_1$ and $p_2$ vertices and $q_1$ and $q_2$ edges, respectively. Also suppose $\mathcal{B}(G \vee v_h)$ is a basis according to Lemma 3.13 for $G \vee v_h$ where $v_h$ is an arbitrary vertex of $H$, $T_H$ is a spanning tree of $H$ and $\mathcal{B}(H)$ is a basis of $H$. Let $H_i = \{u_i \vee e | e \in T_H\}$ and $u_i \in V(G)$, then $\mathcal{B} = \mathcal{B}(G \vee v_h) \cup \mathcal{B}(H) \cup (\bigcup_{i=1}^{p_1} H_i)$ is a cycle basis for $G \vee H$.*

**Proof.** We have $\nu(G \vee H) = q_1 + q_2 - p_1 - p_2 + p_1 p_2 + 1$. Let $u_1, \cdots, u_{p_1}$ and $v_1, \cdots, v_{p_2}$ be the vertices of $G$ and $H$, respectively. Since each cycle in $\mathcal{B}(G \vee v_h)$ has an edge from $G$, $\mathcal{B}(G \vee v_h)$ is independent from $H_i$ and $\mathcal{B}_H$. Each $H_i$ must contain an edge of the form $u_i v_j$ for $i = 1, \cdots, p_1$ and $j = 1, \cdots, p_2$, thus $\bigcup_{i=1}^{p_1} H_i$ is an independent cycle set and is independent from $\mathcal{B}(H)$. Thus $\mathcal{B}$ is independent set and disjoint from the union of $\mathcal{B}(G \vee v_h)$ , $\mathcal{B}(H)$ and $\bigcup_{i=1}^{p_1} H_i$. Therefore $|\mathcal{B}| = |\mathcal{B}(G \vee v_h)| + |\mathcal{B}(H)| + |\bigcup_{i=1}^{p_1} H_i| = q_1 + q_2 - p_2 + 1 + p_1(p_2 - 1) = q_1 + q_2 - p_1 - p_2 + p_1 p_2 + 1 = \nu(G \vee H)$. Hence $\mathcal{B} = \mathcal{B}(G \vee v_h) \cup \mathcal{B}(H) \cup (\bigcup_{i=1}^{p_1} H_i)$ is a cycle basis for $G \vee H$. □

**Theorem 3.15.** *Let $G$ and $H$ be two connected graphs then*

$$\max\{b(G), b(H)\} \leq b(G \vee H) \leq \min\{\max\{b(G)+2, (b(H)+2)p_1, 4\}, \max\{b(H)+2, (b(G)+2)p_2, 4\}.$$

**Proof.** The left inequality is clear. Let $\mathcal{B}_1$ be a $b(G)$-fold basis and $\mathcal{B}_2$ be a $b(H)$-fold basis and $\mathcal{B}$ be a basis of $G \vee H$ as in Theorem 3.14 . Let $e \in G \vee H$. If $e \in E(G)$, by

Lemma 3.13 we have $f_{\mathcal{B}}(e) \leq f_{\mathcal{B}_1}(e) + 2$. If $e \in E(H)$, then

$$f_{\mathcal{B}}(e) \leq (f_{\mathcal{B}_2}(e) + 2)p_1$$

by Lemma 3.13 and the construction of $\mathcal{B}$. If $e \in \{u_i v_j | u_i \in V(G)\}$ then $f_{\mathcal{B}}(e) \leq 4$. If $e \in \{u_i v_j | u_i \in V(G),\ v_j \in V(H)$ and $j \neq k\}$,we have $f_{\mathcal{B}}(e) = 2$. Thus $b(G \vee H) \leq \max\{b(G) + 2, (b(H) + 2)p_1, 4\}$ and by symmetry of join of two graphs, the results holds. $\square$

Determining a upper bound for the co-normal product of graphs remains an open problem.

# CHAPTER IV

# THE MINIMUM CYCLE BASIS OF GRAPHS

The last subject which we study is the minimum cycle basis of graphs.

The length $|C|$ of a cycle $C$ is the number of its edges. The length $\ell(B)$ of a cycle basis $B$ is the sum of the lengths of its generalized cycles, $\ell(B) = \sum_{C\in B} |C|$. A **minimum cycle basis** (MCB) of the graph $G$ is a cycle basis with minimum length. Minimum cycle basis of graphs has variety of applications in science and engineering, for example in structural flexibility analysis [30], electrical networks [8], and in chemical structure storage and retrieval systems [11]. Brief surveys and extensive references can be found in [15], [20], [21], [34].

## 4.1 Minimum cycle basis algorithms

In studying MCB's of graphs, one line of research is finding efficient algorithms to find a MCB . The first polynomial time algorithm to find a MCB of a graph was given by Horton in [21]. Before seeing his algorithm we give some definitions.

**Definition 4.1.** *A matroid is an ordered pair $M = (S, I)$ satisfying the following conditions.*

1. *$S$ is a finite set.*

2. *I is a non-empty family of subsets of $S$, called the independent subsets of $S$, such that if $B \in I$ and $A \subseteq B$, then $A \in I$. Note that the empty set is necessarily a member of $I$ .*

3. *If $A \in I$, $B \in I$, and $|A| < |B|$,then there exists some element $x \in B - A$ such that $A \cup \{x\} \in I$ .*

A weight function $w : S \longrightarrow R^+$ for a matroid $M = (S, I)$ assigns a positive weight to each element of $S$. A matroid with a weight function is called a weighted matroid.
It is well known that a greedy algorithm can be used to find a maximum (minimum)-weight basis of the matroid. It suffices to start with an empty set and add the maximum (minimum)-weight element among the rest of elements that preserve the independence of the augmented set.
The set of generalized cycles of a graphs forms a matroid so a greedy algorithm can be used to find a MCB. The first polynomial time algorithm that is based on this property was introduced by Horton in [21].

Horton's algorithm is based on the following theorem:

**Theorem 4.2.** *[21] Let $x$ be any vertex of any cycle $C$ in a MCB of a graph $G$. There is an edge $e = yz$, where $y, z \neq x$, in $C$ such that $C$ consists of a shortest path from $x$ to $y$, a shortest path from $x$ to $z$ and the edge $yz$.*

In this algorithm one can extract a MCB of a graph with $p$ vertices and $q$ edges from a list of $pq$ candidate cycles by a greedy algorithm say Gaussian elimination in $O(q^3p)$ steps.

1: **procedure** HORTON'S ALGORITHM

Initially the set $\mathcal{C}$ of cycles is empty.

$\forall a, b \in V(G)$ find a shortest path $P(a, b)$ between $a$ and $b$.

**for** all $v \in V(G)$ **do**

  **for** all edges $xy \in E(G)$ **do**

    add to $\mathcal{C}$ the cycle $C = P(v, x) + P(v, y) + (x, y)$

  **end for**

**end for**

Order the set of cycles $\mathcal{C}$ by their length.

Use a greedy algorithm to extract a MCB from $\mathcal{C}$.

**end procedure**

The shortest path between two vertices $P(a, b)$ can be found by Floyd algorithm [13] in $O(p^3)$. Horton's algorithm produces the $pq$ candidate cycles in the main **for** loop in $O(qp^2)$ and then order the cycles by their length in $O(n \log n)$ and in the last line by a greedy algorithm extract MCB in $O(nk)$ operations where $n$ is the number of cycles found in the main loop and $k$ is the number of operations to decide whether a cycle is independent of another given set of cycles or not.

One can consider the cycles as rows of a $0 - 1$ matrix. The columns correspond to the edges of the graph, the rows are the incidence vectors of the cycles. Gaussian elimination using elementary row operation can be applied to the matrix. Each row can be processed in $O(qd)$ operations, where $d$ is the maximum number of independent cycles. Since the total number of cycles that has to be processed is $O(pq)$, last step takes $O(q^2dp)$.

De Pina [10] introduced another approach, that was further improved in [32], to reach a time bound of $O(q^2p + qp^2 log\ p)$. This is faster than Horton's collection approach. In his algorithm, the cycles look such vectors restricted to the coordinates indexed by $\{e_1, \cdots, e_{\nu(G)}\}$. That is, each cycle can be represented as a vector in $\{0, 1\}^{\nu(G)}$, see [10] for more details. The cycles of an MCB are computed sequentially. Assume that

$i-1$ cycles $C_1, C_2, \cdots, C_{i-1}$ of a MCB are already known. In order to compute the cycle $C_i$, we first compute a non-zero vector $S_i \in \{0,1\}^{\nu(G)}$, called a support vector, s.t. $\langle C_j, S_i \rangle = 0$ for all $1 \leq j < i$ (the inner product of two vectors $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ is defined by $\sum_{i=1}^{n} a_i b_i$) . Then cycle $C_i$ is the shortest cycle $C$ in the graph $G$ s.t. $\langle C, S_i \rangle = 1$. The fact that $C_i$ is not orthogonal to $S_i$ ensures linear independence, the shortest cycle computation ensures the optimality of the resulting cycle basis.

Mehlhorn and Michail [40] presented a $O(q^2 p/log\ p + p^2 q)$ algorithm to find a MCB of graphs. They used a minimum decycling set of a graph to reduce the collection of cycles which is a subset of Horton's collection, and still contains an MCB. In fact, they define the reduced Horton collection $\mathcal{R}$ to be all cycles $C[s,e]$ such that $s \in S$ ($S$ is decycling vertex set of $G$) and the endpoints of $e$ lie in different subtrees of $T_s$ where $T_s$ is a minimum distance tree rooted at $s$ and $C[s,e]$ is the cycle consisting of $e$ and the shortest paths from $v$ to the endpoints of $e$ in $G - e$.
They proved that $\mathcal{R}$ contains a MCB and $\mathcal{R} \subseteq \mathcal{H}$ where $\mathcal{H}$ is Horton's candidate collection. Let $\mathcal{A}$ denote the candidate set. The algorithm can be outlined as follows.

**Melhorn's Algorithm** [40] The algorithm is like De Pina's algorithm and computes the next member of a MCB sequentially. The algorithm proceeds as follows. Assume that we have cycles $C_1, \cdots, C_{i-1}$ of an MCB and a non-trivial support vector $S_i \in \{0,1\}^{\nu(G)}$ such that $\langle C, S_i \rangle = 0$ for all $1 \leq j < i$. In order to obtain $C_i$ we look for the shortest cycle $C \in \mathcal{A}$ such that $\langle C, S_i \rangle = 1$.

## 4.2 Minimum cycle basis of the Cartesian product of graphs

The aim of another research line is to find an exact MCB for various families of graphs, especially products of graphs. Hammack in [15] gave a cycle basis for $G\Box H$.

**Hammack's cycle basis for the Cartesian product of graphs:** Suppose $G$ and $H$ are two non-empty graphs and that $T_G$ and $T_H$ are their spanning trees. Let $\mathcal{B}_G$ and $\mathcal{B}_H$ be cycle bases of $G$ and $H$, respectively.
The following sets are pairwise disjoint and their union forms a cycle basis for the Cartesian product of graphs

$$H_1 = \{e\Box f | e \in E(T_G), f \in E(T_H)\}$$
$$H_2 = \{C^y | C \in \mathcal{B}_G, y \in V(H)\}$$
$$H_3 = \{{}^xC | x \in V(G), C \in \mathcal{B}_H\}.$$

Where $C^y = (C\Box H)_{|\{(x,y)|x\in V(C)\}}$ and ${}^xC = (G\Box C)_{|\{(x,y)|y\in V(C)\}}$.

The induced subgraphs $G^y = (G\Box H)_{|\{(x,y)|x\in V(G)\}}$ and ${}^xH = (G\Box H)_{|\{(x,y)|y\in V(H)\}}$ are called the fibers of $G\Box H$ and are copies of graphs $G$ and $H$ respectively.

In general however, Hammack's cycle basis is not a MCB for the Cartesian product of graphs. One counterexample is the graph $K_2\Box C_5$. The Hammack basis consists of two pentagons ${}^xC_5$ and ${}^yC_5$ and 4 squares and $\ell(B) = 5 + 5 + 4 \times 4 = 26$ while a MCB consisting of all five squares and only one of the fibers has length $5\times 4+5 = 25$.

**Lemma 4.3.** *[21] If $\mathcal{B}$ is a cycle basis for a graph, $C \in \mathcal{B}$, and $C = C_1 + C_2$ , then $\mathcal{B} - C + C_i$ is a cycle basis for $i = 1, 2$.*

**Proof.** Both $C_1$ and $C_2$ are generated uniquely by $\mathcal{B}$. Hence for both $i = 1$ and 2, $C_i$ is the sum of a subset of the cycles in $\mathcal{B}$, say $A_i$ and $A_i$ is unique. Since $\sum_{C'\in A_1} C' =$

$C_1 = C + C_2 = C + \sum_{C' \in A_2} C'$, $A_1$ and $A_2$ differ only in that one of them includes $C$ and the other does not. Assume $C \in A_1$ then $\mathcal{B}-C$ generates $C_1$, hence $\mathcal{B}' = \mathcal{B}-C+C_2$ generates all cycles in $\mathcal{B}$ and is a basis. □

**Proposition 4.4.** *[21] A cycle $C$ is contained in some MCB then it cannot be written as a sum of shorter cycles.*

**Proof.** Assume $\mathcal{B}$ is a minimum cycle basis of graph $G$ and $C$ is in $\mathcal{B}$ such that can be written as a sum of shorter cycles i.e. $C = C_1 + C_2$ and $|E(C_i)| < |E(C)|$ for $i = 1, 2$. By Lemma 4.3 we have $\mathcal{B}' = \mathcal{B} - C + C_1$ is a basis for one of $i = 1$ or 2. But $\mathcal{B}'$ has less weight than $\mathcal{B}$. □

Using the Hammack basis, Imrich and Stadler in [23] constructed a MCB for the Cartesian and the strong products of graphs, in terms of MCB's of the factors.

Furthermore, in the general case they constructed a MCB for $G\Box H$ by using the property that the corresponding cycles in different fibers can be transformed into each other by the addition of series of squares of the set $\mathcal{C}_\Box = \{e\Box f | e \in E_G, f \in E_H\}$.

**Lemma 4.5.** *[23] Let $C \in \mathcal{C}(G)$ and $y, z \in V_H$. Then there is a family of squares $\mathcal{C}' \subseteq \mathcal{C}_\Box$ such that we can write:*

$$C^z = C^y + \sum_{C \in \mathcal{C}'} C.$$

The following cycle basis can be obtained by changing the Hammack basis as suggested by the previous lemma.

**Theorem 4.6.** *[23] Suppose $x \in V_G$, $y \in V_H$, and $\mathcal{B}_G$ and $\mathcal{B}_H$ are cycle bases of $G$ and $H$, respectively. Furthermore, let $T_G$ and $T_H$ be spanning trees of $G$ and $H$. Set $B_\Box = \{e\Box f | e \in T_G, f \in E_H\} \cup \{e\Box f | e \in E_G, f \in T_H\}$ where $e\Box f$ is the Cartesian product of the two edges e and $f$. Then*

$$\mathcal{B}^{xy}_{G\Box H} = \{{}^{x}C | C \in \mathcal{B}_H\} \cup \{C^{y} | C \in \mathcal{B}_G\} \cup \mathcal{B}_\Box$$

*is a cycle basis for $G\Box H$.*

Note that it is easy to see that $|\mathcal{B}^{xy}_{G\Box H}| = |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + 1 = \nu(G\Box H)$. It is also proved in [23] that $\mathcal{B}^{xy}_{G\Box H}$ generates the cycle space of the graph $G\Box H$ by a series of lemmas and so it is a cycle basis for graph $G\Box H$. Another series of lemmas is then needed to show that $\mathcal{B}^{xy}$ is a MCB for the Cartesian product of two graphs .

When the graphs contain triangles, with some changes we have the next result.

**Theorem 4.7.** *[23] For two graphs $G$ and $H$ let $\mathcal{B}(G)$ and $\mathcal{B}(H)$ be MCBs of $G$ and $H$. Let $\Delta_G \subseteq \mathcal{B}(G)$ and $\Delta_H \subseteq \mathcal{B}(H)$ be the sets of triangles in the bases. Then, for each $u \in V(G)$ and $v \in V(H)$ there is a minimum cycle basis $\mathcal{B}^*$ of $G\Box H$ containing*

$$\{t^{y} | t \in \Delta_G, y \in V(H)\} \cup \{{}^{x}t | t \in \Delta_H, x \in V(G)\} \cup \{C^{v} | C \in \mathcal{B}_G \backslash \Delta_G\} \cup \{{}^{u}C | C \in \mathcal{B}_H \backslash \Delta_H\}$$

*and a suitable subset of squares $Q \subseteq \mathcal{B}_\Box$.*

There is a relationship between the Cartesian and strong products. This can are exploited to get a MCB for $G \boxtimes H$ from one for $G\Box H$. The strong product of two edges $e = xy$ and $f = uv$ is the complete graph $K_4$ while the Cartesian product of them is $C_4$. In fact, the strong product $G \boxtimes H$ contains two *non- Cartesian*

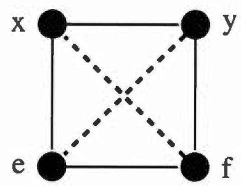



Figure 4.1: Graph $K_2 \boxtimes K_2$.

edges for each square from $\mathcal{B}_\square$, the dashed edges in Figure 4.1. Therefore the strong product $G \boxtimes H$ contains the Cartesian product $G\square H$ as a subgraph, and also has $2|E(G)||E(H)|$ more edges than the Cartesian product with the same vertex set. Hence $\nu(G \boxtimes H) = \nu(G\square H) + 2|E(G)||E(H)|$. One non-Cartesian edge is contained in a triangle the two other edges of which are Cartesian, hence a cycle basis of $G \boxtimes H$ can be constructed starting from a cycle basis of $G\square H$ by adding $2|E(G)||E(H)|$ different cycles, each of which contains exactly one of the non-Cartesian edges, but each of these squares can be produced by two triangles with one non Cartesian edge in common for example in Figure 4.1 we can write $(x, y, v, u) = (u, v, x) + (x, y, v)$. Then all cycles of a $K_4$ can be produced by 3 triangles $(u, v, x), (x, y, u)$ and $(x, y, v)$. We denote the set of all of these triangles in $G \boxtimes H$ by $\mathcal{B}_\boxtimes$.

**Theorem 4.8.** *[23] Suppose $x \in V_G$, $y \in V_H$, and $\mathcal{B}_G$ and $\mathcal{B}_H$ are cycle bases of $G$ and $H$.*

$$\mathcal{B}_\boxtimes^{xy} = \{{}^xC | C \in \mathcal{B}_H\} \cup \{C^y | C \in \mathcal{B}_G\} \cup \mathcal{B}_\boxtimes$$

*is a minimum cycle basis for $G \boxtimes H$.*

## 4.3 Minimum cycle basis of the join of two graphs

In Chapter 3 we found a cycle basis for $G \vee H$ when we wanted to compute the basis number of the join of two graphs in Theorem 3.14. But that cycle basis is not a MCB in general. One counter example is $K_1 \vee C_4$, by Theorem 3.14 $\ell(B) = 3(3) + 4 = 13$ while a cycle basis consisting of 4 triangles $K_1 \vee C_4$ with length of $4(3) = 12$ is a MCB for $K_1 \vee C_4$

**Lemma 4.9.** *Let $u$ be a vertex not in graph $H$ then $D = \cup\{u \vee e | e \in E(H)\}$ is a MCB for $u \vee H$.*

**Proof.** Each triangle $u \vee e$ is different at least in the edge $e$ from other triangles, so $D$ is an independent cycle set. Also $|E(u \vee H)| = q+p$ and so $\nu(u \vee H) = q+p-(p+1)+1 = q = |E(H)| = |D|$. Therefore $D$ is a cycle basis consisting of triangles and hence a MCB of $u \vee H$. $\square$

Now with some changes in Theorem 3.14 and using the above lemma we can construct a MCB for the join of two graphs.

**Theorem 4.10.** *Let $G$ and $H$ be two connected graphs with $p_1$ and $p_2$ vertices and $q_1$ and $q_2$ edges, respectively, and furthermore let $u_g \in V(G)$, $v_h \in V(H)$ and let $T_H$ be spanning tree of $H$. Then a cycle basis, consisting of all triangles of $\mathcal{B} = \{G \vee v_h\} \cup \{u_g \vee H\} \cup (\cup_{l=1, l \neq g}^{p_1} H_l)$ is a minimum cycle basis for $G \vee H$, where $H_l = \{u_l \vee e | e \in T_H\}$ and $u_l \in V(G)$.*

**Proof.** It is easy to see that $\{G \vee v_k\}$, $\{u_g \vee H\}$ and $\cup(\cup_{l=1, l \neq g}^{p_1} H_l)$ are independent sets and that they are independent from each other. On the other hand $|B| = |G \vee v_k| +$

$|u_g \vee H| + |\cup(\cup_{l=1,l\neq g}^{p_1} H_l)| = q_1 + q_2 + (p_1-1)(p_2-1) = \nu(G \vee H)$ and so $\mathcal{B}$ is a MCB for $G \vee H$ because all the cycles are triangles. □

## 4.4 Minimum cycle basis of the direct product of graphs

Direct product doesn't preserve the cycle structure of its factors and it seems that a MCB of the directed product is harder to find. The first results were given by Hammack [16]. He provided a MCB for $G \times H$ when $G$ and $H$ are connected bipartite graphs. He continued this subject to find a MCB consisting of triangles for the direct product of complete graphs in [17]. The latter paper gives a MCB for $K_p \times K_q$ consisting of $2\binom{p}{2}\binom{q}{2} - pq + 1$ triangles, if $p, q \geq 4$, or if $p = 3$ and $q \geq 5$.
The latest result is by Bradshaw and Jaradat in [7]. They found a MCB for $K_2 \times K_n$ and also showed that for $n \geq 3$ the MCB consists of only squares.
Finding a MCB of the direct product of other graph families is an open problem.

## 4.5 Minimum cycle basis of the lexicographic product of graphs

Jaradat presented a MCB for lexicographic product of two connected graphs in [25]. Before giving their result, the next definition is needed.
Consider two graphs $G$ and $H$ and their product $G[H]$. For any edges $ab \in E(G)$, $uv \in E(H)$ and a vertex $w \in V(H) - \{u, v\}$, define the following cycles, Figure 4.2.

$$\mathcal{P}_{ab,w}^{uv} = ((a,w),(b,u),(b,v)).$$

Furthermore for a graph $H$ with $E(H) = \{u_1v_1, u_2v_2, \cdots, u_{|E(H)|}v_{|E(H)|}\}$, let

$$\mathcal{P}_{ab,w}^{H} = \cup_{i=1}^{|E(H)|}\mathcal{P}_{ab,w}^{u_iv_i}.$$

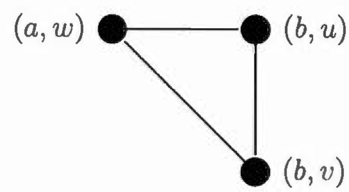



Figure 4.2: $\mathcal{P}^{uv}_{ab,w}$.

Let $T$ be a tree in $H$ of order greater than or equal to 2 and $ab$ be an edge. Then

$$\mathcal{P}^T_{ab} = \left( \bigcup_{w \in V(T)} \mathcal{P}^T_{ab,w} \right) \cup \mathcal{P}^T_{ba,w_0}$$

for some fixed vertex $w_0 \in V(T)$.

Now, let $G$ be a graph with $E(G) = \{a_1 b_1, a_2 b_2, \cdots, a_{|E(G)|} b_{|E(G)|}\}$, also

$$\mathcal{P}^T_G = \bigcup_{i=1}^{|E(G)|} \mathcal{P}^T_{a_i b_i}.$$

They use the following proposition to show that $\mathcal{P}^T_{ab}$ and $\mathcal{P}^T_G$ are linearly independent set.

**Proposition 4.11.** *[25] Let $A$ and $B$ be sets of cycles of a graph $G$, and suppose that both $A$ and $B$ are linearly independent, and $E(A) \setminus E(B)$ induces a forest in $G$ (we allow the possibility that $E(A) \setminus E(B) = \emptyset$). Then $A \cup B$ is linearly independent.*

Let $a_1$ be an end vertex of $T_G$ and $T^*_G$ be a minimum distance tree (breadth-first search tree) rooted at a vertex $a_1$. As in the definition, this induces a breadth-first search numbering of the vertices of $G$, $a_1, a_2, \cdots, a_{|V(G)|}$ with the obvious property that $d(a_1, a_i) \leq d(a_1, a_{i+1})$ if $i \leq j$ and in particular

$$d(a_1, a_i) \leq d(a_1, a_{i+1}) = d(a_1, a_i) + 1.$$

Consider the edges of the rooted tree are the following $e_1 = a_1 b_1, e_2 = a_2 b_2, \cdots, e_{|E(T_G)|} = a_{|E(T_G)|} b_{|E(T_G|}$. In this way it is guaranteed that $V(e_i) \cap V(\bigcup_{j=1}^{i-1} e_j)$ is exactly one vertex,

$a_i$.

For vertex $w_0 \in V(H)$ define

$$\mathcal{A}^{(H-T_H)}_{T^*_G,w_0} = \mathcal{P}^{H-T_H}_{b_1a_1,w_0} \cup \left( \bigcup_{i=1}^{|E(T_G)|} \mathcal{P}^{H-T_H}_{a_ib_i,w_0} \right)$$

Jaradat proved that $A^{(H-T_H)}_{T^*_G,w_0} \cup P^{T_H}_G$ is linearly independent and then

$$A^{(H-T_H)}_{T^*_G,w_0} \cup \mathcal{P}^{T_H}_G \cup B_{G^{x_0}}$$

is a MCB for $G[H]$ where $B_{G\square x_0}$ is a basis for the fiber $G^{x_0}$.

As we saw in the definitions of graph products $G \boxtimes H \subseteq G[H]$. It should be possible to a construct cycle basis of the lexicographic product of two graphs by extension of a MCB of the strong product .

Kaveh and Mirzaie in [31] presented an algorithm to find a MCB of the lexicographic product of graphs by using a MCB of the strong product of graphs.
The difference between the cyclomatic number of the lexicographic product and that of the strong product is the difference in the number of their edges, i.e.

$$\nu(G[H]) - \nu(G \boxtimes H) = E(G[H]) - E((G \boxtimes H)$$

$$= p_1q_2 + p_2^2q_1 - (p_1q_2 + p_2q_1 + 2q_1q_2) = q_1p_2(p_2 - 1) - 2q_1q_2$$

Kaveh and Mirzaie tried to find $q_1p_2(p_2 - 1) - 2q_1q_2$ additional cycles to add to a MCB of the strong product of graphs to obtain a MCB for the lexicographic product of the

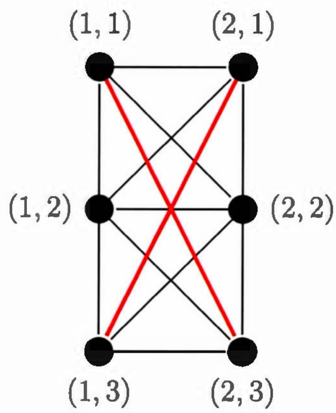



Figure 4.3: $P_2[P_3]$ and $P_2 \boxtimes P_3$ where the edges belonging to $P_2[P_3]$ but not $P_2 \boxtimes P_3$ are in red.

graphs.

The additional cycles of the lexicographic product are constructed by adding an edge of the graph $G$ to the start vertex of each path $P_i$ with length of $i \geq 3$ of a spanning tree of $H$ and then joining the start and end vertices of resulting $P_{i+1}$ by an edge, resulting in a cycle $C_{i+1}$. They do this also for end vertex of $P_i$. They do not consider the path $P_i$ if its end vertices are the same as those of a chord (in spanning tree of $H$).

Kaveh and Mirzaie in [31] presented the following algorithm for finding the additional cycles.

1: **procedure** KAVEH AND MIRZAIE'S ALGORITHM
2: Generate a spanning tree of $H$ and identify its chords.
3: Extract all the paths with length of more than 2 of the spanning tree of graph $H$.
4: If the two end nodes of a $P_i$ are the same as those of a chord, then delete that $P_i$.

**for** all $e \in E(G)$ **do**

place $e$, once in the start vertex of $P_i$ and once in the end vertex of $P_i$. Then join the start and end vertices of resulting $P_{i+1}$ by an edge, resulting in a cycle $C_{i+1}$.

**end for**

**end procedure**

Unfortunately, the next counter example shows that the algorithm does not work. Consider the lexicographic product of the path graphs $P_2$ and $P_3$, Figure 4.3. By the algorithm, the additional cycles of the lexicographic product will be rectangles of

$$C_1 = ((1,1),(1,2),(1,3),(2,3))$$
$$C_2 = ((2,1),(2,2),(2,3),(1,3))$$

But we can write $C_1$ as sum of two triangles

$$C_1 = ((2,1),(2,2),(2,3)) + ((1,2),(2,3),(1,3)).$$

Therefore by Proposition 4.4, $C_1$ cannot be contained in a MCB of $P_2[P_3]$.

## 4.6 A new minimum cycle basis of the lexicographic and co-normal products

By definition, the lexicographic product has edges of the form $(a,u)(b,v)$ such that $ab \in E(G)$ and $uv \notin E(H)$ ($ab \in E(G)$ and $uv \in E(\overline{H})$) in addition to those of the the strong product, these are the red edges in Figure 4.4. We have $|E(\overline{H})| = \frac{p_2(p_2-1)}{2} - q_2$ and so there are $2 \times q_1 \times (\frac{p_2(p_2-1)}{2} - q_2) = q_1(p_2(p_2-1)) - 2q_1q_2$ of such these edges We can construct a cycle basis of the lexicographic product starting from a cycle basis

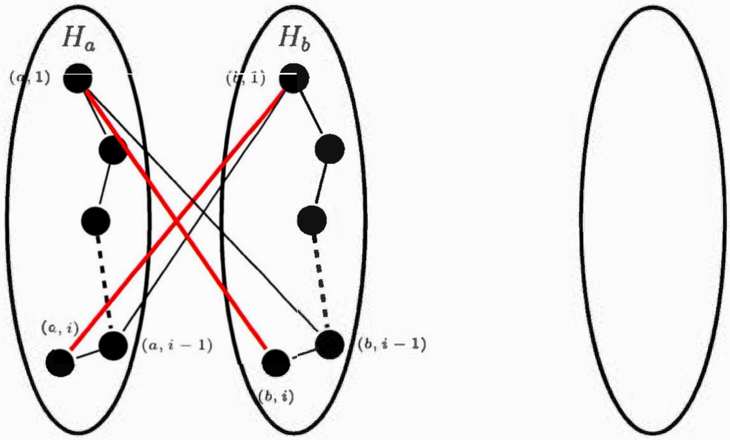



Figure 4.4: Non-strong triangles in $G[H]$

of $G \boxtimes H$ by adding $q_1(p_2(p_2-1)) - 2q_1q_2$ additional cycles, each of which contains exactly one of the "non-strong edges".

Suppose $T_H$ is a spanning tree of $H$. Let path $P_i$ be a path of length of $i$ in $T_H$ between two non-adjacent vertices $u$ and $v$ in $H$. For each $P_i$ it is clear that $i \geq 3$ and also the two end vertices of a $P_i$ are not the same as those of a chord in $H$. Suppose $V(P_i) = \{u = 1, 2, \cdots, i-1, i = v\}, i \geq 3$ and for each edge $ab \in E(G)$ consider the triangles

$$\mathcal{C}_{ab}^{uv} = ((a,1),(b,i),(b,i-1))\,.$$

We define

$$\mathcal{C}_{ab}^{H} = \bigcup_{uv \in E(H)} \mathcal{C}_{ab}^{P_i}$$

and

$$\mathcal{C}_{G}^{H} = \bigcup_{ab \in E(G)} \mathcal{C}_{ab}^{H} \cup \mathcal{C}_{ba}^{H}.$$

**Theorem 4.12.** *Suppose $x \in V_G$, $y \in V_H$, and $\mathcal{B}_{\boxtimes}^{xy}$ is a MCB of the strong product of the graphs $G$ and $H$. Then*

$$\mathcal{B}_{[\,]}^{xy} = \mathcal{B}_{\boxtimes}^{xy} \cup \mathcal{C}_{G}^{H}$$

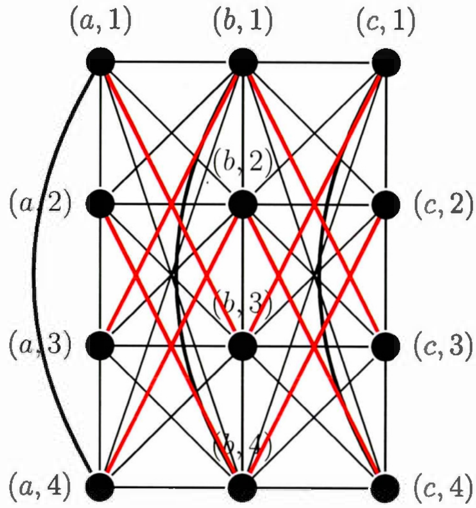



Figure 4.5: $C_4[P_3]$ and $C_3 \boxtimes P_3$ where the edges belonging to $C_4[P_3]$ but not $C_3 \boxtimes P_3$ are in red.

*is a MCB for* $G[H]$.

**Proof.** We prove that $C_G^H$ can be the additional cycles of cycle basis of the lexicographic product.

The total number of such triangles in $G[H]$ is

$$|C_G^H| = q_1(2 \times (\frac{p_2(p_2-1)}{2} - q_2)) = q_1(p_2(p_2-1)) - 2q_1q_2.$$

That is equal to $\nu(G[H]) - \nu(G \boxtimes H)$.

The non-strong triangles are independent because they differ at least at the edge $(a,u)(b,v)$, and also are independent from Imirich's MCB of the strong product.

As we said, Imirich's MCB of the strong product consists of copies of MCB's of the graphs $G$ and $H$ and triangles that construct the strong product squares , $\mathcal{B}_{\boxtimes}$. Each combination of triangles of $C_G^H$ has at least one non-strong product edge, and no combination of strong product triangles contains a non-strong product edge. So $C_G^H$ and

$\mathcal{B}_{\boxtimes}$ are independent. On the other hand the additional triangles $\mathcal{C}_G^H$, by their definition, contain no copies of edges of $G$ so cannot produce any copy of cycles of the graph $G$. So $\mathcal{C}_G^H$ and $G^y$ are independent. Furthermore $\mathcal{C}_G^H$ and ${}^xH$ is also independent because we use those paths of the graph $H$ whose end vertices do not coincide with a chord of a spanning tree of $H$ i.e. no combination of triangles of $\mathcal{C}_G^H$ can produce a cycle of $H$. It follows that $\mathcal{C}_G^H$ can be the additional cycles of the lexicographic product and Imirich's MCB of the strong product union $\mathcal{C}_G^H$ is MCB for the lexicographic product of two graphs. □

**Example.** Suppose $V(P_3) = \{a, b, c\}$ and $V(C_4) = \{1, 2, 3, 4\}$. The additional triangles cycles to construct a MCB for $P_3[C_4]$, Figure 4.5, consists of 8 triangles. Suppose a spanning tree of graph $G$, $T_{C_4}$, is path on the vertices $\{1, 2, 3, 4\}$. Then the set of paths of $T_{C_4}$ with length more than 2 such that the two end vertices of the path are not the same as those of a chord are $P_3' = (1, 2, 3)$ and $P_3'' = (2, 3, 4)$.

Then

$$\mathcal{C}_G^H = \mathcal{C}_{ab}^H \cup \mathcal{C}_{ba}^H \cup \mathcal{C}_{bc}^H \cup \mathcal{C}_{cb}^H = \{C_{ab}^{P_3'}, C_{ba}^{P_3'}, C_{ab}^{P_3''}, C_{ba}^{P_3''}, C_{bc}^{P_3'}, C_{cb}^{P_3'}, C_{bc}^{P_3''}, C_{cb}^{P_3''}\}$$

$$= \Big\{ ((a,1),(b,3),(b,2)), ((b,1),(a,3),(a,2)), ((a,2),(b,4),(b,3)), ((b,2),(a,4),(a,3)),$$

$$((b,1),(c,3),(c,2)), ((c,1),(b,3),(b,2)), ((b,2),(c,4),(c,3)), ((c,2),(b,4),(b,3)) \Big\}.$$

We can apply the same procedure to find a MCB for the co-normal product by starting from a MCB of the lexicographic product. First note that the co-normal product has edges like $(a, u)(b, v)$ such that $ab \notin E(G)$ and $uv \in E(H)$ ($ab \in E(\overline{G})$ and $uv \in E(H)$) in addition to those of the lexicographic product.

Suppose $T_G$ is a spanning tree of graph $G$. Consider all paths $P_i$ , $i \geq 3$ of $T_G$ between two non adjacent vertices of $a$ and $b$ of $G$. Then end vertices are not coincide with chord of graph $G$. Suppose $V(P_i) = \{a = 1, 2, \cdots, i-1, b = i\}$, $i \geq 3$. For each edge $uv \in E(H)$ consider triangles of

$$\mathcal{C}_{uv}^{ab} = ((1, u), (i, v), (i-1, v)).$$

Then we define

$$\mathcal{C}_{uv}^{G} = \bigcup_{ab \in E(G)} \mathcal{C}_{uv}^{ab}$$

and

$$\mathcal{C}_{H}^{G} = \bigcup_{uv \in E(H)} \mathcal{C}_{uv}^{G} \cup \mathcal{C}_{vu}^{G}.$$

**Theorem 4.13.** *Suppose for graphs $G$ and $H$, $x \in V_G$ and $y \in V_H$.*

$$\mathcal{B}_{\bullet}^{xy} = \mathcal{B}_{\boxtimes}^{xy} \cup \mathcal{C}_{G}^{H} \cup \mathcal{C}_{H}^{G}$$

*is a MCB for $G \bullet H$.*

**Proof.** The total number of additional triangles in $G \bullet H$ is

$$|\mathcal{C}_{H}^{G}| = q_2(2 \times (\frac{p_1(p_1-1)}{2} - q_1)) = q_2(p_1(p_1-1)) - 2q_2q_1.$$

That is equal to $\nu(G \bullet H) - \nu(G[H])$ since

$$\nu(G \bullet H) - \nu(G[H]) = p_1q_2 + p_2q_1 + 2q_1q_2 - (p_2^2q_1 + p_1q_2) = q_1(p_2(p_2-1)) - 2q_1q_2.$$

It is easy to prove that $\mathcal{B}_{\boxtimes}^{xy}$ and $\mathcal{C}_{H}^{G}$ are independent with the same argument as in previous theorem. Furthermore each triangle (and also each combination of triangles) of $C_G^H$ contains the copies of some edges of $H$ of the form of $(a,i)(a,i-1)$, Figure 4.4, that does not exist and can not produced by triangles of $C_H^G$, and conversely each triangles (and also each combination of the triangles) of $C_H^G$ contains the copies of edge of $G$ of the form of $(i,v)(i-1,v)$, that does not appear in $C_G^H$ ( and any combination of its elements). So $C_H^G$ and $C_G^H$ are independent and the result follows. □

# CHAPTER V

# FURTHER QUESTIONS FOR RESEARCH

Decycling number of the Cartesian product of two arbitrary graphs is a open problem yet. A tighter upper bound for grid graphs, upper bound for the Cartesian product of other classes of graphs besides paths, cycles and complete graphs can be subject of future researches.

Determining the basis number of the co-normal product of graphs is open. The suggested way is to find a cycle basis of co-normal product by using a tree decomposition introduced by Jaradat in [26]. Then by computation of basis number of this cycle basis we have an upper bound for this product.

Finding a minimum cycle basis of the direct product of graph is open. Since the direct product of graphs does not preserve the structure of its factors, it is more complicated to find a MCB.

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