Summary: Cortical Surface Alignment Using Geometry-Driven Multispectral Optical Flow

We would like to perform a nonrigid registration between two cortical surfaces. The first surface we denote the atlas; e.g., the reference surface. The second we denote the subject.

1 Basic Hypotheses

1.1 Anatomical Assumptions

sulci: narrow grooves on the surface of the brain
gyri: ridges between these grooves

Major sulci are the deeper, longer grooves common to most individuals. "It is believed that many major sulci are linked to the underlying cytoarchitectonic and functional organization of the cortex [16]". So our first problem becomes smoothing out the other, irrelevant convolutions (minor sulci and gyri) in the brain, which vary widely between individuals. The second problem is the identification ("soft" segmentation) of sulci on these smoother manifolds. Once we have a images $\hat{I}_{\text{atlas}}(\mathbf{x})$ and $\hat{I}_{\text{subj}}(\mathbf{x})$, mapping their respective manifolds to an intensity that represents level of membership to a sulcal region, we can try to register these images (and thus the respective major anatomical features).

1.2 Simplifying Steps

"Visualization and analyses on the cortical surface,... is difficult...". Cortical unfolding procedures have been developed to address these problems. ... Preservation of the metric details – i.e., creation of approximately <u>isometric</u> maps – of the 3D surface has been a major goal in flattening". Just a reminder, an isometric map $f : A \to B$ is such that $\forall(x, y) \in A$, $d_A(x, y) = d_B(f(x), f(y))$. In our case, A is the convoluted cortical surface with locally Euclidean distances, and $B = S^2$ is a sphere on which the geodesics (shortest distances between two points) are sections of great arcs.

A technique called hemispherical mapping takes the two smoothed hemispheres, and attempts to isometrically map each cortical "hemisphere" onto its own sphere $S^2 \subset \mathbb{R}^3$. We can then merge the two spheres (the left and right lobes) and register $I_{\text{atlas}}(\mathbf{x})$ and $I_{\text{subj}}(\mathbf{x})$, which are both functions on S^2. Here, we define $I(\mathbf{x}) = \hat{I}(m^{-1}(\mathbf{x}))$ where m is the isometric hemispherical map.

2 Problem setup

2.1 Cortical Surface Reconstruction

"In this paper, we start with a triangle mesh representation of the human brain cortex. We use <u>Cortical Reconstruction Using Implicit Surface Evolution</u> (CRUISE) [5] to find the central surface that lies at the geometric center of the gray matter tissue. ... Each reconstructed central surface is a triangle mesh comprising approximately 300,000 vertices."

2.2 Cortical Normalization

The first part of the Cortical Normalization [15] step, Parametric Surface Relaxation (PSR), smoothes out the minor sulci and gyri. Some registration is performed to align the major sulci into approximately the same areas. Conformal mapping is performed as part of the hemispherical mapping step. The isometric mapping m is estimated in this step.

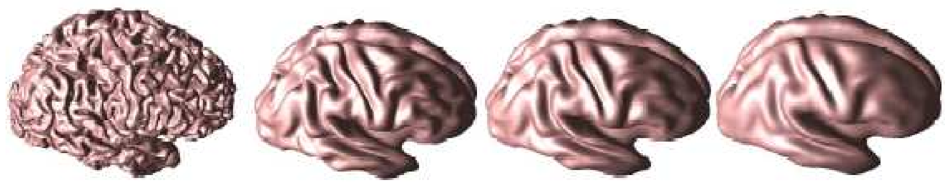

Figure 1: Parametric Surface Relaxation to create a Partially Flattened Surface of the cortex

2.2.1 Principal curvatures and their use in Parametric Surface Relaxation

Consider $\gamma : [0,1] \to S$, where $S \subset \mathbb{R}^n$ is some surface. The curvature of $\gamma(t)$ at point $p \in S$ can be characterized in two compatible ways:

$$\kappa(p) = \lim_{p' \to p} \frac{\alpha(p,p')}{s(p,p')}$$

where α is the angle between the tangents to γ at the points p and p' (and p' is on γ), and $s(p,p')$ is the arclength of the curve between p and p'. An equivalent definition of curvature in this case is:

$$\kappa(p) = \left\| \frac{d^2\gamma(t)}{dt^2} \right\|_2$$

Note that $\kappa(p) = 0 \Leftrightarrow$ the curve coincides with a segment of a straight line in a neighborhood around p.

The principal curvatures $\kappa_1(p)$ and $\kappa_2(p)$, for $p \in S$ derive from this definition. Specifically, consider all curves C_α on S passing through p. Each such curve has an associated curvature $\kappa_\alpha(p)$ at point p. We can then define

$$\kappa_1(p) = \min_\alpha \kappa_\alpha(p) \quad \text{and} \quad \kappa_2(p) = \max_\alpha \kappa_\alpha(p)$$

are the principal curvatures of surface S at p.

Defining the mean curvature at p as $H(p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p))$, Parametric Surface Relaxation smoothes the cortical surface until the L_2 norm of the mean curvature over S is below some predefined threshold. Specifically, until

$$\|H\|_2^2 = \int_S |H(p)|^2 \, dA < \tau^2$$

where S is the cortical surface.

2.3 Useful maps $I(\mathbf{x})$ for the segmentation of sulci

"In [19], shape index (SI) and curvedness (C) measures were introduced as a pair of local shape indicator measures". We define

$$SI = \frac{2}{\pi} \arctan \frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1} \quad \text{and} \quad C = \sqrt{\frac{\kappa_1^2 + \kappa_2^2}{2}}$$

Note that $SI \in [-1, 1]$ and "The extreme values of the shape index represents local shapes that look like either the inside ($SI = -1$) or the the outside ($SI = 1$) of a spherical surface."

We choose to use both of these measures as our "features" image. Specifically, we define $\mathbf{I}(\mathbf{x}) : S^2 \to [-1, 1] \times \mathbb{R}^+$ as:

$$\mathbf{I}(\mathbf{x}) = [w_{SI} I_{SI}(\mathbf{x}), w_C I_C(\mathbf{x})]$$

where the coefficients w_{SI}, w_C are of our choosing.

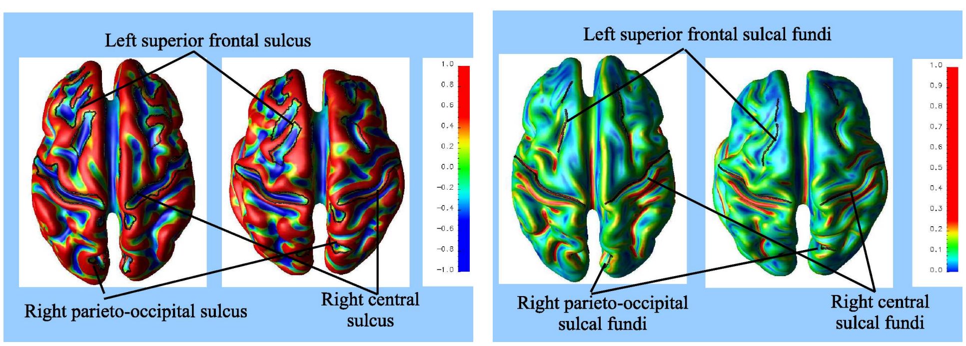

Figure 2: [left] Shape Index ($\hat{I}_{SI}(\mathbf{x})$) and [right] Curvature ($\hat{I}_C(\mathbf{x})$) images on PFS

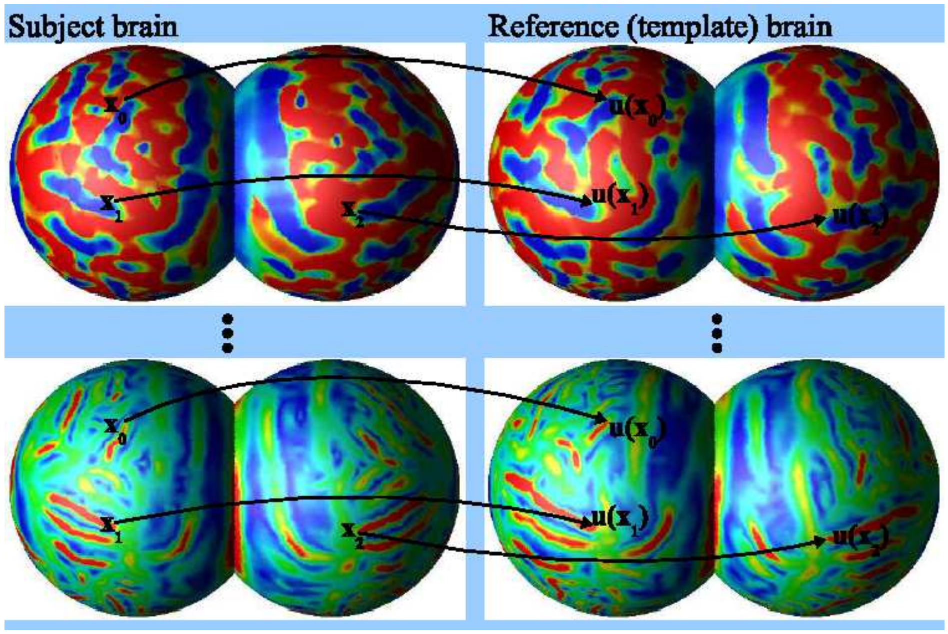

Figure 3: Example matched features on hemispherical maps created from the PFS

2.4 Nonrigid Registration

We try to find a continuous (this needs to be defined) warp from $\mathbf{I}_{\text{subj}}(\mathbf{x})$ to $\mathbf{I}_{\text{atlas}}(\mathbf{x})$. One way is to first define a sequence of images (a warp) over a time interval $t \in [0,1]$, starting with the subject and ending with the atlas. Specifically, we set up a differential equation problem with boundary conditions:

$$\mathbf{I}(\mathbf{x}(t=0), t=0) = \mathbf{I}_{\text{subj}}(\mathbf{x}) \quad \text{and} \quad \mathbf{I}(\mathbf{x}(t=1), t=1) = \mathbf{I}_{\text{atlas}}(\mathbf{x})$$

and we try to calculate a flow field $\frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(t) = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix}$ on $t \in [0,1]$. As this is an ill-posed problem (there are many such solutions), we impose a number of constraints. The first of these is a form of continuity called the optical flow (OF) constraint. The second is incompressible flow.

However, one constraint we immediately identify is that $\forall \mathbf{x} \in S^2$, $\mathbf{u}$ must be such that $\mathbf{x} + \mathbf{u} \in S^2$ (the resulting point must still reside on the sphere). It is equivalent to say $\|\mathbf{x}+\mathbf{u}\|_2 = \|\mathbf{x}\|_2 = \text{const}$.

3 Constraints and properties of the solution vector $\mathbf{u}(\mathbf{x}, t)$

3.1 Optical Flow

Suppose we have a sequence of images $I(\mathbf{x}, t)$ with objects moving in those images. Suppose also that these objects only move around (their intensity values do not change with time). Then $\forall \mathbf{x}, t \in [0, 1]$,

$$I(\mathbf{x}(t) + d\mathbf{x}, t + dt) = I(\mathbf{x}(t), t)$$

where $d\mathbf{x}$ is the distance an object moves from $\mathbf{x}$ in dt.

By Taylor expansion (and the chain rule),

$$I(\mathbf{x} + d\mathbf{x}, t + dt) \approx I(\mathbf{x}, t) + \langle \nabla_{\mathbf{x}} I(\mathbf{x}, t), d\mathbf{x} \rangle + \frac{\partial I}{\partial t} dt$$

where the second order terms are assumed small. Combining the two equations above and dividing through by dt, we get

$$\frac{DI}{Dt} = \left\langle \nabla_{\mathbf{x}} I(\mathbf{x}, t), \frac{d\mathbf{x}}{dt} \right\rangle + \frac{\partial I}{\partial t} = \langle \nabla_{\mathbf{x}} I(\mathbf{x}, t), \mathbf{u}(t) \rangle + \frac{\partial I}{\partial t} = 0$$

where $\frac{DI}{Dt}$ is the total derivative.

In our case, we have a multispectral image, and thus the condition is $\frac{D\mathbf{I}}{Dt} = \mathbf{0}$, or

$$w_j \left(\langle \nabla_{S^2} I_j(\mathbf{x}, t), \mathbf{u}(t) \rangle + \frac{\partial I_j(\mathbf{x}, t)}{\partial t} \right) = 0 \quad \text{for} \quad j = \{SI, C\}$$

where ∇_{S^2} is the gradient on the surface of the sphere.

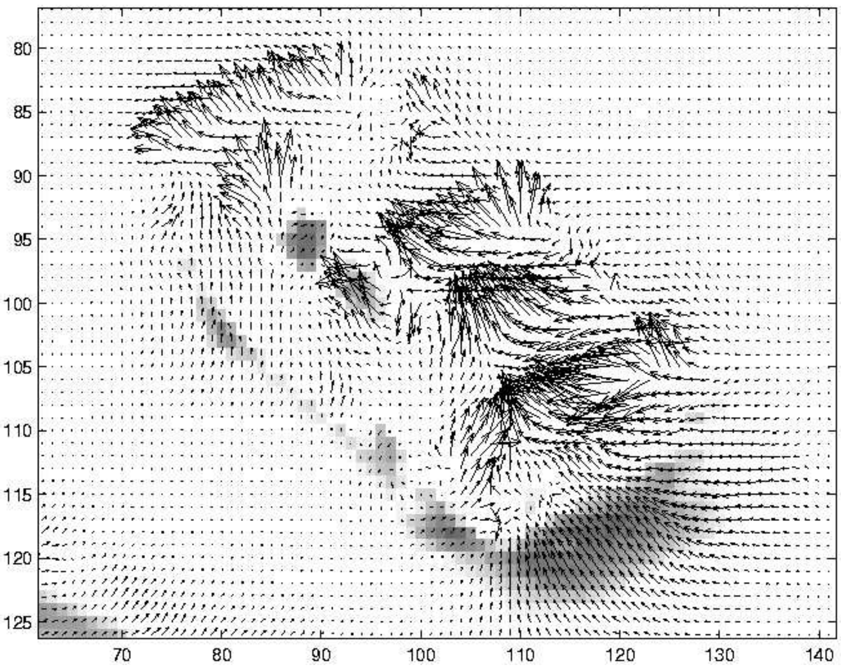

Figure 4: Sample optical flow estimate of a moving car

3.2 Incompressible Flow

The incompressible flow field constraint is stated as:

$$\nabla_{S^2} \cdot \mathbf{u}(\mathbf{x}) \approx \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

Such a field is also called a solenoidal field, as it is known that $\mathbf{u} = \nabla \times \mathbf{v}$ for some $\mathbf{v}$. Note that a zero divergence implies that a flow field has no sources or sinks, so this constraint also forces $\mathbf{u}$ to be what is called a rotational field. No point or set of points on the surface of the sphere can create the movement (warping) of features in all directions; the warp is conserved.

3.3 Smoothness

This soft constraint penalizes large changes in $\mathbf{u}$ in any given direction for nearby points. That is, we want $\mathbf{u}(\mathbf{x}, t) + d\mathbf{u}(\mathbf{x}) \approx \mathbf{u}(\mathbf{x}, t)$. One way of doing this is to minimize an increasing nonnegative function of $\{\|\nabla_{S^2} u_i(\mathbf{x}, t)\|\}_{i=1,2,3}$.

The authors use a "robust Lorentzian error measure" as the increasing nonnegative function mentioned above. Let $\rho(\epsilon, \mu) = \log(1 + \frac{1}{2}(\frac{\epsilon}{\mu})^2)$, for any $x \in S^2$, one can minimize $\sum_{i=1}^{3} \rho(\|\nabla_{S^2} u_i(\mathbf{x}, t)\|_2)$ where μ is some user-chosen regularization coefficient.

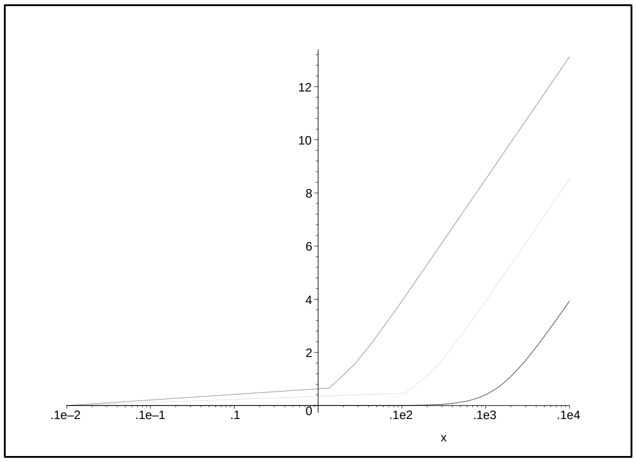

Figure 5: Semilog plots of $\rho(x, \mu)$ for (in the order of rising position, left to right) $\mu = \{1, 10, 100\}$. Realistic values are $\mu \in (0, 1]$

4 Complete problem statement

Instead of setting hard constraints on $\mathbf{u}(\mathbf{x})$ for an uncountable set $\mathbf{x} \in S^2$, we choose to minimize the Lagrangian form of an energy functional over the set of gradient fields on the sphere, $\mathcal{F} = \left\{\mathbf{u} : \forall \mathbf{x} \in S^2, \mathbf{x} + \mathbf{u} \in S^2\right\}$.

$$\arg\min_{\mathbf{u}\in\mathcal{F}} E(\mathbf{u}) = \int_{S^2} \rho\left(\sqrt{\sum_{j=SI,C} w_j^2 \left(\langle \nabla_{S^2} I_j(\mathbf{x}), \mathbf{u}(\mathbf{x})\rangle + \frac{\partial I_j(\mathbf{x})}{\partial t}\right)^2}, \mu\right) d\mathbf{x}$$
$$+\alpha \int_{S^2} \left(\sum_{i=1}^{3} \rho(\|\nabla_{S^2} u_i(\mathbf{x})\|, \mu)\right) d\mathbf{x} + \beta \int_{S^2} \rho(|\nabla_{S^2} \cdot \mathbf{u}(\mathbf{x})|, \mu) d\mathbf{x}$$

where α, β are chosen before the optimization. Note that above, we choose to minimize the average of the error from the constraint over the sphere S^2 by minimizing the integral of the error.

The first term is the optical flow constraint, the second term is the incompressibility constraint, and the third is the smoothness constraint.

5 Solving this optimization problem using variational methods

Suppose there exists a unique, continuous, $\mathbf{u} : S^2 \to \mathbb{R}^3$ that minimizes the integral above. Now, consider possible pertubations (continuous functions) from this optimal solution $\mathbf{u}$. That is, consider some functions $\mathbf{U} : S^2 \to \mathbb{R}^3 \in \mathcal{F}$; "near" $\mathbf{u}$. Then we know $E(\mathbf{U}) \geq E(\mathbf{u})$. We can "parametrize" such pertubations without knowing their explicit forms.

5.1 Pertubations to nearby admissible functions

Consider any continuous (C^∞) function $\phi : S^2 \to \mathbb{R}^3$, which is zero outside any closed subset of S^2. We can construct a pertubation function $\mathbf{U} \in \mathcal{F}$ using ϕ. Specifically, we parametrize $\mathbf{U}(s) = F(s, \phi)$ (where s is any real number) such that:

1. $\mathbf{U}(0) = \mathbf{u}$ (we have the optimizing function at $s = 0$) and
2. $\forall s, \mathbf{U}(s) \in \mathcal{F}$ ($\mathbf{U}(s)$ is admissible).

in the following way:

$$\mathbf{U}(s) = (\mathbf{u} + s\phi(\mathbf{x}) + \mathbf{x}) \frac{\|\mathbf{x}\|}{\|\mathbf{u} + s\phi(\mathbf{x}) + \mathbf{x}\|} - \mathbf{x}$$

because

1. $$\mathbf{U}(0) = \frac{(\mathbf{u} + \mathbf{x})\|\mathbf{x}\| - \mathbf{x}\|\mathbf{u} + \mathbf{x}\|}{\|\mathbf{u} + \mathbf{x}\|} = \frac{(\mathbf{u} + \mathbf{x})\|\mathbf{x}\| - \mathbf{x}\|\mathbf{x}\|}{\|\mathbf{x}\|} = \mathbf{u}$$
   as $\|\mathbf{u} + \mathbf{x}\| = \|\mathbf{x}\|$, and
2. $$\|\mathbf{U}(s) + \mathbf{x}\| = \|\mathbf{u} + s\phi(\mathbf{x}) + \mathbf{x}\| \frac{\|\mathbf{x}\|}{\|\mathbf{u} + s\phi(\mathbf{x}) + \mathbf{x}\|} = \|\mathbf{x}\|$$
   so $\mathbf{U}(s) \in \mathcal{F}$.

Note to self: Is this parametrization unique up to a scaling factor? Uniquely characterizes $\mathcal{F}$? How?

5.2 Finding optima of $E(\cdot)$

For any given ϕ, we can reduce the search for minima of $E(\mathbf{U}(s))$ to a simple search for critical points over s (basic calculus). That is, we look for the point where

$$\frac{d}{ds} E(\mathbf{U}(s)) = 0$$

But we already know where our optimal point is! We constructed $\mathbf{U}(s)$ to give us an optimal point at $s = 0$. That is, we know that $\mathbf{u} = (\mathbf{U}(s = 0))$ must necessarily fulfill the following equation:

$$\left. \frac{d}{ds} E(\mathbf{U}(s)) \right|_{s=0} = 0$$

5.3 Example using a simplified version of the problem

Suppose we do not restrict $\mathbf{u}$ to lie in $\mathcal{F}$, e.g., a small pertubation of $\mathbf{u}$ might be (and we will only use the notation in this section) $\mathbf{U}(s) = \mathbf{u} + s\phi(\mathbf{x})$. We try to optimize

$$E(\mathbf{U}(s)) = \int_{S^2} F(\mathbf{x}, \mathbf{U}(s), \nabla_{S^2} U_1(s), \nabla_{S^2} U_2(s), \nabla_{S^2} U_3(s)) d\mathbf{x}$$

and again we have $\frac{d}{ds} E(\mathbf{U}(s))\big|_{s=0} = 0$. Taking the derivative inside the integral and using the chain rule, we get

$$\int_{S^2} \left(D_{\mathbf{U}(s)}[F(\cdot)] \circ \frac{d}{ds}\mathbf{U}(s) + \sum_{k=1}^{3} D_{\nabla U_k(s)}[F(\cdot)] \circ \frac{d}{ds} \nabla U_k(s) \right) d\mathbf{x} = 0 \tag{1}$$

where $D_{\mathbf{t}}[F]$ is the differential operator on F with respect to parameter $\mathbf{t}$.

Remember that we now have $\nabla \mathbf{U}(s) = \nabla_{\mathbf{x}} \mathbf{u}(\mathbf{x}) + s\nabla_{\mathbf{x}} \phi(\mathbf{x})$. Thus, in this case,

$$\frac{d}{ds}\mathbf{U}(s) = \phi(\mathbf{x}) \quad \text{and} \quad \frac{d}{ds}\nabla U_k(s) = \nabla \phi_k(\mathbf{x})$$

and thus (1) becomes

$$\int_{S^2} \left(\left\langle D_{\mathbf{U}(s)}[F(\cdot)], \phi(\mathbf{x}) \right\rangle + \sum_{k=1}^{3} \left\langle D_{\nabla U_k(s)}[F(\cdot)], \nabla \phi_k(\mathbf{x}) \right\rangle \right) d\mathbf{x} = 0 \tag{2}$$

all evaluated at $s = 0$.

We can further apply Green's theorem in 3 dimensions:

$$\int_R \langle \mathbf{H}(\mathbf{x}), \nabla \phi_k(\mathbf{x}) \rangle \, d\mathbf{x} = -\int_R \phi_k \nabla \cdot \mathbf{H}(\mathbf{x}) d\mathbf{x} + \oint_{\partial R} \phi_k \mathbf{H}(\mathbf{x}) \cdot \hat{\mathbf{n}} dS$$

where R is some region in space and ∂R is its boundary. Note that in our case this simplifies tremendously because $R = S^2$ and $\partial S^2 = \emptyset$. Returning to (2) we can simplify again to

$$\begin{aligned}
0 &= \int_{S^2} \left(\left\langle D_{\mathbf{U}(s)}[F(\cdot)], \phi(\mathbf{x}) \right\rangle + \sum_{k=1}^{3} \phi_k(\mathbf{x}) \nabla_{S^2} \cdot D_{\nabla U_k(s)}[F(\cdot)] \right) d\mathbf{x} \\
&= \int_{S^2} \left(\left\langle D_{\mathbf{U}(s)}[F(\cdot)], \phi(\mathbf{x}) \right\rangle + \left\langle D_{\nabla \mathbf{U}(s)}[F(\cdot)], \phi(\mathbf{x}) \right\rangle \right) d\mathbf{x} \\
&= \int_{S^2} \left(\left\langle D_{\mathbf{U}(s)}[F(\cdot)] + D_{\nabla \mathbf{U}(s)}[F(\cdot)], \phi(\mathbf{x}) \right\rangle \right) d\mathbf{x}
\end{aligned}$$

Finally, there is a lemma in functional analysis that states the following:

If $\int_R \langle \mathbf{H}(\mathbf{x}), \phi(\mathbf{x}) \rangle \, d\mathbf{x} = 0$ for all continuous, differentiable functions ϕ defined on R, then each component $H_k(\mathbf{x})$ must (almost surely) be zero on the entirety of R.

Note that in our case, $\frac{d}{ds}\mathbf{U}(s) \neq \phi(\mathbf{x})$, and so it's not apparent that we get an inner product form with $\phi(\mathbf{x})$ inside our integral $E(\cdot)$. However, we give evidence that this is indeed the case towards the end of the presentation.

5.4 Alternatively, taking derivatives directly (the slightly more way)

Note below that $U_i(s) = U_i(\mathbf{x}, s) = (u_i(\mathbf{x}) + s\phi_i(\mathbf{x}) + \mathbf{x}_i)\frac{\|x\|}{\|\mathbf{u}(\mathbf{x})+s\phi(\mathbf{x})+\mathbf{x}\|} - \mathbf{x}_i$, is simply the ith component of the admissible function.

$$
\begin{aligned}
\frac{d}{ds}E(\mathbf{U}(s)) \;\; = \;\; & \int_{S^2} \Bigg[\left(\frac{\frac{1}{2}\sum_{j=SI,C} 2w_j^2 \left(\langle \nabla_{S^2} I_j(\mathbf{x}), \mathbf{U}(s) \rangle + \frac{\partial I_j(\mathbf{x})}{\partial t} \right) \frac{d}{ds} \left(\langle \nabla_{S^2} I_j(\mathbf{x}), \mathbf{U}(s) \rangle + \frac{\partial I_j(\mathbf{x})}{\partial t} \right)}{\mu^2 + \frac{1}{2}\sum_{j=SI,C} w_j^2 \left(\langle \nabla_{S^2} I_j(\mathbf{x}), \mathbf{U}(s) \rangle + \frac{\partial I_j(\mathbf{x})}{\partial t} \right)^2} \right) \\
& + \;\; \alpha \left(\sum_{i=1}^{3} \frac{\frac{1}{2}\frac{d}{ds} \langle \nabla_{S^2} U_i(s), \nabla_{S^2} U_i(s) \rangle}{\mu^2 + \frac{1}{2}\|U_i(s)\|^2} \right) + \beta \left(\frac{\frac{1}{2}\frac{d}{ds} |\nabla_{S^2} \cdot \mathbf{U}(s)|}{\mu^2 + \frac{1}{2}|\nabla_{S^2} \cdot \mathbf{U}(s)|^2} \right) \Bigg] d\mathbf{x}
\end{aligned}
$$

5.4.1 Derivatives of $U_i(s)$

It can (rather, should) be checked that (and below we use shorthand $\mathbf{u}, \phi, u_i, \phi_i$ for $\mathbf{u}(\mathbf{x}), ...$)

$$
\frac{d}{ds}U_i(s) = \frac{\|\mathbf{x}\|}{\|\mathbf{u} + s\phi + \mathbf{x}\|^3} \Big(\|\mathbf{u} + s\phi + \mathbf{x}\|^2 \phi_i - \langle \mathbf{u} + s\phi + \mathbf{x}, \phi \rangle (u_i + s\phi_i + \mathbf{x}_i) \Big)
$$

and therefore that

$$
\frac{d}{ds}\mathbf{U}(s) = \frac{\|\mathbf{x}\|}{\|\mathbf{u} + s\phi + \mathbf{x}\|^3} \Big(\|\mathbf{u} + s\phi + \mathbf{x}\|^2 \phi - \langle \mathbf{u} + s\phi + \mathbf{x}, \phi \rangle (\mathbf{u} + s\phi + \mathbf{x}) \Big)
$$

further

$$
\frac{d}{ds}U_i(0) = \frac{\|\mathbf{x}\|}{\|\mathbf{u} + \mathbf{x}\|^3} \Big(\|\mathbf{u} + \mathbf{x}\|^2 \phi_i - \langle \mathbf{u} + \mathbf{x}, \phi \rangle (u_i + \mathbf{x}_i) \Big) = \phi_i - \frac{\langle \mathbf{u} + \mathbf{x}, \phi \rangle}{\|\mathbf{x}\|^2}
$$