

# Evolution of Information Gathering Strategies

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## 1 Introduction

Questions:

1. How are the two group properties— $\mathcal{H}_2$  robustness and correlation length—related?
2. What are the equilibrium strategies of individuals seeking to optimize their own fitness?
3. What group properties emerge from individual-level optimization?

## 2 Model

**2.1 Opinion dynamics** In our model, each bird has a scalar value,  $v_i$ , that represents its opinion about the environment. Each bird also has a strategy,  $k_i$ , that represents the number of its peers whose opinions it observes, which it uses to update its own opinion. Specifically, a bird observes the  $k_i$  nearest birds, which form the neighborhood  $\mathcal{N}_i$ , and the focal bird changes its opinion to be close to the average of its neighbors’:

$$\dot{v}_i = \frac{1}{k_i} \sum_{j \in \mathcal{N}_i} (v_j - v_i).$$

The strategy of choosing  $k_i$  nearest neighbors will be the focus of our analysis. It will be useful to write down the matrices describing the interaction between pairs of birds. If

$$A_{ij} = \begin{cases} \frac{1}{k_i} & \text{if } j \text{ is in } \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad L_{ij} = \begin{cases} A_{ij} & \text{if } j \neq i \\ -\sum_l A_{il} & \text{if } j = i \end{cases},$$

then  $L$  can be used to write a system of differential equations describing the vector of opinions  $v = (v_1, \dots, v_N)$ :

$$\dot{v} = Lv.$$

As written, the system will reach consensus with each  $v_i = \alpha$  for some  $\alpha \in \mathbb{R}$ , the exact value of which depends on the initial conditions  $v(0)$ . To incorporate environmental information, we assume that there is an environmental signal such that birds that can perceive the signal change their opinions to be close to 1. Thus, having an opinion closer to 1 means a bird is more informed and having an opinion farther from 1 means a bird is less informed. The opinion dynamics of a bird that do not receive the signal

are unchanged:  $v_i = \sum_j L_{ij} v_j$ . To describe the importance of the environmental signal, we introduce a parameter  $b$ , so that the opinion dynamics of a bird that *do* receive the signal are given by

$$\dot{v}_i = \sum_j L_{ij} v_j + b(1 - v_i).$$

If  $B$  is the diagonal matrix with  $B_{ii} = b$  if  $i$  can perceive the signal and  $B_{ii} = 0$  otherwise, and  $L_s = L - B$ , then

$$\dot{v} = Lv + B(\vec{1} - v) = L_s v + B\vec{1}. \quad (1)$$

Note that  $L\vec{1} = \vec{0}$ , so  $L$  has (at least) one eigenvalue equal to 0. If at least one bird receives an environmental signal, then generally  $L_s$  will have all negative non-zero eigenvalues. Then Claim 1 in the Appendix shows the solution  $v(t)$  to this set of ODEs.

**2.2 Individual-level fitness** To find the optimal strategies from the individuals' perspectives, we need a measure of individual-level fitness. In our model, birds with more information about the environment have higher fitness. A signaling event is determined by the interaction matrix  $A$  and by which birds receive the signal. For a single signaling event, the receivers and the birds to whom they are closest will generally be the most informed. To assess the fitness consequences of the number of nearest neighbors  $k_i$ , we need to determine how well birds with different strategies perform over many signaling events.

We assume that signals appear relatively rarely compared to the timescale on which the birds in a flock move. Specifically, at each signaling event, we will assign all birds a new position by drawing a position randomly from a unit square. Each bird then pays attention to its  $k_i$  nearest neighbors, which determines the adjacency matrix  $A$  and therefore the matrix  $L$ . To determine which birds receive the signal, we pick a focal bird at random. Then, all birds within a radius  $r$  of the focal bird receive the signal, which determines the matrix  $B$ . To fully specify the opinion dynamics, we assume that  $v_i(0) = 0.5$  when the signal appears for all birds. Eq. (1) then determines the opinion dynamics. We assume that the fitness consequences of the opinion dynamics occur a short time after the signal appears. Specifically, fitness is determined at time  $t = 1$ , i.e. by  $v_i(1)$ . We repeat this process 1000 times to model many independent signaling events.

Birds that receive the signal will always be well informed. It is their performance when they do *not* receive the signal that In Claim 1 in the Appendix, we show that, for  $i$  that do not receive the signal,

$$\begin{aligned} v_i(1) &\approx v_i(0) + \frac{1}{2} \frac{\# \text{ of } i\text{'s neighbors that receive the signal}}{k_i} \\ &= \frac{1}{2} \left( 1 + \frac{\# \text{ of } i\text{'s neighbors that receive the signal}}{k_i} \right). \end{aligned}$$

To improve its performance, a bird should increase the probability of having a high proportion of informed neighbors. If  $N_r$  birds receive the signal in a flock of size  $N$  and each bird chooses  $k_i$  neighbors at random (instead of its closet neighbors), the number of informed neighbors bird  $i$  has follows a hypergeometric distribution:

$$P(i \text{ has } j \text{ informed neighbors}) = \frac{\binom{N_r}{j} \binom{N - N_r}{k_i - j}}{\binom{N}{k_i}} \quad (2)$$

This gives us an approximation of the distribution  $P(v_i(1) = x)$  as a function of  $k_i$ , which will be useful in considering the advantages and disadvantages of each strategy.

How exactly environmental information is translated into fitness depends on what the information is about. We consider signals that provide information about two types of content, predators and food, which lets us consider four selection regimes. If the signal is information that a predator is present, we assume that whichever individual is least informed about the signal will be predated. The probability of bird  $i$  being eaten is therefore

$$P(i \text{ is eaten}) = \frac{\# \text{ of signaling events in which } v_i(1) = \min_i \{v_i(1)\}}{1000}.$$

The probability of  $i$  surviving is  $P(i \text{ survives}) = 1 - P(i \text{ is eaten})$  and  $i$ 's fitness is proportional to this probability. If the signal is information that resources are available, we consider two levels of resource abundance. If there are scarce resources, we assume that whichever individual is most informed will get access to the resource. The probability that bird  $i$  gets to eat is therefore

$$P(i \text{ gets to eat}) = \frac{\# \text{ of signaling events in which } v_i(1) = \max_i \{v_i(1)\}}{1000}$$

and  $i$ 's fitness is proportional to this probability. If there are abundant resources, we assume that the first four individuals to the resources can share it, so that the probability that bird  $i$  gets the resource is the probability of that  $v_i(1)$  is one of the four highest opinions over all signaling events and  $i$ 's fitness is proportional to this probability. Finally, we consider a combination of predation and access to scarce resources. To do well, a bird must both survive and get to eat. In this case,  $i$ 's fitness is proportional to  $P(i \text{ survives}) \times P(i \text{ gets to eat})$ .

**2.3 Optimization methods** We use two different methods for analyzing the evolutionary dynamics of the strategies  $k_i$ . First, we identify the evolutionarily stable strategy. The ESS is that strategy such that, if all members of the group use it, no other strategy can invade. Equivalently, if all members of the group use the ESS, no individual has an incentive to use another strategy. The probability of a new strategy invading the group depends on the relative fitness of the invader and resident strategies. The details on to find the ESS in a finite group are provided in the Appendix. Assuming the group all uses the same strategy allows us to exhaustively search over all strategies to find the ESS. However, it is a rather strict constraint. Our second method of optimization allows us to consider how the birds should optimize their strategies in the context of a heterogeneous population. Given a random set of initial strategies, we allow each bird to choose the strategy that would give it the highest fitness, given the rest of the strategies being used. We repeat this process until the birds reach an equilibrium set of strategies. We repeat this over many initial sets of strategies to find average properties of this greedy optimization process.

**2.4 Group-level properties** In our model, the information each bird can gather from its neighbors about an environmental signal has direct consequences on its fitness. It may also be the case that being in a flock whose members have the same opinion is beneficial. There are two ways to measure how well a flock can reach consensus, correlation length and  $\mathcal{H}_2$  robustness, both of which require that we introduce noise into the opinion dynamics of the flock.

In the absence of an environmental signal, the flock will eventually reach a consensus state such that  $\lim_{t \rightarrow \infty} v(t) = \alpha \vec{1}$  for some  $\alpha \in \mathbb{R}$ , where  $\alpha$  depends on the initial conditions. In the presence of an environmental signal,  $\lim_{t \rightarrow \infty} v(t) = \vec{1}$ . If we introduce noise into the opinion dynamics, the steady state opinion vector will no longer be deterministic but will instead be described by a probability distribution. We will take the distribution of the steady state opinion vector,  $\lim_{t \rightarrow \infty} v(t)$ , to be known for the moment and will come back to it shortly.

Since the consensus value  $\alpha$  is not always well-defined and since we are interested in the group's ability to reach consensus, but not the actual value of consensus, we subtract the average opinion from the opinion vector. If  $V = \sum_i v_i/N$ , then we define  $z = v - V\vec{1}$ . At steady state,  $\lim_{t \rightarrow \infty} E[z(t)] = \vec{0}$  as long as the flock is connected. If we know the distribution of opinion vectors at steady state, we can also find the distribution of  $z$  at steady state, and therefore the covariance matrix  $C_z = \lim_{t \rightarrow \infty} E[z(t)z(t)^T]$ . The correlation matrix  $\bar{C}$  is defined by  $\bar{C}_{ij} = C_{zij}/\sqrt{C_{zii}C_{zjj}}$ . By plotting the correlations  $\{\bar{C}_{ij}\}$  against the distance between the birds, we can find the correlation length  $\lambda$  at which the correlations cross 0. (As shown in Fact 4,  $\sum_{ij} C_{zij} = 0$ , so there must be a distance at which the covariances and hence the correlations drop below 0.)

The second group-level property,  $\mathcal{H}_2$  robustness requires another manipulation of the opinion vector. We define a rotation matrix  $R \in \mathbb{R}^N \times \mathbb{R}^N$  such that the last row of  $R$  is equal to  $\frac{1}{\sqrt{N}}\vec{1}^T$  and each of the first  $N - 1$  rows are normal and orthogonal to  $\vec{1}$ . We will denote the first  $N - 1$  rows of  $R$  as  $Q \in \mathbb{R}^{N-1} \times \mathbb{R}^N$ . We now define  $\bar{y} = Rz$ . Then  $\bar{y}_N = 0$  and if  $y = (\bar{y}_1 \dots \bar{y}_{N-1})^T$  then  $y = Qz = Q(v - V\vec{1}) = Qv$ , so that  $y$  represents the component of  $v$  that is orthogonal to the consensus

vector  $\vec{1}$  [2, 3]. The  $\mathcal{H}_2$  robustness  $\rho$  is defined as

$$\rho = \frac{1}{\lim_{t \rightarrow \infty} \sqrt{E[y(t)^T y(t)]}}.$$

$\rho$  measures how far away from consensus the opinion vector is on average at steady state. The close relationship between these two measures is discussed in more detail in the Appendix.

What we have not specified so far is how to find the distribution of the opinion vector at steady state, which is needed to find both  $\rho$  and  $\lambda$ . The same probability distribution could be used to find both. However, the two measures come from two different bodies of literature,  $\lambda$  from statistical physics and  $\rho$  from control theory, and the conventional probability distribution for each measure is different.

The Boltzmann distribution is the standard way in which to write an equilibrium distribution in statistical physics. In physics, those states with the highest energy are least likely. In our system, those sets of opinions that are least in consensus will be less likely. The distribution is only well-defined if there are symmetric interactions. Since  $L$  is only symmetric if  $A$  is, we symmetrize  $A$  by defining  $S = \frac{1}{2}(A + A^T)$  and

$$P_{ij} = \begin{cases} S_{ij} & \text{if } j \neq i \\ -\sum_l S_{il} & \text{if } j = i \end{cases}.$$

If  $A$  is symmetric, then  $S = A$  and  $P = L$ . To incorporate the environmental signal, we can define  $P_s = P - B$ . We can then use the following expressions to represent the “energy” of an opinion vector:  $H(v) = -v^T P v$  without the signal and  $H_s(v) = -(v - \vec{1})^T P_s (v - \vec{1})$  with the signal. This gives the Boltzmann distributions

$$P(v) = \frac{1}{Z} \exp\left(-\frac{1}{2}H(v)\right) \text{ without the signal and } P_s(v) = \frac{1}{Z_s} \exp\left(-\frac{1}{2}H_s(v)\right) \text{ with the signal.}$$

where  $Z = \int_{\mathbb{R}}^N \exp\left(-\frac{1}{2}H(v)\right) dv$  and  $Z_s = \int_{\mathbb{R}}^N \exp\left(-\frac{1}{2}H_s(v)\right) dv$ . Given these probability distributions, we can find  $C_z$  with and without the signal, and therefore the correlation length  $\lambda$  with and without the signal. Details on these calculations are provided in the Appendix.

In control theory, it is conventional to write down the noisy dynamics leading to a steady state distribution rather than to assume the form of the steady state distribution. Specifically, we consider the noisy opinion dynamics of the form

$$\dot{v} = Lv + K\xi \text{ without the signal and } \dot{v} = L_s v + B\vec{1} + D\xi \text{ with the signal,}$$

where  $\xi$  such that  $E[\xi(t)] = \vec{0}$  and  $E[\xi(t)\xi(s)^T] = \frac{1}{2}\delta(t-s)I$  and  $D$  is a diagonal matrix with  $\sqrt{k_i}$  along the diagonal. This choice of  $D$  assumes that a bird with more neighbors makes noisier estimates of the many opinions it is trying to estimate. Young et al. [3] showed that if  $\Sigma_y = \lim_{t \rightarrow \infty} E[y(t)y(t)^T]$  than, in the absence of the signal  $\Sigma_y$  satisfies the Lyapunov equation

$$\bar{L}\sigma_y + \sigma_y\bar{L} = -QD^2Q^T,$$

where  $\bar{L} = QLQ^T$ . A proof of this claim and details on how to calculate  $\Sigma_y$  with a signal present are provided in the appendix. Once we find  $\Sigma_y$ , then

$$\rho = \frac{1}{\lim_{t \rightarrow \infty} \sqrt{E[y(t)^T y(t)]}} = \frac{1}{\lim_{t \rightarrow \infty} \sqrt{E[\text{Tr}(y(t)y(t)^T)]}} = \frac{1}{\lim_{t \rightarrow \infty} \sqrt{E[\text{Tr}(\Sigma_y)]}}.$$

## 3 Results

**3.1 Robustness and correlation length are positively related.** In a homogeneous flock, where every bird is using the same strategy, as both the strategy and the radius of the signal change, both  $\mathcal{H}_2$  and the correlation length are affected. Figure 1 shows that both  $\mathcal{H}_2$  and correlation length are maximized at intermediate strategies when the radius is low and at low strategies when the radius is high. We prove in the Appendix (Appendix Claim 5) that if we use a symmetric matrix to calculate  $\rho$  and all birds have the same number of neighbors  $n$ , then the covariance matrices used to calculate  $\rho$  and  $\lambda$  are identical up to a constant, which suggests that the two properties cannot be optimized independently.

**3.2 The ESS number of neighbors depends on the content of the signal.** Those birds that can perceive the environmental signal will always have better information than those that cannot. Selection, therefore, acts to improve a bird’s ability to learn about the signal when it cannot perceive it. In Appendix Claim 1 we show that an uninformed bird’s opinion can be approximated by  $v_i(1) \approx v(0) + \frac{1}{2} \sum_{\text{informed } j} L_{ij}$ . Being the most well-informed uninformed bird, therefore, amounts to having the highest proportion of informed neighbors and being the least well-informed amounts to having the smallest proportion of informed neighbors. As described above, the hypergeometric distribution is a good approximation for the probability distribution of the number of informed neighbors a bird with strategy  $k_i$  will have. As shown in Figure 4, under the hypergeometric distribution, having many neighbors makes it unlikely that few of those neighbors will be uninformed, but it also makes it unlikely that all of them will be uninformed (Figure 4). Consequently, a bird with more neighbors will be less likely to be the least informed, but will also be less likely to be the best informed.

These observations explain the ESS number of neighbors we find, which depends on which selection pressure is being applied. Figure 2 shows that the ESS number of neighbors when selection is due to access to scarce resources is always 1. The ESS number of neighbors when selection is due to both access to resources and predation is also always 1. On the other hand, the ESS number of neighbors when selection is due to predation is always high, although not necessarily as high as possible. There is no ESS when selection is due to access to abundant resources.

**3.3 The predation ESS increases as the signal becomes more public.** The other feature of the ESS number of neighbors when selection is due to predation is that it increases as the signal becomes more public (Figure 2). When trying to avoid being the least well-informed, having too few neighbors is always disadvantageous because a bird with too few neighbors runs the risk of having too few or no informed neighbors. When the signal is not widely available, having too many neighbors is also disadvantageous because, even if a bird paying attention to many neighbors pays attention to all the informed birds, the total weight given to those informed birds will be lower than a bird with fewer total neighbors. As more can perceive the signal, the costs of paying attention to uninformed birds disappear and the ESS is to pay attention to all other birds in the flock.

**3.4 The birds construct an optimal network when they learn strategies to optimize access to abundant resources.** Learned strategies when selection is due to predation are high and learned strategies when selection is due to access to scarce resources or both resources and predation are low (Figure 3). Whereas there is no ESS when selection is due to access to abundant resources, when we let a heterogeneous flock learn strategies, we find that most birds in the flock decrease their strategies until they have three or four neighbors under this selection regime. As the birds choose new strategies, they change the average properties of the networks they construct. When they choose strategies according to predation, access to scarce resources, or both, both  $\rho$  and  $\lambda$  decrease or, at best, remain constant. However, when they choose strategies to get access to abundant resources, both  $\rho$  and  $\lambda$  increase (Figure 3).

## 4 Discussion

## 5 Appendix

### 5.1 Linear algebra claims and theorems

**Claim 1** Suppose.  $v(t)$  satisfies  $\dot{v} = Av + f$  with  $v(0) = v_0$ , where  $A$  has no 0 eigenvalues. If  $V$  is the matrix of eigenvectors of  $A$  and  $\Lambda$  is diagonal with the eigenvalues of  $A$ , then

$$v(t) = V \exp(t\Lambda) V^{-1} v_0 + V \Lambda^{-1} (\exp(t\Lambda) - I) V^{-1} f. \quad (3)$$

We can approximate  $v(t) \approx V \exp(t\Lambda) V^{-1} v_0 + V(tI + \frac{t^2}{2}A) V^{-1} f$  if the eigenvalues are not too far from 0. In the case that  $A = L_f$  and  $f = B\vec{1}$ , this suggests that for  $i$  that cannot perceive the signal, for small  $t$ ,  $v_i(t) \approx v_i(0) + \frac{t^2}{2} \sum_{\text{informed } j} L_{ij}$ . This approximation is reasonable for  $i$  such that  $B_{ii} = 0$ , but worse for  $i$  such that  $B_{ii} \neq 0$ .

*Proof.* First we show that (3) solves the differential equation. Note that if  $A = V\Lambda V^{-1}$ .

$$\begin{aligned} \dot{v} &= V \Lambda \exp(t\Lambda) V^{-1} v_0 + V \Lambda^{-1} (\Lambda \exp(t\Lambda)) V^{-1} f \\ &= V \Lambda V^{-1} V \exp(t\Lambda) V^{-1} v_0 + V \exp(t\Lambda) V^{-1} f \\ &= AV \exp(t\Lambda) V^{-1} v_0 + V \exp(t\Lambda) V^{-1} f - V V^{-1} f + V V^{-1} f \\ &= AV \exp(t\Lambda) V^{-1} v_0 + V (\exp(t\Lambda) - I) V^{-1} f + f \\ &= AV \exp(t\Lambda) V^{-1} v_0 + V \Lambda V^{-1} V \Lambda^{-1} (\exp(t\Lambda) - I) V^{-1} f + f \\ &= AV \exp(t\Lambda) V^{-1} v_0 + A \Lambda^{-1} (\exp(t\Lambda) - I) V^{-1} f + f \\ &= Av(t) + f. \end{aligned}$$

If  $g(x) = \frac{1}{x}(\exp(ax) - 1)$ , then  $g(x) \approx a + \frac{a^2}{2}x$  for  $x$  close to 0. Then, if the eigenvalues of  $A$  are not too far from 0,  $\Lambda^{-1}(\exp(t\Lambda) - I) \approx tI + \frac{t^2}{2}\Lambda$ . If  $A = L_f$ , then all eigenvalues of  $A$  are negative. If  $N_I$  is the number of informed individuals, i.e.  $i$  such that  $B_{ii} \neq 0$ , then there are  $N_I$  eigenvalues of  $A$  close to  $-2$  such that, if  $k$  is the index of one of these eigenvalues,  $V_{ik}$  is large if and only if  $i$  is informed. The rest of the  $N - N_I$  eigenvalues are between  $-2$  and 0 and, if  $k$  is the index of one of these eigenvalues,  $V_{ik}$  is large if and only if  $i$  is uninformed. Therefore, the eigenvalue approximation is worst for the informed individuals and best for the uninformed individuals. Using the eigenvalue approximation,  $v(t)$  becomes

$$v(t) = V \exp(t\Lambda) V^{-1} v_0 + t V V^{-1} B \vec{1} + \frac{t^2}{2} V \Lambda V^{-1} B \vec{1} = V \exp(t\Lambda) V^{-1} v_0 + t B \vec{1} + \frac{t^2}{2} L_f B \vec{1}$$

Assuming  $t$  is small, for uninformed individuals, this gives  $v_i(t) \approx v_i(0) + \frac{t^2}{2} \sum_{\text{informed } j} L_{ij}$ . ■

**Theorem 2** Given any positive symmetric matrix  $Q > 0$  there exists a unique positive symmetric matrix  $\Sigma > 0$  satisfying  $A\Sigma + \Sigma A^T = -Q$  if and only if the system  $\dot{x} = Ax$  is globally asymptotically stable.

*Proof.* Proof available elsewhere. ■

**Claim 3** Consider the dynamical system,

$$\dot{y} = Ay + v + Q\xi \quad (4)$$

with  $E[\xi(t)] = 0$ ,  $E[\xi(t)\xi(s)^T] = \frac{1}{2}\delta(t-s)I$ , and  $E[y(0)\xi^T(t)] = 0$ . If all eigenvalues of  $A$  are negative and if  $\Sigma_t = E[y(t)y(t)^T]$  and  $\Sigma = \lim_{t \rightarrow \infty} \Sigma_t$ , then  $\Sigma$  satisfies the Lyapunov equation

$$A\Sigma + \Sigma A^T = A^{-1}vv^T + vv^T(A^{-1})^T - QQ^T.$$

Further, if  $C_{ij} = \lim_{t \rightarrow \infty} E[(y_i(t) - E(y_i(t)))(y_j(t) - E(y_j(t)))]$  then

$$AC + CA^T = -QQ^T$$

*Proof.*

$$\begin{aligned}
\Sigma_t &= E[y(t)y(t)^T] \\
\Rightarrow \dot{\Sigma} &= E[\dot{y}(t)y(t)^T + y(t)\dot{y}(t)^T] \\
&= E[(Ay + v + Q\xi)y^T + y(y^T A^T + v^T + \xi^T Q^T)] \\
&= AE[yy^T] + E[yy^T]A^T + vE[y^T] + E[y]v^T + QE[\xi y^T] + E[y\xi^T]Q^T \\
&= A\Sigma_t + \Sigma_t A^T + vE[y^T] + E[y]v^T + QE[\xi y^T] + E[y\xi^T]Q^T.
\end{aligned}$$

Note that we can write a solution to (4) as

$$y(t) = \exp(At)y(0) + \int_0^t \exp(A(t-s))v ds + \int_0^t \exp(A(t-s))Q\xi(s)ds. \quad (5)$$

Eq. (5) allows us to write

$$\begin{aligned}
E[y(t)] &= \exp(At)y(0) + \int_0^t \exp(A(t-s))v ds \\
&= \exp(At)y(0) - A^{-1} \exp(A(t-s))v \Big|_{s=0}^{s=t} \text{ which exists because } A \text{ has no } 0 \text{ eigenvalues} \\
&= \exp(At)y(0) - A^{-1}(I - \exp(At))v.
\end{aligned}$$

Note that if we take the limit as  $t$  goes to infinity,

$$\lim_{t \rightarrow \infty} E[y(t)] = \lim_{t \rightarrow \infty} \exp(At)y(0) - A^{-1}(I - \exp(At))v = -A^{-1}v$$

since all eigenvalues of  $A$  are negative. Eq. (5) also allows us to write

$$\begin{aligned}
E[y\xi^T] &= \exp(At)y(0)E[\xi(t)^T] + \int_0^t \exp(A(t-s))BE[\xi(s)^T]ds + \int_0^t \exp(A(t-s))QE[\xi(s)\xi(t)]ds \\
&= 0 + 0 + \frac{1}{2}Q \\
\Rightarrow \dot{\Sigma} &= A\Sigma_t + \Sigma_t A^T + vE[y(t)^T] + E[y(t)]v^T + \frac{1}{2}QQ^T + \frac{1}{2}QQ^T \\
&= A\Sigma_t + \Sigma_t A^T + vE[y(t)^T] + E[y(t)]v^T + QQ^T \\
\Rightarrow \lim_{t \rightarrow \infty} \dot{\Sigma} &= A\Sigma + \Sigma A^T - vv^T(A^{-1})^T - A^{-1}vv^T + QQ^T \\
\Rightarrow A\Sigma + \Sigma A^T &= vv^T(A^{-1})^T + A^{-1}vv^T - QQ^T \text{ since } \dot{\Sigma} = 0 \text{ at equilibrium.}
\end{aligned}$$

Further, by definition of  $C$ ,

$$\begin{aligned}
C &= \Sigma - \lim_{t \rightarrow \infty} E[y(t)]E[y(t)^T] \\
&= \Sigma - (-A^{-1}v)(-A^{-1}v)^T \\
&= \Sigma - A^{-1}vv^T(A^{-1})^T \\
\Rightarrow AC + CA^T &= A\Sigma + \Sigma A^T - AA^{-1}vv^T(A^{-1})^T - A^{-1}vv^T(A^{-1})^T A^T \\
&= A\Sigma + \Sigma A^T - vv^T(A^{-1})^T - A^{-1}vv^T \\
&= -QQ^T.
\end{aligned}$$

■

**Fact 4** If  $V = \sum_i v_i/N$  and  $C = (E[(v_i - V)(v_j - V)])_{ij}$ , then  $\sum_{ij} C_{ij} = 0$ .

*Proof.*

$$\begin{aligned}
\sum_{ij} E[(v_i - V)(v_j - V)] &= \sum_{ij} (E[v_i v_j] - E[v_i V] - E[v_j V] + E[VV]) \\
&= \sum_{ij} E[v_i v_j] - N \sum_i E[v_i V] - N \sum_j E[v_j V] + N^2 E[VV] \\
&= \sum_{ij} E[v_i v_j] - \sum_{ij} E[v_i v_j] - \sum_{ij} E[v_i v_j] + \sum_{ij} E[v_i v_j] \\
&= 0
\end{aligned}$$

■

In the following sections of the Appendix, transforming an opinion vector  $x$  will be useful. To do so, we define a matrix  $Q \in \mathbb{R}^{N-1} \times \mathbb{R}^N$  such that each row of  $Q$  is orthogonal to  $\vec{1}$  and the rows are pairwise orthogonal. Young et al. show in [3] show that this equivalent to finding  $Q$  such that  $Q\vec{1} = \vec{0}$ ,  $QQ^T = I_{N-1}$ , and  $Q^T Q = I_N - \frac{1}{N} \vec{1}_N \vec{1}_N^T$ . We next define a rotation matrix  $R \in \mathbb{R}^N \times \mathbb{R}^N$  such that the last row of  $R$  is  $\frac{1}{\sqrt{N}} \vec{1}^T$  and the other rows are given by the rows of  $Q$ . Since the rows of  $R$  are orthonormal,  $RR^T = I$  and  $R$  is invertible, which implies that  $R^T = R^{-1}$ . If  $z = Q^T Qx$ , then  $z_i = v_i - \frac{1}{N} \sum_i v_i$ . Similarly, if  $\bar{y} = Rx$ , then  $\bar{y}_N = \frac{1}{\sqrt{N}} \sum_i x_i$  and  $y = (\bar{y}_1, \dots, \bar{y}_{N-1})^T = Qx$  gives the component of  $v$  that is orthogonal to the consensus vector  $\vec{1}$ . We can also translate between  $y$  and  $z$  by noting that  $z = Q^T y$  and  $y = QQ^T z$  or  $y$  is given by the first  $N-1$  coordinates of  $Rz$ .

$$\boxed{V = \frac{1}{N} \sum_i x_i, \quad z = Q^T Qx = x - V\vec{1}, \quad \bar{y} = Rx, \quad y = Qx = \bar{y}_{i=1}^{N-1}} \quad (6)$$

**5.2 Opinion dynamics with noise** In our model, each bird in the flock as an opinion  $x_i$ , giving an opinion vector  $x$ . The  $\mathcal{H}_2$  robustness of the opinion dynamics is defined by the inverse of the expected distance at equilibrium from the opinion vector to a consensus state in which all birds have the same opinion, i.e.  $x_i = \alpha$  for all  $i$  and for some  $\alpha \in \mathbb{R}$ . That is, the dynamics are robust if, at equilibrium, the opinions are close to being in consensus. We can find this distance by rotating the opinion vector into the space orthogonal to the consensus vector,  $\vec{1}$ . If  $y = Qx$ , then  $\|y\| = y^T y$  describes how far away from consensus  $x$  is. Specifically, robustness as in [3] is given by

$$\rho = \lim_{t \rightarrow \infty} \frac{1}{E[\|y\|]} = \lim_{t \rightarrow \infty} \frac{1}{\text{Tr} E[yy^T]}. \quad (7)$$

We also define a full rotation matrix  $R \in \mathbb{R}^N \times \mathbb{R}^N$  such that  $R = \begin{pmatrix} Q \\ \frac{1}{\sqrt{N}} \vec{1}^T \end{pmatrix}$ .

Young et al. [3] model opinion dynamics by assuming that each bird averages the differences between its opinion and each of the opinions of its neighbors and changes its opinion to minimize this difference and that each bird's opinion is subject to independent noise with variance proportional to the number of neighbors each bird has. The equations describing these dynamics can be written as

$$\dot{x}_i = \frac{\sum_{\text{neighbors } j \text{ of } i} (x_j - x_i)}{k_i} + \sqrt{k_i} \xi_i = \sum_j \bar{A}_{ij} x_j - x_i + \sqrt{k_i} \xi_i = \sum_j L_{ij} x_j + \sqrt{k_i} \xi_i. \quad (8)$$

In vector notation this becomes

$$\dot{x} = Lx + D\xi. \quad (9)$$

The equilibrium of these dynamics is a consensus state in which all birds have the some opinion,  $x = \alpha \vec{1}$  for some  $\alpha \in \mathbb{R}$ . In the absence of any environmental signal, the consensus opinion can be any value  $\alpha$ . In the presence of an environmental signal, some birds in the flock perceive the signal and change



their opinions to be closer to 1. The environmental forcing is described by a diagonal matrix  $B$  where  $B_{ii} = b$  if  $i$  perceives the environmental signal and  $B_{ii} = 0$  otherwise and the dynamics then become

$$\dot{x} = Lx + B(\vec{1} - x) + D\xi = L_s x + v + D\xi. \quad (10)$$

where  $L_s = L - B$  and  $v = B\vec{1}$ . If  $b = 0$ , then  $L_s = L$  and  $v = \vec{0}$ , so the case with the signal reduces to the case without the signal.

If  $\bar{y} = Rx$ , then the first  $N - 1$  coordinates of  $\bar{y}$  are  $y = Qx$  and the last is  $\sum_i x_i / \sqrt{N}$ . The dynamics of  $\bar{y}$  are described by

$$\begin{aligned} \dot{\bar{y}} &= R\dot{x} \\ &= RL_s x + Rv + RD\xi. \\ &= RL_s R^T \bar{y} + Rv + RD\xi \\ &= \tilde{L}_s \bar{y} + \tilde{v} + \tilde{D}\xi \end{aligned}$$

where  $\tilde{L}_s = RL_s R^T$ ,  $\tilde{v} = Rv = RB\vec{1}$ , and  $\tilde{D} = RD$ . If  $b \neq 0$ ,  $L_s$  and therefore  $\tilde{L}_s$  have all negative eigenvalues and  $\bar{y} = (0, 0, \dots, 0, \sqrt{N})^T$  is a globally asymptotically stable equilibrium. Therefore, by Theorem 2 and Claim 3,  $\lim_{t \rightarrow \infty} E[\bar{y}(t)\bar{y}(t)^T] = \Sigma_{\bar{y}}$  where  $\tilde{L}_s \Sigma_{\bar{y}} + \Sigma_{\bar{y}} \tilde{L}_s^T = -RD^2 R^T$ . We can find  $\Sigma_y = \lim_{t \rightarrow \infty} E[y(t)y(t)^T]$  by removing the last row and column of  $\Sigma_{\bar{y}}$ .

If  $b = 0$ ,  $L$  has a 0 eigenvalue corresponding to the eigenvector  $\vec{1}$ . Therefore,  $RLR^T$  also has a 0 eigenvalue, so we can therefore no longer find the desired  $\Sigma_{\bar{y}}$ . Since the last row of  $R = \frac{1}{\sqrt{N}}\vec{1}$ , all entries of the the last column of  $LR^T$  are 0. Therefore, all entries of the last column of  $RLR^T$  are 0. In other words, the dynamics of the first  $N - 1$  elements of  $\bar{y}$ , i.e.  $y$ , do not depend on the last. This lets us write

$$\dot{y} = QLQ^T y + Q\xi = \bar{L} + Q\xi$$

where  $\bar{L} = QLQ^T$ . Another way to see this is to write

$$\begin{aligned} \dot{y} &= Q\dot{x} \\ &= QLx + Q\xi \\ &= QL(x - \langle x \rangle \vec{1}) + Q\xi \text{ since } L\vec{1} = \vec{0} \\ &= QLQ^T Qx + Q\xi \\ &= \bar{L}y + Q\xi. \end{aligned}$$

Again, by Claim 3 and Theorem 2, we can now find  $\lim_{t \rightarrow \infty} E[y(t)y(t)^T] = \Sigma_y$  by finding a solution to  $\bar{L}\Sigma_y + \Sigma_y \bar{L}^T = -QD^2 Q^T$ .

To summarize, in the absence of an environmental signal,  $\Sigma_y$  satisfies  $\bar{L}\Sigma_y + \Sigma_y \bar{L}^T = -QD^2 Q^T$  where  $\bar{L} = QLQ^T$  and in the presence of an environmental signal,  $\Sigma_{\bar{y}}$  can be found in the first  $N - 1$  rows and columns of  $\Sigma_{\bar{y}}$  where  $\Sigma_{\bar{y}}$  satisfies  $\tilde{L}_s \Sigma_{\bar{y}} + \Sigma_{\bar{y}} \tilde{L}_s^T = -RD^2 R^T$  where  $\tilde{L}_s = RL_s R^T$ .

**5.3 Probability distribution at equilibrium** A quite different way of modeling social interactions the method used by Bialek et al. in [1], which is to assume the opinions come to equilibrium and write down an expression for the probability of finding the opinion vector in every state. In statistical physics, a Hamiltonian describes the energy of a given state,  $H(v)$ , and the Boltzmann distribution describes a probability distribution in which the most energetic states are least likely. Analogously, we can write a function  $H(v)$  that gives a high energy when the opinions are not in consensus and a low energy when the opinions are in consensus and a probability distribution of opinion vectors that favors less energetic states. In order to do this, we must use a symmetric interaction matrix, rather than the possibly asymmetric  $\bar{A}$  we used above. Bialek et al. study a model in which opinions are a three-dimensional velocity vector and the birds can weight the opinions of their nearest neighbors with any value  $J$ . In order to simplify this model and make it directly comparable to the opinion dynamics model we described above, we only consider a scalar-valued opinion  $x_i$  and we impose the condition

that the weights given to the neighbors' opinions are inversely proportional to the number of neighbors. In particular, we define  $S = \frac{1}{2}(A + A^T)$  and  $P$  such that  $P_{ij} = S_{ij}$  for  $j \neq i$  and  $P_{ii} = -\sum_k S_{ik}$ . This allows us to write down a Hamiltonian

$$H(v) = -v^T P v \quad (11)$$

where the negative sign makes opinions far from consensus give high energies and conversely. The probability distribution over opinion vectors  $v$  then becomes

$$P(v) = \frac{1}{Z} \exp \left( -\frac{1}{2} v^T (-P) v \right), \quad (12)$$

where  $Z = \int_{\mathbb{R}^N} \exp \left( -\frac{1}{2} H(v) \right) dv$ . We can now use this probability distribution to find the covariances and correlations between the birds' fluctuations around consensus. If we let  $V = \sum_i v_i / N = \langle v \rangle$  and  $z = v - V\vec{1}$ ,  $z$  represents the deviations in each bird's opinion from the consensus opinion. We are interested in finding  $C_z = (E[z_i z_j])_{ij}$ . To do this, we first find the probability distribution over  $(z, V)$ :

$$\begin{aligned} P(z, V) &= \frac{1}{Z} \exp \left( -\frac{1}{2} (z + V\vec{1})^T (-P) (z + V\vec{1}) \right) \\ &= \frac{1}{Z} \exp \left( -\frac{1}{2} z^T (-P) z \right) \text{ since } P\vec{1} = P^T\vec{1} = \vec{0}. \end{aligned}$$

Since  $P$  has a 0 eigenvalue, it is not invertible. However,  $z$  is orthogonal to the eigenvector of  $P$  with eigenvalue 0. Therefore, the covariance of  $z$  is  $C_z = -P^\dagger$ . By dividing each entry  $C_{zij}$  of the covariance matrix by  $C_{zii}C_{zjj}$ , the covariance matrix can be used to find the correlation matrix.

The Hamiltonian framework, like the opinion dynamics framework, can also accommodate an environmental signal. In this case, the appropriate Hamiltonian is

$$H(v) = -(v - \vec{1})^T P_s (v - \vec{1}) \quad (13)$$

where  $P_s = P - B$ . It is clear that  $H$  is minimized when  $v = \vec{1}$ . Again, we are interested in the deviations from the consensus opinion. As above, let  $V = \sum_i v_i / N$  and  $z = v - V\vec{1}$  so that  $v = z + V\vec{1}$ .

$$\begin{aligned} P(v) &= \frac{1}{Z} \exp \left( -\frac{1}{2} (v - \vec{1})^T (-P_s) (v - \vec{1}) \right) \\ &= \frac{1}{Z} \exp \left( -\frac{1}{2} (z - (1 - V)\vec{1})^T (-P_s) (z - (1 - V)\vec{1}) \right) \end{aligned}$$

Since the subspace of  $z \in \mathbb{R}^N$  such that  $\sum_i z_i = 0$  is  $N - 1$  dimensional, it will be easier to rotate  $z$  into  $\mathbb{R}^{N-1}$  and integrate over all of  $\mathbb{R}^{N-1}$ . Therefore, we define  $y = Rz = RQ^T Qv = Qv$ . Since  $\sum_i z_i = 0$ ,  $y_N = 0$  so  $V$  describes the mean of  $v$  and  $y$  describes the  $N - 1$  dimensional space of deviations from the consensus opinion.

$$\begin{aligned} P(v) &= \frac{1}{Z} \exp \left( -\frac{1}{2} (R^T y - (1 - V)\sqrt{N}R^T e_N)^T (-P_s) (R^T y - (1 - V)\sqrt{N}R^T e_N) \right) \\ &= \frac{1}{Z} \exp \left( -\frac{1}{2} (y - (1 - V)\sqrt{N}e_N)^T (-RP_s R^T) (y - (1 - V)\sqrt{N}e_N) \right) \\ &= \frac{1}{Z} \exp \left( -\frac{1}{2} (y + \sqrt{N}Ve_N - \sqrt{N}e_N)^T \tilde{P}_s (y + \sqrt{N}Ve_N - \sqrt{N}e_N) \right) \end{aligned}$$

where  $\tilde{P}_s = -RP_s R^T$ . To find  $P(z)$  we can integrate out  $V$ . However, to consider this as a probability density function over  $y$  and  $V$ , we have to consider the determinant of the transformation from  $v$  to  $y$  and  $V$ . The transformation can be represented by  $\bar{R} = \begin{pmatrix} Q \\ 1/\sqrt{N}R_N \end{pmatrix}$ , so that  $\det(\bar{R}) = \frac{1}{\sqrt{N}}\det(R) = \frac{1}{\sqrt{N}}$ .

Therefore,  $P(y, V) = \sqrt{N}P(v)$  is the appropriate density function for  $y$  and  $V$ . Finally, note that  $Z = \sqrt{(2\pi)^N \det(-P_s^{-1})}$ . This gives

$$\begin{aligned}
P(y, V) &= \sqrt{N}P(v) \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} ((V-1)\sqrt{N}e_N + y)^T \tilde{P}_s ((V-1)\sqrt{N}e_N + y) \right) \\
\Rightarrow P(y) &= \int_{\mathbb{R}} P(y, V) dV \\
&= \int_{\mathbb{R}} \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} ((V-1)\sqrt{N}e_N + y)^T \tilde{P}_s ((V-1)\sqrt{N}e_N + y) \right) dV \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} y^T \tilde{P}_s y \right) \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \left( \tilde{P}_{sNN} N (V-1)^2 + (V-1)\sqrt{N} \tilde{P}_{sN} \cdot y + (V-1)\sqrt{N} y^T \tilde{P}_{sN} \right) \right) dV \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} y^T \tilde{P}_s y \right) \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \left( \tilde{P}_{sNN} N (V-1)^2 + 2(V-1)\sqrt{N} \tilde{P}_{sN} \cdot y \right) \right) dV \text{ since } \tilde{P}_s \text{ is symmetric} \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} \left( y^T \tilde{P}_s y - \left( \frac{\tilde{P}_{sN} \cdot y}{\sqrt{\tilde{P}_{sNN}}} \right)^2 \right) \right) \times \\
&\quad \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \left( \tilde{P}_{sNN} N (V-1)^2 + 2(V-1)\sqrt{N} \tilde{P}_{sN} \cdot y + \left( \frac{\tilde{P}_{sN} \cdot y}{\sqrt{\tilde{P}_{sNN}}} \right)^2 \right) \right) dV \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} \left( y^T \left( \tilde{P}_s - \frac{1}{\tilde{P}_{sNN}} \tilde{P}_{sN} \tilde{P}_{sN} \right) y \right) \right) \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \left( \sqrt{\tilde{P}_{sNN} N} (V-1) + \frac{\tilde{P}_{sN} \cdot y}{\sqrt{\tilde{P}_{sNN}}} \right)^2 \right) dV \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} \left( y^T \left( \tilde{P}_s - \frac{1}{\tilde{P}_{sNN}} \tilde{P}_{sN} \tilde{P}_{sN} \right) y \right) \right) \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \tilde{P}_{sNN} N \left( V-1 + \frac{\sum_i \tilde{P}_{sNi} y_i}{\tilde{P}_{sNN} \sqrt{N}} \right)^2 \right) dV \\
&= \frac{\sqrt{N}}{Z} \exp \left( -\frac{1}{2} \left( y^T \left( \tilde{P}_s - \frac{1}{\tilde{P}_{sNN}} \tilde{P}_{sN} \tilde{P}_{sN} \right) y \right) \right) \times \sqrt{\frac{2\pi}{\tilde{P}_{sNN} N}} \\
&= \frac{1}{\sqrt{(2\pi)^{N-1} \det(-P_s^{-1}) \tilde{P}_{sNN}}} \exp \left( -\frac{1}{2} \left( y^T \left( \tilde{P}_s - \frac{1}{\tilde{P}_{sNN}} \tilde{P}_{sN} \tilde{P}_{sN} \right) y \right) \right)
\end{aligned}$$

Therefore,  $E[y_i] = 0$  for all  $i$  and  $E[yy^T] = (\tilde{P}_s - \frac{1}{\tilde{P}_{sNN}} \tilde{P}_{sN} \tilde{P}_{sN})^{-1}$ . Then,

$$E[zz^T] = R^T \left( \tilde{P}_s - \frac{1}{\tilde{P}_{sNN}} \tilde{P}_{sN} \tilde{P}_{sN} \right)^{-1} R = \left( -P_s - \frac{1}{\tilde{P}_{sNN} N} B \tilde{\mathbf{I}} \tilde{\mathbf{I}}^T B \right)^{-1}$$

Let  $N_I$  be the number of birds that can perceive the signal, i.e.  $N_I$  is the number of non-zero elements along the diagonal of  $B$ . Then,  $\tilde{P}_{sNN} = \frac{b}{N} N_I$ . Finally, we find that

$$C_z = \left( -P_s - \frac{1}{b N_I} B \tilde{\mathbf{I}} \tilde{\mathbf{I}}^T B \right)^{-1} \quad (14)$$

If  $P' = P_s + \frac{1}{b N_I} B \tilde{\mathbf{I}} \tilde{\mathbf{I}}^T B$ , then  $P' \tilde{\mathbf{I}} = -B \tilde{\mathbf{I}} - \frac{1}{b N_I} B \tilde{\mathbf{I}} \tilde{\mathbf{I}}^T B \tilde{\mathbf{I}} = -B \tilde{\mathbf{I}} - \frac{b N_I}{b N_I} B \tilde{\mathbf{I}} = \vec{0}$ , so we cannot invert  $-P'$  to find the covariance matrix. However, as above, we can discard the 0 eigenvector since we know that  $z \cdot \tilde{\mathbf{I}} = 0$ . This gives us  $C_z = (-P_s - \frac{1}{b N_I} B \tilde{\mathbf{I}} \tilde{\mathbf{I}}^T B)^\dagger$ .

**5.4 Connecting two frameworks** Table 2 shows how to find the covariance of  $y$  depending on the framework being used and whether or not a signal is present.

**Claim 5** If we use  $P$  instead of  $L$  with the noisy dynamics framework and all birds have the same number of neighbors  $n$ , then  $C_y = \frac{2}{n}\Sigma_y$ .

*Proof.* If we use  $P$  in the noisy dynamics framework,  $\Sigma_y$  satisfies

$$QPQ^T\Sigma_y + \Sigma_yQP^TQ^T = -QD^2Q^T. \quad (15)$$

If all birds have  $n$  neighbors, then  $D^2$  has  $n$  along the diagonal and 0 elsewhere and  $-QD^2Q^T = -nI_{N-1}$ . Now consider

$$\begin{aligned} QPQ^TC_y + C_yQP^TQ^T &= QPQ^T(-QP^\dagger Q^T) + (-QP^\dagger Q^T)QP^T \\ &= -QP(I_N - \frac{1}{N}\vec{1}\vec{1}^T)P^\dagger Q^T - QP^\dagger(I_N - \frac{1}{N}\vec{1}\vec{1}^T)P^T \\ &= -QPP^\dagger Q^T - QP^\dagger P^T Q^T \text{ since } P\vec{1} = \vec{0} \\ &= -Q(I_N - \frac{1}{N}\vec{1}\vec{1}^T)Q^T - Q(I_N - \frac{1}{N}\vec{1}\vec{1}^T)Q^T \\ &= -QQ^T - QQ^T \text{ since } Q\vec{1} = \vec{0} \\ &= -2I_{N-1} \\ \Rightarrow QPQ^T\left(\frac{n}{2}C_y\right) + \left(\frac{n}{2}C_y\right)QP^TQ^T &= -nI_{N-1} \\ \Rightarrow \frac{n}{2}C_y &= \Sigma_y \end{aligned}$$

since there is a unique solution to this Lyapunov equation. ■

A similar proof shows the equivalence of the two covariance matrices in the presence of a signal, as long as  $P_s$  is used in the noisy dynamical system and all birds have the same number of neighbors.

## References

- [1] William Bialek, Andrea Cavagna, Irene Giardina, Thierry Mora, Edmondo Silvestri, Massimiliano Viale, and Aleksandra M. Walczak. Statistical mechanics for natural flocks of birds. *Proceedings of the National Academy of Science*, 109(13):4786–4791, March 2012.
- [2] George F. Young, Luca Scardovi, Andrea Cavagna, Irene Giardina, and Naomi E. Leonard. Starling flock networks manage uncertainty in consensus at low cost. *PLOS Computational Biology*, 9, 2013.
- [3] George Forrest Young, Luca Scardovi, and Naomi Ehrich Leonard. Robustness of noisy consensus dynamics with directed communication. *Proceedings of the American Control Conference*, 2010.

## 6 Figures

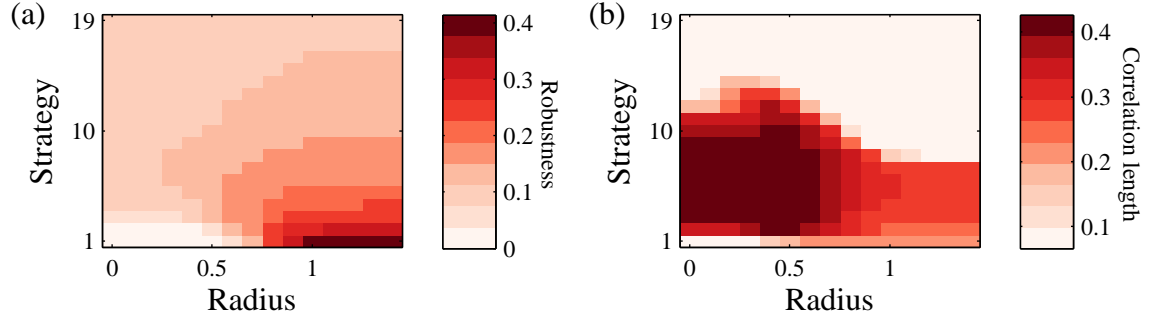


Figure 1: Both  $\mathcal{H}_2$  and correlation length are maximized at intermediate strategies when there are few informed birds and at low strategies when there are many informed birds. In (a), we show the average  $\mathcal{H}_2$  robustness over 1000 signaling events, where all birds use the same strategy and the signal is available to all birds within a circle of a given radius. In (b), we show the average correlation length over 1000 signaling events, where all birds use the same strategy and the signal is available to all birds within a circle of a given radius.

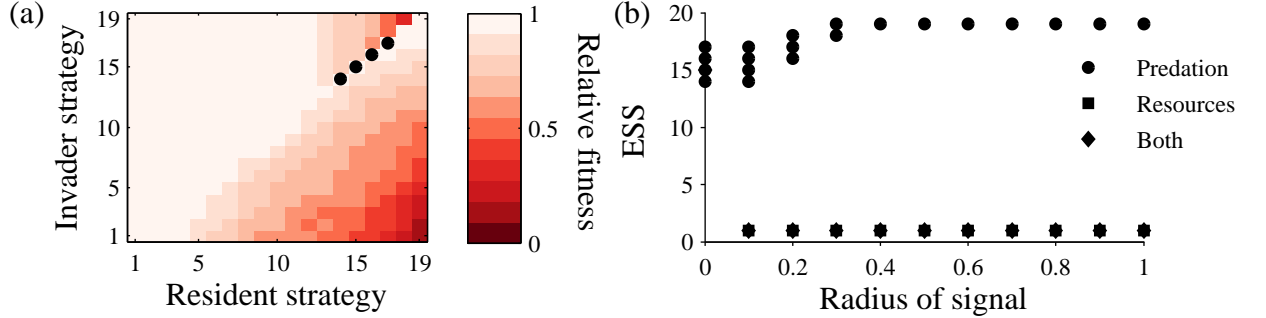


Figure 2: The ESS number of neighbors is always higher if selection is due to predation than due to resources or to both. When selection is due to predation, the (possibly multiple) ESS strategies increase with the radius of the signal. In (a), we show the relative fitness of an invader as a function of the invader and resident strategies when the radius  $r = 0.1$ , where fitness depends on escaping predation. The circles indicate the ESS strategies, i.e. those resident strategies such that no invader has a higher fitness than the resident. In (b), we show the (possibly multiple) ESS strategies as a function of the radius of the signal. The circles show the ESS when selection is due to predation, the squares when selection is due to access to scarce resources, and the diamonds when selection is due to both. The diamonds and squares overlap. There are no ESS when selection is due to access to abundant resources. Parameters: in both panels  $N = 20$ ,  $b = 1$ ,  $T = 1$ .

## 7 Tables

Table 1: Table of matrices and variables used in the text.

$A$	adjacency matrix	$A_{ij} = \begin{cases} 1 & \text{if } i \text{ pays attention to } j \\ 0 & \text{otherwise} \end{cases}$
$\bar{A}$	normalized adjacency matrix	$\bar{A}_{ij} = \begin{cases} \frac{1}{k_i} & \text{if } i \text{ pays attention to } j \\ 0 & \text{otherwise} \end{cases}$
$B$	diagonal signal matrix	$B_{ij} = \begin{cases} b & \text{if } i = j \text{ and } i \text{ perceives the environmental signal} \\ 0 & \text{otherwise} \end{cases}$
$\lambda$	correlation length	
$\Lambda$	diagonal matrix of eigenvalues of $P$	
$L$	opinion dynamics w/o signal	$L = \bar{A} - I$
$L_s$	opinion dynamics with signal	$L_s = L - B$
$\bar{L}$	rotated opinion dynamics w/o signal	$\bar{L} = QLQ^T$
$\tilde{L}_s$	rotated opinion dynamics w/ signal	$\tilde{L} = RL_sR^T$
$S$	symmetrized interaction matrix	$S = \frac{1}{2}(\bar{A} + \bar{A}^T)$
$P$	energy interactions w/o signal	$P_{ij} = \begin{cases} S_{ij} & \text{for } j \neq i \\ -\sum_k S_{ik} & \text{for } j = i \end{cases}$
$P_s$	energy interactions with signal	$P_s = P - B$
$\tilde{P}_s$		
$Q$	$N - 1 \times N$ rotation matrix	$Q \in \mathbb{R}^{N-1} \times \mathbb{R}^N$ such that $Q\vec{1} = \vec{0}, QQ^T = I_{N-1}, Q^T Q = I_N - \frac{1}{N}\vec{1}_N\vec{1}_N^T$
$R$	full rotation matrix	$\begin{pmatrix} Q \\ \frac{1}{\sqrt{N}}\vec{1}^T \end{pmatrix}$
$\rho$	$\mathcal{H}_2$ robustness	$\lim_{t \rightarrow \infty} \frac{1}{E[\ y\ ]}$ where $y = Qx$
$W$	matrix of eigenvectors of $P$	

Table 2:  $P^\dagger$  is the inverse of  $P$  with the 0 eigenvector / value removed.

	Without signal	With signal
From noisy dynamics	$\bar{L}\Sigma_y + \Sigma_y\bar{L}^T = -QD^2Q^T$	$\tilde{L}_s\Sigma_{\bar{y}} + \Sigma_{\bar{y}}\tilde{L}_s^T = -RD^2R^T, \Sigma_y = (\Sigma_{\bar{y}})_{i,j=1}^{N-1}$
From Boltzmann equilibrium	$C_y = -QP^\dagger Q^T$	$C_y = (\tilde{P}_s - \frac{1}{\tilde{P}_{sNN}}\tilde{P}_{s \cdot N}\tilde{P}_{sN \cdot})^\dagger$ or $C_y = Q(-P_s - \frac{1}{bN_1}B\vec{1}\vec{1}^TB)^\dagger Q^T$

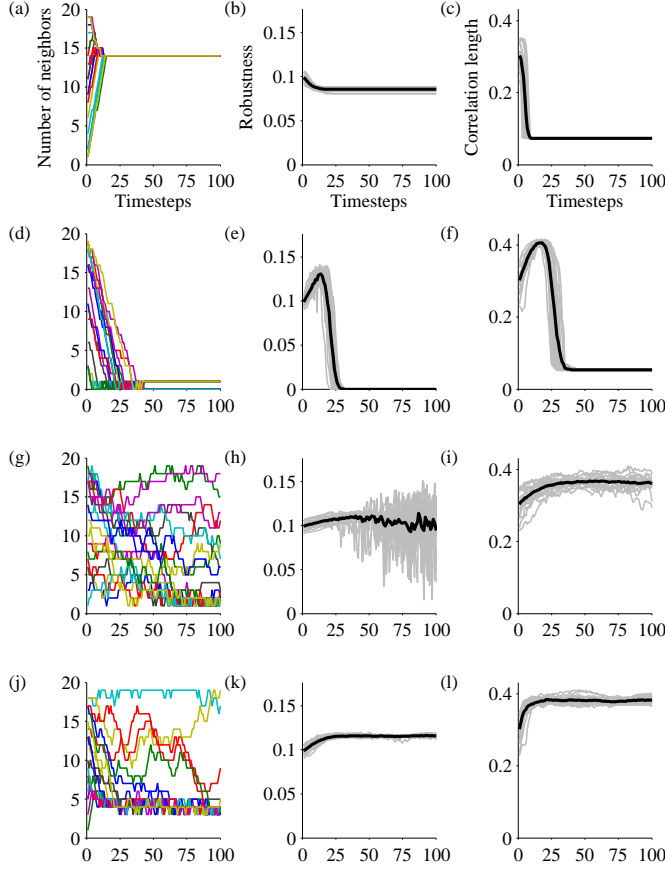


Figure 3: When the birds change their number of neighbors to improve their own fitness, they settle on strategies close to the ESS strategies for that selection regimes. When selection is due to predation, they settle on high strategies; when selection is due to access to few resources or to both resources and predation, they settle on low strategies; when selection is due to access to abundant resources, they settle in intermediate strategies. Only under selection due to generous access to resources does this optimization lead to an increase in both  $\mathcal{H}_2$  robustness and correlation length. The upper row shows results from selection based to predation, the second row shows results from selection based on access to scarce resources, the third row shows results from selection based on both resources and predation, and the last row shows results from selection based on generous to abundant resources. In each panel, the horizontal axis represents the number of times the birds are allowed to choose optimal strategies. The first column shows one example of how the birds' strategies change over time. The second column shows, for various initial conditions, how the  $\mathcal{H}_2$  robustness changes over time, with the black line showing the average of the 20 gray lines. The third column shows, for various initial conditions, how the correlation length changes over time, with the black line shows the average of the 20 gray lines. **NOTE: I'M RUNNING THIS FOR MORE TIMESTEPS TO SEE IF THE BOTTOM TWO ROWS CONVERGE MORE RELIABLY.** Parameters:  $N = 20$ ,  $b = 1$ ,  $r = .1$ ,  $T = 1$ .



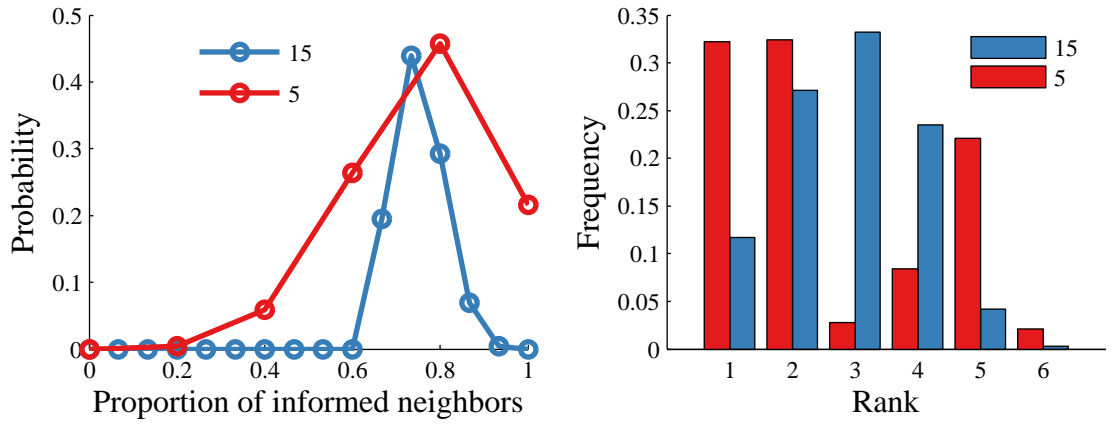


Figure 4: If a fixed number of birds are informed and birds choose neighbors randomly without regard to spatial proximity, the probability of having any number of informed neighbors follows the hypergeometric distribution. A bird with more neighbors is less likely to have a low proportion of informed neighbors, but is also less likely to have a high proportion of informed neighbors. In (a), we show the hypergeometric distribution, giving the probability of having a number of informed neighbors, normalized by the total number of neighbors, for two strategies, 15 in blue and 5 in red. In (b), we consider a group made up of 19 birds using strategy 15 and 1 using strategy 5. For each bird, we draw a number of informed neighbors from the hypergeometric distribution 1000 times and rank the birds according to the proportion of their neighbors that are informed. We show the frequency with which the bird using 5 neighbors and one of the birds using 15 neighbors is ranked 1<sup>st</sup> through 6<sup>th</sup>. The bird with 5 neighbors is ranked 1<sup>st</sup> much more often, but is also ranked 6<sup>th</sup> more often. Parameters:  $N = 20$ .