Stochastic Learning Dynamics with Two Applications

Contents

1 Abstract

Learning usually occurs in noisy environments. In order to make a decision, one must accumulate the evidence for different alternatives and choose one only once the uncertainty about the choice is sufficiently reduced. There are two biological systems in which the variables used to make a decision can be described by a random walk: the firing rates of neural populations in the brain of an individual making a decision about a visual stimulus and the beliefs of monkeys about their relative dominance based on the fights they have engaged in. The same stochastic differential equations can be used to model these quite different systems, which suggests that there are universal principles underlying learning through noisy dynamical systems. In the neural system, the differential equations are often reduced to a one-dimensional model, but the social system is inherently two-dimensional. The thresholds dictating when a decision has been made affect both the accuracy of the decision and the expected time until a decision is reached. We find that the dimensionality of the decision process does not affect its accuracy. We find that if a pair of individuals optimize their thresholds to avoid waiting, a pair making an easy decision will take more time than a pair making a difficult decision. In a group of individuals optimizing their thresholds to avoid waiting, those pairs making the difficult decisions about high value inputs will have to wait the longest. Finally, the skewness of the power distribution in the social system is maximized at intermediate waiting costs and the information content of the power distribution is

higher when the individual maximize their probability of becoming dominant rather than the accuracy of their decisions. In comparing the output of the model of the social system to data, we found that ...

2 Introduction

2.1   Neural Stochastic differential equations have been used to model, explain, and predict learning and decision making in both monkeys and humans [6, 2, 7, 1]. Typically, an experimental subject is presented with a visual stimulus that consists of dots, some percentage of which are moving coherently either left or right and the rest of which are moving randomly. The subject is then asked to decide in which direction the dots were moving. If there are two neural populations, each of which fire in response to a particular direction of motion, the firing rates of the two populations will tend to increase or decrease, depending on the strength and direction of the stimulus. Eventually, either the experimenter forces a decision to be made, in which case whichever population has a higher firing rate determines which decision is made, or the firing rate of one population becomes high enough that the subject decides for that direction. To summarize, there are two decision variables— the firing rates of two neural populations responding to left and right motion— that perform a biased random walk until enough evidence has been accumulated to make a decision one way or the other.

2.2   Social In groups of pigtailed macaques, different animals have different fighting abilities, and when a pair of animals engages in a fight, one animal may be more likely on average to "win" the fight. Over time, as one animal learns that it is likely to lose fights, it will decide to emit a formal subordination signal, communicating its agreeing to the subordinate role in their relationship [9, 11, 10, 5, 4]. The outcome of a fight is affected by each animal's allies, the presence of those allies, motivation levels, external environmental variables, and other factors. Additionally, the number of fights that occur in any period of time depends on similarly stochastic factors. Because of this stochasticity, the number of fights won and lost in any period of time is a random variable. Each animal's opinion about its relative dominance, therefore, will perform a biased random walk until one animal's opinion of itself is low enough to agree to being subordinate.

Similar models have previously been applied to animal conflicts, although in that case the model assumed that the animals had perfect memory of their past interactions [13]

2.3   Extent and Limits of Analogy In Table 1, we list the inputs, decision variables, and outputs of the two systems.

3 The model

3.1   Stochastic differential equations Learning systems have been successfully modeled with stochastic differential equations [6, 2, 7, 1], although the application of this type of model has been quite phenomenological. With two assumptions about the timescales on which probabilistic events occur, a simple mechanistic description of the accumulation of information about random events can be turned into a stochastic differential equation [16].

In the absence of any input, the decision variables leak back towards 0 with rate ℓ. (A table of all variables used in the text is given in Table 1.) If there is no input, then over a period of time of length τ a decision variable would decrease as $X(t+\tau) = (1-\ell\tau)X(t)$. There are two types of inputs: in the neural system, a dot can move either left or right and in the social system a fight can be won by one animal or another. For clarity in the following we will refer to these left or right dots, but the biological interpretation of the two types of inputs depends on the system. Each decision variable responds positively to one type of input and negatively to the other, with the signs of their responds the opposite of each other. Specifically, in the neural case, one neural population responds positively to left dots and negatively to right dots and the other neural population does the opposite. In the social case, one animal's estimate of its own dominance increases when it wins fights and decreases when it loses fights, and the other animal's estimate does the opposite.

To determine the estimate at time $t+\tau$, we can count how many times each of type of input occurred in the time since t and add the changes resulting from these events to the background leaky estimate (as in [16]):

$$X_1(t+\tau) = (1-\ell\tau)X_1(t) + b \times \# \text{ of left dots in } [t, t+\tau] - b \times \# \text{ of right dots in } [t, t+\tau]$$
$$X_2(t+\tau) = (1-\ell\tau)X_2(t) - b \times \# \text{ of left dots in } [t, t+\tau] + b \times \# \text{ of right dots in } [t, t+\tau].$$

where b is the change caused by a single input.

Our first assumption is that the rates at which the two types of inputs occur is constant. We can thus describe the number of each type of event with a Poisson random variable, N_L and N_R. We rewrite our basic equations to give

$$X_1(t+\tau) = (1-\ell\tau)X_1(t) + bN_L - bN_R$$
$$X_1(t+\tau) = (1-\ell\tau)X_2(t) - bN_L + bN_R.$$

If events happen at a rate r and the event is a left dot with probability c and a right dot with probability $1-c$, then the expectation of N_L and N_R in a period of length τ are, respectively, τrc and $\tau r(1-c)$. In the neural case, c is related to the "coherence" of the visual stimulus. A c close to 0 or 1 means that most of the events are of one type or the other and the decision is easier to make than when c is close to .5.

If enough events happen in the period of time from t to $t+\tau$ then we can approximate the Poisson random variables with Normal random variables with mean and variance equal to the mean of the Poisson random variables. Our second assumption, then, is that the period of time of length τ is long enough to make this approximation. Let Z_L and Z_R, be independent standard Normal random variables, i.e. with mean 0 and standard deviation 1. Then our equations for $X_i(t+\tau)$ become

$$X_1(t+\tau) = (1-\ell\tau)X_1(t) + b\left(\tau rc + \sqrt{\tau rc}Z_L\right) - b\left(\tau r(1-c) + \sqrt{\tau r(1-c)}Z_R\right)$$
$$X_2(t+\tau) = (1-\ell\tau)X_2(t) - b\left(\tau rc + \sqrt{\tau rc}Z_L\right) + b\left(\tau r(1-c) + \sqrt{\tau r(1-c)}Z_R\right).$$

Finally, as we make the period of time shorter and shorter, making τ infinitesimally small, these equations become stochastic differential equations,

$$
\begin{aligned}
dX_1 &= \left(-\ell X_1(t) + br(2c-1)\right)dt + \left(b\sqrt{rc}\right)dW_L t - \left(b\sqrt{r(1-c)}\right)dW_R t \\
dX_2 &= \left(-\ell X_2(t) - br(2c-1)\right)dt - \left(b\sqrt{rc}\right)dW_L t + \left(b\sqrt{r(1-c)}\right)dW_R t
\end{aligned}
\tag{1}
$$

where dW_L and dW_R are independent Brownian motions representing, respectively, the left and right inputs. The assumptions about the timescales on which inputs occur are particularly reasonable in the social system, although the successful application of this type of model to neural populations implies they are not unreasonable in that system.

3.2 Reaching a decision Each decision variable, X_1 and X_2, accumulates information about the stimulus. The sign and size of the two variables indicate how much evidence there is for each possible decision outcome. When a lot of evidence for the first possibility has been observed, X_1 will be large and positive, and similarly for the second possibility. If the difference in the variables can be observed, the difference indicates the relative strengths of the evidence for either possibility. If $Y = X_1 - X_2$, when Y is large enough the system decides on X_1 and when Y is low enough the system decides on X_2. Specifically, there are two thresholds, T_2 and T_1 such that if $Y > T_1$ the decision is for X_1 and if $Y < -T_2$ the decision is for X_2. However, it may be the case that the difference in the variables cannot be observed. In this case, the decision depends on whether X_1 or X_2 hits a threshold first. Again, there are two thresholds, T_2 and T_1, and if $X_1 < -T_1$ the decision is for X_1 and if the $X_2 < -T_2$ the decision is for X_2.

In the neural system, the two stochastic equations describing the activity of the two neural populations integrating environmental evidence are often reduced to the one-dimensional equation describing the difference in the activity levels to make the system easier to analyze [2, 1, 7]. However, in the social system we are considering, we cannot make any assumptions about the existence of a third party evaluating the difference in the opinions of the two animals. A "decision," i.e. the emission of a signal from one of the two animals, is only reached when one of the two animals' opinions goes below a certain threshold. The social system is therefore inherently two-dimensional, whereas the neural system may be adequately described by one dimension.

3.3   Utility of the decision The thresholds affect the expected time until a decision is reached (decision time, DT) and the probability that an incorrect decision is made (error rate, ER). The thresholds also affect the probability that either variable will "give up" first. In the social case, this is the probability that each animal will emit a subordination signal (signaling probability, SP_i). In the neural case, ??? These quantities, DT, ER, and SP_i, satisfy partial differential equations with respect to the starting values of the decision variables. In Appendix 6.1, we derive these partial differential equations. While an analytical solution is known in the one-dimensional case, the two-dimensional case is analytically intractable. We therefore numerically solve the PDEs in the two-dimensional case. In Table 2, we compare how the decision is made and how the decision process is optimized in the two systems.

The quality of the decision process depends on each of these three quantities. The lower the decision time and error rate, the better the decision. However, both cannot be decreased simultaneously so there is a tradeoff that depends on how the two quantities are prioritized. Additionally, in the social case, each animal may benefit from lower its signaling probability, which may or may not conflict with lowering the error rate, depending on whether the animal is weaker or stronger than its opponent. To quantify the utility of the decision process, we introduce three weights, w_1, w_2, and w_3, which determine how the three quantities are prioritized. For animal i, we define

$$U_i = w_1 \mathrm{DT} + w_2 \mathrm{ER} + w_3 \mathrm{SP}_i.$$

The decision is better if U_i is lower. In the social system, w_2 can be interpreted as the cost of fighting since when w_2 is higher, the time spent fighting until a decision is reached is more costly. Similarly, w_3 can be interpreted as the benefit from being the dominant animal in a pair.

3.4   Optimal thresholds An individual– either a neural population or a monkey– can be engaged in either a single decision or in many decisions at once. In the social system, for example, an animal wants to decide whether it is dominant or subordinate to every other member of its social group. For any pair of individuals, each has a utility, U_1 and U_2, and consequently each has a Nash equilibrium threshold such that, if they choose those thresholds, neither has an incentive to choose another. Similarly, in a group of individuals, each animal i has a utility U_{ij} from its decision process with animal j. These lead to an optimized set of thresholds such that no individual has an incentive to choose another threshold.

Since an individual is trying to use the same algorithm to learn about every other animal, he is constrained to have only one decision threshold for each of these decision processes. The optimal decision threshold in this group context depends on the distribution of difficulties (c) of the decision an individual will have to make and the thresholds of the other individuals. In the social system, the distribution of difficulties (c) comes from the distribution of fighting abilities of the animals.

We use the following procedure to find optimal thresholds as a function of position in the group. We repeatedly draw a set of fighting abilities from a uniform distribution; we find the optimal thresholds for each set, and then find the average optimal threshold as a function of an animal's order in the set of fighting abilities. To find the optimal thresholds for a given set of fighting abilities, we turn the difference in fighting abilities between each pair into the difficulty of the decision they have to make (c). For an initial set of thresholds, we find U_{ij} for each animal i in each of its decisions and then average $\langle U_{ij} \rangle_j$ across all the animals with which it is engaged in a decision process. We then allow each animal to choose a threshold that optimizes $\langle U_{ij} \rangle_j$ with respect to the rest of the group's thresholds and we iterate these choices until no animal chooses a new threshold.

3.5 **Consensus** Consensus about the value of each individual in a group emerges out of the decision process. In the social system, an animal's social power depends on the consensus in the group about its fighting ability [10, 8, 3]. Given the decisions that each pair of individuals reaches, there are measures of the consensus in the group that successfully predict an animal's social power and also extend to other social systems [3]. For each random set of fighting abilities, we let the animals optimize their thresholds, which give the probability that each pair will reach either decision, and we then generate a set of decisions given by these probabilities. Given this signaling network, we use various formalisms to measure the consensus about each node in the network, i.e. a power score for each animal in the group. For many random sets of fighting abilities, we can find the mutual information between the power score and the true fighting abilities. This tells us how informative observing the outcomes of the decisions is about the underlying fighting abilities. We also find the average skewness of the power distributions.

In the following we consider four types of consensus formalisms. The first and simplest is simply the number of signalers submitting to each animal. The second is the entropy of the distribution of the number of signals sent to each animal, which measures how uniformly all other animals in the group behave with respect to a focal animal. The third is the eigenvector centrality of the signaling network, which measures how central each node is in the global structure of the network. Finally, there are a number of ways to Incorporate the number of signals received into a measure of power, including just counting that number or multiplying that number by the entropy. However, these measures all perform very similarly with respect to how informative they are so we group them together in the following analysis. For more details see [3].

4   Results

4.1 **Decisions are as accurate without an arbiter.** We compare the decision making process in the full two-dimensional system and in a reduced one-dimensional system to see whether decisions are made more accurately or quickly in one or the other. To make the systems directly comparable, we assume in both cases that the thresholds are symmetric, i.e. $T_1 = T_2$. We find that the two-dimensional decision is as accurate as the one-dimensional decision. That is, for a given expected time to reach a decision, the two-dimensional decision will reach the correct decision with the same probability. Accuracy decreases as leak rates (ℓ) become higher, but increases when the strength of the input (c) is higher.

4.2 **Prioritizing signaling probability does not always decrease pairwise decision accuracy.** We first look at the Nash equilibrium thresholds when the animals only care about one property of the decision (error rate, signaling property, or decision time). In these extremes, the Nash thresholds do not depend on how difficult the decision is. The pair can make the most accurate decision if the weaker animal chooses a low threshold and the strong animal chooses a high one, so if only error rate matters ($w_1 = 1$, $w_2 = 0$, $w_3 = 0$), those will be the Nash thresholds. If only signaling probability matters ($w_1 = 0$, $w_2 = 0$, $w_3 = 1$), there is no incentive for the individuals to stop accumulating evidence (fighting, in the social case) so both individuals wait until the other gives up and the Nash thresholds are both high. If only decision time matters ($w_1 = 0$, $w_2 = 1$, $w_3 = 1$), both animals have an incentive to set their thresholds low and reach a decision quickly, whatever that decision may be (Figure 1). Notice that in changing from prioritizing signaling probability ($w_3 = 1$) to prioritizing decision time ($w_2 = 1$), first the weaker animal lowers its threshold and *then* the stronger animal lowers his.

For intermediate optimization weights, the Nash thresholds depend on the difficulty of the decision being made. The Nash thresholds for two animals making a difficult decision (c close to .5) are more similar to each other than those for two animals making an easy decision. For a pair making a difficult decision, their similar Nash thresholds decrease as waiting costs become more important (Figure 1). Once waiting costs are sufficiently high, the tradeoff between error rate and signaling probability does not affect the Nash thresholds.

The thresholds affect the accuracy and decision time with which a pair can reach a decision. For a difficult decision (c close to .5), the accuracy with which a pair using Nash thresholds can reach a

decision is highest when only error rate matters ($w_1 = 1$) and lowest when only signaling probability matters ($w_3 = 1$). Accuracy decreases as the weight given to decision time increases (Figure 1). Since, for sufficiently high waiting costs, the Nash thresholds do not depend on the tradeoff between error rate and signaling probability, neither does the accuracy for a pair using Nash thresholds. In other words, once waiting costs are high, a pair of animals each minimizing error rate and a pair minimizing signaling probability will reach a decision with the same accuracy.

4.3   Intermediately difficult decisions sometimes take the longest. For most optimization weights, it takes more time to make a difficult decision than to make an easy one. However, at intermediate costs, pairs of individuals making intermediate decisions will take the longest to reach a decision. An easy decision can always be reached quickly. Higher waiting costs incentivize pairs making difficult decisions to lower their thresholds more than pairs making intermediately difficult decisions, with the result that they can actually reach a decision more quickly. It is the pairs making intermediately difficult decisions using intermediate thresholds that will take the longest in this case (Figure 1).

4.4   Decisions in a group context take longest between animals with similarly high abilities. After analyzing Nash equilibrium thresholds between pairs of animals, we found the optimal thresholds for a group of animals ($N = 20$) such that no animal had an incentive to choose a different threshold. As with pairwise Nash thresholds, when the animals only care about signaling probability ($w_3 = 1$), all animals in the group choose high thresholds, whereas if they only care about decision time ($w_2 = 1$), all animals choose low thresholds. When the animals only care about accuracy ($w_1 = 1$), all animals choose high thresholds, except the weakest who choose a low threshold. At intermediate weights, thresholds decrease as position in the group decreases (Figure 2).

When waiting costs are low, the pairs that take the longest to reach a decision are those with similar fighting abilities, but pairs with less similar abilities also take a fairly long time. When waiting costs are high, most pairs can reach a decision quite quickly, except those who have similar and high abilities (Figure 2). The average accuracy decreases as waiting costs increase (Figure 2).

As soon as any waiting costs are introduced, as with the pairwise Nash thresholds, the weaker animals have incentives to lower their thresholds to make quicker decisions at the expense of accuracy. This incentive is stronger when error rate is more important than signaling probability. The result is that all the weakest animals set their thresholds as low as possible, making the accuracy of decisions between those animals very poor and decreasing the overall accuracy of the group. In other words, average accuracy *increases* as signaling probability is prioritized over error rate, except when there are no waiting costs (Figure 2). The mutual information of any measure of consensus in the signaling network increases (more or less) with accuracy so it too increases as signaling probability becomes more important.

4.5   The skewness of the power distribution is maximized at intermediate waiting costs. For each measure of consensus in the signaling network, we find the average skewness of the distribution of power scores of a group using optimal thresholds, as they depend on optimization weight. We find that for all measures skewness is maximized at intermediate waiting costs and does not depend strongly on the tradeoff between error rate and signaling probability (Figure 3). (Results for number of signalers are shown in Figure 3 and others are similar). When waiting costs are low and accuracy is high, the signals accurately represent the animals' true fighting abilities so the distribution of number of signalers accurately represents the distribution of abilities, which is not very right-skewed. When waiting costs are high and accuracy is low, all animals receive signals from a similar number of others, giving a distribution that is quite uniform and not right-skewed. At intermediate waiting costs, the signals sent to animals with low to middle abilities are very noisy so that the animals in the bottom and middle of the group receive signals from similar numbers of animals. However, the group accurately signals to the top couple of animals, giving them high scores, resulting in a right-skewed distribution.

4.6   Which measure of consensus is most informative depends on how accurate pairwise decisions are. We consider four types of measures of consensus in the signaling network to evaluate

which is most informative about the underlying fighting abilities. When pairwise decisions are quite accurate, the number of signals received is the most informative about the underlying fighting ability (Figure 4). This makes sense since when each signal accurately reflects the different in ability between receiver and sender, the number of signals is informative about an animal's position in the group. When the pairwise decisions are less accurate, global structure of the signaling network provides information about ability that the individual relationships do not, so eigenvector centrality is the measure that is the most informative about ability.

The number of signalers consistently overestimates animals' ability. When accuracy is fairly high, this bias makes it uninformative about ability. However, when accuracy is quite low, this consistent bias actually make the number of signalers more informative than measures that both underestimate and overestimate ability. When accuracy is lower still, none of the measures are very informative, but entropy correctly gives the lowest animals low scores, giving it an edge over the other measures.

5   Discussion

what's interesting and exciting and new?

we can derive sde's from first principles

prioritizing signaling probability sometimes makes decision more accurate

prioritizing waiting costs change which decisions take the longest

optimization weights change the overall social structure

extension of neural model by removing third party comparison and allow individuals to minimize signaling probability

extension of game theory by introducing heterogeneous abilities

extension of David's paper

explanation for variation across species of macaque

evolutionary implications

Empirically, those pairs of monkeys that take the longest to establish a signaling dynamics are those with the most similar fighting abilities...

skewness they're using eigenvector centrality

Table 1: **Variables and interpretations in neural and social systems.**

Variable	Interpretation	Neural	Social
	input	dots moving left or right	fights won or lost
c	strength of input	coherence of dots	probability of stronger animal winning
X_1, X_2	decision variables	firing rates of neural populations	opinions about relative dominance
	dist. of states of the world	left / right equally likely	distribution of fighting abilities
	decision	subject indicates left or right	one animal emits a subordination signal
	correct decision	subject chooses correct direction	weaker animal signals

Table 2: **Comparison of models applied to neural and social systems.** Differences are highlighted in red.

	Neural	Social
dimensionality	1	2
decision	difference hits a threshold	one var. hits a threshold
optimality criterion	reward from being "right"	reward from being "right"
	decision time	decision time
		reward from receiving signal
optimization depends on	input strength	input strength
	noise	noise
	leak rate	leak rate
		other animal's threshold

6   Appendix

6.1 Derivation of PDEs for waiting time and accuracy

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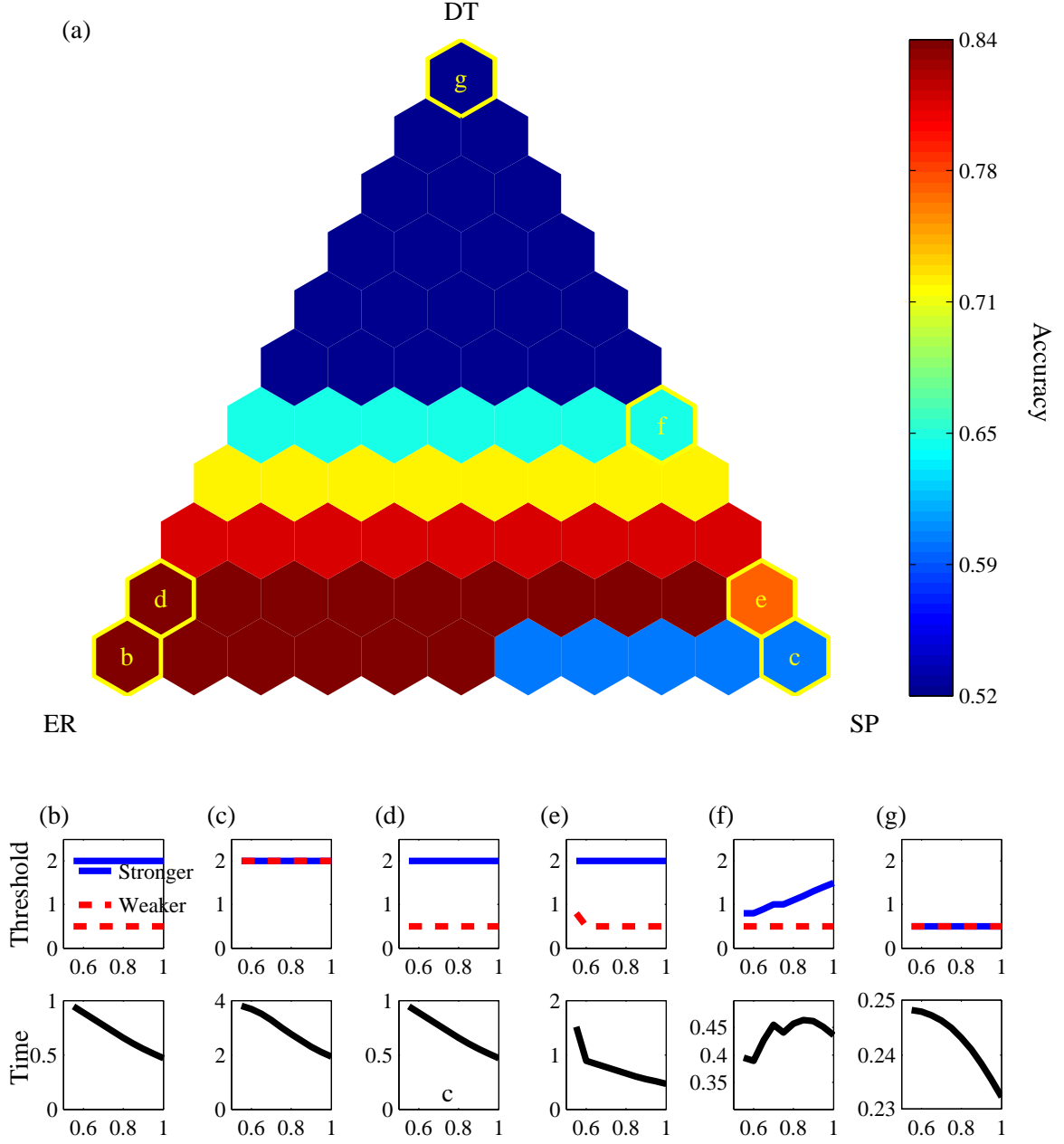
Figure 1: The accuracy of a pair using Nash equilibrium thresholds decreases as the weight given to decision time increases. (a) In the upper panel, the color indicates the accuracy of a decision made by a pair making a difficult decision ($c = .55$) using Nash equilibrium thresholds, as a function of the optimization weights, w_1, w_2, w_3. In the lower left corner of the simplex, only error rate matters ($w_1 = 1$). In the upper corner, only decision time matters ($w_2 = 1$). In the lower right corner, only signaling probability matters ($w_3 = 1$). Accuracy decreases as w_2 increases but is not greatly affected by a tradeoff between w_1 and w_3. In (b)-(g) we show how the Nash equilibrium thresholds (top row) and decision time for a pair using those thresholds (bottom row) depends on the difficulty of the decision (c). The optimization weights for each panel are indicated in the simplex with the corresponding letter. At intermediate waiting costs (f), intermediately difficult decision take the longest.
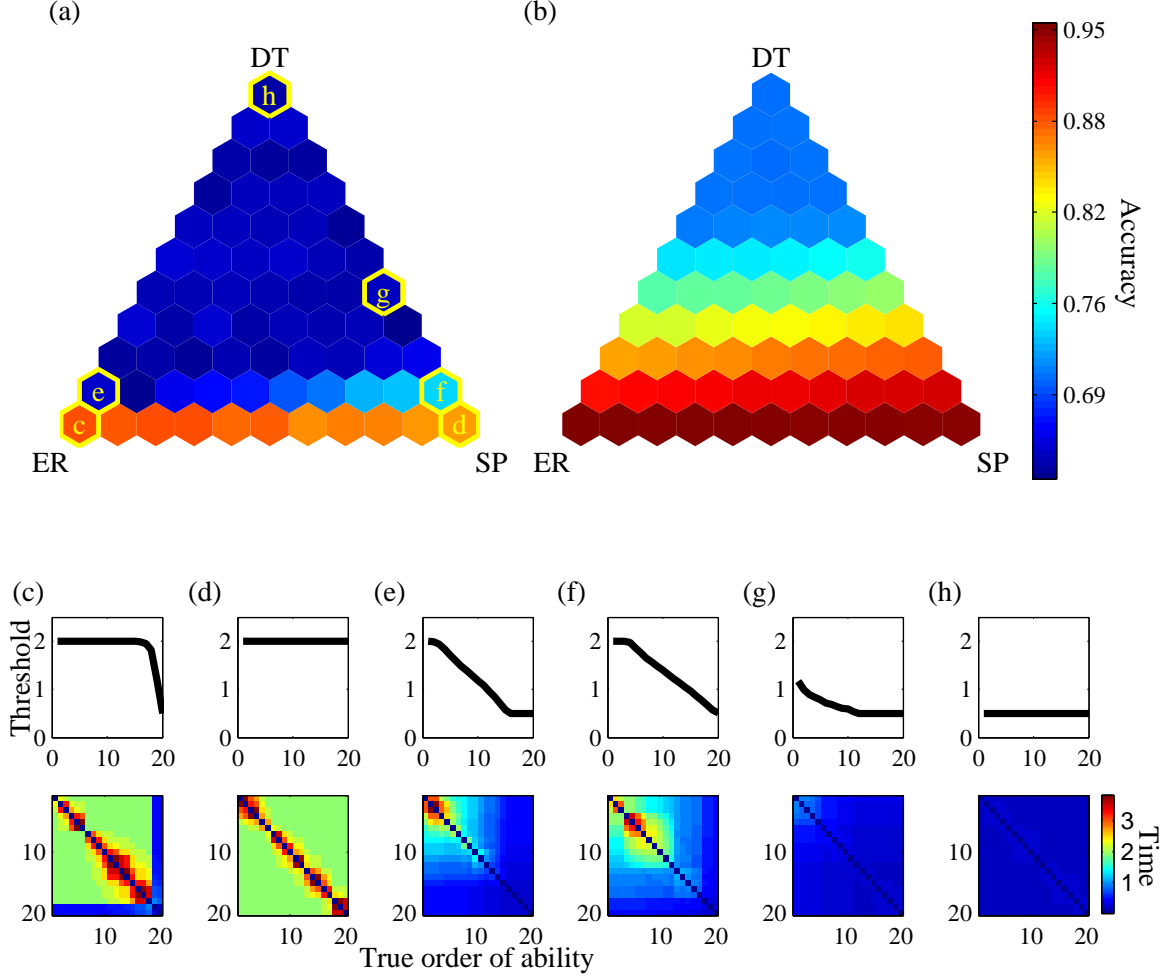
Figure 2: The accuracy of a group using optimal thresholds decreases as the weight given to decision time increases. In the upper panels, the color indicates the average accuracy of the decisions between individuals in the bottom quartile (a) and the average accuracy of the whole group (b) using optimal thresholds, as a function of the optimization weights, w_1, w_2, w_3. In the lower left corner of the simplex, only error rate matters ($w_1 = 1$). In the upper corner, only decision time matters ($w_2 = 1$). In the lower right corner, only signaling probability matters ($w_3 = 1$). Both bottom-quartile and total accuracy decrease as w_2 increases, and the increase in bottom-quartile accuracy as w_3 increases drives a slight increase in total accuracy. In (c)-(h) we show how the group-context optimal thresholds (top row) and decision time for each pair (bottom row) depends on the true order of ability where 1 is the strongest animal and 20 is the weakest. The optimization weights for each panel are indicated in the simplex with the corresponding letter. For low waiting costs (c and d), it takes longest for pairs with similar ability to reach a decision. For slightly higher waiting costs (f and g), it takes longer for pairs with similar and high ability to reach a decision.
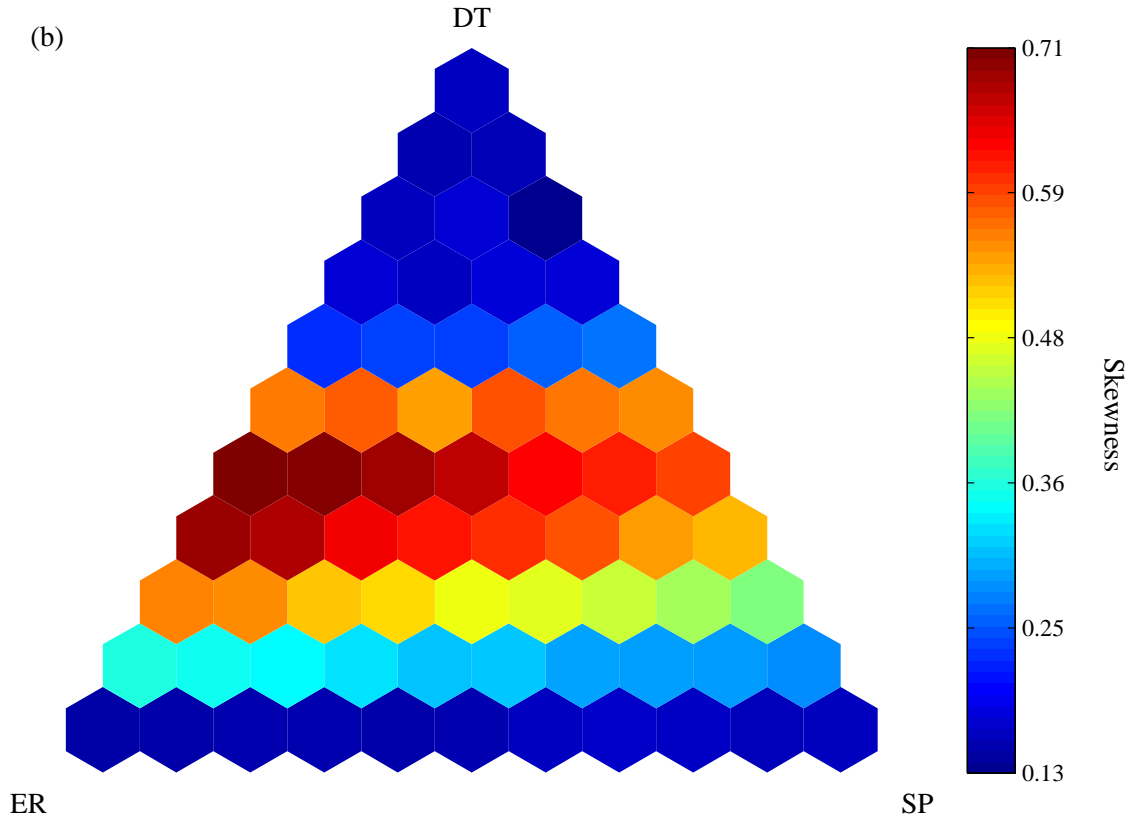
Figure 3: The skewness of the distribution of the number of signalers is maximized at intermediate waiting costs. In the lower left corner of the simplex, only error rate matters ($w_1 = 1$). In the upper corner, only decision time matters ($w_2 = 1$). In the lower right corner, only signaling probability matters ($w_3 = 1$).

(a)

DT

number of signals

entropy

number of signalers
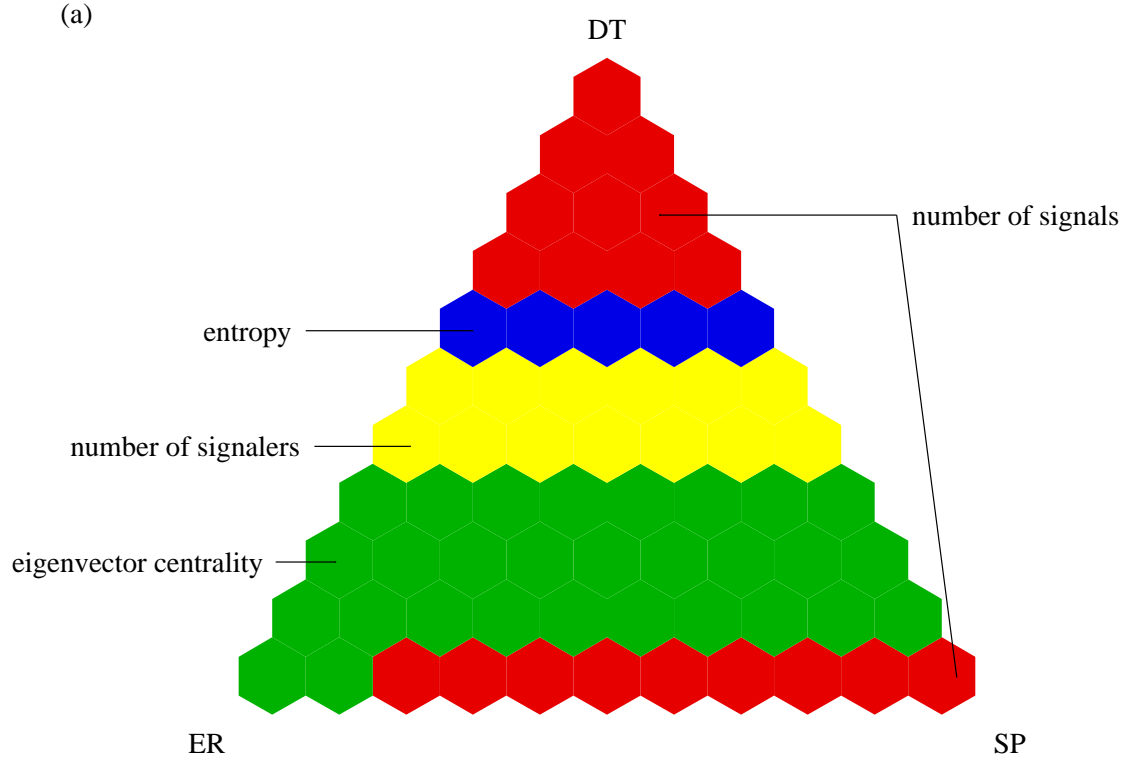
eigenvector centrality

ER

SP

Figure 4: Which measure of consensus is most informative about fighting ability depends on how accurate decision are on average. When decision are very accurate or very inaccurate, measures of the number of signals received are most informative. When accuracy is slightly lower, eigenvector centrality of the signaling network is most informative. When accuracy is slightly lower again, the number of signalers is most informative. When accuracy is lower still, the entropy of signals received is most informative. In the lower left corner of the simplex, only error rate matters ($w_1 = 1$). In the upper corner, only decision time matters ($w_2 = 1$). In the lower right corner, only signaling probability matters ($w_3 = 1$).