J. Appl. Prob. **33**, 164–175 (1996)
Printed in Israel

FIRST PASSAGE TIME DISTRIBUTION OF A WIENER PROCESS WITH DRIFT CONCERNING TWO ELASTIC BARRIERS

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Abstract

We solve the Fokker–Planck equation for the Wiener process with drift in the presence of elastic boundaries and a fixed start point. An explicit expression is obtained for the first passage density. The cases with pure absorbing and/or reflecting barriers arise for a special choice of a parameter constellation. These special cases are compared with results in Darling and Siegert [5] and Sweet and Hardin [15].

FIRST PASSAGE TIME; FIRST EXIT TIME; ELASTIC BOUNDARIES; REFLECTION; ABSORPTION; WIENER PROCESS

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60G40
SECONDARY 60J65; 60J70

1. Introduction

In this paper we consider the first passage time problem through two constant elastic barriers (see Feller [9], [10], and Bharucha-Reid [1]) for the Wiener process. In a variety of engineering, biological and physical problems of interest one can find the evaluation of first passage time probability density functions ([3], [7], [8], [12], [13], [14]). What motivated the present article was a reliability investigation of a system which is gradually influenced by time-dependent damage. For the modeling of the damage process we use a Wiener process (or Brownian motion) $\{X_t;\, t \geqq 0\}$ with drift μ and diffusion σ^2. The first passage (or first exit) time is defined as the random time τ that X_t reaches a given boundary (considered as an absorbing boundary, i.e. the process remains in that state for all subsequent times) ∂G of a domain G for the first time,

$$\tau = \inf_{t \geqq t_0} \{t : X_t \in \partial G \mid X_{t_0} = x;\, x \in G \backslash \partial G\}, \tag{1}$$

where the process starts in some initial state $x \in G\backslash \partial G$ at time $t = t_0$ with probability one. To determine the probability density function $f_\tau(t)$ of the first passage time τ we use the probability that, in $(0, t)$, X_t is never absorbed by the boundary ∂G of G:

Received 23 February 1994; revision received 14 October 1994.
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$$
\begin{aligned}
\Pr(X_t \notin \partial G;\, X_s \in G\backslash\partial G,\, t_0 < s < t \mid x \in G\backslash\partial G) &= 1 - \Pr(\tau < t) \\
&= \int_G p(t, y \mid t_0, x)dy,
\end{aligned} \tag{2}
$$

where $p(t, y \mid t_0, x)$ represents the transition probability density of X_t in $G\backslash\partial G$. On account of (2) we get for the first passage time probability density:

$$
f_\tau(t) = -\frac{\partial \Pr(X_t \notin \partial G;\, X_s \in G\backslash\partial G,\, t_0 < s < t \mid x \in G\backslash\partial G)}{\partial t}. \tag{3}
$$

In the following we take $G = [h_1, h_2]$ and $\partial G = \{h_1\} \cup \{h_2\}$, i.e. we have two barriers h_1 and h_2. There are solutions of the first passage problem in the presence of constant absorbing and/or reflecting (i.e. the process cannot cross the barrier) barriers ([1], [4], [5], [15]). The aim of this paper is to determine the first passage time distribution for the Wiener process X_t with drift in the more general case of two elastic barriers. If the process reaches an elastic barrier, it can be absorbed with probability α or reflected with probability $(1-\alpha)$. The elasticity of the barrier depends on the choice of the parameter α, $0 \leqq \alpha \leqq 1$. As special cases, we have pure absorbing behavior for $\alpha = 1$ and pure reflecting behavior for $\alpha = 0$. So we can get, for example, the following cases: two absorbing barriers, one absorbing barrier and one reflecting barrier, two elastic barriers and so on. After a determination of the probability distribution we shall compare the results for known cases.

2. The Fokker–Planck equation for the Wiener process with drift

On account of Equations (2) and (3) we need the transition probability density of X_t in $G\backslash\partial G$ for the determination of the first passage time probability distribution. The transition probability density of X_t in $G\backslash\partial G$, which is denoted for brevity as $p(t, y) = p(t, y \mid t_0, x)$, satisfies the Fokker–Planck (or Kolmogorov forward) equation

$$
\frac{\partial}{\partial t} p(t, y) = -\mu \frac{\partial}{\partial y} p(t, y) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p(t, y) \tag{4}
$$

and the initial condition (we set $t_0 = 0$)

$$
p(0, y) = \delta(y - x), \tag{5}
$$

where $\delta(y-x)$ is the Dirac delta function. For the elastic barriers we have two boundary conditions of the form

$$
\beta p(t, h_2) - (1-\beta)\left(\mu p(t, h_2) - \frac{\sigma^2}{2} \frac{\partial p(t, y)}{\partial y}\bigg|_{y=h_2}\right) = 0, \tag{6}
$$

$$
\gamma p(t, h_1) + (1-\gamma)\left(\mu p(t, h_1) - \frac{\sigma^2}{2} \frac{\partial p(t, y)}{\partial y}\bigg|_{y=h_1}\right) = 0, \tag{7}
$$

with the elasticity parameters β and γ. This form of consideration contains two important special cases. For $\beta=0$ (respectively $\gamma=0$) one gets a reflecting barrier and for $\beta=1$ (respectively $\gamma=1$) an absorbing barrier. Of course, we will exclude the case of two reflecting barriers, i.e. $\gamma=0$ and $\beta=0$. After determining the first passage time probability density for the general case we shall discuss the special cases in detail.

To solve the Fokker–Planck equation (4) we make some transformations. First, we shift the lower barrier to the origin by changing to the coordinate $u=y-h_1$. Next, we remove the first-order derivative relating to the local coordinate by using the transformation

$$p(t,u)=\exp\left(\frac{\mu}{\sigma^2}u\right)z(t,u). \tag{8}$$

For $z(t,u)$ we get the partial differential equation

$$\frac{2}{\sigma^2}\frac{\partial}{\partial t}z(t,u)=\frac{\partial^2}{\partial u^2}z(t,u)-\frac{\mu^2}{\sigma^4}z(t,u). \tag{9}$$

For the solution of Equation (9) we separate $z(t,u)=T(t)U(u)$ and we get two ordinary differential equations

$$\frac{d}{dt}T(t)+\frac{\sigma^2}{2}\nu T(t)=0, \tag{10}$$

$$\frac{d^2}{du^2}U(u)+\lambda U(u)=0, \tag{11}$$

where ν is the separation parameter and $\lambda=\nu-\mu^2/\sigma^4$. The solution of the first equation has the form $T(t)=C\exp(-\sigma^2\nu t/2)$. The solutions of the second equation with associated boundary conditions

$$[\beta-\tfrac{1}{2}(1-\beta)\mu]U(h_2-h_1)+\tfrac{1}{2}(1-\beta)\sigma^2\frac{d}{du}U(u)\bigg|_{u=h_2-h_1}=0, \tag{12}$$

$$[\gamma+\tfrac{1}{2}(1-\gamma)\mu]U(0)-\tfrac{1}{2}(1-\gamma)\sigma^2\frac{d}{du}U(u)\bigg|_{u=0}=0 \tag{13}$$

depend on the value of λ and are of the form

$$U(u)=\begin{cases} C_1\cos(\sqrt{\lambda}u)+C_2\sin(\sqrt{\lambda}u) & \lambda>0\\ C_3u+C_4 & \lambda=0\\ C_5\cosh(\sqrt{-\lambda}u)+C_6\sinh(\sqrt{-\lambda}u) & \lambda<0. \end{cases}$$

The following abbreviations are used throughout this paper:

$$\kappa = \beta - \tfrac{1}{2}(1-\beta)\mu, \tag{14}$$

$$\omega = \gamma + \tfrac{1}{2}(1-\gamma)\mu, \tag{15}$$

$$\mu_1 = -\frac{\gamma}{1-\gamma} + \frac{\beta}{1-\beta} - \left[\left(\frac{\gamma}{1-\gamma} + \frac{\beta}{1-\beta}\right)^2 + \frac{2\sigma^2}{h_2-h_1}\left(\frac{\beta}{1-\beta} + \frac{\gamma}{1-\gamma}\right)\right]^{1/2}, \tag{16}$$

$$\mu_2 = -\frac{\gamma}{1-\gamma} + \frac{\beta}{1-\beta} + \left[\left(\frac{\gamma}{1-\gamma} + \frac{\beta}{1-\beta}\right)^2 + \frac{2\sigma^2}{h_2-h_1}\left(\frac{\beta}{1-\beta} + \frac{\gamma}{1-\gamma}\right)\right]^{1/2}, \tag{17}$$

$$\mu_3 = -\frac{\sigma^2}{h_2-h_1} - \frac{2\gamma}{1-\gamma}, \tag{18}$$

$$\mu_4 = \frac{\sigma^2}{h_2-h_1} + \frac{2\beta}{1-\beta}. \tag{19}$$

Furthermore, we set $\vartheta = \sqrt{\lambda}(h_2-h_1)$ in the case of $\lambda > 0$ and $\eta = \sqrt{-\lambda}(h_2-h_1)$ in the case of $\lambda < 0$. Using a classification into the following three cases:

$$\text{case } A := \begin{cases} \beta \neq 1;\ \gamma \neq 1;\ \mu_1 < \mu < \mu_2 & \text{or} \\ \beta = 1;\ \gamma \neq 1;\ \mu > \mu_3 & \text{or} \\ \beta \neq 1;\ \gamma = 1;\ \mu < \mu_4 & \text{or} \\ \beta = 1;\ \gamma = 1, & \end{cases}$$

$$\text{case } B := \begin{cases} \beta \neq 1;\ \gamma \neq 1;\ \mu = \mu_1 & \text{or} \\ \beta \neq 1;\ \gamma \neq 1;\ \mu = \mu_2 & \text{or} \\ \beta = 1;\ \gamma \neq 1;\ \mu = \mu_3 & \text{or} \\ \beta \neq 1;\ \gamma = 1;\ \mu = \mu_4, & \end{cases}$$

$$\text{case } C := \begin{cases} \beta \neq 1;\ \gamma \neq 1;\ \mu < \mu_1 & \text{or} \\ \beta \neq 1;\ \gamma \neq 1;\ \mu > \mu_2 & \text{or} \\ \beta = 1;\ \gamma \neq 1;\ \mu < \mu_3 & \text{or} \\ \beta \neq 1;\ \gamma = 1;\ \mu > \mu_4, & \end{cases}$$

which depends on the parameter space, the solutions of Equation (11) are of the type $U(u) = U_n(u) + C(u)$, with

$$C(u) = \begin{cases} 0 & \text{case } A \\ U_{\mathrm{I}}(u) & \text{case } B \\ U_{\mathrm{II}}(u) & \text{case } C, \end{cases}$$

$$U_n(u) = \frac{(1-\gamma)\sigma^2\vartheta_n}{2(h_2-h_1)}\cos\left(\vartheta_n \frac{u}{h_2-h_1}\right) + \omega \sin\left(\vartheta_n \frac{u}{h_2-h_1}\right),$$

$$U_{\mathrm{I}}(u)=u-(h_2-h_1)-\frac{\sigma^2(1-\beta)}{2\kappa}$$

and

$$U_{\mathrm{II}}(u)=\frac{(1-\gamma)\sigma^2\eta}{2(h_2-h_1)}\cosh\left(\eta\frac{u}{h_2-h_1}\right)+\omega\sinh\left(\eta\frac{u}{h_2-h_1}\right).$$

The positive eigenvalues ϑ_n satisfy the equation

(20) $$\tan(\vartheta)=\frac{2[\beta(1-\gamma)+\gamma(1-\beta)]\sigma^2(h_2-h_1)\vartheta}{\sigma^4(1-\beta)(1-\gamma)\vartheta^2-4(h_2-h_1)^2\kappa\omega}.$$

Furthermore, η is the positive solution of

(21) $$\tanh(\eta)=\frac{-2[\beta(1-\gamma)+\gamma(1-\beta)]\sigma^2(h_2-h_1)\eta}{\sigma^4(1-\beta)(1-\gamma)\eta^2+4(h_2-h_1)^2\kappa\omega}.$$

We return to the function $p(t,y)$. To satisfy the initial condition (5) we expand $p(t,y)$ in a Fourier series

(22) $$p(t,y)=\exp\left(\frac{\mu}{\sigma^2}y-\frac{\mu^2 t}{2\sigma^2}\right)\left\{\sum_{n=1}^{\infty}\left[C_n\exp\left(-\frac{\sigma^2\vartheta_n^2 t}{2(h_2-h_1)^2}\right)U_n(y-h_1)\right]+K_1\right\}$$

with

$$K_1=\begin{cases}0 & \text{case } A\\ C_{\mathrm{I}}U_{\mathrm{I}}(y-h_1) & \text{case } B\\ C_{\mathrm{II}}\exp\left(\dfrac{\sigma^2\eta^2 t}{2(h_2-h_1)^2}\right)U_{\mathrm{II}}(y-h_1) & \text{case } C.\end{cases}$$

For the constants C_n, C_{I} and C_{II} we have the equations

(23) $$C_n=\frac{\int_{h_1}^{h_2}\exp(-\mu y/\sigma^2)p(0,y)U_n(y-h_1)dy}{\int_{h_1}^{h_2}U_n^2(y-h_1)dy},$$

$$C_{\mathrm{I}}=\frac{\int_{h_1}^{h_2}\exp(-\mu y/\sigma^2)p(0,y)U_{\mathrm{I}}(y-h_1)dy}{\int_{h_1}^{h_2}U_{\mathrm{I}}^2(y-h_1)dy},$$

and

(24) $$C_{\mathrm{II}}=\frac{\int_{h_1}^{h_2}\exp(-\mu y/\sigma^2)p(0,y)U_{\mathrm{II}}(y-h_1)dy}{\int_{h_1}^{h_2}U_{\mathrm{II}}^2(y-h_1)dy}.$$

Note that the denominators in (23) and (24) can be determined using the relations

$$\frac{\vartheta_n^2}{(h_2-h_1)^2}\int_0^{h_2-h_1} U_n^2\,du = -[U_n U_n']_{u=0}^{u=h_2-h_1} + \int_0^{h_2-h_1} (U_n')^2 du,$$

$$-\frac{\eta^2}{(h_2-h_1)^2}\int_0^{h_2-h_1} U_{\mathrm{II}}^2\,du = -[U_{\mathrm{II}} U_{\mathrm{II}}']_{u=0}^{u=h_2-h_1} + \int_0^{h_2-h_1} (U_{\mathrm{II}}')^2 du.$$

Finally we obtain for the transition probability density in $G\backslash\partial G$:

$$\begin{aligned} p(t,y) &= \exp\left(-\frac{\mu^2 t}{2\sigma^2}+\frac{\mu}{\sigma^2}(y-x)\right) \\ &\quad \times\left\{\sum_{n=1}^{\infty}\left[\frac{1}{M_1}\exp\left(-\frac{\vartheta_n^2\sigma^2 t}{2(h_2-h_1)^2}\right)U_n(y-h_1)U_n(x-h_1)\right]+K_2\right\}, \end{aligned} \tag{25}$$

where

$$K_2=\begin{cases} 0 & \text{case } A \\ \dfrac{1}{M_2}U_{\mathrm{I}}(y-h_1)U_{\mathrm{I}}(x-h_1) & \text{case } B \\ \dfrac{1}{M_3}\exp\left(\dfrac{\sigma^2\eta^2 t}{2(h_2-h_1)^2}\right)U_{\mathrm{II}}(y-h_1)U_{\mathrm{II}}(x-h_1) & \text{case } C, \end{cases}$$

$$\begin{aligned} M_1 &= \frac{(1-\gamma)^2\sigma^4}{8(h_2-h_1)}\vartheta_n^2+\tfrac{1}{2}\omega^2(h_2-h_1)-(\tfrac{1}{2}\cos^2(\vartheta_n)+\sin^2(\vartheta_n)) \\ &\quad\times\frac{4\omega^2\kappa(1-\gamma)\sigma^2(h_2-h_1)^2-(1-\beta)(1-\gamma)^2\sigma^6\vartheta_n^2\omega}{2(1-\beta)(1-\gamma)\sigma^4\vartheta_n^2-8\kappa(h_2-h_1)^2\omega} \\ &\quad-\frac{4\omega^3(1-\beta)\sigma^2(h_2-h_1)^2-(1-\gamma)^3\sigma^6\vartheta_n^2\kappa}{4(1-\beta)(1-\gamma)\sigma^4\vartheta_n^2-16\kappa(h_2-h_1)^2\omega}\cos^2(\vartheta_n), \end{aligned} \tag{26}$$

$$M_2=\frac{(h_2-h_1)^3}{3}+\frac{\sigma^2(1-\beta)(h_2-h_1)^2}{2\kappa}+\frac{\sigma^4(1-\beta)^2(h_2-h_1)}{4\kappa^2} \tag{27}$$

and

$$\begin{aligned} M_3 &= \frac{(1-\gamma)^2\sigma^4}{8(h_2-h_1)}\eta^2-\tfrac{1}{2}\omega^2(h_2-h_1)+(\sinh^2(\eta)-\tfrac{1}{2}\cosh^2(\eta)) \\ &\quad\times\frac{4\omega^2\kappa(1-\gamma)\sigma^2(h_2-h_1)^2+(1-\beta)(1-\gamma)^2\sigma^6\eta^2\omega}{2(1-\beta)(1-\gamma)\sigma^4\eta^2-8\kappa(h_2-h_1)^2\omega} \\ &\quad-\frac{4\omega^3(1-\beta)\sigma^2(h_2-h_1)^2+(1-\gamma)^3\sigma^6\eta^2\kappa}{4(1-\beta)(1-\gamma)\sigma^4\eta^2+16\kappa(h_2-h_1)^2\omega}\cosh^2(\eta). \end{aligned} \tag{28}$$

The above series (25) converges absolutely and uniformly in $(0,\infty)\times[h_1,h_2]$.

3. The first passage time distribution

In this section, we deal with the determination of the first passage time probability distribution. In view of Equation (2), the first passage time probability distribution function $F_\tau(t)$ is

$$F_\tau(t) = 1 - \int_{h_1}^{h_2} p(t, y)dy, \tag{29}$$

where $p(t, y)$ is given in (25). If we integrate out we obtain for $F_\tau(t)$ the result

$$\begin{aligned} F_\tau(t) = 1 - \sum_{n=1}^{\infty} \left[\frac{\sigma^2 C(n, x)}{2[\mu^2(h_2-h_1)^2 + \vartheta_n^2\sigma^4]} \exp\left(-\frac{\vartheta_n^2\sigma^2 t}{2(h_2-h_1)^2} - \frac{\mu^2 t}{2\sigma^2} \right) \right] \\ - \exp\left(-\frac{\mu}{\sigma^2} x - \frac{\mu^2 t}{2\sigma^2} \right) K_3, \end{aligned} \tag{30}$$

where we have for $C(n, x)$ and K_3 the equations

$$\begin{aligned} C(n, x) = \frac{1}{M_1} \exp\left(-\frac{\mu}{\sigma^2} x \right) U_n(x-h_1) \left\{ \exp\left(\frac{\mu}{\sigma^2} h_2 \right) [((1-\gamma)\sigma^4\vartheta_n^2 + 2\omega\mu(h_2-h_1)^2) \right. \\ \times \sin(\vartheta_n) - 2\gamma\vartheta_n(h_2-h_1)\sigma^2\cos(\vartheta_n)] \\ \left. + 2\gamma\vartheta_n(h_2-h_1)\sigma^2 \exp\left(\frac{\mu}{\sigma^2} h_1 \right) \right\}, \end{aligned} \tag{31}$$

$$K_3 = \begin{cases} 0 & \text{case } A \\ \dfrac{\sigma^2}{\mu M_2} U_1(x-h_1) \left\{ \dfrac{\sigma^2}{\mu} \left[\left(\dfrac{\mu}{\sigma^2} h_2 - 1 \right) \exp\left(\dfrac{\mu}{\sigma^2} h_2 \right) \right. \right. & \\ \qquad \left. - \left(\dfrac{\mu}{\sigma^2} h_1 - 1 \right) \exp\left(\dfrac{\mu}{\sigma^2} h_1 \right) \right] & \text{case } B \\ \qquad \left. - \left(h_2 + \dfrac{\sigma^2(1-\beta)}{2\kappa} \right) \left[\exp\left(\dfrac{\mu}{\sigma^2} h_2 \right) - \exp\left(\dfrac{\mu}{\sigma^2} h_1 \right) \right] \right\} & \\ \dfrac{\sigma^2 C^*(\eta, x)}{2[\mu^2(h_2-h_1)^2 - \eta^2\sigma^4]} \exp\left(\dfrac{\sigma^2\eta^2 t}{2(h_2-h_1)^2} \right) & \text{case } C, \end{cases}$$

with

$$C^*(\eta, x) = \frac{1}{M_3} U_{\mathrm{II}}(x-h_1) \left\{ \exp\left(\frac{\mu}{\sigma^2} h_2\right) [(-(1-\gamma)\sigma^4\eta^2 + 2\omega\mu(h_2-h_1)^2)\sinh(\eta) - 2\gamma\eta(h_2-h_1)\sigma^2\cosh(\eta)] + 2\gamma\eta(h_2-h_1)\sigma^2 \exp\left(\frac{\mu}{\sigma^2} h_1\right) \right\}. \tag{32}$$

The density function $f_\tau(t)$ and the expected value $E(\tau)$ of the first passage time may also be obtained (see also [6]). The density is given by

$$f_\tau(t) = \sum_{n=1}^{\infty} \left[\frac{C(n, x)}{4(h_2-h_1)^2} \exp\left(-\frac{\vartheta_n^2 \sigma^2 t}{2(h_2-h_1)^2} - \frac{\mu^2 t}{2\sigma^2} \right) \right] + \exp\left(-\frac{\mu}{\sigma^2} x - \frac{\mu^2 t}{2\sigma^2} \right) K_4, \tag{33}$$

with

$$K_4 = \begin{cases} 0 & \text{case } A \\ \dfrac{\mu}{2M_2} U_{\mathrm{I}}(x-h_1) \left\{ \dfrac{\sigma^2}{\mu} \left[\left(\dfrac{\mu}{\sigma^2} h_2 - 1\right) \exp\left(\dfrac{\mu}{\sigma^2} h_2\right) - \left(\dfrac{\mu}{\sigma^2} h_1 - 1\right) \exp\left(\dfrac{\mu}{\sigma^2} h_1\right) \right] - \left(h_2 + \dfrac{\sigma^2(1-\beta)}{2\kappa}\right) \left[\exp\left(\dfrac{\mu}{\sigma^2} h_2\right) - \exp\left(\dfrac{\mu}{\sigma^2} h_1\right) \right] \right\} & \text{case } B \\ \dfrac{C^*(\eta, x)}{4(h_2-h_1)^2} \exp\left(\dfrac{\sigma^2 \eta^2 t}{2(h_2-h_1)^2} \right) & \text{case } C. \end{cases}$$

Now we will consider the probability density $f_\tau(t)$ of the first passage time for various parameter arrangements. In Figure 1, the function $f_\tau(t)$ is represented in the case of an upper absorbing barrier (i.e. $\beta = 1$) and a reflecting (i.e. $\gamma = 0$), elastic (e.g. $\gamma = 0.5$) or lower absorbing (i.e. $\gamma = 1$) barrier. The functions (II) and (III) are bimodal functions. Note that for this parameter set the first 'peak' essentially describes the first passage behavior concerning the lower barrier h_1. For the function (I) the first 'peak' vanishes, because the lower barrier is a reflecting one. Besides, as was to be expected, the function (II) in the case of an elastic barrier is located between the functions for the absorbing and reflecting case. The same result follows for the expected value. Furthermore, we can also get a unimodal density for example in the case of two absorbing barriers (for

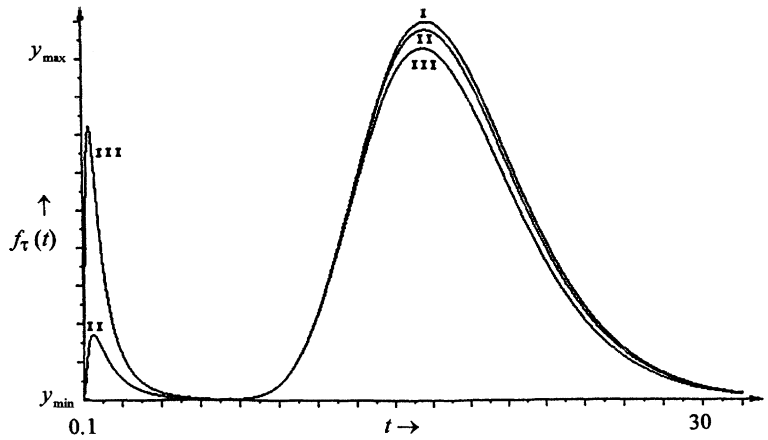

Figure 1. $y_{min}=0$; $h_1=2$; $\gamma_{\mathrm{I}}=0$; $\boldsymbol{E}_{\mathrm{I}}(\tau)=16.9622$; $y_{max}=1.0949\cdot 10^{-1}$; $h_2=40$; $\gamma_{\mathrm{II}}=0.5$ $\boldsymbol{E}_{\mathrm{II}}(\tau)=16.4918$; $\mu=2.1$; $x=4.3$; $\gamma_{\mathrm{III}}=1$; $\boldsymbol{E}_{\mathrm{III}}(\tau)=15.4603$; $\sigma=1.98$; $\beta=1$.

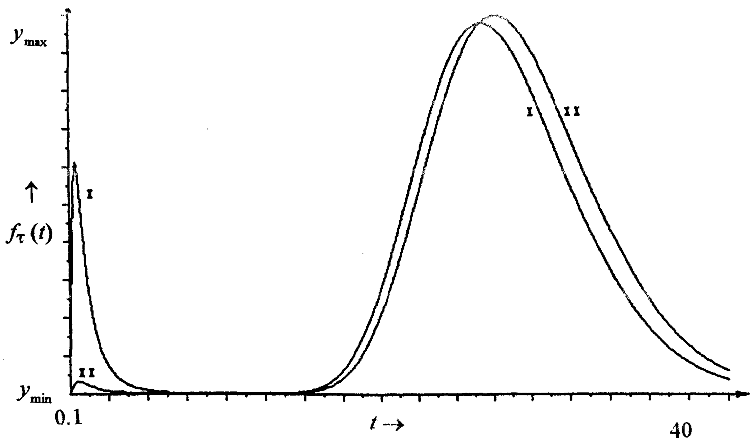

Figure 2. $y_{min}=1.4360\cdot 10^{-6}$; $h_1=2$; $\gamma_{\mathrm{I}}=0.85$; $\boldsymbol{E}_{\mathrm{I}}(\tau)=24.8962$; $y_{max}=8.2498\cdot 10^{-2}$; $h_2=60$; $\beta_{\mathrm{I}}=0.9$; $\boldsymbol{E}_{\mathrm{II}}(\tau)=27.2628$; $\mu=2.1$; $x=4.3$; $\gamma_{\mathrm{II}}=0.1$; $\sigma=1.98$; $\beta_{\mathrm{II}}=0.51$.

instance in the case $\mu>0$, $h_1 \ll x$ and σ is not too large). In Figure 2, we have two examples with two elastic barriers. For the function (II) the barriers have a 'more reflecting' behavior. Among other things, the expected value $\boldsymbol{E}_{\mathrm{II}}(\tau)$ is naturally larger than the expected value $\boldsymbol{E}_{\mathrm{I}}(\tau)$.

4. Concluding remarks

In this section we consider two special cases which we will compare with results in Darling and Siegert [5] and Sweet and Hardin [15].

First we discuss the case of two absorbing barriers (i.e. $\gamma=1$, $\beta=1$). For this case we have from a consideration of (30) and (33) the following equations for $F_\tau(t)$ and $f_\tau(t)$:

$$F_\tau(t)=1-\sum_{k=1}^{\infty}\frac{2\sigma^4 k\pi\tilde{C}(k,x)}{\mu^2(h_2-h_1)^2+\sigma^4k^2\pi^2}\exp\left(-\frac{k^2\pi^2\sigma^2 t}{2(h_2-h_1)^2}-\frac{\mu^2 t}{2\sigma^2}\right), \tag{34}$$

$$f_\tau(t)=\sum_{k=1}^{\infty}\frac{\sigma^2 k\pi\tilde{C}(k,x)}{(h_2-h_1)^2}\exp\left(-\frac{k^2\pi^2\sigma^2 t}{2(h_2-h_1)^2}-\frac{\mu^2 t}{2\sigma^2}\right), \tag{35}$$

with

$$\begin{aligned}\tilde{C}(k,x)=&\left[\exp\left(\frac{\mu}{\sigma^2}h_2\right)(-1)^{k+1}+\exp\left(\frac{\mu}{\sigma^2}h_1\right)\right]\\ &\times\sin\left(k\pi\frac{x-h_1}{h_2-h_1}\right)\exp\left(-\frac{\mu}{\sigma^2}x\right).\end{aligned} \tag{36}$$

The result for the distribution function can be compared with a result in Darling and Siegert [5]. If we set $x=0$ and also use some relations for trigonometric functions it is possible to write the distribution function as

$$\begin{aligned}F_\tau(t)=1-&\frac{\sigma^2\pi}{(h_2-h_1)^2}\sum_{k=1}^{\infty}\frac{k(-1)^{k+1}}{\dfrac{\mu^2}{2\sigma^2}+\dfrac{\sigma^2k^2\pi^2}{2(h_2-h_1)^2}}\exp\left[-t\left(\frac{\mu^2}{2\sigma^2}+\frac{\sigma^2k^2\pi^2}{2(h_2-h_1)^2}\right)\right]\\ &\times\left[\exp\left(\frac{\mu}{\sigma^2}h_1\right)\sin\left(\frac{k\pi h_2}{h_2-h_1}\right)-\exp\left(\frac{\mu}{\sigma^2}h_2\right)\sin\left(\frac{k\pi h_1}{h_2-h_1}\right)\right].\end{aligned}$$

This result is in agreement with that in S.633 of [5] (where two misprints must be corrected). In [5] we have π^2 in the prefactor of the sum instead of π and in the sum $(-1)^k$ instead of $(-1)^{k+1}$. Without these corrections, $F_\tau(t)$ is a strongly decreasing function and always greater than 1.

Observe also that, if we consider $\lim_{h_1\to-\infty}f_\tau(t)$ for $f_\tau(t)$ (see (35)) we arrive at the one barrier problem and we get for the density function of the first passage time the inverse Gaussian distribution

$$\lim_{h_1\to-\infty}f_\tau(t)=\frac{h_2-x}{\sqrt{2\pi\sigma^2t^3}}\exp\left(-\frac{(h_2-x-\mu t)^2}{2\sigma^2 t}\right).$$

Furthermore, we will discuss the case of an upper absorbing barrier and a lower reflecting barrier (i.e. $\gamma=0$, $\beta=1$). The function $f_\tau(t)$ is, in view of (33), of the form

$$f_\tau(t)=\exp\left(-\frac{\mu^2 t}{2\sigma^2}-\frac{\mu}{\sigma^2}x\right)\left\{\sum_{n=1}^{\infty}\left\{\frac{2}{\sigma^4\vartheta_n^2+\mu^2(h_2-h_1)^2+\sigma^2\mu(h_2-h_1)}\right.\right.$$

$$(37)\qquad \times\frac{1}{h_2-h_1}\left[\frac{\sigma^2\vartheta_n}{2(h_2-h_1)}\cos\left(\vartheta_n\frac{x-h_1}{h_2-h_1}\right)+\frac{\mu}{2}\sin\left(\vartheta_n\frac{x-h_1}{h_2-h_1}\right)\right]$$

$$\left.\left.\times\exp\left(-\frac{\vartheta_n^2\sigma^2 t}{2(h_2-h_1)^2}+\frac{\mu}{\sigma^2}h_2\right)[\sigma^4\vartheta_n^2+\mu^2(h_2-h_1)^2]\sin(\vartheta_n)\right\}+K_6\right\}$$

where

$$K_6=\begin{cases} 0 & \mu>-\dfrac{\sigma^2}{h_2-h_1}\\ \dfrac{3(x-h_2)}{2(h_2-h_1)^3}\left[(\sigma^2+\mu(h_2-h_1))\exp\left(\dfrac{\mu}{\sigma^2}h_1\right)-\sigma^2\exp\left(\dfrac{\mu}{\sigma^2}h_2\right)\right] & \mu=-\dfrac{\sigma^2}{h_2-h_1}\\ \dfrac{2}{[\sigma^4\eta^2-\mu^2(h_2-h_1)^2-\sigma^2\mu(h_2-h_1)](h_2-h_1)} & \\ \quad\times\left[\dfrac{\sigma^2\eta}{2(h_2-h_1)}\cosh\left(\eta\dfrac{x-h_1}{h_2-h_1}\right)+\dfrac{\mu}{2}\sinh\left(\eta\dfrac{x-h_1}{h_2-h_1}\right)\right] & \\ \quad\times\exp\left(\dfrac{\eta^2\sigma^2 t}{2(h_2-h_1)^2}+\dfrac{\mu}{\sigma^2}h_2\right)[\mu^2(h_2-h_1)^2-\sigma^4\eta^2]\sinh(\eta) & \mu<-\dfrac{\sigma^2}{h_2-h_1}. \end{cases}$$

The eigenvalues satisfy

$$\tan(\vartheta)=-\frac{\sigma^2}{\mu(h_2-h_1)}\vartheta,\qquad \tanh(\eta)=-\frac{\sigma^2}{\mu(h_2-h_1)}\eta.$$

Now we will compare $f_\tau(t)$ with a result in Sweet and Hardin [15]. They use different parameters. Therefore we make the transformations $\mu=2\sigma^2\tilde{\beta}$, $\sigma^2=2D$, $h_1=0$, $h_2=a$, $\vartheta_n=\lambda_n a$ and $\eta=qa$ in order to compare results. After these transformations we obtain

$$f_\tau(t)=\frac{2D}{a}\exp[-4\tilde{\beta}^2Dt+2\tilde{\beta}(a-x)]$$

$$(38)\qquad \times\sum_{n=1}^{\infty}\left[\exp(-\lambda_n^2Dt)\frac{\lambda_n\sin(\lambda_n a)}{1+(2\tilde{\beta}/a\lambda_n^2)\sin^2(\lambda_n a)}\left(\cos(\lambda_n x)+\frac{2\tilde{\beta}}{\lambda_n}\sin(\lambda_n x)\right)\right]$$

$$+\frac{D}{a}\exp(-4\tilde{\beta}^2Dt+2\tilde{\beta}(a-x))K_7$$

where

$$K_7 = \begin{cases} 0 & -2\tilde{\beta}a < 1 \\ \dfrac{3(x-a)}{a^2}[(1+2\tilde{\beta}a)\exp(-2\tilde{\beta}a)-1] & -2\tilde{\beta}a = 1 \\ \dfrac{2q\exp(Dq^2t)\sinh(qa)}{-1-(2\tilde{\beta}/aq^2)\sinh^2(qa)}\left[\cosh(qa)+\dfrac{2\tilde{\beta}}{q}\sinh(qa)\right] & -2\tilde{\beta}a > 1. \end{cases}$$

This result is in agreement with that in S.426, (2.23) of [15] (where three misprints must be corrected). In [15] we have 4 in the prefactor of the sum instead of 2 (and similarly in the last relation of K_7 for $-2\tilde{\beta}a > 1$) and, in the middle relation (for $-2\tilde{\beta}a = 1$), -1 instead of $[(1+2\tilde{\beta}a)\exp(-2\tilde{\beta}a)-1]$. If one uses the transition probability density function from [15] to determine the distribution function of the first passage time, then $F_\tau(t)$ is negative for a time.

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