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FIRST PASSAGE PROBABILITIES OF A TWO DIMENSIONAL BROWNIAN MOTION IN AN ANISOTROPIC MEDIUM

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SUMMARY. The diffusion equation associated with a two dimensional Brownian motion in an anisotropic medium is solved under the boundary conditions that the solution be zero on the lines $x = a$ and $y = b$ given that the particle starts from the origin. The first passage time density for the particle reaching either of the lines is derived.

1. Introduction

The practitioners of the collective risk theory have been confronted with a diffusion approximation in obtaining the distribution times until ruin. Those working on branching processes were faced with similar problems in finding the distribution of extinction time. Thus for many processes, the time it takes for the process to reach a threshold value is of prime importance. Since Brownian motion is the asymptotic process of so many other processes, first passage time probabilities for Brownian motion are of particular importance.

In the following we find the first passage time of a two dimensional Brownian motion to a boundary consisting of two intersecting lines. This is the same as finding the first time either of a pair of correlated one dimensional Brownian motion processes reach some critical level. A physical representation of this problem is the time it takes for a particle in the interior of a two dimensional semi-infinite crystal bounded by two intersecting faces to reach one of the faces. If the crystal properties depend on direction as they do in graphite, calcite and many other crystals (any non-isometric crystal) the crystal is said to be anisotropic. If the faces are not perpendicular to the principal axes of the crystal, then the one dimensional diffusions toward the faces are correlated.

In order to find the first passage probability density, the transition probability density $f(x, y, t)$ of finding the particle in the region

$\{(x, y) \mid x < a \cap y < b\}$ is found by solving the diffusion equation associated with the Brownian motion. Integrating f over the region yields the probability, $P(t)$, that the particle has not reached the boundary by time t. Hence $G(t) = 1-P(t)$ is the probability the particle has reached the boundary by time t so that

$$g(t) = \frac{dG(t)}{dt} = -\frac{dP(t)}{dt}$$

is the probability density of the particle reaching the boundary at time t. Since the conditions imposed at the boundary are those of an absorbing barrier, $g(t)$ is the first passage time probability density of the particle starting from the origin at $t = 0$ reaching the boundary $\{(x, y) \mid x = a \text{ or } y = b\}$ at time t.

2. Diffusion equation

Let $\{(X(t), Y(t)) \mid t \geqslant 0\}$ be a two dimensional Brownian motion with mean vector $E(X(t), Y(t)) = (0, 0)$ and infinitesimal covariance matrix

$$E((\Delta X, \Delta Y)^T(\Delta X, \Delta Y)) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \Delta t \qquad \begin{array}{r} \sigma_1 > 0 \\ \sigma_2 > 0 \\ -1 < \rho < 1 \end{array}$$

where

$$(\Delta X, \Delta Y) = (X(t+\Delta t), Y(t+\Delta t))-(X(t), Y(t)).$$

If $\sigma_1\sigma_2$ and ρ are constants, then the transition probability density function $f(x, y, t)$ satisfies the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial x^2} + \rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial x \partial y} + \frac{\sigma_2^2}{2}\frac{\partial^2 f}{\partial y^2} \qquad \ldots \quad (1)$$

subject to appropriate conditions (see Buckholtz and Wasan, 1973).

The conditions on (1) appropriate to absorbing barriers at $x = a$ and $y = b$ for a particle starting at the origin at $t = 0$ are

$$f(a, y, t) = f(x, b, t) = f(-\infty, y, t) = f(x, -\infty, t) = 0$$

$$f(x, y, 0) = \delta(x).\delta(y)$$

$$\int_{-\infty}^{a}\int_{-\infty}^{b} f(x, y, t)\,dx\,dy \leqslant 1, \qquad t > 0. \qquad \ldots \quad (2)$$

3. The transition probability density

In order to solve (1) subject to (2), a scale changing transformation makes (1) a diffusion equation of a standard type.

Let

$$u = \frac{x}{\sigma_1}$$

$$v = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{y}{\sigma_2} - \frac{x}{\sigma_1}\right) \tag{I}$$

so that

$$f^*(u, v, t) = f(\sigma_1 u, \sigma_2(\rho u + \sqrt{1-\rho}\, v),\ t)$$

satisfies

$$\frac{\partial f^*}{\partial t} = \frac{1}{2} \frac{\partial^2 f^*}{\partial u^2} + \frac{1}{2} \frac{\partial^2 f^*}{\partial v^2}. \quad \ldots \quad (3)$$

The barriers are now the lines

$$u = \frac{a}{\sigma_1},\ v = \frac{1}{\sigma_2\sqrt{1-\rho^2}}\, b - \frac{\rho}{\sqrt{1-\rho^2}}\, u$$

which are generally not at right angles. The transformations which put the boundaries in a "standard position" are translation of the intersection point to the origin

$$w = u - \frac{a}{\sigma_1}$$

$$z = v - \frac{1}{\sqrt{1-\rho^2}} \left(\frac{b}{\sigma_2} - \frac{a}{\sigma_1}\right) \tag{II}$$

and the rotation through the angle $\beta = \gamma + \pi$ where

$$\gamma = -\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)$$

which gives

$$q = -\sqrt{1-\rho^2}\, w + \rho z$$

$$p = -\rho w - \sqrt{1-\rho^2}\, z. \tag{III}$$

Under these rigid transformations (3) does not change form. The particle now starts from some point (p_0, q_0) away from the origin and is absorbed at the boundaries $p = 0$, $q = -\frac{\rho}{\sqrt{1-\rho^2}}\, p$. The problem now has a certain circular symmetry suggesting transforming to polar coordinates.

Letting

$$p = r\cos\theta \qquad\qquad \text{(IV)}$$
$$q = r\sin\theta$$

(3) becomes

$$2\frac{\partial f}{\partial t} = \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \qquad \begin{array}{l} r < 0 \\ p < \theta < \tan^{-1}\dfrac{-\sqrt{1-\rho^2}}{\rho} \end{array} \qquad \ldots \quad (4)$$

subject to

$$f(r, 0, t) = f(r, \alpha, t) = f(\infty, \theta, t) = 0$$

$$Jf(r, \theta, 0) = \delta(r-r_0)\,\delta(\theta-\theta_0) \qquad \ldots \quad (5)$$

$$\int_0^\alpha\int_0^\infty Jf(r, \theta, t)\, dr d\theta \leqslant 1, \quad t > 0$$

where

$$\alpha = \tan^{-1}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right)$$

and

$$J = r\sigma_1\sigma_2\sqrt{1-\rho^2}$$

is the Jacobian of the overall transformation from (1).

Remark : The transformations I to IV can be thought of as an overall single transformation.

We give here a probabilistic method of obtaining the probability density associated with the solution of (4). Let $X(t) = (r(t), \theta(t))$ be a standard Brownian motion in polar coordinates. We find the density through the joint expectation

$$E_{\theta_0, r_0}[\phi(r(t), \theta(t)) \mid 0 \leqslant \theta(s) \leqslant \alpha \quad \text{for} \quad 0 \leqslant s \leqslant t]$$

$$= \int_0^\infty\int_0^\alpha \phi(r, \theta)\mathrm{f}(r, \theta, t; r_0, \theta_0)\, dr d\theta.$$

We can assume that $\phi(r, \theta)$ is of the form $\phi(r, \theta) = \psi_1(r)\psi_2(\theta)$. Let us compute the conditional expectation given the radial component. In polar coordinates $ds^2 = r^2 d\theta^2 + dr^2$. If r is fixed $ds^2 = r^2 d\theta^2$. If $(x(t), y(t))$ is a two dimensional standard Brownian motion and (a, b) is a unit vector then $\beta(t) = (a, b) \cdot (x(t), y(t)) = ax(t) + by(t)$ is a standard one dimensional Brownian motion. Taking (a, b) to be the unit vector in the direction of increasing θ, $d\beta(t) = r(t)\, d\theta(t)$ is a standard Brownian differential. For $h(t)$ an increasing differentiable function $d\beta(h(t)) = \sqrt{h'(t)}d\beta(t)$. It follows that

$$d\theta(t) = \frac{1}{r(t)}\; d\beta(t) = d\beta\left(\int_0^t \frac{ds}{r^2(s)}\right)$$

is a representation of $\theta(t)$ where $\beta(\cdot)$ is a one dimensional Brownian motion independent of the radial component $r(\cdot)$. For a standard one dimensional Brownian motion $\beta(t)$ such that $\beta(0) = \theta_0$

$$E_{\theta_0}[\psi_2(\beta(t)) \,|\, 0 \leqslant \beta(s) \leqslant \alpha \text{ for } 0 \leqslant s \leqslant t]$$

$$= \int_0^\alpha \psi_2(\theta) f_\alpha(\theta; \theta_0 t) d\theta$$

where

$$f_\alpha(\theta; \theta_0, t) = \frac{2}{\alpha} \sum_{n=1}^\infty \sin \frac{n\pi\theta}{\alpha}\; \sin \frac{n\pi\theta_0}{\alpha} \exp\left[-\frac{n^2\pi^2}{2\alpha^2}\, t\right]$$

is the solution of standard diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\; \frac{\partial f}{\partial \theta^2}\,, \quad 0 \leqslant \theta \leqslant \alpha$$

obtained by separation of variables.

Using the representation of $\theta(t)$ as $\beta\left(\int_0^t \frac{ds}{r^2(s)}\right)$, we have

$$\int_0^\infty \int_0^\alpha \psi_1(r)\psi_2(\theta)\, \mathrm{f}(r, \theta; r_0, \theta_0, t)\, dr d\theta$$

$$= \frac{2}{\alpha} \sum_{n=1}^\infty \left(\int_0^\alpha \sin \frac{n\pi\theta}{\alpha}\; \sin \frac{n\pi\theta_0}{\alpha}\; \psi(\theta) d\theta.\right.$$

$$E_{r_0}\left[\psi_1(r(t)) \exp\left[-\frac{n^2\pi^2}{2\alpha^2} \int_0^t \frac{ds}{r^2(s)}\right]\right)$$

$$= \frac{2}{\alpha} \sum_{n=1}^\infty \int_0^\infty \int_0^\alpha \sin \frac{n\pi\theta}{\alpha}\; \sin \frac{n\pi\theta_0}{\alpha}\; p_n(r; r_0, t) \cdot \psi_1(r)\psi_2(\theta) r\; dr d\theta \qquad \ldots \quad (6)$$

where $p_n(r; r_0, t)$ is the solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2}\left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{n^2\pi^2}{\alpha^2 r^2}\, p\right] \quad \dots \quad (7)$$

with p concentrated at r_0 at $t = 0$.

This equation is related to (4) through the separation of variables

$$f(r, \theta, t) = p(r, t)\, g(\theta).$$

It is easy to check that if $p(r, t)$ is a solution of (7) then

$$p(r, t) = cr^{\frac{n\pi}{\alpha}}\, u(r, t)$$

where $u(r, t)$ is a solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u}{\partial r^2} + \frac{1+\frac{2n\pi}{\alpha}}{2r}\,\frac{\partial u}{\partial r}. \quad \dots \quad (8)$$

The solution of (8) is given in Ito and McKean (1965, p. 60) as

$$u(r, r, t) = \frac{1}{t}\, e^{-\frac{r^2+r_0^2}{2t}}\, (r\, r_0)^{-\frac{n\pi}{\alpha}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r\, r_0}{t}\right)$$

when the Jacobian factor is divided out. $I_\nu(\cdot)$ is the modified Bessel function (see Magnus *et al*, 1966, p. 66). The functions $p_n(r; r_0, t)$ are then

$$p_n(r; r_0, t) = (r\, r_0)^{\frac{n\pi}{\alpha}}\, u(r; r_0, t)$$

$$= \frac{1}{t}\, e^{-\frac{r^2+r_0^2}{2t}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r\, r_0}{t}\right). \quad \dots \quad (9)$$

Equation (7) may be solved alternatively using the separation of variables $p(r, t) = \phi(r)\, \psi(t)$ and the identity (see Magnus *et al*, 1966, p. 93)

$$\int_0^\infty \lambda e^{-\frac{1}{2}\lambda^2 t}\, J_{\frac{n\pi}{\alpha}}(\lambda r) J_{\frac{n\pi}{\alpha}}(\lambda r_0)\, d\lambda = \frac{1}{t}\, e^{-\frac{r^2+r_0^2}{2t}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r\, r_0}{t}\right).$$

It follows from (9) and (6) that the joint probability density in r and θ is

$$\mathrm{f}(r, \theta; r_0, \theta_0, t) = \frac{2r}{\alpha t} e^{-\frac{r^2+r_0^2}{2t}} \sum_{n=1}^{\infty} \sin \frac{n\pi\theta}{\alpha} \sin \frac{n\pi\theta_0}{\alpha} I_{\frac{n\pi}{\alpha}} \left(\frac{r\, r_0}{t} \right).$$

4. The first passage time density

Theorem : *Let $\{X(t), Y(t)) \mid t > 0\}$ be a two dimensional Brownian motion giving rise to the diffusion equation* (1). *The probability density of the first passage time random variable*

$$T = \mathit{inf}\, \{t \mid X(t) = a \bigcup Y(t) = b\}$$

is given by the expression

$$g(t) = -\frac{d}{dt} \left\{ \sum_{n=1,3,5,\dots} \frac{2r_0}{n(2\pi t)^{\frac{1}{2}}} \sin \frac{n\pi}{\alpha} \theta_0 e^{-\frac{3r_0^2}{4t}} \left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)} \left(\frac{r_0^2}{4t}\right) - I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)} \left(\frac{r_0^2}{4t}\right) \right] \right\}$$

where θ_0, r_0 and α are given by the relations

$$r_0 \cos \theta_0 = \frac{a(1-\rho)}{\sigma_1} - \frac{b}{\sigma_2}$$

$$r_0 \sin \theta_0 = \frac{a(\rho^2-\rho-1)}{\sigma_1 \sqrt{1-\rho^1}} - \frac{\rho b}{\sigma_2 \sqrt{1-\rho^2}}$$

$$\alpha = \tan^{-1} \left(-\frac{\sqrt{1-\rho^2}}{\rho} \right).$$

Proof : The probability that the particle is in the region $\{(x, y) \mid x < a \bigcap y < b\}$ is

$$P(t) = \int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} \mathrm{f}(r, \theta, t)\, dr\, dt$$

and so

$$P(t) = \int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} \frac{2r}{\alpha} \sum_{n=1}^{\infty} \sin \frac{n\pi}{\alpha} \theta_0 \sin \frac{n\pi}{\alpha} \theta e^{\frac{r^2+r_0^2}{2t}} I_{\frac{n\pi}{\alpha}} \left(\frac{r\, r_0}{t} \right) dr\, d\theta$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{\alpha} \theta_0 e^{-\frac{r_0^2}{2t}} \int_{\theta=0}^{\alpha} \frac{n\pi}{\alpha} \sin \frac{n\pi}{\alpha} \theta d\theta \int_{r=0}^{\infty} r e^{-\frac{r^2}{2t}} I_{\frac{n\pi}{\alpha}} \left(\frac{r\, r_0}{t} \right) dr.$$

Using the identity

$$\int_0^\infty re^{-\beta r^2} I_\nu(cr)dr = \frac{c}{8\beta}\left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} e^{-\frac{c^2}{8\beta}} \left[I_{\frac{1}{2}(\nu+1)}\left(\frac{c^2}{8\beta}\right) - I_{\frac{1}{2}(\nu-1)}\left(\frac{c^2}{8\beta}\right)\right]$$

one finds that

$$P(t) = \sum_{n=1,3,5,\dots} \frac{2r_0}{n(2\pi t)^{\frac{1}{2}}} \sin\frac{n\pi}{\alpha}\theta_0\, e^{-\frac{3r_0^2}{4t}}$$

$$\left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) - I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right)\right].$$

The probability the particle has reached the boundary is

$$G(t) = P(T < t) = 1 - P(t)$$

so that the first passage time probability density is

$$g(t) = \frac{-dP(t)}{dt}. \quad \square$$

Using well known Bessel function identities and the differentiation formula for $I_\nu(z)$ it is found that

$$g(t) = \sum_{n=1,3,5,\dots} K_n \frac{1}{t^{\frac{3}{2}}} e^{-\frac{3r_0^2}{4t}} \left\{\left(\frac{1}{2} - \frac{3r_0^2}{4t}\right)\left[I_{p+\frac{1}{2}}\left(\frac{r_0^2}{4t}\right) - I_{p-\frac{1}{2}}\left(\frac{r_0^2}{4t}\right)\right]\right.$$

$$\left. -p\left[I_{p+\frac{1}{2}}\left(\frac{r_0^2}{4t}\right) + I_{p-\frac{1}{2}}\left(\frac{r_0^2}{4t}\right)\right]\right\}$$

where

$$K_n = \frac{2r_0}{n\sqrt{2\pi}} \sin\frac{n\pi}{\alpha}\theta_0 \quad \text{and} \quad p = \frac{n\pi}{2\alpha}.$$

In the above, it has been supposed that the lines forming the boundaries meet at right angles. If the lines do not meet at right angles, a rotation will bring them to the form $x = a$, $y = mx+b$ and under the further transformation $u = x$, $v = -mx+y$ the lines become perpendicular. Under these linear transformations, equation (1) is form invariant with only the constants σ_1, σ_2, and ρ taking on new values. Thus solutions for non-perpendicular boundary lines are the same as those for boundary lines at right angles except the constants are different.

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