Conflict and Strategy in Collective Computation

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Supplementary Information

A. Derivation of PDEs for waiting time and accuracy

$$d\mathbf{X}_t = \mu(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{W}_t$$

$$\text{forward equation }: \frac{\partial p(\mathbf{x}, t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\mu_i(\mathbf{x}, t)p(\mathbf{x}, t)\Big) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}\Big(D_{ij}(\mathbf{x}, t)p(\mathbf{x}, t)\Big)$$

$$\text{where } D_{ij}(\mathbf{x}, t) = \sum_{k=1}^{M} \sigma_{ik}(\mathbf{x}, t)\sigma_{jk}(\mathbf{x}, t))$$

backward equation :

(Section 4.3.6 Stochastic Differential Equations, Multivariable Systems in Gardiner) In the following we will assume we have a set of stochastic differential equation written as

$$d\mathbf{x} = \mathbf{A}(\mathbf{x}, t)dt + \mathrm{B}(\mathbf{x}, t)d\mathbf{W}(t), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, $\mathrm{B} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N$, and $d\mathbf{W}(t)$ is an n variable Weiner process. The forward equation describing the probability density $p(\mathbf{x}, t|\mathbf{x}_0, t_0)$ is

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{N} \frac{\partial\big(A_i(\mathbf{x}, t)p\big)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2\big([\mathrm{B}(\mathbf{x}, t)\mathrm{B}^T(\mathbf{x}, t)]_{ij}p\big)}{\partial x_i \partial x_j} \tag{2}$$

Let $G(\mathbf{x}_0, t)$ be the probability that the particle is still in the region at time t given that it started at $\mathbf{x}_0$ so that

$$G(\mathbf{x}_0, t) = \int_R p(\mathbf{x}, t|\mathbf{x}_0, 0)d\mathbf{x}.$$

If T is the time at which the particle hits the boundary ∂R, then $P(T \geq t|x_0) = G(\mathbf{x}_0, t)$. Therefore

$$
\begin{aligned}
E_{\mathbf{x}_0}(T) &= \int_0^\infty tP(T \in [t, t+dt]|\mathbf{x}_0) \\
&= \int_0^\infty t\Big(P(T \geq t|\mathbf{x}_0) - P(T \geq t+dt|\mathbf{x}_0)\Big) \\
&= \int_0^\infty t\Big(G(\mathbf{x}_0, t) - G(\mathbf{x}_0, t+dt)\Big) \\
&= \int_0^\infty t\frac{G(\mathbf{x}_0, t) - G(\mathbf{x}_0, t+dt)}{dt}dt \\
&= -\int_0^\infty t\frac{\partial G}{\partial t}dt \\
&= -\Big([tG(\mathbf{x}_0, t)]_0^\infty - \int_0^\infty G(\mathbf{x}_0, t)dt\Big) \text{ by integration by parts} \\
&= \int_0^\infty G(\mathbf{x}_0, t)dt \text{ if } \lim_{t \to \infty} G(\mathbf{x}_0, t) = 0 \text{ quickly enough}
\end{aligned}
$$

If we let $f(\mathbf{x}_0) = E_{\mathbf{x}_0}(T)$, then

B.   Initial conditions

C.   Analysis of homogeneous abilities

D.   Explanation for most informative measures Which measure is the most informative about the individuals' true values depends on how accurately pairwise decisions tend to be and what kind of errors are most frequent. When decisions are quite accurate ($w_2 < .4$), the finer measures of consensus—weighted in-degree and eigenvector centrality—can use all of the information in the decision network to make fine distinctions between individuals, leading to consensus scores that are more informative than those from the coarser measures—in-degree and entropy. Whether weighted in-degree or eigenvector centrality is more informative depends on what type of error is common. When there are no waiting costs ($w_2 = 0$) and the individuals value their personal preferences over accuracy ($w_1 < .8$), low valued individuals can sometimes wait out higher valued individuals leading to an incorrect decision being made in favor of a low individual at the expense of a high individual. Since eigenvector centrality takes the identity of the source of decisions into account, it will incorrectly assign the low individual a high score, whereas weighted in-degree will ignore the source and assign the low individual an appropriately low score. Therefore, in these circumstances, weighted in-degree is more informative. When accuracy matters ($w_1 \geq .8$, $w_2 = 0$) or when there are low non-zero waiting costs ($0 < w_2 \leq .3$) low value individual are less likely to wait out high value individuals, but there are a moderate amount of mistakes between pairs of individuals in the middle and bottom of the group. Consider two middle value individuals, with A having slightly higher value than B. Another individual might make a mistake with respect to B so that weighted in-degree gives B a slightly higher score than A. But A and B will tend to decide correctly so that, despite errors, B will send more decisions than A. Since eigenvector centrality is affected by the decisions sent as well as received, it will correctly give A a higher score. Therefore, in these circumstances, eigenvector centrality is the most informative. As waiting costs increase ($.4 \leq w_2 \leq .5$) and decisions become even less accurate, even the high value individuals start to make mistakes. Now consider two high value individuals, with A having slight higher value than B. If A errs and decides strongly in favor of a lower value individual, eigenvector centrality will give A a lower score because of those sent decisions and weighted in-degree will give A a lower score because of the decisions it has prevented itself from receiving. However, A is likely to make at most one more mistake than B so that the number of individuals deciding in favor of either will be the same. Thus the coarseness of unweighted in-degree becomes an advantage by allowing it to group high value individuals together when finer measures would make incorrect fine distinctions. When decisions are even less accurate ($.5 < w_2 < .7$), weighted in-degree, unweighted in-degree, and eigenvector centrality can all get confused by incorrect decisions between any pair of individuals. What becomes especially difficult is distinguishing between low value individuals, one of whom might incorrectly receive decisions from higher individuals. If a low value individual receives correct decisions, they will be of similar strengths from even lower individuals. However, if it receives decisions from a higher individual, even though its score from any of the other measures will be increased, entropy decreases its score because of the difference in the strengths of signals coming from different types of individuals. In these circumstances, entropy does well at measuring the lack of consensus of decisions with respect of low individuals. When decisions are very inaccurate ($w_2 > .6$), weighted in-degree becomes the most reliable.

E. Supplemental Figures

1. Draw values.

2. Difficulty of pairwise decisions depends on difference in values.

3. Nash thresholds.

4. Probability of edge forming each direction and expected time for edge to form.
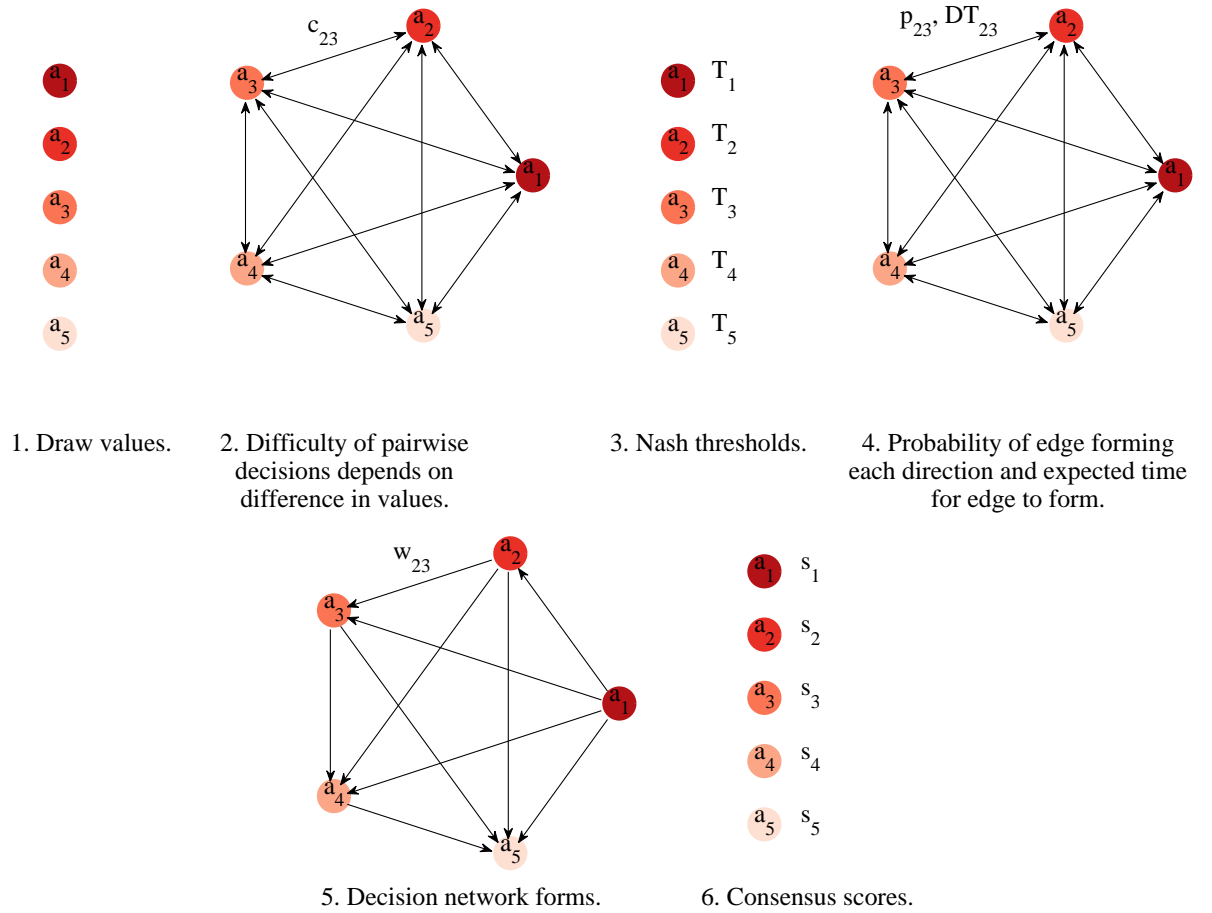
5. Decision network forms.

6. Consensus scores.

Figure S1: Schematic of the development of the decision network. 1. First, each individual's value is drawn from a uniform distribution. 2. The difficulty of each pairwise decision increases with the similarity of values, so that c_{ij} increases as $|a_i - a_j|$ increases. 3. Given these difficulties, each individual has a Nash threshold strategy T_i. 4. Given the set of Nash thresholds, a pair will reach each decision with some probability and in an expected amount of time. 5. We draw the decisions according to the
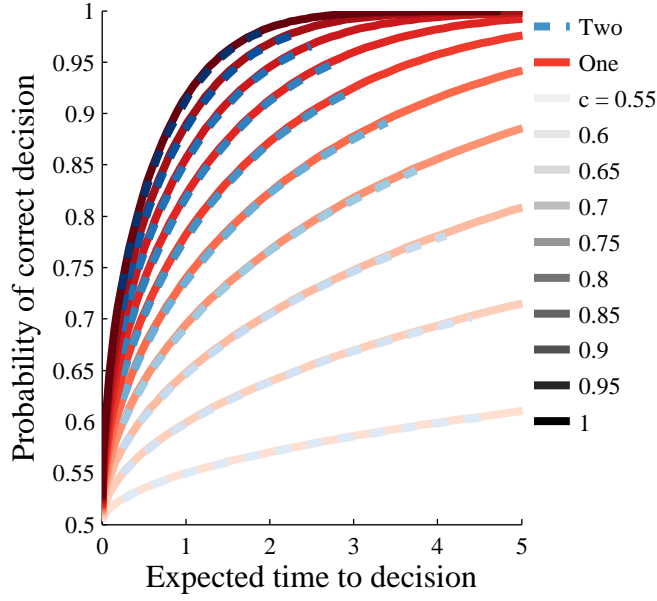
Figure S2: The accuracy of decision-making is the same for the reduced one-dimensional model and the full two-dimensional model. In each version of the model, changing the decision thresholds changes the probability of a correct decision and the expected time to a decision. Here we show the probability of a correct output as a function of the expected time to a decision, as these thresholds are varied. The red lines correspond to the reduced model and the dashed blue lines correspond to the full model. Decisions of different difficulty (c) are represented with lines of different intensity. Parameters: $N = 20$, $b = 1$, $r = 1$, $\ell = 0.1$.
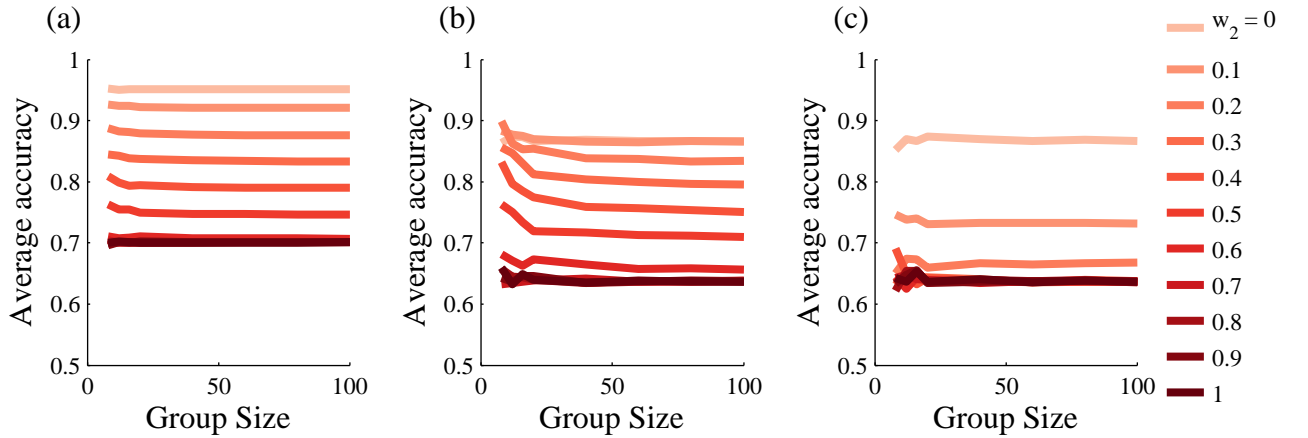

Figure S3: Group size does not affect the accuracy with each pairs of individuals make decisions. In each panel, group size is on the horizontal axis and average accuracy is on the vertical axis. In (a) we show the average accuracy of the whole group, in (b) we show the average accuracy of the top quartile, and in (c) we show the average accuracy of the bottom quartile. Each line corresponds to a different value of w_2. Parameters: $N = 20$, $b = 1$, $r = 1$, $\ell = 0.1$, $w_1 = 0$, $w_3 = 1 - w_2$.
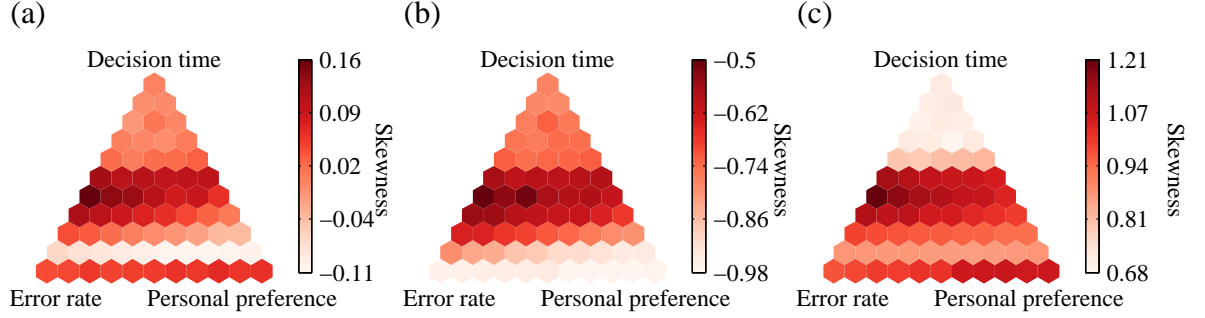
Figure S4: The skewness of the consensus scores is maximized at intermediate waiting costs for every consensus measure. In each panel, the color indicates the average skewness of the consensus scores of a group using Nash thresholds, as a function of the optimization weights, w_1, w_2, w_3. In the lower left corner of the simplex, only error rate matters ($w_1 = 1$). In the upper corner, only decision time matters ($w_2 = 1$). In the lower right corner, only preference matters ($w_3 = 1$). In (a) consensus is given by weighted in-degree, in (b) by entropy, and in (c) by eigenvector centrality. Parameters: $N = 20$, $b = 1$, $r = 1$, $\ell = 0.1$.