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# HOMOGENEOUS GAUSSIAN MARKOV PROCESSES ON GENERAL LATTICES

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**Abstract**

A homogeneous Gaussian Markov lattice-process model has a regression coefficient that determines the extent to which a random variable of a vertex is dependent on those of the neighbors. In many studies, the absolute value of this parameter has been assumed to be less than the reciprocal of the number of neighbors. This condition is shown to be necessary and sufficient for the existence of the Gaussian process satisfying the model equations under some assumptions on lattices using the notion of dual processes. We also give examples of models that neither satisfy the condition imposed on the region for the parameter nor the assumptions on lattices. A formula for autocovariance functions of Gaussian Markov processes on general lattices is derived, and numerical procedures to calculate the autocovariance functions are proposed.

AUTOCOVARIANCE FUNCTIONS; CAR PROCESSES; DUAL PROCESSES; NUMERICAL APPROXIMATION OF THE AUTOCOVARIANCE; ADMISSIBLE REGION FOR PARAMETERS

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## 1. Introduction

Gaussian models on lattices have been utilized to analyze spatial data since the pioneering work by Whittle (1954). A lot of data in the fields of computer graphics, geography and the environment can be modelled in such a way. Homogeneous Gaussian Markov lattice-processes are a simple but rich class of models, useful for this kind of data.

The study of the Gaussian Markov processes on square lattices originated from Rozanov (1967) and Moran (1973a). They are still extensively applied and studied (see Amit and Grenander (1991)). Although square lattices are most frequently used, other lattices are also important to analyze data that are organized on lattices or grids of various meshes on surfaces and spaces. In the present paper we consider Gaussian processes on general lattices such as triangular, square or hexagonal.

A lattice is, intuitively speaking, a graph with a regular pattern. (The precise definition is given in Section 2.) A graph $G$ is defined as a pair $(V, A)$ where $V$ is a set containing at most countably infinite elements and $A$ is a set of pairs of elements of $V$. An element of $V$ and an element of $A$ are called a vertex and an arc,

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respectively. We do not take into consideration directed graphs and graphs in which there is more than one arc between the same two vertices. The vertices $V$ are indicated by integers $i \in \{0, 1, 2, \cdots, p(\text{or } \infty)\}$, where $p+1$ or $\infty$ is the number of the vertices of the lattice. A vertex $j$ is said to be a neighbor of $i$ if $\{i, j\} \in A$. A lattice process $\{x_i : i \in V\}$ is a family of random variables on a probability space $(\Omega, \mathcal{B}, P)$, and is said to be Markov if it satisfies $p(x_i \mid \{x_j : j \neq i\}) = p(x_i \mid \{x_j : \{i, j\} \in A\})$, for all $i$. (Several other different definitions of the Markov property have been proposed. See Moran (1973b).)

From now on, we will only consider homogeneous models. A homogeneous conditional autoregressive (CAR) model $\{x_i\}$ is the Gaussian Markov lattice-process defined by

$$\boldsymbol{E}(x_i \mid x_j; j \neq i) = \sum_{j \text{ nhbr of } i} r x_j$$

$$\text{var}\,(x_i \mid x_j; j \neq i) = \lambda^2 > 0.$$

The simplest example is the CAR model on the one-dimensional lattice defined by

$$\boldsymbol{E}(x_t \mid \{x_t : s \neq t\}) = r x_{t-1} + r x_{t+1}, \qquad \text{var}\,(x_t \mid \{x_t : s \neq t\}) = \lambda^2 > 0.$$

This is simply the first-order autoregressive model, widely known in the field of time series analysis, defined by $x_{t+1} = \alpha x_t + \xi_{t+1}$, $\xi_{t+1} \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2)$, where $\lambda^2 = \sigma^2/(1+\alpha^2)$, $r = \alpha/(1+\alpha^2)$. The regression coefficient $r$ of a CAR model determines the dependence of a vertex's random variable on those of the vertex's neighbors. In many studies, it has been assumed that $|r|$ is less than the reciprocal of the number of neighbors, $\mathcal{N}$. Moran (1973a) constructed a Gaussian Markov process on a square lattice and proved that the condition $|r| < 1/\mathcal{N}$ is sufficient for the existence of the process satisfying the CAR model equations. The necessity of this condition is shown in Moran (1973b). However, there has been no proof that this condition $|r| < 1/\mathcal{N}$ for CAR models on general lattices is necessary and sufficient (see Ripley (1988), p. 11). Our discussion provides a simple method to construct homogeneous Gaussian Markov processes on general lattices, and several results concerning the condition $|r| < 1/\mathcal{N}$ are proved using the notion of dual processes. In particular, if the lattice is a bipartite graph and satisfies another assumption, that the area of the boundary of a ball with radius $l$ grows less exponentially as $l$ goes to infinity, it is proved that $|r| < 1/\mathcal{N}$ is a necessary and sufficient condition for the existence of the process satisfying the CAR model equations. The latter assumption on the boundary growth of a ball in a lattice is satisfied by almost all widely used lattices. Furthermore, we will give examples of models that satisfy neither the condition imposed on the parameter range nor the assumptions on the lattice. A formula for autocovariance functions of CAR processes on general lattices will also be obtained. Practical procedures to calculate them are proposed in the following sections.

In Section 2, notions concerning lattices are prepared, and some basic Gaussian lattice-process models are described. In Section 3, the notion of dual processes, playing a fundamental role in the following discussion, is defined. In Section 4, the results concerning the condition $|r| < 1/\mathcal{N}$ are proved. In Section 5, the formula of autocovariance functions of the Gaussian Markov models on general lattices are derived, and practical methods to calculate autocovariance functions are proposed in Section 6.

## 2. Gaussian lattice-process models

A lattice $L = (V, A, T)$ is defined as a graph $(V, A)$ accompanied by a transformation group $T$ of $V$. Each element of $T$ is a one-to-one mapping of the set $V$ onto itself. A mapping $t \in T$ is assumed to satisfy the following conditions:

(1) $\forall i, j \in V, \quad \exists t \in T$ such that $t(i) = j$;

(2) $\forall \{i, j\}, \{k, l\} \in A, \quad \exists t \in T$ such that $\{t(i), t(j)\} = \{k, l\}$;

(3) $\forall t \in T, \forall \{i, j\} \in A, \quad \{t(i), t(j)\} \in A$;

(4) $\forall t \in T, \forall \{i, j\} \notin A, \quad \{t(i), t(j)\} \notin A$.

The number of neighbors $\mathcal{N}$ of a vertex is assumed to be finite. The number of neighbors of each vertex is the same by the assumptions (1)–(4). Some examples of lattices are square lattices, cubic lattices, triangular lattices, honeycomb lattices, kagome lattices, complete graphs, regular polygons and Bethe lattices (for graphs, see Harary (1971); for triangular and honeycomb lattices, see Horiguchi (1972); Bethe lattices are described in Example 4.2).

A stochastic process $\{x_i\}$ on a lattice is defined to be ($T$-)homogeneous if $\forall t \in T$, $\forall i_1, i_2, \cdots, i_k \in V$, $p(x_{t(i_1)}, x_{t(i_2)}, \cdots, x_{t(i_k)}) = p(x_{i_1}, x_{i_2}, \cdots, x_{i_k})$ holds. When a process on the rectangular lattice is considered, $T$-homogeneity implies that the process is stationary and isotropic. Many of the results described below can obviously be extended to anisotropic models by weakening conditions (1)–(4), while for simplicity only $T$-homogeneous models are taken into consideration in the present paper. For example, the results on the square lattice models can be extended to those on rectangular lattice models by considering the parallel transformation group $\tilde{T}$ that does not satisfy condition (2).

In the following, some basic Gaussian models are defined. For details of these models, see Besag (1974, 1975), Ripley (1981), and Cressie (1991).

If $\{x_i\}$ is a homogeneous Gaussian Markov process, the conditional distribution of $x_i$ relative to $\{x_j : j \neq i\}$ is a normal distribution and the conditional expectation and

variance can be expressed in the forms

$$\boldsymbol{E}(x_i \mid \{x_j : j \neq i\}) = \sum_{j:\{i,j\} \in A} r x_j \tag{5}$$

$$\operatorname{var}(x_i \mid \{x_j : j \neq i\}) = \lambda^2, \qquad \lambda^2 > 0,$$

where the mean value of the process is assumed to be zero without loss of generality. This model with parameters $r$ and $\lambda^2$ is CAR.

Now, we define

$$\varepsilon^i = x_i - \sum_{j:\{j,i\} \in A} r x_j. \tag{6}$$

Then, $\{\varepsilon^i\}$ is also a homogeneous Gaussian process and satisfies

$$\boldsymbol{E}(\varepsilon^i) = 0, \qquad \text{for all } i,$$

$$\operatorname{cov}(\varepsilon^i \varepsilon^j) = \begin{cases} \lambda^2, & \text{for } i = j, \\ -r\lambda^2, & \text{for } \{i, j\} \in A, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $\lambda^2 > 0$. To construct lattice-process models, covariance structures such as (7) are sometimes given directly. Gaussian models constructed in this way are called direct covariance (DC) models in Kiiveri and Campbell (1989).

## 3. Dual processes of Gaussian lattice processes

In this section, we define the dual processes of Gaussian lattice processes. In Rozanov (1981), the corresponding notion for processes on continuous domains is defined.

3.1. *Definition and some properties.* An innovation process cannot be naturally defined for a lattice process because, in general, a lattice does not have a natural specific direction. We consider the dual process defined below instead of the innovation process. Let $\overline{\operatorname{sp}}\{x_i\}$ be the closed span of a Gaussian lattice process $\{x_i\}$, that is the smallest closed subspace of the Hilbert space $L^2(\Omega, \mathcal{B}, P)$. (In the following, we consider real Hilbert spaces.)

*Definition.* Suppose $\{x_i\}$ be a Gaussian lattice process. Let $\varepsilon^i = x_i - \boldsymbol{E}(x_i \mid \{x_j : j \neq i\})$. If

$$\overline{\operatorname{sp}}\{x_i\} = \overline{\operatorname{sp}}\{\varepsilon^i\}, \tag{8}$$

then the process $\{c\varepsilon^i\}$, where $c \neq 0$ is an arbitrary constant, is said to be the dual process of $\{x_i\}$.

Let $\langle u, v \rangle$ denote $\operatorname{cov}(u, v)$. If a lattice process $\{x_i\}$ is homogeneous, we have the relation $\langle \varepsilon^i, x_j \rangle = \lambda^2 \delta^i_j$, where $\lambda^2 = \operatorname{var}(\varepsilon^i)$ and $\delta^i_j$ denotes the Kronecker delta. Let $\tau^2$ denote $\operatorname{var}(x_i)$. If $\{\varepsilon^i\}$ is the dual process of $\{x_i\}$, then $(\lambda^2/\tau^2)x_i = \varepsilon^i - \boldsymbol{E}(\varepsilon^i \mid \{\varepsilon^j : j \neq i\})$

because of the relations $\langle [\varepsilon^i - (\lambda^2/\tau^2)x_i],\ (\lambda^2/\tau^2)x_i\rangle = 0$, $\langle \varepsilon^j,\ (\lambda^2/\tau^2)x_i\rangle = 0$, for $i \neq j$.

We shall now prove the following theorem on the relation between CAR models and their dual processes.

*Theorem* 3.1. There exists a Gaussian process $\{\bar{x}_i\}$ satisfying the model Equations (5) if and only if there exists a Gaussian process $\{\varepsilon_i\}$ that satisfies Equation (7) and the condition $\varepsilon^i - \boldsymbol{E}(\varepsilon^i \mid \{\varepsilon^j : j \neq i\}) \neq 0$. Furthermore, in this case there exists $\{x_i\}$ that is the dual process of $\{\varepsilon^i\}$ and it satisfies (5).

*Proof.* It has been already shown that there exists a Gaussian process $\{\varepsilon_i\}$ satisfying equation (7) if there exists a Gaussian process $\{\bar{x}_i\}$ satisfying the model Equations (5). Since $\langle \varepsilon^0, \bar{x}_0\rangle = \lambda^2 > 0$ and $\langle \varepsilon^i, \bar{x}_0\rangle = 0$ for $i \neq 0$, $\varepsilon^0$ is not included in $\overline{\mathrm{sp}}\{\varepsilon^i : i \neq 0\}$.

Assume that there exists a Gaussian process $\{\varepsilon^i\}$ satisfying Equation (7) and the condition $\varepsilon^i - \boldsymbol{E}(\varepsilon^i \mid \{\varepsilon^j : j \neq i\}) \neq 0$. Put $\tilde{x}_i = \varepsilon^i - \boldsymbol{E}(\varepsilon^i \mid \{\varepsilon^j : j \neq i\})$ and $\tilde{\tau}^2 = \boldsymbol{E}(\tilde{x}_i^2)$. Let $\tilde{\varepsilon}^0 = \bar{x}_0 - r\sum_{j:\{j,0\}\in A} \tilde{x}_j$. Then the following relations hold:

$$\langle \tilde{\varepsilon}^0, \varepsilon^0\rangle = \tilde{\tau}^2 = \frac{\tilde{\tau}^2}{\lambda^2}\langle \varepsilon^0, \varepsilon^0\rangle,$$

$$\langle \tilde{\varepsilon}^0, \varepsilon^i\rangle = -r\tilde{\tau}^2 = \frac{\tilde{\tau}^2}{\lambda^2}\langle \varepsilon^0, \varepsilon^i\rangle \qquad \text{for } i \text{ such that } \{0, i\} \in A,$$

$$\langle \tilde{\varepsilon}^0, \varepsilon^i\rangle = 0 = \frac{\tilde{\tau}^2}{\lambda^2}\langle \varepsilon^0, \varepsilon^i\rangle \qquad \text{for } i \neq 0 \text{ such that } \{0, i\} \notin A.$$

Thus, we obtain $\tilde{\varepsilon}^0 = (\tilde{\tau}^2/\lambda^2)\varepsilon^0$. Hence $\langle \tilde{\varepsilon}^0, \tilde{x}_j\rangle = 0$ for all $j \neq 0$ such that $\{0, j\} \in A$. Therefore, the process $\{(\lambda^2/\tilde{\tau}^2)\tilde{x}_i\}$ forms a Markov process and satisfies the model Equations (5). Further, $\{(\lambda^2/\tilde{\tau}^2)\tilde{x}_i\}$ is the dual process of $\{\varepsilon^i\}$.

3.2. *Dual processes on the one-dimensional lattice.* A stochastic model on the one-dimensional lattice is simply a time series model. In this section, dual processes of time series models are considered in order to clarify the meanings of the conditions $\lambda^2 = \mathrm{var}\,(\varepsilon^t) > 0$ and $\overline{\mathrm{sp}}\{x_t\} = \overline{\mathrm{sp}}\{\varepsilon^t\}$. The meanings of these conditions are similar to the conditions of non-deterministic and purely non-deterministic time series, when we consider them in the context not of prediction but of interpolation. A time series $\{x_t\}$ is said to be non-deterministic if $\sigma^2 = \mathrm{var}\,(x_t - \boldsymbol{E}(x_t \mid \{x_s : s < t\})) > 0$ (i.e. $x_t \notin \overline{\mathrm{sp}}\{x_s : s < t\}$), and is said to be purely deterministic if $\sigma^2 = 0$. If $x_t \notin \overline{\mathrm{sp}}\{x_s : s < t\}$, by Wold's decomposition, $x_t$ can be expressed as $x_t = \sum_{i=0}^{\infty} a_i \xi_{t-i} + v_t$, where $\xi_t = x_t - \mathrm{P}\overline{\mathrm{sp}}\{x_s : s < t\}x_t$ and $v_t$ is purely deterministic. If $v_t = 0$, the process $\{x_t\}$ is said to be purely non-deterministic (for details, see Hannan (1970), p. 136). Hence, $\overline{\mathrm{sp}}\{x_s : s < t\} = \overline{\mathrm{sp}}\{\xi_s : s < t\}$ if and only if a time series $\{x_t\}$ is purely non-deterministic. These conditions can be expressed in terms of the spectral

distribution function $F(\theta)$ of the process $\{x_t\}$. The spectral density of the dual process $\{\varepsilon^t\}$ can also be expressed by using $F(\theta)$.

In the time series context, $\varepsilon^t = x_t - \boldsymbol{E}(x_t \mid \{x_s : s \neq t\})$ is simply the error of interpolation when $x_t$ is not observed and the other $x_s$ are observed, and $\lambda^2 = \mathrm{var}(\varepsilon^t)$ is the mean square error of the interpolation. It is known that $\lambda^2 > 0$ if and only if $1/F'(\theta)$ is integrable, where $F'(\theta)$ is the Radon–Nikodym derivative of the absolutely continuous part of $F(\theta)$ (see Grenander and Rosenblatt (1957), Hannan (1970)). If it is integrable, then

$$\lambda^2 = \frac{4\pi^2}{\int_{-\pi}^{\pi} \frac{1}{F'(\theta)} d\theta}.$$

For the condition $\overline{\mathrm{sp}}\{x_t\} = \overline{\mathrm{sp}}\{\varepsilon^t\}$, the following result can easily be proved.

*Proposition* 3.1. The equality $\overline{\mathrm{sp}}\{x_t\} = \overline{\mathrm{sp}}\{\varepsilon^t\}$ holds if and only if the spectral function of $\{x_t\}$ has a density $f(\theta)$ and $1/f(\theta)$ is integrable. If this condition is satisfied, the dual process $\{\varepsilon^t = x_t - \boldsymbol{E}(x_t \mid \{x_s : s \neq t\})\}$ has the spectral density

$$\left( \frac{2\pi}{\int_{-\pi}^{\pi} \frac{1}{f(\theta)} d\theta} \right)^2 \frac{1}{f(\theta)}.$$

*Proof.* By the isomorphism between $\overline{\mathrm{sp}}\{x_t\}$ and $L^2(dF)$, $x_t$ and $\varepsilon^t$ correspond to $e^{it\theta}$ and $\kappa e^{it\theta}/f(\theta)$, where we put

$$\kappa = \frac{2\pi}{\int_{-\pi}^{\pi} \frac{1}{f(\theta)} d\theta}$$

(see Grenander and Rosenblatt (1957), Hannan (1970)). If the spectral function $F(\theta)$ has a singular part, then $\overline{\mathrm{sp}}\{x_t\} \supsetneq \overline{\mathrm{sp}}\{\varepsilon_t\}$ and (8) is not satisfied. Assume the spectral function $F(\theta)$ is absolutely continuous. If $1/f(\theta)$ is not integrable, $\varepsilon^t = 0$ for all $t$. If $1/f(\theta)$ is integrable, $x_t$ is included in $\overline{\mathrm{sp}}\{\varepsilon^t\}$. Therefore, $\overline{\mathrm{sp}}\{x_t\} = \overline{\mathrm{sp}}\{\varepsilon^t\}$ if and only if the spectral function is absoutely continuous and $1/f(\theta)$ is integrable.

The autocovariance function of $\{\varepsilon^t\}$ is

$$\boldsymbol{E}(\varepsilon^{t+h}\varepsilon^t) = \int_{-\pi}^{\pi} e^{i(t+h)\theta} \frac{\kappa}{f(\theta)} e^{-it\theta} \frac{\kappa}{f(\theta)} f(\theta)\, d\theta = \int_{-\pi}^{\pi} e^{ih\theta} \frac{\kappa^2}{f(\theta)} d\theta.$$

Hence, the spectral density is

$$\frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\theta} \int_{-\pi}^{\pi} e^{ih\bar{\theta}} \frac{\kappa^2}{f(\bar{\theta})} d\bar{\theta} = \left( \frac{2\pi}{\int_{-\pi}^{\pi} \frac{1}{f(\bar{\theta})} d\bar{\theta}} \right)^2 \frac{1}{f(\theta)}.$$

*Remark.* By Proposition 3.1, the dual process of an ARMA $(p, q)$ process is an ARMA $(q, p)$ process.

## 4. Existence of the Gaussian processes satisfying the model equations

In this section, we shall prove some results concerning the condition $|r| < 1/\mathcal{N}$, which has been assumed in many studies.

Before stating the following theorem, we give some terms and notations. The distance $\rho(i, j)$ between the vertices $i$ and $j$ is defined by:

$$\rho(i, j) = \min \{n \mid n \text{ is the number of arcs in a path connecting } i \text{ and } j\}.$$

We define a ball $B_l(i)$ and a surface of it $\partial B_l(i)$ as follows:

$$B_l(i) = \{j : \rho(i, j) \leqq l\},$$

$$\partial B_l(i) = \{j : \rho(i, j) = l\}.$$

A graph is said to be bipartite if its vertices can be divided into two disjoint sets, where every arc has one of its vertices in one set and its other vertex in the other set. In the following, an infinite sum of random variables such as $\sum_{i=0}^{\infty} a^i x_i$ denotes $\text{l.i.m.}_{n\to\infty} \sum_{i=0}^{n} a^i x_i$, where l.i.m. denotes limit in mean square of the sequence of random variables; $\log 0$ is defined as $-\infty$; $\#(\cdot)$ denotes the number of the elements of a set.

Now, we shall prove the theorem.

*Theorem* 4.1. (i) If $|r| < 1/\mathcal{N}$, then there exists a unique homogeneous Gaussian lattice process $\{x_i\}$ satisfying the model Equations (5). In this case, the equality $\overline{\text{sp}}\{x_i\} = \overline{\text{sp}}\{\varepsilon^i\}$ holds.

(ii) Suppose $\limsup_{l\to\infty} \log \#\{\partial B_l(x_i)\}/l \leqq 0$. Then the condition $r < 1/\mathcal{N}$ is necessary for a Gaussian lattice process $\{x_i\}$ satisfying the model Equations (5) to exist.

(iii) Suppose that the lattice is a bipartite graph and $\limsup_{l\to\infty} \log \#\{\partial B_l(x_i)\}/l \leqq 0$. Then the condition $|r| < 1/\mathcal{N}$ is necessary and sufficient for a Gaussian lattice process $\{x_i\}$ satisfying the model Equations (5) to exist.

No proof that the condition $|r| < 1/\mathcal{N}$ is necessary and sufficient for the existence of the process has been available (Ripley (1988), p. 11). This theorem shows this under the conditions assumed in (iii). It is proved in Moran (1973a) that the condition $r < 1/\mathcal{N}$ is sufficient for the process on the square lattice to exist. The necessity of this condition is shown in Moran (1973b). It can be seen from Theorem 4.1 that the condition $r < 1/\mathcal{N}$ is necessary and sufficient for this process on the square lattice to exist since the square lattice is a bipartite graph and the condition $\limsup_{l\to\infty} \log \#\{\partial B_l(x_i)\}/l = 0$ holds. There are counterexamples that satisfy the model equations and do not satisfy the assumptions in (iii) (see Examples 4.1, 4.2 below). Intuitively speaking, the condition on the boundary growth of a ball is

required because the effect of the boundary cannot be ignored if it is not satisfied. The growth of the boundary area $\#\{\partial B_l(x_i)\}$ is the same order as that of the ball's volume $\#\{B_l(x_i)\}$ if $\#\{\partial B_l(x_i)\}$ increases exponentially as the radius $l$ increases.

*Proof of Theorem* 4.1. (i) By Theorem 3.1, a Gaussian process $\{x_i\}$ satisfying (5) exists if and only if a Gaussian process $\{\varepsilon^i\}$ satisfying (7) exists.

If $|r| < 1/\mathcal{N}$, we can construct a process $\{\varepsilon^i\}$ satisfying (7) as follows:

$$\varepsilon^i = \begin{cases} \lambda\left[\sqrt{1-\mathcal{N}r}\, Z(i) + \sum_{j:\{j,i\}\in A} \sqrt{r}\,\mathrm{sign}\,(i-j) Z(\{i,j\})\right], & \text{for } r > 0 \\ \lambda\left[\sqrt{1-\mathcal{N}|r|}\, Z(i) + \sum_{j:\{j,i\}\in A} \sqrt{|r|}\, Z(\{i,j\})\right], & \text{for } r \leqq 0, \end{cases}$$

where sign $(\cdot)$ denotes a function defined by sign $(x) = 1$ for $x \geqq 0$ and sign $(x) = -1$ for $x < 0$, and $Z(i)$ and $Z(\{i,j\})$ follow independently $N(0,1)$. Therefore, the condition $|r| < 1/\mathcal{N}$ is sufficient for the lattice process $\{x_i\}$ to exist.

If a Gaussian process $\{\bar{x}_i\}$ satisfies (5), the process $\{\varepsilon^i = \bar{x}^i - \boldsymbol{E}(\bar{x}_i \mid \{\bar{x}_j : j \neq i\})\}$ satisfies (7). From Theorem 3.1, we can construct a Gaussian process $\{x_i\}$ that is the dual process of $\{\varepsilon^i\}$ and satisfies (5). Relation (7) uniquely determines the distribution of the process $\{\varepsilon^i\}$. Therefore, to prove that the process satisfying (5) is unique and the equality $\overline{\mathrm{sp}}\,\{x_i\} = \langle \overline{\mathrm{sp}} \rangle \{\varepsilon^i\}$ holds, it is sufficient to show that $\bar{x}_i = x_i$ almost surely for all $i$ if a process $\{\bar{x}_i\}$ satisfies (5).

Let $\{0, i\} \in A$ for $i = 1, 2, \cdots, \mathcal{N}$ without loss of generality. Then from (5), $x_0 = r(x_1 + \cdots + x_{\mathcal{N}}) + \varepsilon^0$, $\bar{x}_0 = r(\bar{x}_1 + \cdots + \bar{x}_{\mathcal{N}}) + \varepsilon^0$. Put $z_i = x_i - \bar{x}_i$. Then $\{z_i\}$ is also a homogeneous process with mean zero. We have $z_0 = r(z_1 + \cdots + z_{\mathcal{N}})$. Thus, $\boldsymbol{E}(z_0^2) = \boldsymbol{E}(\{r(z_1 + \cdots + z_{\mathcal{N}})\}^2)$. Put $v^2 = \boldsymbol{E}(z_i^2)$, which does not depend on $i$. Since $|\boldsymbol{E}(z_i z_j)| < v^2$, $v^2 \leqq r^2 \mathcal{N}^2 v^2$. Since $|r| < 1/\mathcal{N}$, $v$ must be zero. Hence, $\bar{x}_i = x_i$ almost surely for all $i$.

(ii) Assume $r \geqq 1/\mathcal{N}$ and a homogeneous Gaussian lattice process satisfying (5) exists. The condition $\limsup_{l\to\infty} \log \#\{\partial B_l(x_i)\}/l \leqq 0$ is equivalent to the condition $\lim_{l\to\infty} \#\{\partial B_l(x_i)\}\mathrm{e}^{-\alpha l} = 0$ for all $\alpha > 0$. Since $\lim_{l\to\infty} \#\{\partial B_l(x_i)\}\mathrm{e}^{-\alpha l} = 0$ for all $\alpha > 0$,

$$\sum_{i=0}^{\infty} \mathrm{e}^{-\alpha\rho(0,i)} = \sum_{l=0}^{\infty} \mathrm{e}^{-\alpha l} \#\{\partial B_l(x_0)\}$$

$$\leqq K \sum_{l=0}^{\infty} \mathrm{e}^{-\alpha l} \mathrm{e}^{-\frac{1}{2}\alpha}$$

$$= \frac{K}{1 - \mathrm{e}^{-\frac{1}{2}\alpha}} < \infty,$$

where $K>0$ is an appropriate constant. Hence, the numerical sequence $\{e^{-\alpha\rho(0,i)}\}$ is included in $l^1$. Thus, $\sum_{i=0}^{\infty} e^{-\alpha\rho(0,i)}\varepsilon^i \in \overline{\mathrm{sp}}\{x_i\}$ for all $\alpha>0$. From (7),

$$\left\langle \sum_{i=0}^{\infty} e^{-\alpha\rho(0,i)}\varepsilon^i, \sum_{j=0}^{\infty} e^{-\alpha\rho(0,j)}\varepsilon^j \right\rangle = \sum_{\{i,j\}\in A} \lambda^2\left(\frac{1}{\mathcal{N}}(e^{-\alpha\rho(0,i)})^2 \frac{1}{\mathcal{N}}(e^{-\alpha\rho(0,j)})^2 - 2re^{-\alpha\rho(0,i)}e^{-\alpha\rho(0,j)}\right)$$

$$\leqq \sum_{\{i,j\}\in A} \frac{\lambda^2}{\mathcal{N}}(e^{-\alpha\rho(0,i)} - e^{-\alpha\rho(0,j)})^2$$

$$\leqq \frac{\lambda^2}{\mathcal{N}} \sum_{l=0}^{\infty} \mathcal{N}\{e^{-\alpha l} - e^{-\alpha(l+1)}\}^2 \#\{\partial B_l(x_0)\}$$

$$\leqq \bar{K}\lambda^2 \sum_{l=0}^{\infty} e^{-2\alpha l}(1-e^{-\alpha})^2 e^{\alpha l}$$

$$= \bar{K}\lambda^2(1-e^{-\alpha}),$$

where $\bar{K}>0$ is an appropriate constant. The right-hand side tends to zero as $\alpha\to 0$. Thus, because $e^{-\alpha\rho(0,0)}=1$, $\langle \varepsilon^0 - \boldsymbol{E}(\varepsilon^0 \mid \{\varepsilon^i : i\neq 0\}),\ \varepsilon^0 - \boldsymbol{E}(\varepsilon^0 \mid \{\varepsilon^i : i\neq 0\})\rangle \leqq 0$. This contradicts Theorem 3.1. Therefore, the condition $r<1/\mathcal{N}$ is necessary for a homogeneous Gaussian lattice process satisfying the model equations (5) to exist.

(iii) When the lattice is a bipartite graph, the vertices can be divided into two disjoint sets, as described before the theorem. We adopt a numbering of the vertices such that if $i$ is odd the vertex $i$ is included in one set and if $i$ is even it is included in the other set. It is sufficient to prove that $r>-1/\mathcal{N}$ is necessary for a homogeneous Gaussian lattice process satisfying the model Equations (5) to exist. We can prove this in the same way that (ii) was proven, by considering a vector $\sum_{i=0}^{\infty}(-1)^i e^{-\alpha\rho(0,i)}\varepsilon^i \in \overline{\mathrm{sp}}\{x_i\}$.

Most commonly used lattices satisfy the condition: $\limsup_{l\to\infty} \log \#\{\partial B_l(x_i)\}/l \leqq 0$, which is assumed in (ii) and (iii) of Theorem 4.1.

In the following, some homogeneous Gaussian lattice processes that do not satisfy the conditions in (iii) of Theorem 4.1 are exhibited.

*Example* 4.1. *Regular triangles.* A triangle is not a bipartite graph. The covariance matrix $G$ corresponding to a regular triangle with parameters $r$ and $\lambda^2$ is determined by

$$\frac{1}{\lambda^2}G^{-1} = \begin{pmatrix} 1 & -r & -r \\ -r & 1 & -r \\ -r & -r & 1 \end{pmatrix}.$$

The process satisfying (5) exists if and only if $G$ is positive definite. Since the eigenvalues of the matrix on the right-hand side are $r+1$ (multiple root) and $1-2r$, this matrix is positive definite if and only if $-1<r<1/2$. Thus, there exists a

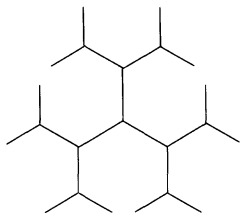

Figure 1. The Bethe lattice with $\mathcal{N} = 3$.

process satisfying the model equations with a parameter value of $r$ that does not satisfy $-1/2 < r < 1/2$.

*Example* 4.2. *Bethe lattices.* Bethe lattices (or Cayley trees) are lattices in which each vertex has $\mathcal{N}$ neighbors and there are no closed loops (see Figure 1). These lattices do not satisfy the condition $\limsup_{l\to\infty} \log \#\{\partial B_l(x_i)\}/l \leqq 0$ assumed in (ii) and (iii) of Theorem 3.1. A homogeneous Gaussian Markov model with the parameter range

$$-\frac{1}{2\sqrt{\mathcal{N}-1}} < r < \frac{1}{2\sqrt{\mathcal{N}-1}}$$

can be constructed on a Bethe lattice as follows.

Let $p$ denote the larger solution of the equation $p(1-p) = (\mathcal{N}-1)r^2$. Put

$$\frac{\varepsilon^0}{\lambda} = \sqrt{\frac{\mathcal{N}p-1}{\mathcal{N}-1}}\, Z_0 - \frac{r}{|r|} \sum_{k:0\to k} \sqrt{\frac{1-p}{\mathcal{N}-1}}\, Z(\{0,k\})$$

$$\frac{\varepsilon^i}{\lambda} = \sqrt{p}\, Z(\{j,i\}: j\to i) - \frac{r}{|r|} \sum_{k:i\to k} \sqrt{\frac{1-p}{\mathcal{N}-1}}\, Z(\{i,k\}), \qquad \text{for } i \neq 0,$$

where $Z_0$ and $Z(\{j,k\})$ are random variables that independently follow $N(0,1)$, and correspond to the vertex 0 and the arc $\{j,k\}$; the notation $i \to j$ denotes the ordered pair satisfying both $\{i,j\} \in A$ and $\rho(0,i) < \rho(0,j)$. Then the following relation holds:

$$\langle \varepsilon^i, \varepsilon^j \rangle = \begin{cases} \lambda^2, & \text{for } i = j, \\ -\dfrac{r}{|r|}\lambda^2 \sqrt{\dfrac{p(p-1)}{\mathcal{N}-1}} = -r\lambda^2, & \text{for } \{i,j\} \in A, \\ 0, & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 3.1, we obtain the desired process $\{x_i\}$ by putting $x_i = \varepsilon^i - \boldsymbol{E}(\varepsilon^i \mid \{\varepsilon^j : j \neq i\})$. In particular,

$$\begin{aligned} x_0 &= \varepsilon^0 - \boldsymbol{E}(\varepsilon^0 \mid \{\varepsilon^i : i \neq 0\}) \\ &= \varepsilon^0 + \sum_{l=1}^{\infty} \left\{ \left( \frac{r}{|r|} \sqrt{\frac{1-p}{p(\mathcal{N}-1)}} \right)^l \sum_{i:\rho(0,i)=l} \varepsilon^i \right\} \\ &= \sqrt{\frac{\mathcal{N}p-1}{\mathcal{N}-1}}\, Z_0. \end{aligned}$$

Thus,

$$\langle x_i, x_i \rangle = \frac{\mathcal{N}p - 1}{\mathcal{N} - 1}.$$

Let $\{x_i\}$ be the Gaussian Markov process on a Bethe lattice with $r = 1/\mathcal{N}$ and $u$ be a Gaussian random variable with a mean of zero. The random variables $\{x_i\}$ and $u$ are assumed to be independent. Then the Gaussian process $\{\bar{x}_i = x_i + u\}$ also satisfies the model equation (5). So a homogeneous Gaussian lattice process satisfying the model equations is not unique.

## 5. Derivation of autocovariance functions

Moran (1973a) obtained the autocovariance functions of Gaussian Markov processes on the two-dimensional square lattice. In his paper, the processes on the square lattice are defined as the limits of those on a torus and investigated by the frequency domain approach. Besag (1981) obtained the autocovariance functions of the models on the two-dimensional rectangular lattice by a more direct way using recurrence equations. Our approach is closer to that of Besag than of Moran. We obtain the autocovariance functions of Gaussian Markov processes on general lattices using dual processes. An intuitively appealing formula of autocovariance functions of the models on general lattices is given.

From now on, we assume the condition $|r| < 1/\mathcal{N}$ and denote $\overline{\mathrm{sp}}\{x_i\} = \overline{\mathrm{sp}}\{\varepsilon^i\}$ as $H$.

First, we prove the following theorem on the convergence of a series in $H$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on a linear space $X$ are said to be equivalent if there exist positive constants $\delta$ and $\delta'$ such that $\delta \|a\|_2 \leqq \|a\|_1 \leqq \delta' \|a\|_2$ for all $a \in X$.

*Theorem* 5.1. Suppose $|r| < 1/\mathcal{N}$. Every element $a \in H$ can be uniquely expressed in the forms $\sum_{i=0}^{\infty} a^i x_i$ and $\sum_{i=0}^{\infty} a_i \varepsilon^i$ with $\sum_{i=0}^{\infty} (a^i)^2 < \infty$ and $\sum_{i=0}^{\infty} (a_i)^2 < \infty$. Conversely, for all numerical sequences $a(i)$ such that $\sum_{i=0}^{\infty} \{a(i)\}^2 < \infty$, there exist $\sum_{i=0}^{\infty} a(i) x_i$ and $\sum_{i=0}^{\infty} a(i) \varepsilon^i$ in $H$. Further, the three norms defined by $\|a\|_H = \sqrt{\langle a, a \rangle}$, $\|a\|_2^{(\varepsilon)} = \sqrt{\sum_{i=0}^{\infty} (a_i)^2}$, $\|a\|_2^{(x)} = \sqrt{\sum_{i=0}^{\infty} (a^i)^2}$ are equivalent.

*Proof.* For a sequence $\{a(i)\}$ with $a(i) = 0$ for $i < n_1$ or $i > n_2$,

$$\begin{aligned}
\left\langle \sum_{i=n_1}^{n_2} a(i)\varepsilon^i, \sum_{j=n_1}^{n_2} a(j)\varepsilon^j \right\rangle &= \lambda^2 \sum_{\{i,j\} \in A} \left\{ \frac{1}{\mathcal{N}} a(i)^2 + \frac{1}{\mathcal{N}} a(j)^2 - 2r a(i) a(j) \right\} \\
&\leqq \frac{\lambda^2}{\mathcal{N}} \sum_{\{i,j\} \in A} \{a(i)^2 + a(j)^2 + \mathcal{N}|r|\, a(i)^2 + \mathcal{N}|r|\, a(j)^2\} \\
&= \lambda^2 \frac{1 + \mathcal{N}|r|}{\mathcal{N}} \sum_{\{i,j\} \in A} \{a(i)^2 + a(j)^2\} \\
&= \lambda^2 (1 + \mathcal{N}|r|) \sum_{i=n_1}^{n_2} \{a(i)\}^2.
\end{aligned} \tag{9}$$

From (9), if $\sum_{i=0}^{\infty}\{a(i)\}^2<\infty$, $\sum_{i=0}^{n} a_i\varepsilon^i$ for $n=1, 2, 3, \cdots$ forms a Cauchy sequence in $H$. Thus $\sum_{i=0}^{\infty} a(i)\varepsilon^i \in H$ exists. Hence, $\{\sum_{i=0}^{\infty} a(i)\varepsilon^i : \sum_{i=0}^{\infty}\{a(i)\}^2<\infty\} \subseteq H$. Similar to the derivation of (9), the following inequality is obtained:

$$\left\langle \sum_{i=n_1}^{n_2} a(i)\varepsilon^i, \sum_{j=n_1}^{n_2} a(j)\varepsilon^j \right\rangle \geqq \lambda^2(1-\mathcal{N}|r|)\sum_{i=n_1}^{n_2}\{a(i)\}^2.$$

Hence,

$$\lambda^2(1-\mathcal{N}|r|)\sum_{i=0}^{\infty}\{a(i)\}^2 \leqq \left\langle \sum_{i=0}^{\infty} a(i)\varepsilon^i, \sum_{j=0}^{\infty} a(j)\varepsilon^j \right\rangle \leqq \lambda^2(1+\mathcal{N}|r|)\sum_{i=0}^{\infty}\{a(i)\}^2.$$

Thus, under the condition $|r|<1/\mathcal{N}$, the linear space $\{\sum_{i=0}^{\infty} a(i)\varepsilon^i : \sum_{i=0}^{\infty}\{a(i)\}^2<\infty\}$ is complete since $l^2$ is complete. Therefore, $H=\overline{\mathrm{sp}}\{\varepsilon^i\}$ is identical to the linear space $\{\sum_{i=0}^{\infty} a(i)\varepsilon^i : \sum_{i=0}^{\infty}\{a(i)\}^2<\infty\}$. As a result, every element $a \in H$ can be expressed in the form $a=\sum_{i=0}^{\infty} a_i\varepsilon^i$, $\sum_{i=0}^{\infty}(a_i)^2<\infty$, and the norms $\|\cdot\|_H$ and $\|\cdot\|_2^{(\varepsilon)}$ are equivalent.

By the relation (6), it can be easily seen that every element $a \in G$ can be expressed in the form $a=\sum_{i=0}^{\infty} a^i x_i$, $\sum_{i=0}^{\infty}(a^i)^2<\infty$, and the norm $\|\cdot\|_2^{(x)}$ is equivalent to the two norms above.

Now, we define a linear operator $C$ on $H$ into $H$ by $Cx_i=\sum_{j:\{j,i\}\in A} x_j$. This operator $C$ has the following properties:

$$C\varepsilon_i = \sum_{j:\{j,i\}\in A}\varepsilon_j \tag{10}$$

$$\forall u, v \in H, \qquad \langle u, v\rangle = \lambda^2(u, (I-rC)v)^{(\varepsilon)},$$

where $(\sum a_i\varepsilon^i, \sum b_j\varepsilon^j)^{(\varepsilon)}$ denotes $\sum_{i=0}^{\infty} a_i b_j$.

Since $(u, Cv)^{(\varepsilon)}$ equals $(Cu, v)^{(\varepsilon)}$, $\langle u, Cv\rangle$ equals $\langle Cu, v\rangle$. Hence, $C$ is a symmetric operator both for $H$ and $l_2^{(\varepsilon)}$. Here $l_2^{(\varepsilon)}$ is defined to be the Hilbert space that consists of the elements of $H$ and has an inner product defined by $(\sum_{i=0}^{\infty} a_i\varepsilon^i, \sum_{j=0}^{\infty}\beta_j\varepsilon^j)^{(\varepsilon)} = \sum_{i=0}^{\infty} a_i\beta_i$. In the same way as (9) was derived, we obtain $(\sum_{i=0}^{\infty} a_i\varepsilon^i, C(\sum_{j=0}^{\infty} a_j\varepsilon^j))^{(\varepsilon)} \leqq \mathcal{N}\|a_i\varepsilon^i\|_2^{(\varepsilon)}$. Hence, we have $\|C\|_2^{(\varepsilon)} \leqq \mathcal{N}$, where $\|\cdot\|_2^{(\varepsilon)}$ denotes the operator norm that corresponds to the inner product $(\cdot,\cdot)^{(\varepsilon)}$.

The random variable $x_i$ can be expressed by the following series:

$$\begin{aligned} x_i &= (I-rC)^{-1}\varepsilon^i \\ &= \varepsilon^i + rC\varepsilon^i + r^2C^2\varepsilon^i + \cdots, \end{aligned}$$

since $\varepsilon^i = x_i - rCx_i = (I-rC)x_i$ and $\|rC\|_2^{(\varepsilon)}<1$. Let $R_{i,k}(n)$ be the number of paths of length $n$ that connect the vertices $i$ and $k$. We have $C^n\varepsilon^i=\sum_{k=0}^{\infty} R_{i,k}(n)\varepsilon^k$. Thus, $x_i=\sum_{n=0}^{\infty} r^n\sum_{k=0}^{\infty} R_{i,k}(n)\varepsilon^k$. Hence, the autocovariance can be expressed as follows:

$$\begin{aligned} \langle x_i, x_j\rangle &= \left\langle \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} r^n R_{i,k}(n)\varepsilon^k, x_j \right\rangle \\ &= \sum_{n=0}^{\infty} R_{i,j}(n)r^n. \end{aligned} \tag{11}$$

This formula shows explicitly that the autocovariance between two vertices is determined as a summation of the contributions from all the paths connecting the two vertices, and the contribution from a path of length $n$ is $r^n$. Besag (1981) pointed out that, in the case of a square lattice, the autocovariance decays very slowly with increasing $r$. From (11), we can obtain an upper bound of the autocovariance decay:

$$\langle x_i, x_j\rangle \leqq \sum_{n=\rho(i,j)}^{\infty} \mathcal{N}^n r^n = \frac{(\mathcal{N}r)^{\rho(i,j)}}{1-\mathcal{N}r} \tag{12}$$

by using the obvious inequality $R_{i,j}(n) \leqq \mathcal{N}^n$. Of course, the inequality (12) is valid for processes on general lattices.

*Example* 5.1. *The autocovariance functions of Gaussian Markov processes on the d-dimensional cubic lattice.* The $d$-dimensional cubic lattice frequently appears in applications. Denote the coordinates of the vertices on the lattice by $(t_1, t_1, \cdots, t_d)$. If the process is homogeneous, the covariance $\langle x(t_1, t_2, \cdots, t_d),\ x(s_1, s_2, \cdots, s_d)\rangle$ depends only on $(t_1 - s_1,\ t_2 - s_2, \cdots, t_d - s_d)$. We denote this covariance function by $\phi(t_1 - s_1,\ t_2 - s_2, \cdots, t_d - s_d)$; let $R_{(t_1,t_1,\cdots,t_d)}(n)$ be the number of paths of length $n$ connecting the vertices $(0, 0, \cdots, 0)$ and $(t_1, t_2, \cdots, t_d)$. Then, we have

$$R_{(t_1,t_2,\cdots,t_d)}(n)$$

$$= \begin{cases} 0, & n < \sum |t_i| \\ 0, & n - \sum |t_i| = 2m-1,\ m = 1, 2, \cdots \\ \displaystyle\sum_{i_1+i_2+\cdots+i_d=m} \frac{\left(2m + \sum_{k=1}^{N} |t_k|\right)!}{\prod_{k=1}^{N} (i_k + |t_k|)! \cdot \prod_{k=1}^{N} i_k!}, & n - \sum |t_i| = 2m,\ m = 0, 1, 2, \cdots. \end{cases}$$

Hence, by (11), we obtain

$$\phi(t_1, t_2, \cdots, t_d) = \sum_{n=0}^{\infty} R_{(t_1,t_2,\cdots,t_d)}(n) r^n$$

$$= \sum_{m=0}^{\infty} \sum_{i_1+i_2+\cdots+i_d=m} \frac{\left(2m + \sum_{k=1}^{d} |t_k|\right)!}{\prod_{k=1}^{d} (i_k + |t_k|)! \cdot \prod_{k=1}^{d} i_k!} r^{2m+\sum|t_k|} \tag{13}$$

$$= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \frac{\left\{\sum_{k=1}^{d} (2i_k + |t_k|)\right\}!}{\prod_{k=1}^{d} (i_k + |t_k|)! \cdot \prod_{k=1}^{d} i_k!} r^{\sum_{k=1}^{d}(2i_k+|t_k|)}.$$

It should be noted that this formula can be extended to anisotropic cases in appropriate ways. We can also obtain the autocovariance functions on other lattices in similar ways. These functions are called lattice Green functions in the field of statistical physics. For the lattice Green functions of rectangular lattices, see Katsura and Inawashiro (1971); see Horiguchi (1972) for triangular and honeycomb lattices. In their paper these functions are represented by using hypergeometric functions and the first and second kinds of complete elliptic integrals.

In the following, we shall see the special cases of dimension $d = 1$ and 2. When $d = 1$, the model is nothing but the first order AR model:

$$x_{t+1} = \alpha x_t + \xi_{t+1}, \qquad \xi_{t+1} \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2).$$

It is known that if $|\alpha| < 1$ the model is stationary, and the stationary distribution of $x_t$ is $N(0, \sigma^2/(1-\alpha^2))$. In the notation of the previous discussion, the relations $\tau^2 = \sigma^2/(1-\alpha^2)$, $\lambda^2 = \sigma^2/(1+\alpha^2)$ and $r = \alpha/(1+\alpha^2)$ hold.

By (13), the autocovariance function is

$$\phi(t) = \sum_{m=0}^{\infty} \frac{(2m+|t|)!}{(m+|t|)!\, m!} r^{2m+|t|} = \sum_{m=0}^{\infty} \binom{2m+|t|}{m} r^{2m+|t|}.$$

This function can be expressed in the form

$$\tau^2 = \phi(0) = \frac{r^t}{\sqrt{1-4r^2}},$$

$$\phi(t) = \phi(0)\alpha^{|t|}.$$

When $|r| \to 1/2$, $\tau^2$ tends to $\infty$. This corresponds to the stationarity condition $|\alpha| < 1$.

When $d = 2$, the autocovariance function in (13) is

$$\begin{aligned}
\phi(t_1, t_2) &= \sum_{m=0}^{\infty} \sum_{i_1+i_2=m} \frac{(2m+|t_1|+|t_2|)!}{(i_1+|t_1|)!(i_2+|t_2|)!i_1!i_2!} r^{2m+|t_1|+|t_2|} \\
&= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{\{2(i_1+i_2)+|t_1|+|t_2|\}!}{(i_1+|t_1|)!(i_2+|t_2|)!i_1!i_2!} r^{i_1+i_2+|t_1|+|t_2|} \\
&= \sum_{m=0}^{\infty} \binom{2m+|t_1|+|t_2|}{m+|t_1|} \binom{2m+|t_1|+|t_2|}{m+|t_1|+|t_2|} r^{2m+|t_1|+|t_2|}.
\end{aligned}$$

This equals the formula by Besag (1981) and also agrees with the following expression of Moran (1973a):

$$\phi(t_1, t_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos t_1\theta_1 \cos t_2\theta_2}{1 + 2r\cos\theta_1 + 2r\cos\theta_2} d\theta_1\, d\theta_2. \tag{14}$$

In particular,

$$\tau^2 = \phi(0, 0) = \sum_{m=0}^{\infty} \binom{2m}{m}^2 r^{2m}, \tag{15}$$

which agrees with the first-kind complete elliptic integral:

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{1-16r^2\sin^2\theta}}\mathrm{d}\theta.$$

The radius of convergence $\bar{r}$ of series (15) is

$$\bar{r}=\limsup_{m\to\infty}\sqrt{\frac{\binom{2m}{m}^2}{\binom{2(m+1)}{m+1}^2}}=\frac{1}{4}.$$

When $|r|\to 1/4$, $\tau^2$ tends to $\infty$. This corresponds to the fact that the homogeneous Gaussian process exists if and only if $|r|<1/4$.

## 6. Numerical approximation of autocovariance functions by iterative methods

In this section, numerical methods to calculate the autocovariance functions of Gaussian Markov lattice processes are suggested. In the case of a square lattice process, the autocovariance function is given by (14). Since the variance of $x_i$ can be expressed by the first kind of complete elliptic integral, several practically useful methods, such as taking the summation of the series using a Randen transformation, calculating the arithmetic–geometrical mean, or numerical integration can be used to calculate it (for the Randen transformation and the arithmetic–geometric mean, see Abramowitz and Stegun (1964)). The covariance between the random variables of two adjacent vertices can also be easily calculated (Besag and Moran 1975). However, practical methods to calculate covariances between the random variables of two separate vertices are not obvious because the integrand of (14) oscillates harder as $t_1$ and $t_2$ increase. Therefore several methods to approximate (14) have been discussed (Besag 1981). In the following, we shall show that the autocovariance functions can be calculated in practice by computing series (16) below with simple acceleration methods. It should be noted that the methods introduced here can be equally available to the processes on general lattices, while more efficient methods or modifications may exist for some specific lattices.

The covariance $\langle x_0, x_i\rangle=\sum_{n=0}^{\infty}R_{o,i}(n)r^n$ can be approximated by the finite sum $\sum_{n=0}^{l}R_{0,i}(n)r^n$. We adopt the $l_\infty$-norm to evaluate the accuracy of approximation of the autocovariance function. Let $\|\cdot\|_\infty^{(\varepsilon)}$ denote $l_\infty$-norm of a vector defined by $\|\sum a_i\varepsilon^i\|_\infty^{(\varepsilon)}=\sup_i|a^i|$ and the corresponding norm of an operator. Here, $\|C\|_\infty^{(\varepsilon)}=\mathcal{N}$. Since $|r|<1/\mathcal{N}$, the Neumann series

$$(I-rC)^{-1}\varepsilon^0=\varepsilon^0+rC\varepsilon^0+(rC)^2\varepsilon^0+\cdots \tag{16}$$

converges. The radius of convergence $\bar{r}$ of this series is determined by

$$\frac{1}{\bar{r}}=\limsup_{n\to\infty}\sqrt[n]{\|C^n\|_\infty^{(\varepsilon)}}=\limsup_{n\to\infty}\sqrt[n]{\mathcal{N}^n}=\mathcal{N}.$$

The accuracy of approximation of the autocovariance function by the finite sum of the first $l$ terms can be evaluated as

$$\max_i \left| \langle x_0, x_i \rangle - \sum_{n=0}^{l} R_{0,i}(n) r^n \right| = \| (I - rC)^{-1} \varepsilon^0 - (I + rC + r^2 C^2 + \cdots + r^l C^l) \varepsilon^0 \|_\infty^{(\varepsilon)}$$

$$\leqq \lim_{n \to \infty} \sum_{k=l+1}^{n} (|r| \mathcal{N})^k = \frac{(|r| \mathcal{N})^{l+1}}{1 - |r| \mathcal{N}}.$$

In principle, we can obtain the autoregressive function by calculating the expansion above. However, the speed of convergence is slow when $\mathcal{N}|r|$ is close to 1. We consider acceleration methods in the following.

While the autocovariance functions can take non-zero values at all points, we can adopt acceleration for iterative methods similar to those used to solve linear equations, because the covariance between the random variables of two vertices sufficiently far apart is close to zero. For this reason, it is sufficient to consider the vertices in an appropriate size ball $B_l(0)$. The three methods described below correspond to the Jacobi, Gauss–Seidel and SOR methods, respectively. Each method has its own feature corresponding to that of its finite-dimensional version (for details, see Varga (1962)). The calculation can be very efficient using parallel computing.

In the following, we put $\lambda^2 = 1$ without loss of generality.

*Procedure* (i). The following step is iterated:

$$a_{0i}(n+1) = \delta_{0i} + \sum_{j:\{j,i\} \in A} r a_{0j}(n), \qquad \text{for } i = 1, 2, \cdots,$$

where $a_{0i}(n)$ denotes the $n$th step approximation of $\langle x_0, x_i \rangle$, and $\delta_{0i}$ is defined as 1 for $i = 0$ and 0 for $i \neq 0$. If $a_{0i}(0) = \delta_{0i}$ is adopted as the initial value, the $n$th step approximation by this procedure is equal to the summation of the first $n$ terms of the series (11).

In actual calculations we adopt an appropriate numbering of the vertices and renew the values of the $a_i$ in the order of the numbering. Since at the $n$th step $a_{0i} = 0$ outside of the ball $B_n(0)$, only the $\alpha_{0i}$ in $B_{n+1}(0)$ are to be renewed at the $(n+1)$th step. This procedure is more efficient than calculating the series $\sum_{n=0}^{l} R_{0,i}(n) r^n$ for all the $i$ $(i = 0, 1, 2, 3, \cdots)$, because this procedure simultaneously carries out some computations required to obtain covariances of different pairs of vertices. However, the speed of convergence of this procedure is not fast if $|r| \mathcal{N}$ is close to 1.

*Procedure* (ii). The following step is iterated:

$$a_{0i} \Leftarrow \delta_{0i} + \sum_{j:\{j,i\} \in A} r a_{0j} \qquad \text{for } i = 0, 1, \cdots,$$

where $\Leftarrow$ denotes substitution. This method is quite simple to implement. The convergence of this procedure may also be slow if $|r| \mathcal{N}$ is close to 1.

TABLE 1
The numbers of iterations required by procedures (i), (ii) and (iii) to approximate autocovariance functions of CAR processes with several $r$ to accuracies of 0.01 and 0.001. A dash '—' indicates that an approximation with required accuracy is not obtained after 300 iterations.

| | | Parameter $r$ | | | |
|---|---|---|---|---|---|
| Accuracy | Procedure | 0.2400 | 0.2450 | 0.2490 | 0.2499 |
| 0.01 | (i) | 57 | 115 | — | — |
| | (ii) | 30 | 59 | 292 | — |
| | (iii) | 12 | 18 | 41 | 130 |
| 0.001 | (i) | 102 | 206 | — | — |
| | (ii) | 53 | 105 | — | — |
| | (iii) | 20 | 28 | 64 | 204 |

*Procedure* (iii). The following step is iterated:

$$a_{0i} \Leftarrow a_{0i} + p\left(\delta_{0i} + \sum_{j:\{j,i\}\in A} r a_{0j} - a_{0i}\right) \qquad \text{for } i = 0, 1, \cdots.$$

Here, $p$ is a given acceleration constant.

In our terminology, if the lattice is a bipartite graph, it is known that the best acceleration constant to solve the linear equation is $p = 2/(1 + \sqrt{1 - \|rC\|})$ (Varga 1962) with a suitable numbering of the vertices. Using this constant $p$ and a suitable numbering, we can calculate the autocovarinace functions of CAR lattice processes effectively, unless $|r|$ is very close to $1/\mathcal{N}$.

Table 1 gives the numbers of iterations required by procedures (i), (ii) and (iii) to approximate the autocovariance functions of square-lattice CAR processes with several values of $r$ to accuracies of 0.01 and 0.001. As $r$ tends to $0.25 = 1/\mathcal{N}$, the convergence of approximated values becomes slow. The result shows the speed of convergence is significantly improved by the acceleration.

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