Section 2

Green's Functions

In this section we show how the Green's function may be used to derive a general solution to an inhomogeneous Boundary Value Problem.

Boundary Value Problems and Linear Superposition

Definition 2.1*:* *A linear boundary value problem (BVP) for an ordinary differential equation (ODE) of at least second order, consists of the following:*
(a) *an unknown function $u(x)$ defined and continuous on an interval $x \in [a, b]$ where a, b are real constants and x is a real variable $a \le x \le b$;*
(b) *an ordinary differential equation (ODE) for $u(x)$,*

$$\mathcal{L}u(x) = f(x) \tag{2.1}$$

on (a, b), where $\mathcal{L}$ is a differential operator

$$\mathcal{L} = \sum_{i=0}^{N} a_i(x)\frac{d^i}{dx^i}, \qquad N \ge 2 \tag{2.2}$$

$$= a_N(x)\,u^{(N)}(x) + a_{N-1}(x)\,u^{N-1}(x) + \ldots + a_1(x)\,u'(x) + a_0(x)\,u(x)$$

*and $f(x)$ (the **forcing function, inhomogeneous term**), $a_i(x)$ (**coefficients**), $0 \le i \le N$ are known functions of x.*
(c) *N **boundary conditions/data** (BCs) to be satisfied by $u(x)$ and/or its derivatives at $x = a$, $x = b$.*

Typical BCs are $u(a) = \alpha_1$, $du/dx\,(b) = \alpha_2$ etc. where α_i are known numbers. For given BCs written in some fixed order, form the vector $\{\alpha_1, ..., \alpha_N\} = \boldsymbol{\alpha}$ and call the set $\{f(x); \boldsymbol{\alpha}\}$ the **data**.
Solve the BVP in a way that exhibits dependence on the data – it is then easy to change the data. **Linearity** of (2.1) allows this.

Example 1: Let

$$x\frac{d^2u}{dx^2} + 2x^2\frac{du}{dx} + 4u = \exp x$$

on the interval $[0, 1]$ subject to

$$u(0) = 1, \qquad \frac{du}{dx}(1) = 2,$$

then

$$a = 0, \qquad b = 1.$$

$$\mathcal{L} = x\frac{d^2}{dx^2} + 2x^2\frac{d}{dx} + 4,$$

$$f(x) = \exp x,$$

$$\alpha_1 = 1, \alpha_2 = 2$$

and the data is $\{\exp x; 1, 2\}$.

Theorem 2.2: *This concerns* ***superposition*** *of the data. Let $u_1(x)$ be a solution for data*

$$\{f_1(x); \boldsymbol{\alpha}_1\}$$

and $u_2(x)$ be a solution for data

$$\{f_2(x); \boldsymbol{\alpha}_2\}.$$

Then

$$Au_1(x) + Bu_2(x)$$

is a solution for the data

$$\{Af_1(x) + Bf_2(x); A\boldsymbol{\alpha}_1 + B\boldsymbol{\alpha}_2\}$$

for real known constants A, B.

Note 1: Theorem 2.2 extends to N solutions and N sets of data and to data $\{f(x,\theta); \boldsymbol{\alpha}(\theta)\}$ depending on a continuous parameter θ. This is superposition corresponding to an integral with respect to θ.
Note 2: Theorem 2.2 allows decomposition of data into simple sets; the corresponding simple BVPs solved and reassembled into the solution for the original data – only linearity allows this!
One way to do this is to decompose the data $\{f(x); \boldsymbol{\alpha}\}$ into $\{f(x); \mathbf{0}\}$ and $\{0; \boldsymbol{\alpha}\}$:
$\{f(x); \mathbf{0}\}$ - **inhomogeneous equation (particular solution), homogeneous boundary data**;
$\{0; \boldsymbol{\alpha}\}$ - **homogeneous equation (complementary function solution), inhomogeneous boundary data**.
Another way is to split the data $\{0; \alpha_1, \dots, \alpha_n\}$ into N simpler problems $\{0; \alpha_1, 0\}, \dots, \{0; 0, \alpha_N\}$ - each with one inhomogeneous BC, all remaining BCs homogeneous.

Example 2: Take

$$\frac{d^2u}{dx^2} - \frac{du}{dx} - 2u = x$$

on the interval $[0,1]$ subject to

$$u(0) = 1, \qquad u(1) = 2.$$

Split the solution into a sum of three parts, say, with respective data

$$\{x, (0,0)\}, \quad \{0, (1,0)\}, \quad \{0, (0,2)\}.$$

A Green's function is a solution to an associated BVP in which $f(x)$ takes a special form (it is the Dirac delta 'function') and $u(x)$ is subject to *homogeneous boundary data*, i.e. the prescribed values $\boldsymbol{\alpha}$ of the BCs at $x = a$, $x = b$ are zero. As we shall see, the Green's function may be used to solve the general BVP stated above.

Dirac Delta Function

Definition 2.3: *The* Dirac Delta Function, $\delta(x-y)$, *is defined such that*

1. $\delta(x-y) = 0$ *if* $x \neq y$,

2. $\int_{y-c}^{y+d} \delta(x-y)\, dx = 1$ *for any* $c, d > 0$ *(i.e. the integral encloses the point* $x = y$*).*

Note 1: The Dirac delta $\delta(x-y)$ is a concentrated 'impulse' of unit strength.
Note 2: The Dirac delta, $\delta(x-y)$, is **not** a function $\delta(\cdot): \mathbb{R} \to \mathbb{R}$ but may be treated like one for some purposes. It is called a generalized function, and can be thought of as the limit of a sequence of functions, e.g. if

$$\delta_k(x-y) = \begin{cases} 2^k & \text{if} \quad |x-y| \le \frac{1}{2^{k+1}}, \\ 0 & \text{if} \quad |x-y| > \frac{1}{2^{k+1}}, \end{cases}$$

then the area under each curve is 1 and

$$\delta(x-y) \ \text{``}=\text{''} \ \lim_{k \to \infty} \delta_k(x-y).$$

Other examples of such sequences are

$$\delta_k(x-y) = \begin{cases} k & \text{if} \quad |x-y| \le \frac{1}{2k}, \\ 0 & \text{if} \quad |x-y| > \frac{1}{2k}, \end{cases}$$

$$\delta_k(x-y) = \frac{1}{\pi} \frac{k}{1 + k^2 (x-y)^2},$$

$$\delta_k(x-y) = \frac{k}{\sqrt{\pi}} \exp\left[-k^2 (x-y)^2\right].$$

Definition 2.4*: The* ***Heaviside (unit) step function****,* $H(x-y)$*, is defined by*

$$H(x-y) = \begin{cases} 1 & \textit{if} \quad x > y, \\ 0 & \textit{if} \quad x < y. \end{cases}$$

At $x = y$*,* $H(0)$ *may be left undefined or given some value* k*, e.g.* $k = 0, \frac{1}{2}, 1$.

Proposition 2.5*: (a)* $H'(x-y) = \delta(x-y)$ *provided* $x \ne y$*, where* $'$ *denotes* $\frac{d}{dx}$*;*
(b) $H(x-y) = \int_{y-c}^{x} \delta(z-y)\, dz$ *for any* $c > 0$.

Proof: (a) Informally - if $x \ne y$ both sides equal zero. If $x = y$ both sides are undefined. A more formal proof is given shortly after the following theorem.
(b) If $x < y$ both sides equal zero. If $x > y$ both sides equal 1.

Theorem 2.6*: We can define the* ***Sifting Property:***

$$\int_{x-c}^{x+d} f(y)\, \delta(x-y)\, dy = f(x),$$

for any $c, d > 0$.

Proof: Integrating the left-hand side by parts, let

$$u = f(y), \qquad \frac{dv}{dy} = \delta(x-y),$$

then

$$\frac{du}{dy} = f'(y), \qquad v = -H(x-y),$$

and so

$$\int_{x-c}^{x+d} f(y)\,\delta(x-y)\,dy = \left[-H(x-y)\,f(y)\right]_{y=x-c}^{y=x+d} - \int_{x-c}^{x+d} -H(x-y)\,f'(y)\,dy$$

$$= \{-H(-d)\,f(x+d)\} - \{-H(c)\,f(x-c)\} + \int_{x-c}^{x+d} H(x-y)\,f'(y)\,dy.$$

Now

$$H(-d) = 0, \qquad H(c) = 1,$$

and

$$H(x-y) = \begin{cases} 1 & \text{if} \quad x-c \le y < x, \\ 0 & \text{if} \quad x < y \le x+d. \end{cases}$$

Hence

$$\begin{aligned} \int_{x-c}^{x+d} f(y)\,\delta(x-y)\,dy &= f(x-c) + \int_{x-c}^{x} f'(y)\,dy \\ &= f(x-c) + \left[f(y)\right]_{x-c}^{x} = f(x). \end{aligned}$$

□

Note: Employing the usual ***inner product*** for two real functions, f and g say:

$$\langle g, f\rangle = \int_a^b f(x)\,g(x)\,dx$$

for $a < x < b$, we may write

$$\langle \delta(x-y), f(y)\rangle = \langle f(y), \delta(x-y)\rangle = f(x). \tag{2.3}$$

We can now prove Proposition 2.5(a) more formally. Indeed for a function $\phi(x)$ with sufficient decay at infinity, we have, initially using integration by parts

$$\begin{aligned} \langle H'(x), \phi(x)\rangle &= -\langle H(x), \phi'(x)\rangle, \\ &= -\int_0^\infty \phi'(x)dx, \\ &= \phi(0) \end{aligned}$$

where we used the fundamental theorem of calculus. Finally we know that

$$\phi(0) = \langle \delta(x), \phi(x)\rangle$$

and therefore we must have $H'(x) = \delta(x)$.

Green's function

Example 3: Take the BVP

$$x\frac{d^2u}{dx^2} + 2x^2\frac{du}{dx} + 4u = f(x),$$

$$u(a) = 2, \qquad \frac{du}{dx}(b) = 1.$$

Let $G(x,y)$ be a function of two variables x, y and apply the differential operator $\mathcal{L}$ to $G(x,y)$. Write

$$\mathcal{L}_x G(x,y) = \left\{ x\frac{d^2}{dx^2} + 2x^2\frac{d}{dx} + 4 \right\} G(x,y)$$

to emphasise that $\mathcal{L}$ acts on G as a function of x (i.e. differentiate with respect to (w.r.t.) x) and that y is a parameter. Then $G(x,y)$ satisfies the BVP with homogeneous BCs:

$$\mathcal{L}_x G(x,y) = \delta(x-y), \qquad y \in [a,b],$$

$$G(a,y) = 0, \qquad \frac{dG}{dx}(b,y) = 0.$$

Definition 2.7: *A **Green's function** $G(x,y)$ for any BVP satisfies the equation*

$$\mathcal{L}_x G(x,y) = \delta(x-y), \tag{2.4}$$

for some operator $\mathcal{L}_x$ with homogeneous BCs, i.e. it is the solution corresponding to the data $\{\delta(x-y); \mathbf{0}\}$.

Thus, $G(x,y)$ is the response, under homogeneous BCs, to a forcing function consisting of a concentrated unit impulse (or inhomogeneity) at $x = y$.

Theorem 2.8: *Let $G(x,y)$ be known for a homogeneous BVP (2.4), then the solution to the inhomogeneous equation*

$$\mathcal{L}u(x) = f(x),$$

where $f(x)$ is a given function, subject to homogeneous boundary conditions, may be written

$$u(x) = \int_a^b G(x,y) f(y)\, dy.$$

Proof: Differentiating under the integral sign

$$\mathcal{L}u(x) = \int_a^b \mathcal{L}_x G(x,y) f(y)\, dy$$

or by (2.4)

$$= \int_a^b \delta(x-y) f(y)\, dy$$

and hence using (2.3)

$$= f(x).$$

□

Let us now show how construction of a Green's function is achieved for a specific example.

Example 4: Find the general solution (i.e. the solution for arbitrary $f(x)$) of

$$\mathcal{L}u(x) = f(x) \tag{2.5}$$

in the case where

$$\mathcal{L} = \frac{d^2}{dx^2}$$

subject to the BC's $u\,(0) = u'\,(1) = 0$.
To find the general solution we know that we have to determine the Green's function, which is the solution to the following BVP:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x-y)$$

with $G(0,y) = 0, \partial G(1,y)/\partial x = 0$. In order to find the Green's function we split up the problem on two two domains, $x \in [0,y)$ and $x \in (y,1]$. On those domains, we know that since $x \neq y$,

$$\frac{\partial^2 G}{\partial x^2} = 0$$

One can consider this like an ODE in x but whose "constants" are actually functions of y. Indeed, this equation has simple solutions

$$G\,(x,y) = A(y)x + B(y), \tag{2.6}$$

where A and B are arbitrary functions of y. In the left hand interval $[0,y)$ we require the Green's function to satisfy the homogeneous governing equation (except at $x = y$) **and** the left hand boundary condition. Similarly, in the right hand interval, $(y,1)$, $G(x,y)$ satisfies the homogeneous governing equation **and** the right hand boundary condition.
Denoting the left hand solution as $G_1(x,y) = A_1(y)x + B_1(y)$, with $G_1(0,y) = 0$ this gives $B_1 = 0$. so that $G_1(x,y) = A_1(y)x$. Similarly in the right domain, with $G_2(x,y) = A_2(y) + B_1(y)$ we find that $G_2'(1,y) = 0$ gives $A_2 = 0$ so that $G_2(x,y) = B_1(y)$. Hence the Green's function can be written as

$$G\,(x,y) = \left\{ \begin{array}{lll} A_1(y)u_1(x) = A_1(y)x & \text{for} & 0 \leq x < y, \\ B_2(y)u_2(x) = A_2(y) & \text{for} & y < x \leq 1, \end{array} \right. \tag{2.7}$$

Now, $G\,(x,y)$ must be continuous at $x = y$, which means that

$$A_1\,(y)\,y = B_2\,(y)\,. \tag{2.8}$$

A second condition can be found by integrating the governing equation

$$\mathcal{L}_x G\,(x,y) = \frac{\partial^2 G\,(x,y)}{\partial x^2} = \delta\,(x-y)$$

from $y - \varepsilon$ to $y + \varepsilon$. This yields

$$\left[\frac{\partial G\,(x,y)}{\partial x}\right]_{x=y-\varepsilon}^{x=y+\varepsilon} = 1, \tag{2.9}$$

which means that the derivative of the Green's functions is discontinuous across the unit impulse, with jump unity. From (2.7) we have

$$\frac{\partial G}{\partial x} = \left\{ \begin{array}{ll} A_1\,(y) & \text{for} \quad 0 < x < y, \\ 0 & \text{for} \quad y < x < 1, \end{array} \right.$$

and hence, from (2.9),

$$0 - A_1(y) = 1.$$

Substituting this into (2.8) reveals

$$A_1(y) = -1, \qquad B_2(y) = -y$$

or

$$G(x,y) = \begin{cases} -x & \text{for} \quad 0 \le x \le y, \\ -y & \text{for} \quad y \le x \le 1. \end{cases}$$

Now, recall the Heaviside step function

$$H(t) = \begin{cases} 1 & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t < 0, \end{cases}$$

and so the Green's function may be expressed as

$$G(x,y) = -xH(y-x) - yH(x-y).$$

The general solution of the ODE (2.5) is, from Theorem 2.8,

$$u(x) = \int_0^1 G(x,y) f(y)\, dy$$

$$= \int_0^1 [-xH(y-x) - yH(x-y)] f(y)\, dy$$

$$= -x \int_x^1 f(y)\, dy - \int_0^x y f(y)\, dy.$$

This completes the solution for this example; however, later we will revisit it and apply a couple of checks to ensure that it is correct.

Analogous problem

In section 0, we stated the Green's function for the steady state heat equation,

$$\frac{d^2u}{dx^2} = f(x)$$

subject to $u(0) = 0 = u(L)$. Derive this using the approach given above.

Construction of Green's Functions for Sturm-Liouville Operators

Let us now consider the construction of $G(x,y)$ for the general Sturm-Liouville operator:

$$\mathcal{L} = \frac{1}{r(x)} \left\{ \frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right\}. \tag{2.10}$$

This operator, and special cases of it, occur when solving PDEs by separation of variables. We will see that the general construction of $G(x,y)$ follows the same route as for the above specific example.

From (2.4), $G(x,y)$ satisfies the homogeneous equation

$$\mathcal{L}_x G(x,y) = 0$$

except at $x = y$. Consider the two sub-intervals (a,y) and (y,b) of (a,b) separately.
The operator $\mathcal{L}$ is second order and so the general solution of $\mathcal{L}u(x) = 0$ contains two arbitrary constants A, B say. Write

$$u(x) = Av(x) + Bw(x).$$

The BVP has two homogeneous boundary conditions, one at $x = a$ and the other at $x = b$. For the Green's function then since $\mathcal{L}_x G = 0$, we can once again split the problem up into two domains:

(i) In $[a,y)$, use the BC at $x = a$ to obtain a solution containing one arbitrary "constant" and which satisfies the BC at $x = a$.
Write the solution as $G_1(x,y) = c_1(y)u_1(x)$.

(ii) In $(y,b]$ use the BC at $x = b$ to obtain a solution containing one arbitrary constant and which satisfies the BC at $x = b$.
Write the solution as $G_2(x,y) = c_2(y)u_2(x)$.

The constants c_1, c_2 depend (continuously) upon the parameter y, i.e. $c_1 = c_1(y)$ and $c_2 = c_2(y)$. We have

$$G(x,y) = \begin{cases} c_1(y)\,u_1(x) & \text{for} \quad a \le x < y, \\ c_2(y)\,u_2(x) & \text{for} \quad y < x \le b. \end{cases} \tag{2.11}$$

To find $c_1(y), c_2(y)$ we need to find two conditions on $G(x,y)$ at $x = y$. From (2.4) and (2.10) the function $G(x,y)$ satisfies

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,y)}{\partial x}\right) + q(x)\,G(x,y) = \delta(x-y)\,r(x). \tag{2.12}$$

Theorem 2.9: *The Green's function $G(x,y)$ is continuous (w.r.t. x) at $x = y$:*

$$c_1(y)\,u_1(y) = c_2(y)\,u_2(y). \tag{2.13}$$

Proof: Suppose it is not continuous, then, at best $G(x,y)$ has a step discontinuity (like $H(x-y)$) at $x = y$. The LHS of (2.12) therefore contains the second derivative of a step function, i.e. the first derivative of a δ-function. The RHS does not have such, and so the equation cannot balance. There is a contradiction, so G must be continuous.
Now, let

$$y_- = \lim_{\varepsilon \to 0}(y - \varepsilon), \qquad y_+ = \lim_{\varepsilon \to 0}(y + \varepsilon)$$

then $G(x,y)$ is continuous $\Rightarrow$

$$G(y_-, y) = G(y_+, y) = G(y,y).$$

From (2.11) at $x = y$,

$$c_1(y_-)\,u_1(y) = c_2(y_+)\,u_2(y).$$

The $c_1(y)$, $c_2(y)$ are continuous in y, and so

$$c_1(y)\,u_1(y) = c_2(y)\,u_2(y).$$

$\square$

Theorem 2.10: *The Derivative of $G(x,y)$,*

$$\frac{\partial G(x,y)}{\partial x}, \text{ is discontinuous at } x=y, \text{ and}$$

$$\left.\frac{\partial G(x,y)}{\partial x}\right|_{x=y_-}^{x=y_+} = c_2(y)\,u_2'(y) - c_1(y)\,u_1'(y) = \frac{r(y)}{p(y)}. \tag{2.14}$$

Proof: The LHS of (2.12) contains the derivative of a step function, i.e. a δ-function. So does the RHS, so the equation balances and everything is consistent.
Integrate (2.12) from $y-\varepsilon$ to $y+\varepsilon$,

$$\left[p(x)\frac{\partial G(x,y)}{\partial x}\right]_{y-\varepsilon}^{y+\varepsilon} + \int_{y-\varepsilon}^{y+\varepsilon} q(x)\,G(x,y)\,dx = \int_{y-\varepsilon}^{y+\varepsilon} \delta(x-y)\,r(x)\,dx$$

$$= r(y)$$

by the sifting property. Let $\varepsilon \to 0$,

$$p(y_+)\frac{\partial G(y_+,y)}{\partial x} - p(y_-)\frac{\partial G(y_-,y)}{\partial x} + 0 = r(y).$$

However, $p(x)$ is continuous, so

$$\frac{\partial G(y_+,y)}{\partial x} - \frac{\partial G(y_-,y)}{\partial x} = \frac{r(y)}{p(y)}. \tag{2.15}$$

From (2.11)

$$\frac{\partial G(x,y)}{\partial x} = \begin{cases} c_1(y)\,u_1'(x) & \text{for} \quad x<y, \\ c_2(y)\,u_2'(x) & \text{for} \quad x>y, \end{cases}$$

and hence

$$c_2(y)\,u_2'(y) - c_1(y)\,u_1'(y) = \frac{r(y)}{p(y)}.$$

□

Solving the equations (2.13),

$$c_1(y)\,u_1(y) - c_2(y)\,u_2(y) = 0,$$

and (2.14),

$$c_2(y)\,u_2'(y) - c_1(y)\,u_1'(y) = \frac{r(y)}{p(y)},$$

for $c_1(y)$ and $c_2(y)$ gives, by elimination,

$$c_1(y) = \frac{r(y)\,u_2(y)}{p(y)\,W(y)}, \qquad c_2(y) = \frac{r(y)\,u_1(y)}{p(y)\,W(y)} \tag{2.16}$$

where

$$W(y) = \begin{vmatrix} u_1(y) & u_2(y) \\ u_1'(y) & u_2'(y) \end{vmatrix}$$

is called the *Wronskian* of $u_1(y)\,,\, u_2(y)\,.$

Therefore for equations of Sturm-Liouville type, the Green's functions can be found in a rather straightforward manner by using the explicit formulae above.

Let us return to the BVP in Example 4. In that example we determined the Green's function directly, and showed that it had the form

$$G(x,y) = \begin{cases} -x, & 0 \le x \le y, \\ -y, & y \le x \le 1 \end{cases} \tag{2.17}$$

. Alternatively, we could have derived this solution using the explicit formulae above because the operator is clearly of Sturm-Liouville form, with $r(x) = 1, p(x) = 1, q(x) = 0$. Let's do this, using $u_1(x) = x$ and $u_2(x) = 1$ which as always have to be determined from solutions to the homogeneous ODE with the necessary boundary conditions at the left and right ends. We use (2.16), finding that the Wronskian is $W = -1$ and therefore

$$c_1(y) = -1, \qquad\qquad c_2(y) = -y$$

which therefore gives the same result as the direct method in (2.17) above.
We can write this (as above) as

$$G\left(x,y\right) = -xH\left(y-x\right) - yH\left(x-y\right)$$

and so the solution to the BVP can be written in the general form

$$u(x) = -x\int_x^1 f\left(y\right)dy - \int_0^x yf\left(y\right)dy.$$

CHECKS!
We can impose two checks on the calculation. The first on the Green's function and the second on the solution $u(x)$.
Check 1: We have

$$G\left(x,y\right) = -xH\left(y-x\right) - yH\left(x-y\right),$$

so

$$\frac{\partial G\left(x,y\right)}{\partial x} = -H\left(y-x\right) - x\left(-1\right)\delta\left(y-x\right) - y\delta\left(x-y\right) = -H\left(y-x\right),$$

and

$$\frac{\partial^2 G\left(x,y\right)}{\partial x^2} = \delta\left(y-x\right).$$

Check 2: Recall the 'fundamental theorem of integral calculus'. If

$$F\left(x\right) = \int_a^x f\left(y\right)dy \quad \text{then} \quad F'\left(x\right) = f\left(x\right).$$

We have

$$u\left(x\right) = -x\int_x^1 f\left(y\right)dy - \int_0^x yf\left(y\right)dy,$$

and so

$$u'\left(x\right) = -\int_x^1 f\left(y\right)dy - x\left[-f\left(x\right)\right] - xf\left(x\right) = -\int_x^1 f\left(y\right)dy$$

together with

$$u''(x) = -[-f(x)] = f(x).$$

Thus, $u(x)$ satisfies the requisite ODE. Checking the boundary values

$$u(0) = -0 \times \int_0^1 f(y)\,dy - \int_0^0 yf(y)\,dy = 0,$$

and

$$u'(1) = -\int_1^1 f(y)\,dy = 0,$$

so these are satisfied too.

We can now make a number of remarks about what we have achieved. First, in the above example, $G(x,y)$ is symmetric in x,y: $G(y,x) = -yH(x-y) - xH(y-x) = G(x,y)$. This is true for any Sturm-Liouville problem for which $r(x)$ is constant. See a later section for further discussion of this issue.
Second, the Green's function $G(x,y)$ may be regarded as the response at the point x to a unit impulse at the point y. The general inhomogeneous term f on $0 < y < 1$ can be regarded as a set of impulses, with $f(y)$ giving the magnitude of the impulse at point y.
Third, note that given that the Green's function permits the solution to a problem involving *homogeneous* boundary conditions with inhomogeneous forcing (RHS in the ODE), we can also consider now how to solve the inhomogeneous boundary condition problem. Well, this is relatively simple. We use linear superposition, determining a solution of

$$\mathcal{L}u = 0$$

subject to the inhomogeneous BCs (e.g. $u(a) = \alpha, u(b) = \beta$). Let us call this the complementary function $u_{CF}(x)$ and therefore the solution to the full BVP

$$\mathcal{L}u = f(x)$$

subject to inhomogeneous BCs (e.g. $u(a) = \alpha, u(b) = \beta$) is

$$u(x) = u_{CF}(x) + \int_a^b G(x,y)f(y)dy$$

In Example 4 we worked out the Green's function directly using boundary conditions, continuity of $G(x,y)$ at $x = y$ and a condition on the *discontinuity* of the derivative of $G(x,y)$ at $x = y$. We noted later that since the operator was of Sturm-Liouville form, we could have also used the explicit formulae (2.16) in order to determine the Green's function. When the problem is *not* of Sturm-Liouville form, then we have no option but to determine the Green's function directly. We now consider an example of this type.

Example 5:
Determine the Green's function associated with part of Q1 from Sheet 2, i.e. that associated with

$$x^2u''(x) - xu'(x) - 3u(x) = x - 3$$

with $u(1) = 0, u(2) = 0$, confirming that the solution you obtain is equivalent to that which you obtained in that question using standard direct ODE methods.

Let us determine the GF associated with this equation, satisfying

$$x^2\frac{\partial^2 G}{\partial x^2} - x\frac{\partial G}{\partial x} - 3G = \delta(x-y) \tag{2.18}$$

and $G(1,y) = G(2,y) = 0$. Since this is an Euler equation (x^n times nth derivative), by posing solutions of the form x^n, we know that the solutions to the homogeneous problem with no forcing are x^3 and x^{-1}. As above we go about finding the Green's function by splitting the function into two domains, one to the left of $x = y$ ($G_1(x,y)$) and one to the right ($G_2(x,y)$). We write

$$\begin{aligned} G_1(x,y) &= A_1(y)x^3 + \frac{B_1(y)}{x}, \qquad && 1 \le x \le y, \\ G_2(x,y) &= A_2(y)x^3 + \frac{B_2(y)}{x}, \qquad && y \le x \le 2. \end{aligned}$$

Imposing BCs gives $B_1 = -A_1$ and $B_2 = -16A_2$ so that

$$\begin{aligned} G_1(x,y) &= A_1(y)\left(x^3 - \frac{1}{x}\right), \qquad && 1 \le x \le y, \\ G_2(x,y) &= A_2(y)\left(x^3 - \frac{16}{x}\right), \qquad && y \le x \le 2. \end{aligned}$$

Now impose continuity at $x = y$ which gives

$$A_2 = A_1\left(\frac{y^4-1}{y^4-16}\right). \tag{2.19}$$

Finally we need a "jump" condition. Integrate the equation (2.18) between y_- and y_+, taking the limit as $\epsilon \to 0$:

$$\int_{y_-}^{y_+} x^2\frac{\partial^2 G}{\partial x^2}dx - \int_{y_-}^{y_+} x\frac{\partial G}{\partial x}dx - 3\int_{y_-}^{y_+} Gdx = \int_{y_-}^{y_+} \delta(x-y)dx = 1.$$

Use integration by parts so that the first term is of the form

$$\int_{y_-}^{y_+} x^2\frac{\partial^2 G}{\partial x^2}dx = \left[x^2\frac{\partial G}{\partial x}\right]_{y-}^{y+} - 2\int_{y_-}^{y^+} x\frac{\partial G}{\partial x}dx$$

We can combine this with the second term and noting that the term involving only an integral of the Greens function (continuous at $x = y$) must be zero (continuity) we find that

$$\left[x^2\frac{\partial G}{\partial x}\right]_{y-}^{y+} - 3\int_{y_-}^{y_+} x\frac{\partial G}{\partial x}dx = 1.$$

Since x^2 is continuous at $x = y$ the first term reduces to

$$y^2\left[\frac{\partial G}{\partial x}\right]_{y-}^{y+} - 3\int_{y_-}^{y_+} x\frac{\partial G}{\partial x}dx = 1.$$

and using integration by parts in the second term we find that

$$\int_{y-}^{y+} x\frac{\partial G}{\partial x}dx = [xG]\int_{y-}^{y+} - \int_{y-}^{y+} Gdx = 0$$

by continuity. Therefore the jump condition for this problem is

$$\left[\frac{\partial G}{\partial x}\right]_{y-}^{y+} = \frac{1}{y^2}.$$

Applying this, using

$$\begin{aligned} \frac{\partial G_1}{\partial x} &= A_1(y)\left(3x^2 + \frac{1}{x^2}\right), && 1 \le x \le y, \\ \frac{\partial G_2}{\partial x} &= A_2(y)\left(3x^2 + \frac{16}{x^2}\right), && y \le x \le 2. \end{aligned}$$

we find that, upon using (2.19)

$$A_1(y) = \frac{y^4 - 16}{60y^4}$$

and therefore

$$A_2(y) = \frac{y^4 - 1}{60y^4}.$$

The Greens function is therefore

$$G(x,y) = \begin{cases} G_1(x,y) = \left(\frac{y^4-16}{60y^4}\right)\left(x^3 - \frac{1}{x}\right), & 1 \le x \le y, \\ G_2(x,y) = \left(\frac{y^4-1}{60y^4}\right)\left(x^3 - \frac{16}{x}\right), & y \le x \le 2, \end{cases} \tag{2.20}$$

which can therefore be written as

$$G(x,y) = H(y-x)\left(\frac{y^4-16}{60y^4}\right)\left(x^3 - \frac{1}{x}\right) + H(x-y)\left(\frac{y^4-1}{60y^4}\right)\left(x^3 - \frac{16}{x}\right). \tag{2.21}$$

and the solution of the BVP can therefore be written

$$u(x) = \int_1^2 G(x,y)f(y)dy = \int_1^2 G(x,y)(y-3)dy.$$

We'll return to this shortly.

Let's first perform some checks on the Green's function derived in (2.20). Boundary conditions? $G(1,y) = 0$, yes. $G(2,y) = 0$, yes. Check continuity at $x = y$, yes. We note that the Green's function is NOT symmetric - the equation is not self-adjoint. Check for yourself that the Green's function satisfies the equation with delta function forcing. Remember to use the form in (2.21) and the fact that $H'(x-y) = \delta(x-y)$, etc.).

Finally, let's check that the solution in integral form is the same as that which we evaluated directly in Q1, sheet 2. We have

$$\begin{aligned} u(x) &= \int_1^2 G(x,y)(y-3)dy, \\ &= \int_1^x \left(\frac{y^4-1}{60y^4}\right)\left(x^3 - \frac{16}{x}\right)(y-3)dy + \int_x^2 \left(\frac{y^4-16}{60y^4}\right)\left(x^3 - \frac{1}{x}\right)(y-3)dy \end{aligned}$$

Bringing out the terms depending on x, performing the integration in y and simplifying, we obtain (after a lot of algebra)

$$u(x) = 1 - \frac{x}{4} - \frac{1}{60}x^3 - \frac{11}{15x}$$

which we note is the correct solution as found in Q1, Sheet 2 with homogeneous boundary conditions.

Further worked examples for you to put in your notes: construct the Green's function associated with, and hence solve the following BVPs:

$$x^2\frac{d^2u}{dx^2} - 4x\frac{du}{dx} + 6u = f(x), \qquad u(1) = 1, u'(2) = 2.$$

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = f(x), \qquad u(0) = 1, u(1) = 0.$$

Symmetry and the Adjoint Green's Function

We have said that

$$u(x) = \int_a^b G(x,y)f(y)dy, \tag{2.22}$$

indeed we have shown this in Theorem 2.8, because we can operate on this equation with $\mathcal{L}$ to give

$$\begin{aligned}\mathcal{L}u &= \int_a^b \mathcal{L}G(x,y)f(y)dy,\\ &= \int_a^b \delta(x-y)f(y),\\ &= f(x)\end{aligned}$$

as we require.
However in Section 0, we showed from first principles, by integrating combinations of the ODEs that u and G satisfy, that, in fact (note the position of the x and y in the argument of the Green's function)

$$u(x) = \int_a^b G(y,x)f(y)dy$$

for the operator $\mathcal{L} = d^2/dx^2$. Since for this operator, $\mathcal{L}^* = \mathcal{L}$ (self-adjoint), we argued that the Green's function is symmetric $G(x,y) = G(y,x)$ and thus

$$u(x) = \int_a^b G(x,y)f(y)dy.$$

But what if the operator is *not* self-adjoint? We have already seen lots of such examples and in this case the Green's function is *not* symmetric. So we cannot use this argument.

We can avoid this difficulty using the definition of the Adjoint operator and Adjoint Greens function. In particular, from the definition of the Adjoint operator, i.e. $\langle \mathcal{L}^*v, u\rangle = \langle v, \mathcal{L}u\rangle$ for the usual inner product and arbitrary functions u and v taken from the appropriate space, it is evident that

$$\int_a^b \Big[(\mathcal{L}^*v(x))u(x) - v(x)\mathcal{L}u(x)\Big]dx = 0. \tag{2.23}$$

This prompts the definition of the *adjoint Green's function* $G^*(x,y)$, satisfying

$$\mathcal{L}_x^* G^*(x,y) = \delta(x-y). \tag{2.24}$$

Consider the original problem ODE:

$$\mathcal{L}u(x) = f(x). \tag{2.25}$$

Combine these in a similar manner to the approach in Section 0, i.e. $u(x)(2.24) - G(x,y)(2.25)$ and integrate over $x \in [a,b]$. This gives

$$\int_a^b [(\mathcal{L}^* G^*)u - G^* \mathcal{L}u]dx = u(y) - \int_a^b G^*(x,y)f(x)dx$$

From (2.23), the LHS is zero and thus

$$u(y) = \int_a^b G^*(x,y)f(x)dx.$$

Now interchange x and y to give

$$u(x) = \int_a^b G^*(y,x)f(y)dy. \tag{2.26}$$

But how does this relate to what we have shown is our solution, i.e. (2.22)? We need a relationship between the Green's function and the Adjoint Green's function. Fortunately we have the following neat theorem:

Theorem 2.11*:* *The Green's function and the Adjoint Green's function are related via* $G(x,y) = G^*(y,x)$.

Proof: We know that

$$\mathcal{L}G = \delta(x-y), \tag{2.27}$$

$$\mathcal{L}^* G^* = \delta(x-y). \tag{2.28}$$

From the definition of $\mathcal{L}^*$ we have

$$\int_a^b G^*(x,y)\mathcal{L}_x G(x,t)dx = \int_a^b \mathcal{L}_x^* G^*(x,y)G(x,t)dx$$

and using (2.27) and (2.28) we can write

$$\int_a^b G^*(x,y)\delta(x-t)dx = \int_a^b \delta(x-y)G(x,t)dx$$

and therefore using the filtering property of the Dirac Delta function, $G^*(t,y) = G(y,t)$ as required. □

Note that for self-adjoint operators, $\mathcal{L} = \mathcal{L}^*$ and then $G^* = G$ and so $G(x,y) = G(y,x)$, the Green's function is symmetric.

However, this is not required for the general result (2.22). We merely need to use Theorem 2.11 in (2.26) to give

$$u(x) = \int_a^b G(x,y)f(y)dy.$$

Summary

The general solution to the BVP

$$\mathcal{L}u = f(x)$$

subject to homogeneous boundary conditions, say here $u(a) = 0, u(b) = 0$ is

$$u(x) = \int_a^b G(x,y)f(y)dy$$

where $G(x,y)$ is the Green's function, i.e. the solution to the BVP

$$\mathcal{L}_x G(x,y) = \delta(x-y)$$

subject to the corresponding boundary conditions, here $G(a,y) = G(b,y) = 0$.

The Green's function can be determined directly (apply boundary conditions, continuity at $x = y$ and determine the appropriate jump condition on the derivative of $G(x,y)$ to apply at $x = y$) or *if* the operator is of Sturm-Liouville form, then the solution can be obtained by first applying boundary conditions (to find $u_1(x)$ and $u_2(x)$) and the use the explicit forms for $c_1(y)$ and $c_2(y)$ as given in (2.16).

Finally to solve a problem with *inhomogeneous boundary conditions* we can determine the complementary function for the homogeneous ode (with f(x)=0), say this is $u_{CF}(x)$ and then we solution to the full BVP

$$\mathcal{L}u = f(x)$$

subject to homogeneous boundary conditions, say here $u(a) = \alpha, u(b) = \beta$ is

$$u(x) = u_{CF}(x) + \int_a^b G(x,y)f(y)dy.$$