

# NUMERICAL SOLUTION OF PROBLEMS ON UNBOUNDED DOMAINS. A REVIEW \*

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**Abstract.** While numerically solving a problem initially formulated on an unbounded domain, one typically truncates this domain, which necessitates setting the artificial boundary conditions (ABC's) at the newly formed external boundary. The issue of setting the ABC's appears most significant in many areas of scientific computing, for example, in problems originating from acoustics, electrodynamics, solid mechanics, and fluid dynamics. In particular, in computational fluid dynamics (where external problems represent a wide class of important formulations) the proper treatment of external boundaries may have a profound impact on the overall quality and performance of numerical algorithms and interpretation of the results.

Most of the currently used techniques for setting the ABC's can basically be classified into two groups. The methods from the first group (global ABC's) usually provide high accuracy and robustness of the numerical procedure but often appear to be fairly cumbersome and (computationally) expensive. The methods from the second group (local ABC's) are, as a rule, algorithmically simple, numerically cheap, and geometrically universal; however, they usually lack accuracy of computations. In this paper we first present an extensive survey and provide a comparative assessment of different existing methods for constructing the ABC's. Then, we describe a new ABC's technique proposed in our recent work and review the corresponding results. This new technique enables one to construct the ABC's that largely combine the advantages relevant to the two aforementioned classes of existing methods.

Our approach is based on application of the difference potentials method by V. S. Ryaben'kii. This approach allows one to obtain highly accurate ABC's in the form of certain (nonlocal) boundary operator equations. The operators involved are analogous to the pseudodifferential boundary projections first introduced by A. P. Calderon and then also studied by R. T. Seeley. In spite of the nonlocality, the new boundary conditions are geometrically universal, numerically inexpensive, and easy to implement along with the existing solvers.

**Key words.** infinite-domain problems, artificial boundary conditions, pseudodifferential equations, difference potentials method, auxiliary problem, boundary equations with projections.

**AMS subject classifications.** 65-02, 31C20, 65N99, 76M25

## 1. Introduction.

**1.1. Preliminaries.** Artificial boundary conditions (ABC's) furnish a widely used approach for the numerical treatment of boundary-value problems initially formulated on unbounded domains. These boundary conditions are typically set at the external boundary of a finite computational domain once the latter is obtained from the original unbounded domain by means of truncation. Implementation of the ABC's enables one to complete the "truncated problem" and therefore, makes this problem available for solution on the computer.

As has been repeatedly shown by different authors both theoretically and experimentally, the overall accuracy and performance of numerical algorithms, as well as interpretation of the results, strongly depend on the proper treatment of external boundaries. Consequently, the potential range of applications for different ABC's techniques appears broad. Aerodynamics, in which external problems represent a wide class of important formulations, especially when it comes to the analysis of three-dimensional configurations, constitutes a fraction of this range. Besides the hydro- and aerodynamic problems (external flows, duct flows, reacting flows, jets, boundary layers, free surfaces, etc., with aerospace, marine/naval, automotive, meteorological, industrial, and environmental applications), the entire range includes the flows in porous media, filtration (with applications, e.g., to oil recovery), magneto-hydrodynamic flows, plasma (e.g., solar wind), the problems of solid mechanics (in particular, elasticity and aeroelasticity), and the problems of wave propagation (electromagnetic, acoustic, seismic), just to name a few.

For almost any problem formulated on an unbounded domain, there are, generally speaking, many different ways of closing its truncated counterpart. In other words, the choice of the ABC's is never unique. Clearly, the minimal necessary requirement of ABC's is to ensure the solvability of the truncated problem. If, however, we restrict ourselves to this requirement only, then we cannot guarantee that the solution found inside the computational domain will be anywhere close to the corresponding fragment of the solution to the original (infinite-domain) problem. Therefore, we must additionally require of the ABC's that the two solutions be in a certain sense close to each other on the truncated domain. An ideal case here would obviously be an exact coincidence of these two solutions, which leads us to formulating the concept of *exact*

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*ABC's*. Namely, we will refer to the ABC's as being exact if one can complement the solution calculated inside the finite computational domain to its infinite exterior so that the original problem is solved. The concept of exact ABC's appears useful for the theoretical analysis of infinite-domain problems.

**1.2. Model Examples.** To provide a simple one-dimensional example of exact ABC's, we consider a half-line problem with the compactly supported right-hand side (RHS)  $f(x)$ :

$$(1.1a) \quad \frac{d^2 u}{dx^2} - \mu^2 u = f, \quad x \geq 0,$$

$$\text{supp } f(x) \subset [0, X_0],$$

$$(1.1b) \quad u(0) = 0,$$

$$(1.1c) \quad u(x) \longrightarrow 0, \quad \text{as } x \longrightarrow +\infty.$$

Equation (1.1a) is homogeneous for  $x \geq X_0$ ; it has two linearly independent eigensolutions. The first eigensolution  $u^{(1)}(x)$  vanishes as  $x \longrightarrow +\infty$  and the second one,  $u^{(2)}(x)$ , infinitely grows as  $x \longrightarrow +\infty$ . Boundary condition (1.1c) can be met if and only if the increasing mode (eigensolution)  $u^{(2)}(x) = e^{|\mu|x}$  does not contribute to the solution of (1.1a) on the entire semi-infinite interval  $[X_0, +\infty)$ . To prohibit this growing mode and to allow only the decaying one,  $u^{(1)}(x) = e^{-|\mu|x}$ , we require that the two-dimensional vector  $\left[ u(x), \frac{du(x)}{dx} \right]^t$  be parallel to the vector  $\left[ u^{(1)}(x), \frac{du^{(1)}(x)}{dx} \right]^t$  at  $x = X_0$ . The latter requirement can be written as

$$(1.2a) \quad \det \begin{bmatrix} u(x) & u^{(1)}(x) \\ \frac{du(x)}{dx} & \frac{du^{(1)}(x)}{dx} \end{bmatrix} \Big|_{x=X_0} = 0,$$

which is equivalent to the following first-order homogeneous differential relation:

$$(1.2b) \quad \frac{du}{dx} \Big|_{x=X_0} + |\mu| u \Big|_{x=X_0} = 0.$$

Formula (1.2b) obviously yields the desired exact ABC at the artificial boundary  $x = X_0$ . Let us emphasize that relation (1.2b) exactly transfers boundary condition (1.1c) from infinity to the finite boundary  $x = X_0$ . In other words, relation (1.2b) takes into the full consideration the structure of the exterior solution to problem (1.1) without performing any explicit calculations for  $x > X_0$ .

We also mention here another way of looking at the selection of appropriate modes for the far-field solution. The differential operator of (1.1a) can be factorized

$$(1.3) \quad \frac{d^2}{dx^2} - \mu^2 = \left( \frac{d}{dx} - |\mu| \right) \left( \frac{d}{dx} + |\mu| \right),$$

where the first bracket on the right-hand side of formula (1.3) turns into identity for the increasing mode  $u^{(2)}(x)$  and cancels out  $u^{(1)}(x)$ , and the second bracket, conversely, becomes identity for  $u^{(1)}(x)$  and annihilates  $u^{(2)}(x)$ ; therefore, this second bracket and the left-hand side of boundary condition (1.2b) are the same.

Although the previous example provides some insight into how the exact ABC's can be obtained, it still lacks comprehension because of its one-dimensional nature. Therefore, let us consider another example: the Poisson equation on  $R^3$  driven by some compactly supported RHS:

$$(1.4a) \quad \Delta u = f(x), \quad x \in R^3, \quad \text{supp } f(x) \subset B \subset \{|x| < R_0\}.$$

The boundary condition for equation (1.4a) consists of the requirement that its solution vanish at infinity,

$$(1.4b) \quad u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty.$$

Clearly, the three-dimensional problem (1.4) is uniquely solvable and its solution can, in fact, be represented as the Newton volume potential with density  $f(x)$ . For the purpose of actually calculating this solution, we first formally expand it along with the RHS in terms of spherical functions and obtain the following family of ordinary differential equations (ODE's) that describe the radial modes ( $r \equiv |x|$ ):

$$(1.5) \quad \frac{d}{dr} \left( r^2 \frac{d\hat{u}_l}{dr} \right) - l(l+1)\hat{u}_l = \hat{f}_l, \quad l = 0, 1, 2, \dots$$

As all  $\hat{f}_l(r) \equiv 0$  for  $r \geq R_0$ , the solution to each of the equations (1.5),  $l = 0, 1, 2, \dots$ , on the interval  $r \geq R_0$  is actually a superposition of the two linearly independent eigensolutions:  $\hat{u}_l^{(1)}(r) = r^{-(l+1)}$  (vanishes as  $r \rightarrow \infty$ ) and  $\hat{u}_l^{(2)}(r) = r^l$  (does not decrease as  $r \rightarrow \infty$ ). To satisfy boundary condition (1.4b), we need to prohibit the non-decreasing mode and to leave only the decaying one for each  $l$  as  $r \rightarrow \infty$ . Therefore, we require that for all  $l$ ,  $l = 0, 1, 2, \dots$ , the two-dimensional vector  $\left[ \hat{u}_l(r), \frac{d\hat{u}_l(r)}{dr} \right]^t$  be parallel to the vector  $\left[ \hat{u}_l^{(1)}(r), \frac{d\hat{u}_l^{(1)}(r)}{dr} \right]^t$  at  $r = R_0$ , which yields the following countable set of relations (compare to (1.2a)):

$$(1.6a) \quad \det \begin{bmatrix} \hat{u}_l(r) & \hat{u}_l^{(1)}(r) \\ \frac{d\hat{u}_l(r)}{dr} & \frac{d\hat{u}_l^{(1)}(r)}{dr} \end{bmatrix} \bigg|_{r=R_0} = 0, \quad l = 0, 1, 2, \dots,$$

equivalent to another set (compare to (1.2b)):

$$(1.6b) \quad \frac{d\hat{u}_l}{dr} \bigg|_{r=R_0} + \frac{(l+1)}{r} \hat{u}_l \bigg|_{r=R_0} = 0, \quad l = 0, 1, 2, \dots$$

Formulae (1.6b) provide the exact ABC's for equation (1.4a) at the spherical artificial boundary  $\{r = R_0\}$ . In other words, the solution of problem (1.4a), (1.6b) on the ball  $\{r \leq R_0\}$  is exactly the same as the accordingly truncated solution to the original infinite-domain problem (1.4) (i.e., the aforementioned Newton potential) would be if the latter problem was solved first. (Note, there are  $2l+1$  linearly independent spherical functions that correspond to each  $l$ ,  $l = 0, 1, 2, \dots$ ; this does not influence the construction of ABC's (1.6b) since the boundary condition is simply the same for all  $2l+1$  components.) One can easily see that boundary conditions (1.6b) are spatially nonlocal (in the original variables) because the inverse Fourier transform of (1.6b) yields a global expression (along the spherical surface  $r = R_0$ ) that contain some pseudodifferential operator ( $\Psi$ DO).

We additionally note that factorization of the differential operator analogous to (1.3) is possible for (1.5) as well:

$$(1.7) \quad \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - l(l+1) = r^2 \frac{d^2}{dr^2} + 2r \frac{d}{dr} - l(l+1) = \left( r \frac{d}{dr} - l \right) \left( r \frac{d}{dr} + (l+1) \right);$$

the operator on the left-hand side of boundary condition (1.6b) can be obtained from the second bracket on the right-hand side of (1.7) by dividing by  $r$ .

**1.3. General Comments.** We emphasize here that the situation illustrated in the last example of Section 1.2 is fairly general. For many different problems, including those that originate from physical applications, *the exact ABC's are nonlocal*, for steady-state problems in space and for time-dependent problems also in time. The exceptions are rare and, as a rule, restricted to one-dimensional model examples. Moreover, as the standard apparatus for deriving the exact ABC's involves integral transforms (along the boundary) and pseudodifferential operators, such boundary conditions can be obtained easily only for the boundaries of regular shape. (Indeed, returning to the second example above we see that if the shape of the artificial boundary were not spherical, then the separation of variables based on the Fourier expansion with respect to the spherical functions would not be possible.)

From the viewpoint of practical computing, the aforementioned nonlocality of the exact ABC's may imply cumbersomeness and high computational cost. Moreover, geometric restrictions that are typically relevant to the exact ABC's also limit their practical use. Therefore, in spite of the extensive study of various ABC's

methodologies over the last two decades, and even given the high demand for accurate ABC's in many areas of scientific computation, the construction of the ideal boundary conditions, i.e., the exact ABC's that would at the same be computationally inexpensive, easy to implement, and geometrically universal, still remains a fairly remote possibility.

As the exact ABC's are not attainable routinely, the alternative is provided by various approximate local methods, which typically meet the other usual requirements of ABC's besides minimization of the error associated with the domain truncation. As mentioned above, these other requirements are low computational cost, geometric universality (i.e., applicability to a variety of irregular boundaries often encountered in real-life settings), and implementation without difficulties, in particular, readiness in combining the ABC's with the existing (interior) solvers. However, the basic trend in terms of accuracy remains the following: higher accuracy for the boundary procedure requires more of the nonlocal nature of exact ABC's to be somehow taken into account.

In fact, almost any numerical algorithm for setting the ABC's can be thought of as a compromise between the two foregoing groups of requirements that in a certain sense contradict one another. Shifting the balance towards locality and practical efficacy often implies insufficient accuracy; shifting it to the other side, towards highly accurate nonlocal techniques, may often yield cumbersome and all but impractical algorithms. It is not surprising, therefore, that the treatment of external boundaries in modern production computations typically follows the first, local, path. In *computational fluid dynamics (CFD)*, for example, only a few ABC's methodologies out of the wide variety proposed to date can be regarded as the commonly used tools. All of them are either based on the essential model simplifications, e.g., local quasi-one-dimensional treatment in the vicinity of the artificial boundary, or obtained as a localization of some nonlocal ABC's. To meet the overall accuracy requirements when using such simple boundary procedures, one often has to choose the excessively large computational domains.

This paper is a review of different ABC's methodologies that have been published in the literature over the recent years. The review consists of two parts. In the first part (Section 2), we survey the work conducted by different authors in the field of constructing the ABC's, i.e., describe, compare, and assess various available techniques that yield both local and global boundary conditions.

In the second part of this paper, we first briefly comment on the main concepts associated with the generalized potentials and the difference potentials method (DPM) by Ryaben'kii (Section 3). In Section 4, we survey the new ABC's methodology that is based on application of the DPM; it, in particular, includes our own recent work. The DPM-based boundary conditions are global. When applied to solving the steady-state external problems in CFD, they, however, combine the advantages relevant to both global and local approaches. In other words, the principal gain from using the DPM is that the method allows us to simultaneously meet the high accuracy standards of ABC's and the requirements of geometric universality and easiness in implementation.

## 2. General Survey.

**2.1. Global Methods.** In work [1–3], Engquist and Majda develop the time-dependent ABC's for some wave propagation problems, in particular, those described by the wave equation and first-order hyperbolic systems, see [1, 2], and also for the linearized potential transonic flows, see [3]. Their approach is based on representing the solution as a superposition of waves and eliminating all the incoming waves from the solution at the artificial boundary. Since the incoming waves can be interpreted as the reflections from the boundary, the ABC's that prohibit such waves are often called the non-reflecting boundary conditions (NRBC's). The idea of [1–3] essentially implies that the exterior solution is sought for in the class of functions that can be composed of the outgoing waves only; this property, in fact, constitutes the desired far-field behavior of the solution and can therefore be interpreted as the boundary condition at infinity; this boundary condition is then replaced by the ABC's at some finite artificial boundary.

Elimination of the outgoing waves in work [1–3] is done exactly, using the apparatus of integral (Fourier) transforms. In the Cartesian coordinates  $(x, y)$  with the time  $t$ , let us consider the plane wave solution  $u(x, y, t) = u(t\omega + x\xi + y\eta)$ ,  $\omega > 0$ , of the two-dimensional wave equation

$$(2.1) \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The dispersion relation for the solution of this type is  $\omega^2 = \xi^2 + \eta^2$ . The constant phase surfaces for the plane wave are given by  $t\omega + x\xi + y\eta = \text{const}$ , therefore the two-dimensional vector  $(-\xi, -\eta)$  determines

the direction of the wave propagation. Let now the computational domain be the half-plane  $x \leq 0$ . Then, the incoming waves will be those that propagate in the negative  $x$ -direction, i.e., those that have  $\xi > 0$ , and the outgoing waves will be those that have  $\xi < 0$ . Assuming that  $\omega^2 - \eta^2 > 0$ , we can derive the one-way dispersion relation

$$(2.2a) \quad \xi = -\sqrt{\omega^2 - \eta^2}$$

for the waves traveling to the right (outgoing) and the one-way dispersion relation

$$(2.2b) \quad \xi = \sqrt{\omega^2 - \eta^2}$$

for the waves traveling to the left (incoming). In accordance with the selected geometric configuration (planar artificial boundary  $x = 0$ ) we now consider equation (2.1) Fourier-transformed only with respect to the transversal spatial coordinate  $y$  and time  $t$  with the longitudinal coordinate  $x$  left in the physical space, here  $\hat{u}(x) \equiv \hat{u}_{\eta, \omega}(x)$ :

$$(2.3) \quad \frac{d^2 \hat{u}}{dx^2} + \xi^2 \hat{u} = 0.$$

Two linearly independent eigensolutions of equation (2.3) are  $\hat{u}^{(1)}(x) = e^{-i|\xi|x}$  and  $\hat{u}^{(2)}(x) = e^{i|\xi|x}$ . For the waves traveling to the right  $-|\xi| = \xi$  because of relation (2.2a) and therefore, the mode  $\hat{u}^{(1)}(x)$  is outgoing; analogously, formula (2.2b) implies  $|\xi| = \xi$  and consequently, the mode  $\hat{u}^{(2)}(x)$  is incoming. The exact ABC's at  $x = 0$  should explicitly prohibit all the incoming waves, therefore we require that the two-dimensional vector  $\left[ \hat{u}(x), \frac{d\hat{u}(x)}{dx} \right]^t$  be parallel to the vector  $\left[ \hat{u}^{(1)}(x), \frac{d\hat{u}^{(1)}(x)}{dx} \right]^t$  at  $x = 0$ , which, similarly to formulae (1.2a) and (1.6a), can be written as the equality of the Wronskian to zero:

$$(2.4a) \quad \det \begin{bmatrix} \hat{u}(x) & \hat{u}^{(1)}(x) \\ \frac{d\hat{u}(x)}{dx} & \frac{d\hat{u}^{(1)}(x)}{dx} \end{bmatrix} \bigg|_{x=0} = 0,$$

and then reduced to the following first-order homogeneous differential relation (compare to (1.2b) and (1.6b)):

$$(2.4b) \quad \frac{d\hat{u}}{dx} \bigg|_{x=0} + i\sqrt{\omega^2 - \eta^2} \cdot \hat{u} \bigg|_{x=0} = 0,$$

which is local in the transformed space. Boundary condition (2.4b) leaves only the outgoing waves in the exterior solution analogously to how boundary conditions (1.2b) and (1.6b) (see Section 1.2) leave in the exterior solution only the decaying modes. (Note, if one considers the computational domain  $x \geq 0$  instead of  $x \leq 0$  and would accordingly like to cancel out all the waves that propagate in the positive  $x$ -direction, it would merely require changing the sign in formula (2.4b).) Local relations of type (2.4b) are written separately for each plane wave, i.e., for each Fourier-component  $\hat{u} \equiv \hat{u}_{\eta, \omega}$  of the solution. Then, using the inverse Fourier transform one represents the solution as a superposition of plane waves, and in so doing the family of relations (2.4b) (the family parameterized by  $\omega$  and  $\eta$ ) transforms into the nonlocal in both space and time relation that contains a  $\Psi$ DO. The square root in formula (2.4b) will be the symbol of this  $\Psi$ DO. Let us note that the one-way dispersion relations of type (2.2) typically contain some non-rational functions of the wavenumbers; therefore, the formulation of the corresponding exact ABC's (see [1–3]) in physical variables routinely requires the usage of  $\Psi$ DO's, i.e., the boundary conditions appear nonlocal. At the same time, global ABC's of [1–3] can be localized, the localizations have been constructed using various approximations both by Engquist and Majda and by other authors, see Section 2.2.

Let us also note that the operator of (2.3) can be factorized similarly to (1.3) and (1.7):

$$(2.5) \quad \frac{d^2}{dx^2} + \xi^2 = \left( \frac{d}{dx} - i|\xi| \right) \left( \frac{d}{dx} + i|\xi| \right),$$

and boundary condition (2.4b) can be obtained using factorization (2.5). The extension of the approach to the first-order hyperbolic systems is straightforward; the boundary conditions may basically be seen as a requirement that the corresponding vector of unknowns belong to a certain invariant subspace of solutions with the desired far-field behavior; clearly, there are several equivalent linear algebra techniques that allow one to explicitly formulate the latter requirement. Finally, extending the foregoing methodology by Engquist and Majda to the case of variable coefficients requires the use of the approximate factorizations of type (2.5) and formal asymptotic expansions of the symbols of corresponding  $\Psi$ DO's.

Gustafsson in work [4] analyzes another hyperbolic problem, which presents an additional complication from the standpoint of constructing the ABC's. Namely, the initial data and the RHS source terms are no longer required to concentrate inside the computational domain (as in [1–3]) but can also spread beyond the artificial boundary. The separation of variables using Laplace transform in time and Fourier transform in space yields in this case an inhomogeneous (unlike (2.4b)) relation that after the inverse transform serves as a nonlocal exact ABC at the planar artificial boundary.

Halpern in work [5] considers even wider class of problems, namely, the so-called incompletely parabolic systems, i.e., the (small) higher-order parabolic-type perturbations of hyperbolic systems. A particular example of the system from this class is the Navier-Stokes equations for viscous fluid flows. In work [5], the ABC's for the linear half-space incompletely parabolic problem (planar Cartesian artificial boundary) are again obtained by first implementing the Fourier-Laplace transform and separating the variables and then explicitly selecting in the transformed space only those modes that comply with the desired behavior of the solution near infinity. Note, similarly to boundary conditions of [1–3] the ABC's of [4] and [5] can be approximated by some local relations, see the discussion in Section 2.2.

In a series of papers [6–10], Sofronov constructs the exact three-dimensional ABC's for the wave equation at the spherical artificial boundary. Similarly to the work mentioned previously, the approach of [6–10] is also based on using the separation of variables. Spherical coordinates and Fourier's expansion with respect to the spherical functions are employed in space; Laplace's transform is implemented in time. For the finite-difference formulation, the continuous spherical functions are substituted by the eigenvectors of the discretized Beltrami operator on the sphere. These eigenvectors form an orthonormal basis and are called the difference spherical functions, see [11]. The same type of approach has been carried out by Sofronov in work [12, 13] for computation of the inviscid compressible flows in cylindrical wind tunnels. The inflow and outflow boundaries are planar cross-sections of the wind tunnel normal to its axis; the governing Euler equations are linearized outside the computational domain. Obviously, different geometry and different governing equations in work [12, 13] necessitate the use of different spatial eigenfunctions (compared to those of [6–10]) for the separation of variables. The resulting boundary conditions of [12, 13] include the explicit formulas that connect the flow variables at the inflow and outflow artificial boundaries. An important part of work by Sofronov [6–10, 12, 13] is his approach to temporal localization of the exact ABC's that we discuss later.

Note, all the methodologies for exact ABC's presented in [1–10, 12, 13] have rather strict limitations on the shape of the artificial boundary. The limitations are accounted for by the very nature of the techniques used for constructing the ABC's; these techniques involve integral transforms along the boundary and are essentially based on the separation of variables. As a way to achieve more geometric flexibility, the author of [9, 10], for example, proposes to first enclose the actual artificial boundary (not necessarily regular) between two additional spherical boundaries and then use interpolation. We also note that many local methods surveyed in Section 2.2 are much less restrictive from the standpoint of geometry. A highly accurate nonlocal ABC's methodology, which would at the same time be geometrically universal, can be constructed using difference potentials method (DPM, see [14–17]); it is surveyed in Sections 3 and 4 of the paper.

It should finally be noticed that any system of partial differential equations (PDE's) with constant coefficients admits the separation of variables in the Cartesian coordinates, whereas for other coordinates it may or may not be the case for each specific PDE/system. This makes planar Cartesian boundaries the most universal configuration for constructing the nonlocal exact ABC's. Another advantage of the Cartesian coordinates for obtaining boundary conditions of type (1.2b), (1.6b), or (2.4b) and factorizations (1.3), (1.7), or (2.5), is that in the Cartesian case the corresponding eigenmodes ( $u^{(1)}$  and  $u^{(2)}$  for the second-order equations/systems) can always be obtained explicitly as the exponential functions. This is important even in comparison with the other cases, for which the shape of the artificial boundary does admit the separation of variables. Indeed, finding the eigenmodes for these other cases may require the analytical solution of ODE's with variable coefficients, like in the second example of Section 1.2 for the Laplace equation in spherical coordinates. As has been seen, these solutions (i.e., eigenmodes) explicitly enter both the boundary

conditions of type (1.2b), (1.6b), or (2.4b) and the factorizations of type (1.3), (1.7), or (2.5). Note, Lončarić in work [18] had obtained the factorization similar to (1.7) for two-dimensional Laplace's operator in polar coordinates.

Hagstrom and H. Keller in work [19] suggested to consider the exact ABC's as a characterization of data at the artificial boundary in terms of belonging to certain admissible subspaces. The latter are specially defined so that to ensure the solvability of the exterior problem (i.e., the one formulated outside the computational domain) in an initially prescribed class of functions, e.g., those bounded or vanishing at infinity. (Note, both examples of Section 1.2, see formulae (1.2b) and (1.6b), as well as the time-dependent example of [1], see (2.4b), provide for this type of classification by selecting only the appropriate modes in the transformed space as  $x \rightarrow \infty$  or  $r \rightarrow \infty$ .) For some problems formulated on cylinders, Hagstrom and H. Keller calculate these admissible subspaces on the plane artificial boundary normal to the cylinder element. They assume that the coefficients of PDE's in the far field may depend only on the transversal but not the longitudinal coordinate and use the separation of variables based on expanding the solution in terms of the transversal eigenfunctions. The latter appear to be Fourier's harmonics for the simplest case of constant coefficients. In a later work, Hagstrom (see [20, 21]) and Hagstrom and H. Keller (see [22]) further develop the technique of [19] and extend its area of applications. Moreover, following the methodology of [19, 20], Dgaygui and Joly in work [23] develop the exact boundary conditions for calculating the propagation of gravity driven waves in the ocean of constant finite depth (inviscid incompressible irrotational flow with free boundary); the artificial boundary in [23] is assumed linear and vertical, i.e., normal to the bottom of the ocean.

Givoli and J. Keller in work [24] construct the nonlocal exact ABC's for the Laplace equation and for some problems in elasticity. In work [25], J. Keller and Givoli develop similar boundary conditions for the Helmholtz equation, which is then solved by the finite elements method (FEM). The approach of [24, 25] is based on application of the so-called Dirichlet-to-Neumann (DtN) maps, which express the normal derivative of the solution at the artificial boundary in terms of the boundary values of the solution itself so that the particular desired asymptotics of the solution at infinity is enforced. In work [24], this desired asymptotics actually corresponds to the zero limit of the solution at infinity; for the Helmholtz equation (work [25]), the Sommerfeld radiation boundary condition at infinity is specified. The DtN maps can be considered as particular classes of the so-called Poincaré-Steklov operators, these classes correspond to the initially prescribed type of the solution behavior at infinity. The Poincaré-Steklov operators in general are described, for example, in work by Agoshkov [26] (see also the bibliography there). The DtN maps of [24, 25] are first obtained in the general form using Green's formula (integral representation of the solution), for some particular formulations (circular and spherical artificial boundaries) the operators are then calculated analytically by explicitly selecting only the appropriate modes in the Fourier space. The way the selection is performed for each harmonic can actually be interpreted as using the relations similar to (1.6); in the case of the Helmholtz equation, the corresponding (modified) Hankel functions are substituted for the eigensolutions  $\hat{u}_l^{(1)}(r)$  so that the Sommerfeld radiation condition at infinity be met. For example, if the number of space dimensions is three, then  $\hat{u}_l^{(1)}(r) = r^{-1/2} H_{l+1/2}^{(1)}(\mu r)$ , where  $\mu^2$  is the constant (i.e., squared wavenumber) in the Helmholtz equation  $\Delta u + \mu^2 u = 0$ .

We should also note here that the nonlocal exact ABC's for Helmholtz's equation on spherical artificial boundaries have, in fact, been known in the literature; for example, boundary conditions based on the use of Wronskians of the type (1.6a) with the appropriate homogeneous eigensolutions given by the modified Hankel functions as the entries, can be found in book [27] by Zavadsky. Similar boundary conditions have also been used by Ryaben'kii and Sofronov in work [28] as an element of the more elaborate construction associated with the DPM (see Sections 3 and 4). An alternative variational derivation of the DtN maps for Helmholtz's equations (identical to those of [25]) has been given by Harari in a later paper [29]; in this work, the solution is also partitioned by the artificial boundary into the finite interior and infinite exterior parts and the continuity across the artificial boundary is enforced weakly.

In work [30], Givoli and Vigdergauz apply the DtN-based approach to a geometrically more complicated problem; they consider the setup typical for geophysics and use the DtN maps to obtain the exact ABC's for Helmholtz's equation and for the elastostatics system. The computational domain in work [30] is a semi-disk, the exterior domain is accordingly a semi-plane without semi-disk, the artificial boundary is composed of a semi-circle and two semi-infinite straight lines. Another infinite-domain problem for Helmholtz's equation associated with the Earth sciences has been studied in collection volume [31], it involves electromagnetic diffraction on prolonged periodic and quasi-periodic surfaces (like the ocean surface); the corresponding exact nonlocal ABC's can be found in [31, Part II, Chapter 4].

Harari and Hughes in work [32] and Grote and J. Keller in work [33] specially analyze the truncated DtN maps for the Helmholtz equation. Truncation means taking only a certain number of leading terms in the infinite Fourier series that represents the original map, it is often done when discretizing and implementing the DtN map in practice, for example, in the FEM framework. For the Helmholtz equation, this truncation may disturb solvability of the problem and/or result in non-uniqueness of the solution for higher modes, which, in fact, is associated with the possible resonances of the complementary (interior) domain. The authors of [32] and [33] propose different recipes to avoid this undesirable phenomenon. In particular, the uniqueness is guaranteed if a sufficiently high number of terms is taken into account before the truncation (see [32]) or if, for example, the Sommerfeld radiation condition is imposed on higher modes at the finite artificial boundary rather than at infinity (see [33]). Additionally, Grote and J. Keller in work [33] construct the Helmholtz DtN maps for other coordinates besides the cylindrical and spherical, namely, for the elliptic and spheroidal systems; this widens the pool of admissible shapes for the artificial boundaries.

Deakin and Rasmussen in work [34] study the truncated DtN map for the Laplace equation on a cylindrical domain. Their primary concern is to obtain a special form of the boundary operator that would be convenient to incorporate into the sparse finite-difference solver. The sparse modification of the truncated DtN map is obtained in [34] by clustering the coefficients of the original map along both circumferential and axial lines of the grid on the lateral surface of the cylinder; the modified operator does approach the original one as the radius of the cylinder increases.

Generally, we should mention that certain computational requirements originating from the interior solvers may necessitate introducing some changes in nonlocal boundary operators. For example, besides the sparse solvers, another widely used class, the multi-block solvers, may require similar clustering. When developing the clustering algorithm, one can take advantage of the special structure typical for nonlocal boundary operators, we briefly comment on this issue in Section 4.

We also note that the DtN-based approach can be effectively employed not only for solving the infinite-domain problems but for constructing the interface boundary conditions in domain decomposition methods as well. In work [35,36], Engquist and H.-K. Zhao demonstrate that implementation of the appropriate DtN boundary conditions instead of the standard Dirichlet boundary conditions in Schwarz alternating method can noticeably speed up its convergence and moreover, guarantee the convergence even for the case when the neighboring subdomains do not overlap.

In work [37], Givoli generalizes and extends the original DtN-based methodology to allow for the treatment of time-dependent problems. The idea of [37] consists of analyzing (by means of the DtN) the “steady-state” system that arises on the upper time level when one integrates the time-dependent problem by an implicit method. The resulting boundary conditions (see [37]) appear nonlocal in space and local in time; in this particular case the localization in time implies that the overall ABC’s can be referred to only as approximate rather than exact.

Practical implementation of the true exact ABC’s for time-dependent problems generally presents more substantial difficulties if compared to the steady-state case. The primary reason for that is the aforementioned nonlocality of the exact ABC’s not only in space but also in time (see, e.g., [1,2,4–10,12,13]). This nonlocality may cause severe computational problems, mainly because when the numerical solution advances in time the amount of information to be stored in the computer memory for updating the boundary conditions will need to constantly increase. There are, however, some particular classes of problems (e.g., linear hyperbolic systems in odd-dimensional spaces), for which one can restrict the temporal nonlocality of the ABC’s by a fixed time interval. In this connection, we mention work [38] by Givoli and Cohen; they analyze some essentially time-dependent problems for the wave equation and elastodynamics system. The approach of [38] requires introducing two artificial boundaries, one inside another, and directly employs the Kirchhoff integral to calculate the values of the solution on the upper time level at the exterior boundary using the available data at the previous moments of time on the interior boundary as the source terms for the corresponding retarded potentials. Since the fundamental solution of the three-dimensional wave equation is concentrated on the surface of the characteristic cone only (see, e.g., [39]), the integration along the temporally aligned cylindrical artificial boundary needs to be performed only for some finite time interval in the past provided that the size of the computational domain is finite as well (which is always the case). Basically, the approach of [38] is not subject to those geometric limitations (on the shape of artificial boundary) that are typically relevant to other exact time-dependent ABC’s (see [1–10,12,13]). It, however, requires the explicit knowledge of the fundamental solutions, i.e., kernels for the Kirchhoff integrals. Moreover, the implementation of the ABC’s of [38] appears to require that a certain amount of the numerical dissipation be introduced into the interior scheme, otherwise some instabilities may be encountered. We also note that the idea of using



two embedded artificial boundaries and applying the Kirchhoff formula in the framework of time-explicit integration has been earlier used for setting the exact ABC's in scattering problems by Ting and Miksis, see work [40]. In this work, the interval for integration in time was finite as well. A similar approach has later been employed by De Moerloose and De Zutter, see work [41], for solving the time-dependent Maxwell equations on exterior domains. The implementation of the staggered finite-difference strategy by Yee [42] allows the authors of [41] to construct the boundary conditions independently for each component of the electromagnetic field, this, in particular, involves choosing different integration surfaces for different components. Among other approaches based on the explicit use of Kirchhoff's integral formulas we mention work by Mathews and Newhouse [43] and Nedelec [44].

Another technique for setting the exact time-dependent ABC's that involves integration in time has been recently proposed by Radvugin and Zaitsev in work [45]. They consider cylindrical artificial boundaries aligned with the time axis and after Fourier transforming the wave equation in space, end up with the family of one-dimensional equations (with the radius  $r$  and the time  $t$  as independent variables). External boundary conditions for each of the latter are obtained with the help of the specially chosen Riemann function, which is constructed on the  $(r, t)$  characteristic triangle adjacent to the artificial boundary  $r = R$  and can be thought of as a hyperbolic analogue to the classical Green functions. The exact ABC's of [45] contain time integrals of the solution multiplied by some derivatives of the Riemann function for each Fourier mode. Similarly to many other exact ABC's (see, e.g., [1–5]) the boundary conditions of [45] can be approximately localized, see Section 2.2.

As previously mentioned, the fundamental solution of the three-dimensional wave equation is zero both inside and outside the characteristic cone and differs from zero only on its surface. In other words, the fundamental solution has lacunas. These lacunas can be considered a particular manifestation of the general property that if the number of space dimensions is odd and the constant-coefficient hyperbolic system does not have any lower-order terms, then the solution driven by any compactly supported source (initial data and/or RHS) will become zero at any point of space after a finite period of time. The lacunas have, in fact, been used both by Givoli and Cohen in work [38] and by Ting and Miksis in work [40] for restricting the nonlocality of the exact time-dependent ABC's in time. The general possibility of using the lacunas to effectively reduce the required computational effort when calculating the exact ABC's for time-dependent problems has also been pointed out by Ryaben'kii in work [46] (see more comments in Section 4).

Yet another approach to temporal localization of the exact time-dependent ABC's has been proposed by Grote and J. Keller in work [47, 48]. They study the three-dimensional wave equation outside a sphere and first use the expansion in spherical harmonics to reduce the problem to a family of one-dimensional “radial” equations  $\hat{u}_{tt} = (r^2 \hat{u}_r)_r - l(l+1)\hat{u}$  with variable coefficients. Then, a reciprocal of the special transform (see Lamb [50]) that involves both the unknowns and the independent variables is used to reduce each of these equations to a standard one-dimensional wave equation with respect to a new function  $\hat{v}$ :  $\hat{v}_{tt} = \hat{v}_{rr}$ . Finally, the same idea that the exterior solution is to consist of outgoing waves only is employed, and the incoming wave, i.e., the incoming Riemann variable, is explicitly prohibited at the artificial boundary  $r = R_0$  by imposing the following obvious first-order boundary condition:

$$(2.6) \quad \left. \frac{\partial \hat{v}}{\partial t} + \frac{\partial \hat{v}}{\partial r} \right|_{r=R_0} = 0.$$

Indeed, formula (2.6) turns into identity for the waves traveling to the right  $\hat{v} = \hat{v}(t-r)$ , i.e., for the outgoing ones, whereas the waves traveling to the left  $\hat{v} = \hat{v}(t+r)$ , i.e., the incoming ones, are not admitted by (2.6) (the latter would rather satisfy the similar relation but with the opposite sign). Clearly, relation (2.6) is analogous to formula (2.4b). To actually obtain the ABC's for the wave equation in physical variables, the authors of [47] perform the sequence of inverse transforms, which eventually yields the boundary conditions that are local in time and global in space. Unlike the boundary conditions of [37], which are also local in time and global in space, the ABC's of [47] are exact (for any prescribed number of the leading Fourier modes). These ABC's contain first derivatives with respect to  $t$  and higher-order derivatives with respect to  $r$ ; for convenience of the practical implementation the calculation of the higher-order spatial derivatives is replaced by the calculation of the same number of auxiliary quantities that depend only on time and are obtained by integrating in time the special auxiliary system of ODE's. These ODE's, in turn, are driven by the source terms that depend on the solution of the actual problem itself. This circumstance, in fact, does take into account the temporal history of the time-dependent process that is studied in [47, 48] (as is generally relevant to the time-dependent processes). However, the specific setting considered in [47, 48]

allows the authors to effectively eliminate the temporal nonlocality of the ABC's. In particular, on the implementation stage the numerical integration of the aforementioned ODE's does not require storing the solution in the computer memory for more than a few time levels, see work by Grote and J. Keller [48], in which the issues of combining the ABC's of [47] with both the finite-difference and finite-element interior solvers and applying them to actual computations are delineated. In work [49], Grote and J. Keller have generalized the approach of [47, 48] to treating the full three-dimensional system of Maxwell's equations (the shape of artificial boundary is spherical); an interesting feature of this work is that the ABC's have been obtained with the help of decomposing the field into the transverse electric and transverse magnetic components (two vector quantities) rather than reducing the Maxwell equations to a collection of six scalar wave equations (for each field component), the first approach is obviously advantageous from the standpoint of computational cost.

It is also interesting to mention that as shown by Sofronov in work [10], his boundary conditions [6–10] for the wave equation are equivalent to those proposed by Grote and J. Keller in work [47, 48]. The equivalence itself, is, of course, natural, because both approaches yield the exact ABC's, which are unique. On the other hand, the methodology proposed by Sofronov for localizing in time the exact ABC's for the wave equation (see [6–10]) and similar ABC's for the inviscid flows (see [12, 13]) is entirely different from the one of [47, 48]. Sofronov develops the special recurrence formulas for calculating the convolutions in time. This again yields the exact ABC's that are global in space and local in time. Moreover, an important feature of this approach is that the locality in time is not affected by the number of space dimensions. In other words, the ABC's of [6–10, 12, 13] remain local in time for even-dimensional spaces, when Huygen's principle (see [39]) would not hold.

Gustafsson in work [51] and Ferm and Gustafsson in work [52] calculate an inviscid flow of compressible gas in a plane duct with parallel walls. Linearizing the steady-state Euler equations in the far field and Fourier transforming these equations in the cross-stream direction, they obtain the exact nonlocal ABC's at the linear outflow boundary normal to the duct walls. The ABC's of [52] use the integral principle of conservation of mass and guarantee boundedness of the solution downstream of the artificial boundary up to infinity. The boundedness is ensured by selecting only the decaying modes in the Fourier space. In addition, boundary conditions of [52] have been directly applied to treating the corresponding linearized time-dependent problem and the well-posedness has been established using energy method. In work [53], Ferm extends the methodology of [52] and constructs the nonlocal exact ABC's for both upstream and downstream boundaries in the channel. Both boundaries are again linear and perpendicular to the duct walls, although the cross sections of the duct at inflow and outflow may be different. In work [54], Ferm follows the methodology proposed by Engquist and Halpern in work [55] and combines the stationary nonlocal ABC's of [52, 53] with some local time-dependent boundary conditions based on the analysis of characteristics (see, e.g., formula (2.6) and also the comments in Section 2.2). This combination results in a significant speedup of convergence of the pseudo-time iterations to steady state.

The technique by Engquist and Halpern [55] provides the far-field boundary conditions for (dispersive) hyperbolic problems. First, the exact ABC's for the spatial part of the corresponding differential operator are constructed; they can be thought of as boundary conditions expressing the incoming characteristic variables in terms of the outgoing ones. At the same time, the exact NRBC's, i.e., the boundary conditions that fully eliminate the incoming waves, can be thought of as the conditions stipulating that no incoming quantities change in time at the artificial boundary. Indeed, as no changes occur in the incoming variables, this essentially means that nothing can propagate from the boundary into the domain (see also the discussion on local wave models in Section 2.2). Therefore, these NRBC's (in the transformed space) can be written as the requirement that the time derivatives of the incoming Riemann variables be zero. In the combined methodology of [55], these time derivatives are added to the stationary ABC's with a certain weight, which does not affect the corresponding steady-state solution. Numerical experiments by Ferm (see [54]) corroborate that if employed with the exact stationary ABC's for the linearized Euler equations in the duct, the technique by Engquist and Halpern [55] is capable of noticeably increasing the convergence rate of transient iterations to steady state. In work [56], an approach similar to the one of [53] has been proposed by Ferm to construct the exact ABC's at elliptic artificial boundaries for computation of the two-dimensional external Euler flows. In work [57], the technique to speed up the convergence of pseudo-time iterations has been developed for the ABC's of [56]. However, as the methodology by Engquist and Halpern [55] appears not as effective for the external flows as it is for the flows in ducts (probably, because the predominant direction of the wave propagation in the duct can to a reasonable degree of accuracy be regarded as longitudinal/streamwise and therefore, normal to the boundary, whereas for the external flows it may not be so) the modification of [57]

differs from the one of [54] and is based on a slight perturbation of the free-stream Mach number. In work [58], Ferm further studies the optimal balance between the local time-dependent and nonlocal stationary ABC's for the Euler flows in ducts (following [52–54]) from the standpoint of maximizing the convergence rate of transient iterations. It has been shown that for the moderate subsonic Mach numbers the parameters that yield optimal convergence are actually determined by the time required for a small perturbation to travel back and forth along the duct between the inflow and outflow artificial boundaries.

As shown in work [54, 57, 58], when solving a steady-state flow problem by pseudo-time iterations the direct implementation of stationary exact ABC's may result in a relatively slow convergence. Therefore, the author of [54, 57, 58] suggests that the special acceleration techniques (e.g., the one of [55]) be employed in order to obtain an algorithm that would simultaneously be highly accurate (exact ABC's) and computationally effective (fast convergence to steady state). However, when implemented along with some multigrid interior solver for the Euler equations, the nonlocal ABC's of [56] no longer slow down the convergence and therefore, do not require any additional acceleration techniques. This interesting phenomenon has been observed by Ferm in work [59]; the number of multigrid cycles required for reducing the initial error by a prescribed factor appears approximately the same for both the ABC's of [56] and some local characteristic boundary conditions (see Section 2.2). Computational experiments conducted in our own work support and further extend the aforementioned results. As this work is actually surveyed in Section 4, we only note here that we have constructed and implemented in practice the nonlocal highly accurate ABC's for two-dimensional and three-dimensional steady-state external viscous flows calculated on the domains of arbitrary geometry. Our approach is based on application of the DPM by Ryaben'kii, see [14–17]. When combined with the multigrid interior solver by Jameson, Schmidt, Turkel, and Swanson [60–63] in two dimensions and the solver by Vatsa, et al. [64, 65] in three dimensions, the DPM-based ABC's can drastically increase, by up to a factor of three, the convergence rate of the multigrid iteration procedure. A more detailed description of the corresponding results can be found in Section 4, as well as in the publications referenced there.

In work [66], Verhoff and Stookesberry construct the nonlocal ABC's for calculation of the two-dimensional compressible flows in ducts. In work [67], Verhoff, Stookesberry, and Agrawal construct the similar ABC's for inviscid compressible external flow computations. The flow at and beyond the artificial boundary is always assumed subsonic. An interesting feature of the approach of [66, 67] is that although the Euler equations are linearized in the far field, which is typical when constructing the ABC's for external flows, the background solution for linearization is not constant across the exterior domain. More precisely, whereas the far-field background pressure is assumed constant everywhere, the background entropy is allowed to vary in the cross-stream direction. This makes the coefficients of the linearized far-field system variable as well. This system is solved in [67] by means of a Fourier transform along the boundary combined with a certain iteration technique; in so doing, the inflow and outflow parts of the artificial boundary should be explicitly decoupled. This Fourier-based iterative algorithm ensures the right asymptotic behavior of the solution to the linearized equations at infinity and eventually yields the highly accurate nonlocal ABC's for Euler flows; the shape of the artificial boundary in [67] is limited to a composition of the parabolic (upstream) and linear (downstream) segments. In a series of later papers [68–70], Verhoff had upgraded the approach of [66, 67]. A special change of variables allows the full nonlinear thermodynamic relations to be retained in the far-field equations of [68–70], although the linearization against the constant-pressure state is still in effect and the entire resulting system with respect to the new variables is linear. The iterative solution algorithm of [67] has been eliminated in [68–70]. Basically, the non-constant coefficients that appear in the far-field equations of [66, 67] for the case of non-isentropic flows, as well as the original nonlinear thermodynamic relations that are implicitly present in the linear far-field system of [68–70], enable one to accurately take into account the entropy-wake solutions (i.e., the rotational effects) that are relevant to the inviscid treatment of the far field. The shape of the artificial boundary for the O-grid configurations of [68, 70] is presumed circular, and the shape of the C-grid boundaries of [69, 70] is again composed of a parabolic and linear segments. Finally, in work [71], Verhoff had extended the approach of [68–70] to treating the three-dimensional inviscid flows on the domains with specially oriented planar boundaries. Let us also note that as concerns the validity of far-field linearization for different flow cases, some general discussion can be found in work by Verhoff [68, 69], as well as later on in this paper, see Sections 2.2 and 4.

A concise yet very clear description of the concept of nonlocal exact ABC's (based on integral transforms and the separation of variables) for hyperbolic systems on the domains with planar boundaries has been given by Giles in work [72]. He had also considered some local approximations to the nonlocal boundary conditions (see Section 2.2) and described the numerical computations for two-dimensional Euler equations in the context of solving the problems in turbomachinery.

Among other papers devoted to constructing the (nonlocal) exact ABC's, we mention work by Fix and Marin [73], in which the authors solve the Helmholtz equation in an axially symmetric duct and construct the exact ABC's on its lateral boundary expanding the solution in terms of the trigonometric and Hankel functions; work by Poezd and Yakunin [74], in which the time-dependent propagation of waves through a semi-infinite cylindrical guide is considered and the exact solution of the wave equation is used to set the ABC's at the artificial boundary normal to the cylinder axis; work by Zorumski, Watson, and Hodge [75–78], in which the exact ABC's at the transversal planar artificial boundary are developed (using separation of variables) for the Helmholtz equation in a semi-infinite rectangular constant-section duct and then numerically compared (see [78]) against some localized versions of the boundary conditions by Engquist and Majda [1] and Giles [72]; work by Watson and Zorumski [79], in which the method of [75–77] is extended to treat the one-dimensional time-periodic duct acoustic phenomena described by the linearized Euler equations in the far field (the one-dimensional methodology of [79] does admit the multidimensional generalization); work by Jiang and Wong [80], in which the exact NRBC that contains a  $\Psi$ DO is obtained at the planar artificial boundary for a general second-order hyperbolic equation provided that the corresponding dispersion relation is known (the technique of [80] is analogous to that of [1, 2]); and work by Guillaume and Masmoud [81], in which the boundary conditions for Helmholtz-type equations are obtained on a planar cross section of a rectangular waveguide. Hagstrom and Lorenz in work [82] construct the nonlocal exact ABC's at the planar cross-stream artificial boundary for the unsteady compressible isentropic Navier-Stokes equations linearized against the constant background for the case of low speed flows; again, the boundary conditions are obtained by separating the variables and then explicitly eliminating the growing modes in both upstream and downstream directions. Nataf in work [83] calculates the external incompressible viscous flow past a finite body using the vorticity-stream function formulation of the Navier-Stokes equations. The linearized equations are solved outside the rectangular computational domain by the separation of variables. The resulting solution is then used to construct the exact nonlocal ABC's for the Poisson equation that describes the far-field stream function. The boundary conditions for vorticity used in work [83] are local; they have been proposed by Halpern in work [84]. Note, the far-field vorticity is described by the linear advection-diffusion equation; first, the exact nonlocal ABC's for this equation are derived in [84] using cross-stream Fourier's transform and the separation of variables in the Cartesian coordinates; then, these boundary conditions are approximated by some local relations (see Section 2.2). Actually, work by Halpern [84] precedes [5].

A special group of ABC's techniques for elliptic equations is based on the potential theory, the resulting boundary conditions are primarily intended for implementation in the variational framework (FEM). In the methods of this type, the desirable far-field behavior of the solution is guaranteed by the structure of the potentials used for its representation. The corresponding ABC's can be obtained in the form of integral relations along the boundary. An important advantage that one gains using such methods is that the shape of the boundary should not necessarily be regular. The literature on the methods of this kind is broad, we will not concentrate much on it in this paper and rather mention only work by MacCamy [85], in which the global integral boundary condition is obtained for the two-dimensional Helmholtz equation that originates from considering the time-harmonic electromagnetic field, and work by Berger, Warnecke, and Wendland [86], in which the authors solve the transonic full-potential equation for airfoil by FEM and couple its solution at the external artificial boundary with the boundary element solution (solution to the discretization of the boundary integral equation obtained via the Green representation theorem) to the linearized exterior problem for the Prandtl-Glauert equation, see Section 4. In the context of geometrically universal global boundary conditions we also refer the reader to Sections 3 and 4 of this paper, in which we describe the methodology based on the generalized potentials and DPM [14–16].

Han, Lu, and Bao in work [87] study the flow of incompressible viscous fluid in an infinite two-dimensional duct. The computational domain is finite, the adjacent semi-infinite upstream and downstream portions of the duct have linear parallel no-slip walls; both upstream and downstream artificial boundaries are linear and normal to the walls; the flow limits both far upstream and far downstream are the parabolic Poiseuille profiles. The nonlocal ABC's of [87] are constructed directly for the finite-difference formulation of the problem, which is convenient for practice. One interesting feature of the approach of [87] is that the separation of variables is not employed, the authors rather use a successive stream-wise elimination algorithm with matrix coefficients calculated recurrently, which after taking into account the asymptotic values of the solution provides the closure, i.e., the ABC's, for the difference system at a finite artificial boundary. (The iteration count, i.e., the number of elimination steps actually performed, determines how close the resulting nonlocal ABC's will be to the exact ones.) In work [88], Bao and Han set the nonlocal ABC's for the same formulation of the flow problem but this time they do use the separation of variables and then a spectral algo-

rithm for solving the corresponding eigenvalues/eigenfunctions problem. We, however, note that the matrix elimination algorithm of [87] may become prohibitively expensive for three space dimensions and also that the linearization of Navier-Stokes' equations against the Poiseuille flow in work [87] seems erroneous, which may, in our opinion, jeopardize some of the results, although the authors have been able to demonstrate numerically the performance of their boundary conditions.

Clearly, if the exact ABC's are implemented for computations, then the resulting accuracy of the numerical solution should not be affected by the domain truncation. In other words, the exact ABC's can completely eliminate the domain truncation as a factor influencing the results and consequently, make the final accuracy depend only on the properties of the interior algorithm. This, of course, can be very attractive for computational practice. As, however, mentioned above, the exact ABC's are rarely attained in reality. Still, in a more favorable case the exactness is not compromised severely. First, it may occur when the continuous global relations need to be discretized for practical purposes (e.g., truncation of the DtN maps, see above). In so doing, the corresponding error can be controlled and kept sufficiently low so that it does not exceed the discretization error of the interior algorithm. Moreover, the nonlocal exact ABC's can be constructed directly for the discrete formulation of the problem (e.g., by applying the DPM, see Sections 3 and 4) so that the exactness is regarded within the accuracy of discretization and the natural increase of accuracy can be expected as the discrete approximations approach their continuous origins (e.g., when the grid size shrinks). Second, the exactness may be slightly compromised by simplifying the model. For example, if the exterior linearity is not pertinent to the problem but rather introduced as an additional requirement (e.g., far-field linearization for the external flows), then the exactness can only be regarded within the accuracy of linearization. Of course, it often happens that both situations take place for the same formulation. However, numerous computational experiments (see the aforementioned work by Ferm, Givoli, Grote, Gustafsson, Hagstrom, Harari, Hodge, Hughes, H. Keller, J. Keller, Sofronov, Verhoff, Watson, Zorumski, and others, as well as the description of DPM-based approach in Sections 3 and 4) consistently corroborate that the nonlocal exact ABC's, even regarded within the accuracy of discretization and/or linearization, still substantially outperform the simpler local methods (see Section 2.2) from the standpoint of accuracy of the numerical solution. This is, of course, the key advantage of using the global boundary conditions for computations.

In a less favorable and more frequently encountered case, the exactness of the boundary treatment can be compromised to a greater extent. Typically, this implies loss of the global structure. First, it may happen when the geometric limitations relevant to the exact ABC's that are obtained with the help of integral transforms and the separation of variables, become unacceptable for practice. Indeed, the shape of the artificial boundary is typically defined by the discretization, e.g., by the grid, used inside the computational domain for each specific case. As a rule, the grid is generated to reflect some essential geometric elements of the particular problem, for example, it can be fitted to the (inner) solid boundary. In so doing, the shape of the (external) artificial boundary may appear rather complicated and not suitable for successful implementation of the Fourier and/or Laplace transforms. Moreover, very often these shapes cannot be easily modified for convenience of setting the ABC's. This issue, however, can be completely overcome using the DPM, see Sections 3 and 4. Second, even for those situations when the exact nonlocal ABC' can be obtained (regular boundaries), the amount of computer resources required for their implementation may appear considerable, especially as it concerns the analysis of time-dependent problems. These arguments have provided a motivation for the numerous attempts to construct the approximate local ABC's.

**2.2. Local Methods.** The first obvious idea on how to obtain the local ABC's is to develop an approximation to the previously derived global boundary conditions. As mentioned above, the nonlocality of the exact ABC's is typically caused by the fact that these boundary conditions in the transformed space contain special expressions that cannot be mapped back into regular differential operators and give rise to the  $\Psi$ DO's instead. One example is given by formula (1.6b), in which  $(l+1)$  would not correspond to any local operator in physical variables once the inverse Fourier transform in spherical functions has been done. Similarly, the exact ABC's for two-dimensional Poisson equation on the disk look very much like (1.6b) with the only difference that in the circumferential direction one uses classical Fourier series with respect to the complex exponents and the absolute value of the wavenumber  $|l|$  replaces  $(l+1)$  in the corresponding analogue to formula (1.6b) (see, e.g., [15] or [18]). Another example is given by the time-dependent boundary condition (2.4b), in which the square root does not have any local operator as its prototype in physical space. Special expressions of this kind (absolute values, non-integer powers) can basically be seen as symbols of the  $\Psi$ DO's.

Using various asymptotics in the transformed space, one can develop rational (e.g., Taylor or Padé) approximations to these symbols, then the resulting boundary conditions in physical variables become dif-

ferential, i.e., local. This idea has been implemented by many authors for different problems. For example, under the assumption of long transversal waves/high temporal frequencies, i.e.,  $\eta/\omega \ll 1$  (see notations in the beginning of Section 2.1), Engquist and Majda in work [1] obtained a sequence of local boundary conditions. The first two members of this sequence are

$$(2.7a) \quad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \Big|_{x=0} = 0$$

and

$$(2.7b) \quad \frac{\partial^2 u}{\partial t^2} - \frac{1}{2} \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial t} \Big|_{x=0} = 0.$$

Boundary condition (2.7a) correspond to the first-order Taylor approximation of the square root of (2.4b):  $\sqrt{\omega^2 - \eta^2} = \omega (1 + O(\eta^2/\omega^2))$ , and boundary condition (2.7b) correspond to the second-order Taylor approximation:  $\sqrt{\omega^2 - \eta^2} = \omega (1 - \frac{1}{2}\eta^2/\omega^2 + O(\eta^4/\omega^4))$ ; in both cases for obtaining formulas (2.7) the remainder terms of the Taylor expansion are neglected. The next member of the sequence (2.7a), (2.7b) is obtained in [1] using the Padé approximation of order  $O(\eta^6/\omega^6)$ .

Notice that boundary conditions (2.6) and (2.7a) formally coincide, although the former is set in the transformed space and the latter in physical space. In fact, boundary condition (2.6) is an exact NRBC for the one-dimensional wave equation and therefore leads to the exact ABC's after the inverse transform as well, whereas boundary condition (2.7a) is exact for the plane waves propagating along the  $x$  axis and for these waves only. In other words, it completely disregards the transversal coordinate  $y$ , which is one way of physically interpreting the first-order Taylor expansion used for deriving (2.7a). The second-order Taylor expansion leads to boundary condition (2.7b), which contains second-order derivatives. This situation is typical for the approaches based on the rational approximation to the symbols of  $\Psi$ DO's, the resulting local differential expressions after the inverse transform may contain derivatives of the solution up to the order of approximation. The corresponding local boundary conditions are often said to have the same order as well.

By implementing different rational approximations in the space of dual variables (wavenumbers/frequencies), Gustafsson in work [4] and Giles in work [72] develop the approximate local ABC's for hyperbolic systems on the domains with planar boundaries. A similar approach is also employed by Jiang and Wong in work [80] for second-order hyperbolic equations, Halpern in work [84] for advection-diffusion equations, Jin and Braza in work [89] for incompressible shear flow calculations, Blaschak and Kriegsmann [90] for the Maxwell equations (more precisely, for the Maxwell equations inside the computational domain coupled with the wave equation in its exterior), Kröner in work [91] for the linearized Euler equations, Dgaygui and Joly in work [23] for the linear water waves in the ocean of a finite depth, Johnsen and Lynch in work [92] for the shallow water problems, Hadley in work [93] and Schmidt and Deuffhard in work [94] for the problems of light beam propagation in optical fibers (Schrödinger's and Fresnel's equations). Note, work by Dgaygui and Joly [23], Johnsen and Lynch [92], Hadley [93], and Schmidt and Deuffhard [94] deal with the case of dispersive waves, this case has also been studied by Higdon in work [95] (see below) and by Engquist and Halpern in work [55]. Clement in work [96] uses the low-frequency asymptotics to derive the so-called piston-like absorbing boundary conditions to calculate the time-dependent free-surface potential flow; the piston-like boundary conditions are local in time. (For the high-frequency part of the wave spectrum Clement in [96] uses the methodology of dumping layers, see the discussion in Section 2.3.) Mur in work [97] employs the approximate local ABC's identical to those of [1] for solving the external problems of electromagnetic scattering. Again, as the scheme by Yee [42] is used in [97] for computing the electromagnetic fields, the external boundary conditions need to be set for the tangential components of the electric field only; each of the latter independently satisfies the three-dimensional wave equation.

Approximations with respect to multiple parameters (not only dual variables but also coefficients of the equations) are often used as well. For example, Halpern in work [5] first requires that the multiplier before the parabolic terms (i.e., viscosity) in the incompletely parabolic system (e.g., linearized Navier-Stokes' equations) tend to zero, which still leaves the boundary conditions of [5] nonlocal but already approximate rather than exact, and then uses the approach similar to the one of [1] to obtain local boundary conditions for the system under consideration. The approach by Halpern [5] has been actually applied by Tourrette in work [98] to solving a number of model one- and two-dimensional problems for the linearized compressible

Navier-Stokes equations by finite differences; the performance of the corresponding boundary conditions (obtained on the basis of the low viscosity/ small spatial wavenumbers/high temporal frequencies approximation) has been studied experimentally in [98] for these model cases (the two-dimensional problems were considered on rectangular domains). Hagstrom and Lorenz in work [82] also use a succession of several different approximations to derive the local ABC's for low speed viscous flows. Their approach requires that the Reynolds number be high and the Mach number be small; in addition, they propose a few ways to approximate the standard “hyperbolic” square root (see (2.2)) so that the approximation be accurate not only for the high temporal frequencies but for the long time intervals as well. It should be noted, however, that the problem of constructing the local time-dependent ABC's that would sufficiently accurately approximate the nonlocal exact boundary conditions for all time scales generally appears hard and for some particular two-dimensional settings may not have a solution at all, see [82] for more comments. The question of constructing the so-called doubly asymptotic approximations, i.e., local ABC's that would apply for both the case of short waves,  $(ka)^2 \gg 1$ , and long waves,  $(ka)^2 \ll 1$ , here  $k$  is the characteristic wavenumber and  $a$  is the characteristic size, has also been discussed by Geers in [99] in the context of three-dimensional exterior acoustics.

Obviously, the ABC's obtained using rational approximations to the symbols of  $\Psi$ DO's inherit the same geometric limitations as relevant to the original nonlocal ABC's. However, locality of the approximate ABC's typically provides for more convenience from the standpoint of computing.

Convergence of the local approximations to the corresponding  $\Psi$ DO's has been studied by Hagstrom. In particular, in work [100] he had shown that over the finite intervals of time the sequence of Padé approximations used by Engquist and Majda in work [1] does converge to the original operator as the approximation order increases.

Typically, high-order local boundary conditions can be factorized into a product of the first-order terms. For example, substituting the wave equation  $u_{yy} = u_{tt} - u_{xx}$  itself into formula (2.7b), we obtain

$$(2.7c) \quad \frac{\partial^2 u}{\partial t^2} - \frac{1}{2} \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial t} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x^2} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right)^2,$$

which yields the square of the operator on the left-hand side of (2.7a). In fact, Higdon in work [101] had demonstrated that any high-order boundary condition of [1] can be represented as a power of (2.7a). Clearly, representation (2.7c) implies that similarly to the first-order boundary condition (2.7a), the second-order boundary condition (2.7b) is completely transparent only for the waves with normal incidence (i.e., those traveling along the  $x$  axis) and any wave with oblique incidence will necessarily cause some reflections. Although Engquist and Majda in work [1] have shown that for any given angle of incidence  $\alpha \neq \pi/2$  boundary condition (2.7b) will generate less reflections than boundary condition (2.7a), the very possibility of factorizing the high-order operators suggests another viewpoint on the structure of local boundary conditions.

This viewpoint has been taken by Higdon in work [101]. He suggested to construct the approximate local ABC's so that they be transparent for an initially prescribed finite set of the angles of incidence  $\alpha_j$  (rather than for all angles, as in the case of exact ABC's). His approach results in boundary conditions of the type

$$(2.8a) \quad \prod_j \left( \cos \alpha_j \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right) \Big|_{x=0} = 0,$$

where each term of the product in (2.8a) annihilates the following two and only these two plane waves:  $u = u(t - \cos \alpha_j x - \sin \alpha_j y)$ , which travels in the direction  $(\cos \alpha_j, \sin \alpha_j)$ , i.e., at the angle of incidence  $\alpha_j$  to the boundary  $x = 0$  and  $u = u(t - \cos \alpha_j x + \sin \alpha_j y)$ , which travels in the direction  $(\cos \alpha_j, -\sin \alpha_j)$ , i.e., at the angle of incidence  $-\alpha_j$  to the boundary  $x = 0$ . Indeed,

$$(2.8b) \quad \left( \cos \alpha_j \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right)^2 = \left( \frac{\partial u}{\partial t} + \cos \alpha_j \frac{\partial u}{\partial x} + \sin \alpha_j \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial t} + \cos \alpha_j \frac{\partial u}{\partial x} - \sin \alpha_j \frac{\partial u}{\partial y} \right),$$

and the first bracket on the right-hand side of (2.8b) is exactly the same as the left-hand side of (2.7a) only for the direction of spatial differentiation  $x' = x \cos \alpha_j + y \sin \alpha_j$  rather than  $x$ , whereas the second

bracket corresponds to  $x' = x \cos \alpha_j - y \sin \alpha_j$ . Under boundary conditions (2.8a), the pre-selected plane waves with the angles of incidence  $\pm \alpha_j$  travel freely through the boundary and leave the computational domain without reflections, other waves generally produce some reflections from the artificial boundary. In certain cases (see [101]), the overall reflection coefficient for boundary condition (2.8a) (distributed angles of incidence) can be lower than the one for the same order boundary condition of [1] (normal incidence). In the subsequent work [102], Higdon has shown that the boundary conditions similar to (2.8a) will also apply to the vertical artificial boundary in the medium with horizontal stratification (a collection of layers with different wave speeds, which is frequently the case of interest for geophysics). Indeed, boundary conditions (2.8a) have been obtained for the wave speed  $c = 1$ , in the case of different wave speeds the factor  $c$  should multiply the spatial derivative in (2.8a); the applicability of these ABC's to the case with stratification is accounted for by the fact that if the plane wave is transmitted through the horizontal interface from the layer with the propagation speed  $c$  to the layer with the propagation speed  $c'$ , then the incidence angle changes accordingly so that  $c / \cos \alpha_j = c' / \cos \alpha'_j$ . Furthermore, as shown by Higdon in [95, 103], boundary conditions similar to (2.8a) will also apply to the case of dispersive waves, for example those governed by the equation  $u_{tt} = c^2 \Delta u - f^2 u$ . In this case the author recommends  $\alpha_j = 0$ , however, the user should instead pick up the set  $c_j$  of group velocities (and accordingly, frequencies), for which the boundary conditions

$$(2.8c) \quad \prod_j \left( \frac{\partial u}{\partial t} + c_j \frac{\partial u}{\partial x} \right) \bigg|_{x=0} = 0$$

will guarantee the exact absorption. The implementation of (2.8c) therefore requires some a priori knowledge of the range of dominant group velocities and frequencies for the problem, which, however, does not appear restrictive for applications.

The results similar to those of Higdon [101] can be found in a more general form in work by Jiang and Wong [80]. We also emphasize that the ABC's by Higdon [101] have actually been constructed directly for the discrete formulation of the problem, on the basis of the dispersion relation for the finite-difference scheme. (Formula (2.8a) is, in fact, the corresponding differential approximation.) This feature is advantageous as it eliminates the entire step from the general numerical procedure, namely, the need to discretize the continuous boundary conditions.

An alternative approach to approximating the exact ABC's consists of retaining only a few leading terms in the far-field asymptotic expansion of the solution in physical space and then using the obtained truncated expansion to set the ABC's. This technique may essentially reduce the required computational effort in comparison with the cost of the original exact ABC's. The idea of the this type was employed by Sa and Chang in work [104] to set the ABC's for vorticity when integrating the incompressible Navier-Stokes equations in the exterior of a cylinder. Burkhart in work [105] and Burkhart et al. in work [106] derive an asymptotic expansion for the finite-difference fundamental solution of the three-dimensional Laplace operator on a Cartesian grid and then use several leading terms of this expansion to set the ABC's for a discrete external flow problem solved within the full-potential framework. Wubs, Boerstoele, and Van der Wees in work [107] use a Fourier representation of the far-field solution to the two-dimensional Laplace equation (circumferential modes are the complex exponents and radial modes are the decaying power functions) to calculate the potential flow around an airfoil. The ABC's of [107] are again derived from the first few leading terms of the expansion; as the artificial boundary approaches the airfoil, more terms are required to maintain the accuracy. The so-called point-vortex model, which has been described by Thomas and Salas in work [108] and which is extensively used in the today's CFD, is also based on the idea of asymptotics. Specifically, the leading term of the far-field expansion for the linearized flow potential is used to calculate the velocity projections at the external boundary when computing the two-dimensional compressible lifting flows. This leading term is proportional to the circulation of the flow. Finally, we note that even earlier Klunker in work [109] had studied the three-dimensional asymptotic expansion for the potential of velocity perturbations in the compressible flow around a thin lifting wing and, in particular, suggested that the leading term of the expansion (the so-called "horseshoe vortex", see Section 4) be used for setting the far-field boundary conditions. However, Klunker's work [109] does not address the actual numerical implementation of the algorithm.

Asymptotic methods may often require explicit knowledge of the coefficients that multiply the corresponding terms in the expansion (the ones that are retained in the far-field representation). In CFD, these coefficients are typically obtained through the boundary conditions on the surface of the immersed body.



For example, the value of the circulation for the point-vortex model of [108] is proportional to the aerodynamic lift, which is calculated by integrating the pressure along the surface. Moreover, asymptotic methods are typically constructed on the basis of the linear (or linearized) equations. In certain cases, however, one takes into account the nonlinear corrections as well. For example, when analyzing the transonic limit  $M_0 \rightarrow 1$  ( $M_0$  is the free-stream Mach number) for the compressible potential flow in small perturbations, some second-order terms with respect to the perturbations should formally be retained in the governing differential equation along with the first-order terms (see, e.g., book by Cole and Cook [110]). This leads to the nonlinear Kármán–Guderley equation rather than the linear Prandtl–Glauert equation (the latter is valid for smaller  $M_0$ ). For two-dimensional external flows (e.g., flows around airfoils) that are described by the Kármán–Guderley equation, it turns out that the nonlinear corrections to the leading linear lift-based term  $\sim -\Gamma\theta/2\pi$  (see [108]) in the far-field expansion of the potential ( $\Gamma$  is the flow circulation,  $\theta$  is the polar angle) contain the terms proportional to  $\log r/r$  ( $r$  is the polar radius), which formally decay slower than the next linear term  $\sim 1/r$  as  $r \rightarrow +\infty$ . This circumstance gave reasons to Drela in work [111] and Giles and Drela in work [112] to include the leading nonlinear correction terms in their simplified far-field potential model for the compressible airfoil calculations. (Note, the entire series that represents the behavior at infinity of the potential function of a two-dimensional subsonic compressible flow has been accurately constructed by Ludford in work [113] using the hodograph plane techniques.) More discussion on the far-field linearization for transonic flows can be found in Section 4 of the paper.

Another group of local ABC's techniques that use the asymptotic form of the solution has been introduced by Bayliss and Turkel in work [114–116] for time-dependent problems (wave-type and Euler's equations) and Bayliss, Gunzburger, and Turkel in work [117] for steady-state problems (Helmholtz's and Laplace's equations). The proposed methodology is independent and does not relate directly to local approximations of the previously constructed exact ABC's. Moreover, unlike some of the aforementioned asymptotic methods, the approach of [114–117] does not require the explicit knowledge of the coefficients in the asymptotic expansion. In fact, the authors of [114–117] do not use the asymptotic form of the solution directly to set the ABC's, they rather construct a set of special local differential relations that identically cancel out a prescribed number of leading terms in the corresponding series. Being applied at the artificial boundary, these relations provide for the approximate local ABC's. For example, for the three-dimensional wave equation  $u_{tt} = \Delta u$ , the far-field asymptotic solution in spherical coordinates  $(r, \theta, \phi)$  is given by

$$(2.9a) \quad u(t, r, \theta, \phi) = \sum_{j=1}^{\infty} \frac{u_j(t - r, \theta, \phi)}{r^j}$$

and the  $m$ -th member of the family of operators

$$(2.9b) \quad B_m = \prod_{l=1}^m \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{2l-1}{r} \right)$$

can be shown to generate the error of order  $r^{-2m-1}$  when applied to the series (2.9a):  $B_m u = O(r^{-2m-1})$ . Clearly, if a sufficiently high number of leading terms in the expansion (2.9a) is to be canceled out, then the boundary condition  $B_m u|_{r=R_0} = 0$  would contain derivatives of the accordingly high order even so the order of the original wave equation is only two. This is consistent with the foregoing results obtained using rational approximation to the  $\Psi$ DO's symbols: higher-order approximations yield higher derivatives in the corresponding ABC's. Let us also note that for the two-dimensional wave equation the first operator  $B_1$  of the sequence analogous to (2.9b) would read  $B_1 = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r} \right)$  (see [116]); the subsequent boundary condition  $B_1 u|_{r=R_0} = 0$  coincides with the one obtained by Engquist and Majda in work [1] using the approximate factorization of type (2.5) for the wave equation in polar coordinates and the asymptotic expansion of the corresponding  $\Psi$ DO symbol. Boundary conditions of [114–117] have later been modified by Pinsky and Abboud for both physical and adjoint formulation and employed in work [118] for calculation of the time-dependent acoustic fields in the variational context of FEM.

The approach similar to the one of [114–117] has been developed by Peterson in work [119] and Mittra et al. in work [120] for the problems of computational electro-magnetics (CEM). For spherical artificial boundaries, Peterson (see [119]) had constructed a family of local ABC's that apply to solving the vector analogue of Helmholtz's equations, which describe the time-harmonic three-dimensional electric or magnetic

field. Mittra et al. in work [120] developed both scalar and vector local ABC's analogous to those of [117] for Helmholtz-like equations considered outside a sphere; they also proposed a modification (based on the use of local near-boundary coordinate systems) to handle non-spherical boundaries for the case of slender scatterers.

Tam and Webb in work [121] used the  $O(r^{-3/2})$  far-field asymptotics for the solution of the two-dimensional linearized Euler equations and constructed the first-order local ABC's for calculation of acoustic fields in uniformly moving media. The inflow and outflow boundaries in work [121] are treated differently because the entropy and vorticity waves are present only at the latter (entropy and vorticity are transported downstream along the streamlines). For the stationary medium, the purely acoustic (i.e., inflow) part of the boundary conditions of [121] actually reduces to the earlier obtained first-order boundary conditions by Bayliss and Turkel [116] or Engquist and Majda [1]; in fact, for this case (stationary) the acoustic boundary conditions of [121] could have been derived by first obtaining the wave equation for pressure from the linearized Euler system and then exactly repeating the derivation of the first term of [116]. In work [122], Tam and Webb used the approximate far-field asymptotic representation  $O(r^{-5/2})$  of the finite-difference solution to the two-dimensional Helmholtz equation on a square-cell grid and constructed the ABC's that for the short waves perform better than those obtained on the basis of asymptotic representation for the continuous solution (see [117]). Actually, this result is by no means unexpected because the short waves on the grid (i.e., the waves that have only a few nodes per wavelength) always differ significantly from the corresponding continuous modes and therefore, specifying the continuous boundary conditions for these Fourier components will inevitably introduce some inconsistencies. On the other hand, if both the interior solver and the boundary conditions are constructed in the same finite-difference framework, then for any particular grid there are no inconsistencies of the aforementioned type and at the same time one can expect convergence of the difference solution to the continuous one as the grid size shrinks provided that the finite-difference far-field asymptotics used for the ABC's does converge to the corresponding continuous asymptotics. (We should note, however, that the issue of grid convergence has not been studied in work [122].) In the context of constructing the ABC's directly for the discrete scheme as opposed to deriving the continuous boundary conditions and then discretizing, we also mention work by Colonius [123], in which the author shows that for the dispersive discretizations (which is very often the case in computational practice) the ABC's must account for the fact that different waves travel with different speeds on the grid even when the original continuous model has only one group velocity. The specific example considered in [123] is the one-dimensional advection equation, and it turns out that for the poorly resolved waves, their speed of propagation may even change sign. The methodology of [123] takes this behavior into account by explicitly splitting the difference solution into two components traveling in the opposite directions.

Note, asymptotic boundary conditions of [114–117], as well as other related methodologies (see [119–122]) are derived in spherical coordinates. Bielak, Kallivokas, and MacCamy in work [124, 125] have been able to essentially reduce this geometric limitation; for a convex smooth artificial boundary in three dimensions they introduce a new coordinate system, for which this artificial boundary is a coordinate surface and the third coordinate direction is given by the normals to the artificial boundary. In these new coordinates the variables for the wave equation separate and the authors of [124, 125] can explicitly write down the first three members of the sequence of operators similar to  $B_m$  (see (2.9b)); these operators, of course, contain metric coefficients of the surface (artificial boundary). For improving the stability properties of the resulting algorithm, a special artificial parameter (that essentially controls the dissipation) has to be introduced into the structure of boundary conditions [124, 125]. We also note that when the artificial boundary reduces to a plain sphere, the boundary operators of [124, 125] do reduce to the genuine  $B_m$  of (2.9b). More details on the derivation and implementation of asymptotic boundary conditions for the three-dimensional wave equation in non-spherical geometries can be found in work by Kallivokas, Bielak, and MacCamy [126], which, in turn, is based on the earlier two-dimensional approach by Barry, Bielak, and MacCamy [127] and Kallivokas and Bielak [128]. Work by Kallivokas, Tsikas, and Bielak [129] uses the method of images and extends the methodology of [124–126] to the case of scattering in a semi-space (rather than truly unbounded 3D space) with a planar boundary of the free surface type.

In work [130], Abarbanel, Bayliss, and Lustman used the linearization of Navier-Stokes' equations against the approximate wake-type downstream solution to construct local ABC's for the viscous flow over a flat plate. The ABC's of [130] are based on selecting the long-wave modes from the asymptotic expansion of the solution to the corresponding linearized system. The approach of [130] has been further generalized by Danowitz in work [131] for a wider class of external viscous flows than only the flows over a flat plate.

Let us also note that the apparatus of asymptotic expansions for constructing the approximate ABC's has

been extensively used by Hagstrom in work [20, 21, 132], Hagstrom and H. Keller in work [22], and Hagstrom and Hariharan in work [133]. In the recent work [134] (as well as in an earlier paper [135]), Hariharan and Hagstrom propose the asymptotic approach that they call progressive wave expansions; this approach yields a sequence of successively more accurate local boundary conditions analogous to those of Bayliss and Turkel, see formula (2.9b). An important difference, however, is that boundary conditions of [134, 135] do not contain the derivatives of order higher than the first in the direction normal to the artificial boundary (instead, they contain higher-order tangential derivatives), which is convenient for implementation.

An alternative to deriving a sequence of successively more accurate boundary conditions that contain the derivatives of higher and higher orders (like the operators  $B_m$ , see (2.9b)) has been proposed by Safjan in [136]; instead, he suggested to consider a sequence of problems with inhomogeneous local low-order (Robin) boundary conditions and solve the entire problem iteratively: the solution from the previous level drives the boundary conditions on the next level. The methodology has been actually implemented in the FEM framework and did not require more than three to four iterations for achieving the accuracies on the level of a fraction of one percent.

For the class of time-dependent boundary conditions that involve the integration in time, there are also ways of constructing the successively more accurate families of local approximations. For example, the nonlocal exact ABC's by Radvogin and Zaitsev [45] are based on the use of Riemann's function (see Section 2.1). The first- and second-order approximations to the exact Riemann function plugged into the integral term of the exact ABC's yield in work [45] two different local boundary conditions. The first-order one coincides with the radiation (inflow) boundary condition by Tam and Webb [121], which, in turn, is the same as the two-dimensional boundary condition  $B_1 u|_{r=R_0}$  of [116] and [1] for the case of stationary medium.

Truncation of the DtN maps (see, e.g., [32, 33]) that often arises as a natural way to discretize the exact nonlocal DtN-based ABC's can also be thought of as retaining only a certain number of leading terms in the far-field asymptotic representation of the solution. However, the "genuine" asymptotic methods typically use much smaller numbers of terms than the methods associated with the DtN truncation do; in the latter case, the number of the retained terms is prescribed by the discretization (the truncated map maintains exactness for as many leading terms of the expansion as, for example, the number of circumferential grid nodes is). Moreover, even when the map is truncated the original DtN-based methodology still involves the Fourier transform and therefore preserves nonlocality of the ABC's, whereas many genuine asymptotic methods, as mentioned above, explicitly use the coefficients of the expansion, which still implies (indirectly) the calculation of some integrals (those that actually replace the Fourier coefficients, as does the aerodynamic lift in work [108]) but in the end formally leaves the ABC's local.

However, local ABC's equivalent to the truncated DtN maps can be developed as well; these ABC are exact for an initially prescribed number of Fourier modes, after which the map is truncated. Boundary conditions of this type have been constructed by Givoli and J. Keller in work [137] for the time-harmonic elastodynamics equations and by Harari and Hughes in work [32] for the simpler case of scalar Helmholtz's equation (in both cases the artificial boundary is circular because the methodology first requires construction of the exact DtN-based ABC's). Recall, the exact DtN-based ABC's are given by the trigonometric series for the normal derivative of the solution; the coefficients of this series are chosen so that the desirable far-field behavior of the solution be guaranteed. For each harmonic, these coefficients can be obtained similarly to how it has been done for the Laplace equation in Section 1.2, see the factors  $(l+1)/r$  in formula (1.6b). For Helmholtz's equation, the homogeneous eigensolutions  $\hat{u}_l^{(1)}(r)$  and  $\hat{u}_l^{(2)}(r)$  are the modified Hankel functions, therefore, the coefficients of the DtN map involve Hankel's functions and their first derivatives (see Section 2.1). The truncated DtN map is actually a finite trigonometric series of the same type and it appears possible (see [32, 137]) to rewrite it as a (weighted) finite sum of the high-order tangential derivatives of the solution (i.e., those calculated with respect to the polar angle). This, in turn, requires recalculation of the coefficients, which for the truncated case results in inverting the matrix of the same order as the level of truncation is. Having the high-order derivatives in the ABC's is an inevitable consequence of the localization in the approach of [32, 137].

In work [138], Givoli and J. Keller discuss the construction of special finite elements with the increased regularity on the boundary that would potentially allow for an effective numerical implementations of local ABC's with high-order tangential derivatives. In so doing, the smoothness of the elements across the computational domain is not uniform, less regularity is required away of the boundary in the interior. The performance of the methodology of [138] has been numerically tested by Givoli, Patlashenko, and J. Keller

in work [139] for two-dimensional Laplace's and Helmholtz's equations and in work [140, 141] for three-dimensional Helmholtz's equation both on the open exterior domains and in the wave guides with parallel planar walls.

An alternative approach to localizing the DtN boundary conditions has been recently proposed by Givoli and Patlashenko, see [142]. Given a special form of local differential operator on the boundary (with first-order radial and higher-order tangential derivatives), they optimize its initially undetermined coefficients so that in the sense of  $L_2$  norm it provides for the best approximation of the original DtN map. Two versions of the technique are proposed: solution-independent and solution-weighted, in the latter case some a priori information is required. When the number of coefficients to be determined by optimization is the same as the number of harmonics in the solution prescribed by the discretization, then the approach of [142] recovers the same truncated DtN map as obtained in [32, 137]. Another topic mentioned by Givoli during the presentation of [142] is the implementation of DtN boundary conditions for nonlinear problems. For some particular classes of problems, the small (lower-order) nonlinearities are actually treated as the source terms. Their contribution is taken into account by calculating the volume integrals using Laguerre-type quadratures; then, the resulting nonlinear system on the boundary is solved iteratively.

Let us also note that for sufficiently high orders of approximation, different localization methodologies generally lead to different ABC's. For example, the first two boundary conditions of [114] and the first two polar boundary conditions of [1] (with time derivatives replaced by  $-i\omega$ ) can be reduced to the corresponding localized DtN form of [137]. As, however, mentioned in [138], the form of boundary conditions suggested in [137] is symmetric, whereas the boundary conditions of [114] are not symmetric for the orders higher than two. This, apparently, reflects the flexibility that exists in constructing the annihilating operators for higher orders of approximation. On the other hand, local ABC's by Feng Kang [143] appear to have the same form as the later obtained boundary conditions of [32, 137]; because of its symmetry, this form fits better into the FEM framework. Another approach to localization that cannot be reduced to those previously mentioned for high orders of approximation has been proposed by Grote and J. Keller in work [33]. Instead of the standard system  $r^{-1/2}H_n^{(1)}(\mu r)$ ,  $n = 0, 1, 2, \dots$  of radial modes for the two-dimensional Helmholtz equation they use an alternative form of expansion with respect to  $r^{-n}H_0^{(1)}(\mu r)$  and  $r^{-n}H_1^{(1)}(\mu r)$ ,  $n = 0, 1, 2, \dots$ , which leads to a different form of annihilating operators and provides better performance (compared to [117]) for small  $\mu r$  ( $\mu$  is the wavenumber in the Helmholtz equation). The annihilating operators of [33] contain high-order radial derivatives. Localizations of the time-dependent ABC's that require high-order spatial derivatives have also been obtained and tested by Grote and J. Keller in work [47, 48].

The primary reason for using local as opposed to global ABC's is greater convenience of implementation, including readiness in combining the boundary conditions with the existing interior solvers. Of course, in so doing numerical accuracy of the resulting solution is expected to be somewhat lower than the one provided by the exact ABC's. In addition, higher-order approximate boundary conditions are generally expected to provide better accuracy than the lower-order ones; the accuracy gradually improves as the approximate ABC's approach the exact boundary conditions. These expectations have, in fact, been corroborated computationally in the experiments conducted by the authors of the aforementioned papers. As concerns practical algorithmic issues (easiness in implementation), all foregoing approaches to constructing local ABC's (the one based on rational approximation to the  $\Psi$ DO's symbols, the one that explicitly uses asymptotic representation of the solution, and the one of [45]) provide an essential simplification of the numerical algorithm compared with the direct implementation of nonlocal ABC's, as well as a substantial reduction in the required computer effort. However, one restriction, namely, the requirement that the artificial boundary be of some simple shape (more precisely, be a coordinate surface that would allow the separation of variables in the governing equation(s)), still holds for most of the previously mentioned approximate ABC's. Indeed, boundary conditions obtained by localization of global ABC's with  $\Psi$ DO's would formally apply only to the boundaries, for which the original method has been constructed; local boundary conditions equivalent to the truncated DtN maps would apply only to those boundaries, for which the original DtN map has been obtained. At the same time, asymptotic methods based on canceling out a certain number of leading terms in the expansion can be considered somewhat less restrictive from the geometric standpoint, although many implementations still involve only regular boundaries. Low order asymptotic methods based on the explicit knowledge of the coefficients (aerodynamic lift in [108]) may not have serious geometric limitations at all.

An important issue associated with the implementation of local (or localized) ABC's is well-posedness of the truncated formulations and computational stability of the resulting algorithm. These questions deserve particular attention (see, e.g., [1, 24, 48, 114] and many other publications) and require separate theoretical

or experimental study for any truncated problem supplemented by the ABC's. For example, the ABC's obtained using rational approximation to a  $\Psi$ DO symbol may yield both ill-posed and well-posed truncated problems. As shown by Engquist and Majda (see [1]), the first-order Taylor boundary condition (2.7a) leads to a well-posed problem whereas the second-order Taylor boundary condition (2.7b) produces an ill-posed truncated formulation. Giles in work [72] had also demonstrated the instability originating from the second-order Taylor approximation when constructing the boundary conditions for the linearized Euler equations. On the other hand, the third-order boundary condition for the wave equation obtained using the  $O(\eta^6/\omega^6)$  Padé approximation has been proven well-posed (see [1]). Givoli, Patlashenko, and J. Keller in work [139] have shown that the localized DtN-based conditions of odd order lead to stable finite-element discretizations whereas similar boundary conditions of even order may produce instabilities. Higdon in work [101] points out that stability analysis for the ABC's that admit the factorization of type (2.8a) can essentially be reduced to the analysis of first-order boundary conditions (a single bracket in formula (2.8a)). Trefethen and Halpern in work [144] specially study and compare different ways of approximating the  $\Psi$ DO's symbol by rational functions (Padé, Chebyshev, least-squares) and characterize stability of the resulting problems in terms of algebraic properties of the rational approximant (locations and multiplicities of poles and zeros). For the more difficult case of hyperbolic systems with non-zero initial data outside the computational domain, the question of well-posedness of the truncated problem has been studied by Gustafsson in work [4] and, at a greater detail, in work [145]. For the case of incompletely parabolic systems, the construction of ABC's by Halpern in work [5] is followed by the rigorous analysis of well-posedness for the resulting half-space problems in weighted integral norms.

We now describe another, independent, approach to developing the approximate local ABC's. Unlike the previously mentioned methods, it requires neither the preceding construction of exact ABC's for further localization nor, formally, the explicit knowledge of far-field asymptotic representation of the solution. Because of its algorithmic simplicity, low computational cost, and geometric universality (though achieved at a sacrifice in accuracy), this approach has become very popular in practical computing. At present, it is most widely used in different applied areas, in particular, CFD.

The methodology is based on the analysis of characteristics for unsteady (hyperbolic) problems. Although it has originally been developed for time-dependent formulations, it is extensively used for steady-state problems as well when the method for calculating the steady solution involves some kind of pseudo-time integration (this is true for most cases).

In its simplest interpretation, the methodology can be considered as another way of implementing the principle of “no reflection” that has already been discussed in this section, as well as in Section 2.1 of the paper. Recall, this principle essentially means that if the solution to the problem of interest is dominated by waves, then the ABC's should guarantee for any wave traveling toward the boundary that it will leave the computational domain without reflections. (Of course, in every specific case the statement “dominated by waves” will require a substantial clarification as to the type of the waves, i.e., their physical nature and/or mathematical properties, shape of the fronts and the speed of propagation, interaction between different waves, etc.) As the incoming wave can basically be treated as reflections from the artificial boundary, the no reflection principle implies that the solution we are looking for exists in the class of functions that can be described as a collection of outgoing waves only. It turns out that this indeed is true for many model and applied problems. Nonetheless, we emphasize that the principle of no reflection is not universal. It is one of the selection principles used for determining the right far-field behavior of the solution when constructing the ABC's. We will see that some versions of the characteristics-based boundary conditions that we are going to describe allow for taking into account the incoming waves as well if the latter are present in the solution.

In fact, the no reflection principle has already been implemented with the help of integral transforms and the separation of variables, formulas (2.4b) and (2.6), for example, are its particular realizations for the transformed spaces; in physical space the corresponding exact ABC's appear nonlocal (see Section 2.1). On the other hand, the first-order localization (2.7a) of boundary condition (2.4b) can be seen as a direct (local) implementation of the method of characteristics in physical space, because formula (2.7a) actually implies that any plane wave traveling normal to the boundary will leave the computational domain  $x \leq 0$  without reflections. Therefore, the method of characteristics in physical space can be considered an alternative direct means of implementing the principle of no reflection. For multi-dimensional problems the resulting ABC's may appear substantially less accurate than the exact ones (see below). Moreover, as shown by Kreiss and Gustafsson in work [146], their simplest version may not account for the exterior forcing terms (i.e., sources of the incoming waves outside the computational domain). However, they are typically much simpler from

the algorithmic standpoint because of locality, can, under certain conditions, be applied to nonlinear cases, and as a result, appear very attractive for practical computing.

Hedstrom in work [147] considers one-dimensional Euler equations (gasdynamic system) on a finite interval. Near edges, he calculates the number of characteristics entering the computational domain (interval) and the number of characteristics leaving this domain. The characteristics that enter the domain correspond to the incoming waves because the information is transported inward along these characteristics. Analogously, the characteristics that leave the domain correspond to the outgoing waves since the information in this case is transported outward. As far as setting the boundary conditions is concerned, the quantities propagating outward should be extrapolated along the characteristics (that have some finite slope on the  $(x, t)$  plane) from inside the domain to its boundary (in fact, this is what formula (2.6) does in the transformed space and formula (2.7a) does in physical space). It makes the boundary transparent for the outgoing waves, and in the special case of supersonic outflow (when there are no incoming characteristics), it enables one to complete the formulation of the problem. The situation is different for the subsonic case, when for both inflow and outflow types of the boundary there are always incoming, as well as outgoing, characteristics. Obviously, specifying only the outgoing quantities would leave the problem subdefinite; therefore, the incoming data must be prescribed as well.

The aforementioned principle of no reflection stipulates that there are no incoming waves in the solution at all. For one second-order equation (wave equation)  $u_{tt} = u_{xx}$ , boundary condition (2.7a) is transparent for the outgoing waves and at the same time prohibits the incoming ones because its general solution is given by  $u = u^{(1)}(x - t) + u^{(2)}(x + t)$  and the outgoing wave  $u^{(1)}(x - t)$  turns (2.7a) into an identity whereas for the incoming wave  $u^{(2)}(x + t)$  we have  $u^{(2)'} = 0$ , which means  $u^{(2)} = \text{const}$  and, without loss of generality,  $u^{(2)} = 0$ . If, however, we had rewritten the second-order equation as an equivalent system of two first-order equations:  $\mathbf{u}_t = \mathbf{A} \mathbf{u}_x$ , where  $\mathbf{u} = [v, w]^t \equiv [u_t, u_x]^t$  and  $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , then the characteristic variables  $\varrho^- = v - w$  and  $\varrho^+ = v + w$  would have satisfied the decoupled equations  $\varrho_t^- = -\varrho_x^-$  and  $\varrho_t^+ = \varrho_x^+$ , respectively, and according to the no reflection principle, we would have had to explicitly prescribe the zero value of the incoming Riemann variable  $\varrho^+$  at the boundary  $x = 0$  of the computational domain  $x \leq 0$ . At the same time, the first equation  $\varrho_t^- = -\varrho_x^-$  coincides with (2.7a), it corresponds to the outgoing wave and merely says that the variable  $\varrho^-$  does not require a prescribed value at the boundary  $x = 0$  of the computational domain  $x \leq 0$ .

Explicit specification of the incoming quantities is typical when setting the ABC's for the first-order systems. Returning to the Euler equations, we note that having no incoming waves in the solution is actually equivalent to saying that instead of having some finite slope, the incoming characteristic(s) become(s) vertical (i.e., parallel to the time axis) at the boundary, which means nothing can propagate into the domain. Therefore, it is the same as to set the quantities that propagate along the incoming characteristics constant and equal to those at infinity (free-stream values). This is exactly what has been done by Hedstrom in work [147]. It is quite clear that first, this treatment can be fully appropriate only for one-dimensional problems (the difficulties arising from multidimensionality are discussed below). Second, strictly speaking the treatment is relevant only for those cases when all the waves can be fully decoupled (as described above). This is always possible for linear problems; as concerns the nonlinear ones, only some particular formulations, for example, isentropic flows, admit decoupling. In this case, the quantities previously referred to as those propagating along the characteristics, actually become the Riemann invariants, they correspond to the finite-amplitude waves that decouple in characteristic variables. For the general case of non-isentropic flows, the finite Riemann invariants do not exist (see, e.g., [148]) and only the infinitesimal-amplitude waves decouple in characteristic variables. This circumstance does not fully prevent the use of characteristic boundary conditions in the finite form for non-isentropic flows but apparently accounts for cubically small reflections found by Hedstrom (see [147]) for the weak shocks propagating through the boundary; at the same time, the rarefaction waves, for which Riemann's invariants are available, caused zero reflection. Characteristic boundary conditions in the infinitesimal form, i.e., characteristic form of the differential equations, can still be used in non-isentropic case for setting the ABC's (although there are no finite Riemann invariants), this may require using one-sided spatial approximations.

Another version of characteristics-related treatment for boundary conditions has been proposed by Atkins and Casper in work [149]. Their approach is primarily designed for the acoustic problems. To update the boundary data, the authors of [149] actually solve (at the boundary) a particular class of Riemann problems, namely those, for which the solution is composed of the acoustic and convection entropy waves only. This

does not lead out of the class of problems with existing Riemann's invariants and therefore ensures the nonlinear (finite-amplitude) wave decoupling. At the same time, the approach of [149] allows the incoming waves to be explicitly included in the solution when/if necessary. Similarly to the results of [147], shock waves crossing the boundary will produce cubic reflections when treated by the methodology of [149].

The ABC's obtained using wave decoupling in characteristic variables and based on the no reflection principle are sometimes called the radiation boundary conditions. As previously mentioned, this decoupling implemented in the transformed space after the separation of variables is a key element in obtaining the exact multidimensional ABC's (local in Fourier space and global in physical space). If, however, the original problem is one-dimensional, then there are no other space coordinates besides the direction of wave propagation and consequently, no need to perform integral transforms in space at all. Therefore, wave decoupling using characteristics will yield the exact one-dimensional ABC's. On the other hand, the ABC's based on the direct implementation of characteristic wave decoupling are widely used in practice for the case of more than one space dimension as well. In this case, the locality is typically rendered through pre-selecting a specific direction for spatial differentiation near the boundary (it, though, may not be the same for different boundary points) and considering locally one-dimensional problems that would formally allow the characteristics-based treatment similar to the one of [147]. As has already been pointed out, the direct application of the one-dimensional characteristics-based approach in physical space actually disregards the transversal coordinate(s), i.e., corresponds to the uniform behavior of the solution in the other space direction(s), and can therefore be seen as the first, roughest, approximation to the exact ABC's. From this observation it is clear that the errors associated with multidimensionality are unavoidable for local radiation boundary conditions and have the fundamental nature. Another way of understanding that is to notice that characteristics for multidimensional problems are surfaces rather than one-dimensional curves and therefore, the very concept of transport along the characteristics becomes inherently nonlocal. As concerns practical computing, the aforementioned selection of a designated direction for spatial differentiation near the boundary actually prescribes the predominant direction(s) of wave propagation in the model. If the outgoing wave happens to propagate in some different direction, then it will partially be reflected back to the domain, which effectively produces the wrong data coming from the boundary, and partially get through, which causes the irreversible loss of information that should have otherwise been taken into account for specifying the incoming quantities at later times. Therefore, unless a thorough justification for the existence of such predominant direction(s) is available, it immediately makes the multidimensional local radiation boundary conditions approximate at best.

Atkins and Casper in work [149] give several examples, for which the predominant direction of wave propagation can be chosen and justified reasonably. In particular, they consider the propagation of sound in the subsonic gas flow far away of the small source where the fronts are essentially spherical (cylindrical in two dimensions). Thompson in work [150] gives another example of implementing the characteristic radiation boundary conditions for multidimensional problems. For inviscid external flow calculations, he always chooses the predominant spatial direction locally normal to the boundary. Although the author of [150] attempts to additionally account for multidimensionality by some kind of the dimension-wise operator splitting, the proposed treatment is essentially one-dimensional at each boundary point and effectively disregards the other space coordinates when conducting the characteristics analysis of type [147] (characteristic form of the differential equations is used for non-isentropic case). In so doing, one cannot generally expect the boundary to be fully transparent for the outgoing waves with incidence angles that differ from  $\pi/2$ . Higher the discrepancy of the incidence direction from normal, more significant reflections one can expect. In work [151], Vanajakshi, Thompson, and Black apply the similar approach to magnetohydrodynamic problems and in work [152], Thompson considers some other types of boundaries (besides the external far-field) in the framework of local characteristics-based approach.

As mentioned above, the predominant direction of wave propagation that one has to specify for constructing local radiation boundary conditions may actually vary from one spatial point to another. Luchini and Tognaccini in work [153] suggested that this direction can also be chosen adaptively to the solution structure at every moment of time. Unlike the work [149], in which the solution structure is assumed globally known (cylindrical wave), the authors of [153] consider the locally planar wave fronts and align their differentiation with the corresponding gradients computed numerically. In so doing, they have been able to demonstrate a decrease in the reflection levels over the first- and second-order boundary conditions of [1]. However, the approach of [153] requires the introduction of nonlinearities even if the original formulation is linear (wave equation for the specific examples from [153]). We also note that Mazaheri and Roe in work [154] suggest that given the incidence angle for the outgoing waves, the corresponding reflection angle can be calculated

using simple geometric considerations (for the moving medium, this is not a mirror reflection) and then incorporated into the construction of local absorbing boundary conditions; the authors of [154], however, do not show any results of implementation of their approach.

Basically, if the locally one-dimensional radiation boundary conditions for Euler's equations are applied to solving the external flow problems, it turns out that they can provide sufficient accuracy only if the artificial boundary is located far enough from the source of perturbations (e.g., from the immersed body). This statement has been repeatedly corroborated by the numerical experiments, see, for example, the results presented in Section 4 and in the related publications. In practice, this circumstance may require choosing the excessively large computational domain in comparison with what would have sufficed if the exact ABC's were used. For each specific case, it is always a separate question whether or not the extra computational effort originating from solving the flow on this extended domain can be compensated for by the simplicity and low cost of the local ABC's algorithm itself. Generally, local radiation boundary conditions have drawn the attention of many researchers over the recent years. In this context, we also mention work by Watson and Myers [155, 156], in which a certain version of the approach has been proposed and implemented for calculation of duct acoustics, and work by Guo and Zeng [157] and Tang and Grimshaw [158], in which different versions of the local characteristics-based approach are studied for the shallow water equations.

Before proceeding to the description of local boundary conditions for viscous flow computations we note that for this particular area the terminology has been least established in the literature and therefore reemphasize that we consider the entire issue of ABC's as a numerical artefact. In fact, the ABC's are the closing procedure for the truncated problem, and from this standpoint any element of this procedure qualifies as a boundary condition. In particular, the aforementioned one-dimensional radiation boundary conditions for inviscid flows involve, in our view, both specification of the incoming quantities and extrapolation of the outgoing ones along characteristics. Moreover, as previously mentioned extrapolation of the outgoing quantities along characteristics can be reformulated as solving the differential equations on the boundary using some specially chosen one-sided spatial discretization. In respect to the aforementioned closure of the truncated problem, this one-sided treatment is also an element of the boundary procedure. Contrary to that, many authors consider only the number of incoming characteristics as the number of the required boundary conditions at each specific type of the boundary. We prefer to distinguish between the construction of boundary conditions, i.e., the closure algorithm, and the number of quantities to be prescribed at the boundary as a part of this closure so that the interior problem not be subdefinite.

Having said that we note that setting the boundary conditions for viscous flow computations involves analysis of the so-called incompletely parabolic PDE's (which have already been mentioned in connection with work by Halpern [5]). The Navier-Stokes equations represent one particular example of this kind. The general well-posedness analysis of boundary-value problems for these systems would, of course, apply to different kinds of boundaries. When it comes to treatment of the artificial boundaries, the same basic idea of decoupling and dealing separately with the incoming and outgoing quantities is used; in particular, the no reflection principle may apply to some formulations. In the case of locally one-dimensional implementation of this principle similar to the foregoing radiation boundary conditions for Euler's equations, the same kind of deficiencies caused by disregarding the multidimensionality will be relevant to the Navier-Stokes approach as well.

A thorough study of well-posedness for some incompletely parabolic problems has been conducted by Strikwerda in work [159] and by Gustafsson and Sundström in work [160] (see also work by Halpern [5]). Two specific examples considered in [160] are the linearized Navier-Stokes equations and the linearized shallow water equations. The approach of [160] first employs the Fourier transform in all but one Cartesian directions and then uses the apparatus of energy estimates to treat the resulting one-dimensional systems; in so doing the artificial boundary is assumed planar and normal to the designated direction  $x$ . Using energy method and considering the local first-order differential relations (differentiation in  $x$ ) as the initial "pool" of admissible boundary conditions, the authors of [160] can select in this pool those boundary conditions that would guarantee the well-posedness of the half-space problem. Later in work [161], Nordström had conducted the similar type of analysis for the linearized two-dimensional Navier-Stokes equations and a wider class of boundary conditions, namely, local second-order differential relations. In addition, Nordström in [161] had studied the convergence rate of the transient (i.e., pseudo-time) iterations as it depends on the type of boundary conditions. This study included the implementation of Euler boundary conditions for the Navier-Stokes calculations. The latter approach often provides for good results in practice although does not have a thorough theoretical justification. (It should be noted, that as the Navier-Stokes equations are of higher order compared to the Euler equations, then even the numbers of quantities to be prescribed at the boundaries



so that the problem not be subdefinite appear different for these two cases. For one-dimensional Euler's equations, the number of variables that one has to specify at the boundary and the number of variables that one has to bring from inside the computational domain are given by the numbers of incoming and outgoing characteristics, respectively (see above), whereas for the Navier-Stokes equations the non-hyperbolic viscous terms will contribute to these numbers as well.) In work [162], Nordström continues the study of [161] and for the linearized constant-coefficient Navier-Stokes equations obtains the boundary conditions that guarantee the fastest convergence of the transient procedure to a steady state. These boundary conditions are referred to as maximally dissipative in the sense that they maximize negative definiteness of the boundary terms that come out of the integration by parts in the energy method. The performance of boundary conditions of [162] has been actually tested by Nordin and Nordström in work [163] in combination with a production flow solver.

The original system in [162] is one-dimensional, although the number of dependent variables is five as all three velocity components are included. For the energy method to apply, the analysis of [162] involves diagonalization of the inviscid hyperbolic part. This is the same as the decoupling of waves in characteristic variables for the truly inviscid Euler equations (see the previous discussion). In this respect, and also in the view of the fact that the Navier-Stokes boundary conditions of [162] do converge to the Euler radiation boundary conditions as the inverse Reynolds number  $Re^{-1}$  vanishes, the former can be called the characteristic boundary conditions for the Navier-Stokes equations, although the characteristic variables used for the analysis actually relate to the corresponding Euler system. Note, the constructions of [160, 161] also yield the Euler radiation boundary conditions in the limit  $Re \rightarrow +\infty$ . We additionally mention that the set of boundary conditions similar to the one of [162] has been obtained by Hesthaven and Gottlieb in work [164], although the implementation of boundary conditions in work [164] is done through the penalty methods in the framework of Legendre collocation discretization and therefore differs significantly from the implementation of [162], which is done in the finite-volume framework.

Historically, one of the first approaches that specifically addressed both the construction and practical implementation of ABC's for viscous flow computations was work by Rudy and Strikwerda [165, 166]. They developed a special "hyperbolic-type" (i.e., inviscid) local NRBC and applied it to calculating the flow over a flat plate in the framework of the two-dimensional compressible Navier-Stokes equations. By adjusting a semi-empiric parameter involved in the NRBC of [165, 166], Rudy and Strikwerda have been able to optimize the convergence rate of transient process to a steady state and subsequently demonstrate the superiority of NRBC's over the simplest Dirichlet-type outflow boundary conditions for the Navier-Stokes computations.

Among the broad literature on local characteristic boundary conditions for Navier-Stokes computations we mention work by Poinso and Lele [167], in which the inviscid locally one-dimensional characteristics-based treatment of boundary conditions essentially follows Thompson [150], and for the viscous enhancements, an extensive description and comparison of different available options is provided, moreover, other types of boundaries besides the artificial far-field ones are considered (see also work by Thompson [152]); work by Baum, Poinso, and Thévenin [168], in which they extend the results of [167] for the case of realistic (non-ideal) thermodynamic relations and reacting flows; work by Grinstein [169], in which the local ABC's for the boundaries that intersect the regions of turbulent mixing layers are analyzed; work by Hayder and Turkel [170, 171], in which the authors experimentally study the performance of different local boundary conditions, including those of [72], [91], and [116], several types of radiation boundary conditions (purely inviscid and those that take into account some viscous corrections), as well as a "sponge layer" (see Section 2.3), for unsteady laminar flat plate flows (boundary layers) and laminar planar and axially symmetric jets (free shear layers); work by Scott, Mankbadi, Hayder, and Hariharan [172], in which the performance of different local outflow boundary conditions, including those of [133] and [152], is assessed and compared for a supersonic axially symmetric viscous jet artificially perturbed by a small harmonic oscillation to model the jet noise; and work by Hayder and Hagstrom [173], in which the local characteristic boundary conditions for the similar type of flow (see [172]) are constructed using linear perturbation theory so that the incoming quantity at subsonic outflow boundary is prescribed from the eigenfunctions of the linearized problem provided that the single dominant mode of disturbances at the boundary is known.

The aforementioned characteristic boundary conditions are primarily intended for treating the time-dependent problems, although, as has already been mentioned, they are also used with pseudo-time, i.e., transient, solvers, when the final solution is steady. If this is the case, the extrapolation of outgoing quantities along characteristics eventually reduces to mere spatial extrapolation for the steady state as the time essentially disappears from the model. Often, extrapolation of flow variables from inside the computational domain to its boundary is used as the ABC's throughout the entire transient process. This technique can

probably be referred to as one of the simplest, although numerical experiments produce mixed results in regard to its performance. For example, Rudy and Strikwerda in work [166] have shown that for subsonic outflow boundaries in two space dimensions the extrapolation of all four flow variables may lead to ill-posedness (no convergence to steady state), which is consistent with the predictions based on characteristics analysis, whereas if only three quantities are extrapolated it does provide convergence to steady state, which is, however, slower than the one obtained with the hyperbolic-type NRBC of [165, 166]. The difficulties with ensuring convergence for extrapolation boundary conditions have been also addressed by Nordström in work [161]. On the other hand, the standard treatment of outflow boundaries in the production Navier-Stokes code TLNS3D, see [64, 65], which has, in particular, been used in our computations reviewed in Section 4, is based on the extrapolation of all flow variables from inside the computational domain to the downstream artificial boundary, and this treatment has been found to provide better convergence to steady state than the radiation ABC's. At any rate, even if the convergence is satisfactory, one cannot expect obtaining high accuracy from this procedure unless the computational domain is sufficiently large (see Section 4 for more specific data). However, there is a particular class of flow problems, the flows with strong cross-stream, i.e., transversal, gradients, for which the extrapolation boundary conditions appear appropriate. The examples of such flows are boundary layers, wakes, free mixing layers, etc. When the extrapolation boundary conditions are used for problems of this type, the well-posedness can be proven for both steady-state (see work by Gustafsson and Nordström [174]) and time-dependent (see work by Nordström [175]) linearized Navier-Stokes equations. The difference in comparison with the previously mentioned results for nearly uniform far-field flows is accounted for by the structure of the linear system as it is now determined by the strong non-uniformities in the background solution, against which the linearization is performed. Moreover, Gustafsson and Nordström have been able to demonstrate numerically that if the entire outflow boundary is subsonic, then the extrapolation conditions are still inapplicable, whereas if the external flow is supersonic and the subsonic part of the flow is concentrated only in the boundary layer (which implies presence of the strong transversal gradients), then the extrapolation conditions do apply and produce good results. The approach of [174, 175] has been further developed by Nordström in work [176] for the finite-difference formulation of the problem and then summarized and presented systematically in work [177]. We should note that the case of intersection of the artificial boundary and the solid boundary (which means that the artificial boundary crosses the boundary layer) presents one of the difficult situations from the standpoint of constructing the ABC's for fluid dynamics. In the meantime, we are unaware of any other sufficiently robust technique for setting the ABC's in this case except for the one based on extrapolation. We additionally note that the extrapolation boundary conditions of [174] can also be seen as the technique for suppressing the non-physical boundary layers that are generally relevant to the near-boundary behavior of the solutions to the Navier-Stokes equations. The idea of suppressing the non-physical boundary layers that may arise near the inflow and outflow artificial boundaries is also the central one in work by Johansson [178], where the set of ABC's that contain higher-order extrapolation conditions as a part was derived and numerically implemented for some incompressible Navier-Stokes computations.

**2.3. Absorbing Layers.** Finally, we briefly comment here on another group of methods that apply to wave dominated problems. Under the same assumption that the exterior solution is composed of outgoing waves only, one surrounds the computational domain by a finite-thickness layer of the specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain. The parameters of the layer (i.e., the governing equations for the model medium) should basically be chosen so that the wave either never reaches its external boundary or even if it does and reflects back, then by the time the reflected mode approaches the boundary between the absorbing layer and the interior computational domain, its amplitude will already be so small that it would not essentially contaminate the solution. The boundary between the computational domain and the layer should also cause minimal and ideally zero reflections, the latter case is called the perfectly matched layer (PML). In a certain sense, this type of treatment cannot exactly be called the boundary conditions because the layers must always have finite thickness; one can also say that the methodology of absorbing layers occupies an intermediate position between local and nonlocal approaches because on one hand, there are no global integral relations along the boundary and when the numerical computations are conducted, the model equations inside the layer are solved by some method close to (or exactly the same as) the one employed inside the computational domain, but on the other hand, a certain amount of nonlocality is still present because the computational domain is artificially enlarged.

Over the recent years, various constructions of absorbing (or damping, or sponge) layers for slowing down

or attenuating the outgoing waves have been proposed in the literature. For example, a PML technique for the Maxwell equations has been developed and implemented by Berenger in work [179] for two-dimensional case and in work [180] for three-dimensional case. Berenger's approach is based on introducing some additional artificial field components in the layer and adjusting the accordingly enlarged set of electric and magnetic conductivities so that on one hand, the electromagnetic waves rapidly attenuate in the layer and on the other hand, the boundary between the layer and the vacuum is completely transparent. Geometrically, the computational domain used by Berenger is a Cartesian rectangle in two dimensions and a parallelepiped in three dimensions. Numerically, Berenger had been able to demonstrate the superiority of his methodology over some local boundary conditions (in particular, those of [90], see [179]). On the other hand, in the recent work [181] Abarbanel and Gottlieb have shown that the set of model equations (with the split field components) used by Berenger for his PML is, in fact, weakly ill-posed. At the same time, there are other versions of PML, e.g., those with the unsplit field components, that actually lead to the well-posed formulations; these versions have been arrived at starting from both purely mathematical, see Abarbanel and Gottlieb [184], and physically motivated, see Petropoulos [182] and L. Zhao and Cangellaris [183], considerations.

In this paper, we do not concentrate on describing the boundary treatment techniques based on absorbing layers. We only mention some work done in this area by different authors. Besides the previously discussed work by Clement [96], Hayder and Turkel [170], Berenger [179, 180], Abarbanel and Gottlieb [181, 184], Petropoulos [182], and L. Zhao and Cangellaris [183], we additionally mention work by Israeli and Orszag [185], R. Kosloff and D. Kosloff [186], Karni [187], Hu [188], Collino [189], Jin and Chew [190], Kantartzis and Tsiboukis [191], Hayder, Hu, and Hussaini [192], Freund [193], Monk and Collino [194], Hayder and Atkins [195], Watanabe and Utsunomiya [196], Bonnet and Poupaud [197], and Petropoulos, et al. [198–201]. Instead of delineating the absorbing layers approach, we refer the reader to review papers by Abarbanel and Gottlieb [202] and Turkel and Yefet [203] (see this issue of the Journal), in which more information on PML and related methods can be found.

**2.4. Survey Summary.** In Section 2 of the paper, we have surveyed the work conducted by different authors over the recent years in the field of constructing the ABC's for various model and applied problems in scientific computing. We have been focusing primarily on the basic ideas and key elements of the algorithms while leaving out the specific implementation details in each particular case. The corresponding numerical results have been mentioned very briefly and mostly as the common tendencies relevant to the entire groups of methods. This has been done for the sake of keeping the volume of the paper low; at the same time, the level of detail in the current presentation is still sufficient, in our opinion, for reflecting and justifying the major trends in ABC's algorithm development.

The basic conclusion that can be drawn from the survey is that any algorithm for setting the ABC's should, generally speaking, satisfy two groups of requirements, which may to a certain extent be contradictory. The requirements of the first group mainly address the computational accuracy; as mentioned above, pursuing the accuracy goal typically leads to nonlocal ABC's. These ABC's, however, have their own limitations along with the advantages. Namely, such boundary conditions can be accurately derived only for the linear governing equations (in most cases) and for some simple geometries; moreover, the discrete implementation of nonlocal ABC's is not always easy, even when their continuous formulation is available. The second group of requirements combines algorithmic simplicity, geometric universality, and low computational cost; the approximate approaches that yield local ABC's are usually much better than the exact ones from this standpoint. On the other hand, the accuracy provided by local ABC's is often insufficient. Therefore, when constructing a specific numerical algorithm, one every time has to choose an intermediate computational strategy that would be optimal for the particular case under study.

In our own view, modern trends in the development of numerical methods are actually making higher and higher requirements for the accuracy of computations. Consequently, we expect the practical demand for highly accurate ABC's to grow in the future. This will obviously necessitate paying more attention to constructing such highly accurate boundary conditions that would at the same time be robust and computationally efficient. A few particular examples of this kind are described in Section 4 of this paper.

When writing a review on different ABC's techniques, we have tried to make it as comprehensive as possible within the limits of a journal publication. At the same time we clearly realize that no survey paper of this kind can be completely thorough. Therefore, we mention here work by Givoli [204, 205] that presents a detailed review of the state of the area that dates a few years back. We also mention that the previous, shorter version of this review was published as NASA paper [206]. In addition, a survey of the ABC's

techniques for CEM that apply to solving the time-dependent Maxwell equations on exterior domains is given by Taflöv in work [207]. A detailed comparative assessment of the computational complexity for different types of time-dependent ABC's including those of [1, 2, 38, 40, 47, 48, 114–116, 179], has been recently given by Hagström in [208]. We should also note that in our survey we have completely left out the entire group of methods for treating infinite-domain problems; these methods are based on the special (singular) coordinate transformations that map finite intervals onto the infinite ones. Typically, the techniques of this type are incorporated into the FEM framework and accordingly called “infinite elements,” see, for example, work by Zeng, Kallivokas, and Bielak [209], Harari [29], J. Bettess and P. Bettess [210], Astley [211], Demkowicz and Ihlenburg [212], Burnett [213], Gerdes [214], Oberai, Malhorta, and Pinsky [215] and the bibliographies quoted there. Mappings between the finite and infinite intervals are also used for constructing the appropriate basis functions for spectral methods on unbounded domains, see, e.g., work by Matsushima and Marcus [216]. Referring the reader to the aforementioned publications (see also [72, 78, 120, 167, 170–172, 187, 202, 203]) for more review and comparison information, we now proceed to describing the DPM [14–16] and the DPM-based ABC's, which are the focus of the second part of this paper.

**3. The Generalized Potentials and the Difference Potentials Method.** Our objective is to construct the new ABC's that would, to a certain extent, combine the advantages of local and nonlocal approaches. Namely, the new methodology should provide for high accuracy relevant to the nonlocal techniques and, at the same time, possess geometric universality and algorithmic simplicity relevant to many local methods. Our main tool in achieving this goal is the apparatus of the generalized potentials and the DPM [14, 15].

In this section of the paper, we use a model example for the Poisson equation to show the principle elements of the construction of generalized potentials and their implementation for setting the ABC's. In so doing, we only outline the key concepts; a delineation of the ideas associated with the generalized potentials, boundary equations with projections, and their numerical implementation (DPM), as well as many examples, can be found in the original work by Ryaben'kii [14, 15]. Section 4 of the paper will be devoted to the specific constructions of ABC's based on the DPM.

Let us first return to the second example of Section 1. Then, we have truncated the original infinite problem (1.4) and obtained the exact ABC's (1.6b) at the spherical artificial boundary  $\{r = R_0\}$ . The problem obviously becomes more difficult if, for some reason, we need to consider an artificial boundary with a more complicated shape. In practice, the shape of the artificial boundary is often prescribed by the interior algorithm (more precisely, by the grid used inside the computational domain); some real examples will be given in Section 4. As concerns the model case studied here, we simply assume that there is an irregular artificial boundary  $\Gamma$  that separates the finite computational domain  $D_{in}$  from its infinite exterior  $D_{ex}$ ; we also assume that  $D_{in}$  entirely contains the support  $B$  of the RHS  $f(x)$  (see (1.4a)). The geometric setup (projection onto the plane) is schematically shown in Figure 3.1, the closed dashed line in this figure represents  $\Gamma$ . The sphere  $\{r = R_0\}$  is needed as an artificial element in further consideration; without loss of generality, we may always assume that  $D_{in} \subset \{r \leq R_0\}$ .

We now remind the classical Green formula for harmonic functions. Namely, let  $u_{ex}(x)$ ,  $x \in R^3$ , be harmonic in  $D_{ex}$  and have zero limit at infinity,  $u_{ex}(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then,

$$(3.1) \quad u_{ex}(x) = -\Theta_\Gamma(x) \frac{u_{ex}(x)}{2} + \int_\Gamma \left( \frac{\partial G(x, y)}{\partial n_y} u_{ex}(y) - G(x, y) \frac{\partial u_{ex}(y)}{\partial n_y} \right) ds_y, \quad x \in \bar{D}_{ex},$$

where  $\Theta_\Gamma(x)$  is the characteristic function of set  $\Gamma$ ,  $\Theta_\Gamma(x)$  is equal to one on  $\Gamma$  and zero on  $R^3 \setminus \Gamma$ ;  $G(x, y) = -(4\pi)^{-1}|x - y|^{-1}$ ;  $n$  is the outward normal to  $\Gamma$ ;  $s$  is the surface area; and the subscript  $y$  denotes the variable of differentiation or integration. Note, the first term on the right-hand side of (3.1) takes into account the jump of the double-layer potential across the boundary, the addition of this term makes (3.1) valid on the closed domain  $\bar{D}_{ex}$ .

We emphasize that the representation of  $u_{ex}(x)$  as a sum of the double-layer and single-layer potentials according to (3.1) holds only for the harmonic functions on  $D_{ex}$ . If, however, we specify two arbitrary functions on  $\Gamma$  and substitute them into (3.1) as densities of the potentials, then the resulting function will obviously be harmonic on  $D_{ex}$ , but its boundary values, as well as boundary values of its normal derivative, will not, generally speaking, coincide with the original densities of the double-layer and single-layer potentials, respectively.

Let us now define the generalized potential with vector density  $\xi_\Gamma \equiv (\xi_0, \xi_1)$  specified on  $\Gamma$ . We will use the formula analogous to (3.1) but will not require in advance that  $\xi_0$  and  $\xi_1$  be the boundary values of some

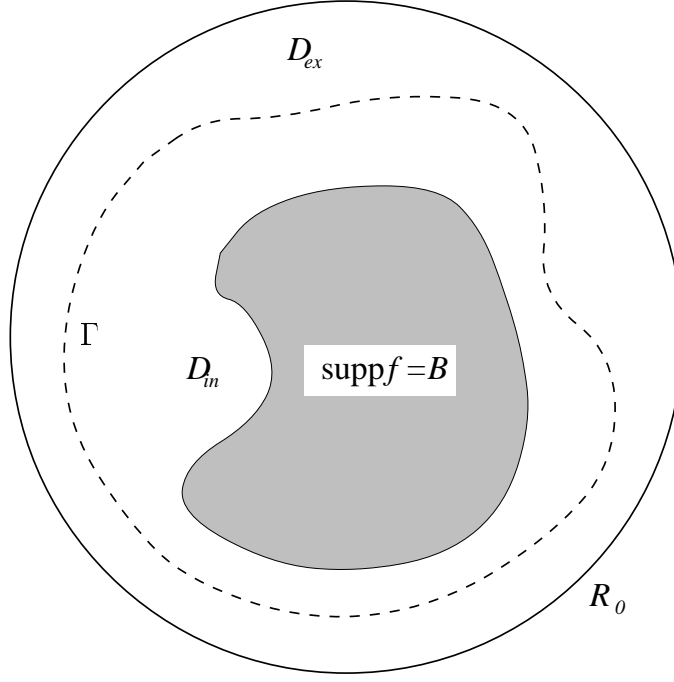


FIG. 3.1. *Example for Poisson equation in three dimensions: schematic geometric setup (projection onto the plane).*

harmonic function and its normal derivative, respectively. Specifically, the generalized potential is given by the following expression:

$$(3.2) \quad u_{ex}(x) = \mathbf{P}_{ex} \xi_\Gamma \equiv -\Theta_\Gamma(x) \frac{\xi_0(x)}{2} + \int_\Gamma \left( \frac{\partial G(x, y)}{\partial n_y} \xi_0(y) - G(x, y) \xi_1(y) \right) ds_y, \quad x \in \bar{D}_{ex}.$$

We also define the operation of taking the (vector) boundary trace of the function defined on  $D_{ex}$ :

$$(3.3) \quad \xi_\Gamma = \mathbf{Tr} u_{ex}(x) \equiv \left( u_{ex}, \frac{\partial u_{ex}}{\partial n} \right) \Big|_\Gamma.$$

Finally, we define the boundary operator  $\mathbf{P}_\Gamma$  that maps the space of traces  $\xi_\Gamma$  onto itself; this operator is a composition of the generalized potential  $\mathbf{P}_{ex}$  and the trace  $\mathbf{Tr}$ ,

$$(3.4) \quad \mathbf{P}_\Gamma \xi_\Gamma \equiv \mathbf{Tr} \mathbf{P}_{ex} \xi_\Gamma.$$

Referring the reader to work by Ryaben'kii [14, 15] for further details, we only note that the structure of the operator  $\mathbf{P}_\Gamma$  of (3.4) is nontrivial since, in particular, it contains the normal derivative of the double-layer potential. The corresponding singularity may noticeably complicate the direct calculation of  $\mathbf{P}_\Gamma \xi_\Gamma$ ; therefore, the actual calculation requires the development of a special alternative approach described below. However, the operator  $\mathbf{P}_\Gamma$  itself appears to be most useful in the analysis of boundary-value problems, it plays a fundamental role in our further consideration.

We note that the operators analogous to  $\mathbf{P}_\Gamma$  have been first studied by Calderon in work [217] and then by Seeley in work [218]. It turns out that  $\mathbf{P}_\Gamma$  is a projection,  $\mathbf{P}_\Gamma = \mathbf{P}_\Gamma^2$ ; sometimes it is called the Calderon boundary projection. Ryaben'kii (see [14–16] and the bibliographies there) had modified and generalized the original Calderon's construction and developed an effective way for numerical treatment of the boundary projection operators.

It can be shown that those and only those vector functions  $\xi_\Gamma$  that satisfy the following boundary equation with projection (BEP)

$$(3.5) \quad \mathbf{P}_\Gamma \xi_\Gamma = \xi_\Gamma$$

admit a harmonic complement  $u_{ex}$  on  $D_{ex}$  that vanishes at infinity and has the trace  $\xi_\Gamma$  on  $\Gamma$ ,  $\mathbf{Tr} u_{ex} = \xi_\Gamma$ . Indeed, if  $u_{ex}(x)$  is harmonic on  $D_{ex}$  and vanishes at infinity, then we rewrite Green's formula (3.1) as  $u_{ex}(x) = \mathbf{P}_{ex} \left( u_{ex}, \frac{\partial u_{ex}}{\partial n} \right) \Big|_\Gamma$  and, applying  $\mathbf{Tr}$  of (3.3), obtain (3.5). Conversely, if equality (3.5) holds for some  $\xi_\Gamma$ , then the harmonic function  $\mathbf{P}_{ex} \xi_\Gamma$  vanishes at infinity (as a sum of two classical potentials) and has the trace  $\xi_\Gamma$ . In other words, those and only those  $\xi_\Gamma$ 's that belong to the image of the projection operator  $\mathbf{P}_\Gamma$ ,  $\xi_\Gamma \in \text{Im} \mathbf{P}_\Gamma$ , can be complemented on  $D_{ex}$  so that the complement is harmonic, vanishes at infinity, and has the trace  $\xi_\Gamma$ .

Thus, equation (3.5) acquires a key importance since it provides for a complete classification of those and only those vector densities  $\xi_\Gamma$  that have a harmonic continuation on  $D_{ex}$ . In other words, equation (3.5) is equivalent to the Laplace equation on  $D_{ex}$  along with the condition of vanishing of its solution at infinity. Therefore, equation (3.5) provides us with the desired exact ABC at the irregular boundary  $\Gamma$ . Rather than solving problem (1.4) on  $R^3$  or equation (1.4a) on  $\{r \leq R_0\}$  with boundary conditions (1.6b), we may now solve equation (1.4a) with boundary conditions (3.5) and obtain exactly the same solution on  $D_{in}$ .

The operator  $\mathbf{P}_\Gamma$  is obviously nonlocal. Moreover, we have not yet addressed the aforementioned computational difficulties associated with the singularities in its structure. Therefore, we will now redefine  $\mathbf{P}_\Gamma$  in a more practical way. First, let us recall the classical Green identity ( $\sigma$  is the volume):

$$(3.6) \quad \begin{aligned} -\Theta_\Gamma(x) \frac{u_{ex}(x)}{2} + \int_\Gamma \left( \frac{\partial G(x, y)}{\partial n_y} u_{ex}(y) - G(x, y) \frac{\partial u_{ex}(y)}{\partial n_y} \right) ds_y = \\ = u_{ex}(x) - \int_{D_{ex}} G(x, y) \Delta_y u_{ex}(y) d\sigma_y, \quad x \in \bar{D}_{ex}, \end{aligned}$$

which holds for any function  $u_{ex}(x)$  such that the convolution in the second term on the right-hand side of (3.6) exists. Since (3.6) is an identity, and its left-hand side coincides with the Green formula for harmonic functions (see (3.1)), we can use (3.6) to construct a new definition of the generalized potential. To do so, for any prescribed vector density  $\xi_\Gamma$  we first take some function  $u(y)$  defined on  $R^3$  and such that  $\mathbf{Tr} u = \xi_\Gamma$ . Then, we consider the following function:

$$(3.7) \quad g_{ex}(y) = \begin{cases} \Delta_y u(y), & y \in D_{ex} \\ 0, & y \in \bar{D}_{in} \end{cases}$$

and easily derive

$$(3.8) \quad \mathbf{P}_{ex} \xi_\Gamma = u(x) - \int_{R^3} G(x, y) g_{ex}(y) d\sigma_y, \quad x \in \bar{D}_{ex}.$$

Indeed, formula (3.8) immediately follows from definition (3.2) and identity (3.6) since, according to (3.7), the right-hand sides in (3.6) and (3.8) are the same for  $x \in \bar{D}_{ex}$ . As concerns the choice of  $u(y)$  from (3.7), the only essential requirement is that  $\mathbf{Tr} u = \xi_\Gamma$ ; this requirement is always easy to meet. At the same time, we can easily ensure the existence of convolution from (3.8), since the function  $u(y)$  (and, consequently,  $g_{ex}(y)$ ) can, for example, be always chosen compactly supported without loss of generality.

Using new definition of the generalized potential (3.8), we construct the boundary projection  $\mathbf{P}_\Gamma$  in the same way as before, i.e., in accordance with formula (3.4). We emphasize that the new definitions of  $\mathbf{P}_{ex}$  and  $\mathbf{P}_\Gamma$  no longer require the calculation of surface integrals; instead, one needs to calculate the volume Newton potential (see (3.8)).

As mentioned above,  $u(y)$  and consequently,  $g_{ex}(y)$  of (3.7), can always be chosen compactly supported, and without loss of generality,  $\text{supp } g_{ex}(y) \subset \{r \leq R_0\}$ . Therefore, we can introduce the spherical artificial boundary  $\{r = R_0\}$  (see Figure 3.1) and rewrite (3.8) as

$$(3.9) \quad \mathbf{P}_{ex} \xi_\Gamma = u(x) - \int_{\{r \leq R_0\}} G(x, y) g_{ex}(y) d\sigma_y, \quad x \in \bar{D}_{ex}.$$

Clearly, boundary projections (operators  $\mathbf{P}_\Gamma$ ) that can be obtained on the basis of formulas (3.8) and (3.9) are exactly the same. The last step is to realize that basically, we do not need to know the generalized

potential everywhere on  $D_{ex}$ . In particular, to construct the exact ABC's of type (3.5) we need to know the potential only in the vicinity of  $\Gamma$  to calculate the operation  $\mathbf{P}_\Gamma$ . Therefore, we merely notice that the second term on the right-hand side of (3.9) is the solution of a certain *auxiliary problem* (AP) formulated on the ball  $\{r \leq R_0\}$ . Indeed, this term obviously solves on  $\{r \leq R_0\}$  the Poisson equation

$$(3.10) \quad \Delta u = -g_{ex}(x), \quad x \in \{r \leq R_0\},$$

supplemented by boundary conditions (1.6b). Therefore, AP (3.10), (1.6b) can be used instead of formula (3.9) to calculate the generalized potential on  $\bar{D}_{ex} \cap \{r \leq R_0\}$  for this specific example. The use of AP (3.10), (1.6b) instead of convolution (3.9) is a major simplification from the algorithmic standpoint since the AP is formulated on a simple domain and admits an effective solution by the separation of variables.

The concept of AP is another principal element in our consideration. Indeed, we see that the exact ABC's at the irregular artificial boundary  $\Gamma$  are obtained in the form of BEP (3.5). To calculate the projection  $\mathbf{P}_\Gamma$ , we first needed the surface integrals (3.2), then we replaced the surface integrals with the volume potential (3.8), and finally, we have found the way to calculate the generalized potential and the boundary projection using AP (3.10), (1.6b), which can be easily solved by the separation of variables. In the next section, we provide some practical examples of appropriate AP's, and generalized boundary projections  $\mathbf{P}_\Gamma$  as they apply to setting the ABC's for infinite-domain problems.

To conclude this section, we first note that metric properties of the potentials and projections (well-posedness of BEP's) for different classes of problems have been thoroughly studied by Ryaben'kii in work [14, 15], Reznik in work [219], and Sofronov in work [220]. Second, we briefly address the numerical aspects of calculating the generalized potentials and projections, i.e., the DPM itself. The key element of the DPM is the finite-difference counterpart to Calderon's boundary projection. To construct this object, we start with formulating the difference AP, which should be uniquely solvable and well-posed and should also admit easy numerical solution, e.g., by the separation of variables in the discrete framework. Of course, the difference AP used for constructing the ABC's should contain some element analogous to boundary conditions (1.6b) so that the desired behavior of the exterior solution be properly taken into account; several examples of such AP's are given in Section 4. The grid for the AP is usually regular (e.g., Cartesian, polar, spherical, etc.), and no grid adaptation to the shape of  $\Gamma$  is required. Instead, we consider a special subset of nodes of this grid called the grid boundary. The grid boundary  $\gamma$  is located near the continuous boundary  $\Gamma$ ; the structure of  $\gamma$  depends on the stencil used for the difference approximation of the differential operator of AP. Specifically, the set  $\gamma$  is built as follows. The entire regular AP's grid is split by  $\Gamma$  into two subsets of nodes; the first subset belongs to  $D_{in}$ , and the second one belongs to  $D_{ex}$ . These subsets have no common points. We apply the stencil to every node of each of the two above-mentioned subsets; clearly, the stencil always sweeps a wider grid area than the original subset. The two grid areas swept by the stencil obviously have now a non-empty intersection; the latter is called the grid boundary  $\gamma$ . Typically, it is a multi-layered fringe of nodes of the regular grid concentrated near  $\Gamma$ . Instead of the vector densities  $\xi_\Gamma$ , we consider scalar grid functions on  $\gamma$ ; since  $\gamma$  is multi-layered, the former can in a certain sense be modeled by the latter. The formal construction of generalized difference potentials and difference projections consists of the same elements as above. We merely plug the corresponding difference objects instead of the continuous ones into the aforementioned procedure. For example, the corresponding subsets of the regular grid are substituted for  $D_{in}$  and  $D_{ex}$ ; the grid boundary  $\gamma$  fills in for  $\Gamma$ ; the finite-difference operators replace the differential ones; the solution to the difference AP is used instead of the solution to the continuous AP. In so doing, we obtain difference analogues to the potentials and projections; the issues of consistency and convergence are delineated by Ryaben'kii in [15]. The full scheme of the DPM (and the construction of the DPM-based ABC's) may also require the operation of continuing the boundary data from  $\Gamma$  to  $\gamma$ , as well as some interpolation operations. The continuation of data from  $\Gamma$  to  $\gamma$  is usually performed using several leading terms of the Taylor expansion, the details can be found in [14, 15]. Once the discrete Calderon's projection has been constructed, it can be implemented for setting the ABC's either in the framework of BEP (see (3.5)) or directly, as will be seen in the next section.

**4. The DPM-Based Nonlocal ABC's.** Basic theoretical foundations for constructing the DPM-based ABC's are provided by Ryaben'kii in work [46, 221]. In this work, BEP's of type (3.5) have for the first time been used as the ABC's to be set at the outer boundary when truncating an external problem for the purpose of its numerical solution.

The author of [46, 221] studies a general unsteady problem formulated on some infinite domain. From

the very beginning, the problem is discretized on some grid, so that only the finite-difference formulation is considered. This allows one to obtain the ABC's directly for the specific numerical algorithm rather than for the original continuous formulation, which is convenient in practice. Outside some finite grid domain (i.e., the computational domain), the problem is assumed linear and homogeneous, the original boundary conditions at infinity are also assumed homogeneous. No restrictive assumptions are made in regard to the problem inside the computational domain; it, in particular, can be nonlinear. The only important requirement of the overall formulation is that the problem be uniquely solvable and well-posed. Such a consideration is relevant to many applications, for example, solid mechanics, where one often has some bounded region with the strong deformations (like plasticity or destruction) surrounded by an extended medium where the deformations are small and, therefore, governed by the linear elasticity equations. The shape of the computational domain in [46, 221] is arbitrary and no adaptive/boundary fitted grids are required.

The ABC's of [46] are obtained in the form of a difference BEP analogous to (3.5). The difference BEP is written with respect to the grid potential density defined on the grid boundary  $\gamma$ . In the case of time-dependent problems,  $\gamma$  is a multi-layered cylindrical fringe (aligned with the time axis) composed of grid nodes; it is analogous to the continuous cylindrical boundary of the type  $\Gamma \times [0, T]$ . The AP used by Ryaben'kii in work [46] is also time-dependent; it is solved by some evolution technique. Since the Green operator of the AP is incorporated in the structure of the boundary projection, the resulting ABC's in [46] appear nonlocal in both space and time. We emphasize that these ABC's are exact in the sense mentioned above: the discrete solution found inside the computational domain with the help of these boundary conditions is the same as it would be if the original (infinite-domain) problem was solved first and the solution then truncated.

To reduce the computational cost associated with nonlocality, which can be rather high for the general formulation, Ryaben'kii (see [46]) proposes several different approaches that are relevant to some particular classes of problems. If the coefficients of the linear system, as well as the discretization parameters in space, do not depend on time, then the corresponding Green operator becomes invariant with respect to shifting along the time axis. This provides a noticeable economy when calculating and storing the coefficients of the Green operator. If the problem under study is parabolic, then the coefficients of its Green operator become small as the corresponding time interval becomes large. Therefore, only some fixed number of these coefficients is effectively needed to be taken into account for each specific moment of time, whereas the remainder of the coefficients can be neglected. This obviously leaves the ABC's nonlocal in time but only for some finite interval in the past. For some hyperbolic problems, it is also possible to effectively "cut off the tail" in time using the existence of lacunas. As has already been mentioned in Section 2.1, if the number of space dimensions is odd, the coefficients of a linear hyperbolic system are constant, and there are no lower order terms in the equations, then the solution to the corresponding Cauchy problem has lacunas. In particular, it means that if the initial data are compactly supported, then the solution becomes zero in any fixed (spatial) domain over a sufficiently long but still finite time interval. The similar property should asymptotically hold for the difference Cauchy problem as well; the latter can be used as an AP. In so doing, we again can neglect those coefficients of the Green operator that correspond to the large time intervals and, therefore, effectively restrict the nonlocality of the ABC's in time. These techniques, as well as some others presented in work [46], can substantially reduce the required computer effort when calculating the nonlocal exact ABC's. Finally, it is worth mentioning that the DPM-based ABC's of [46] can be considered as a certain type of "standard software." Indeed, the parameters of the process inside the computational domain (even the governing equations) can be changed, but as long as the overall solvability is not compromised and the exterior linear problem and the shape of the artificial boundary remain the same, the ABC's also remain exactly the same. Therefore, these boundary conditions can be effectively used for solving families of similar problems.

As mentioned previously, the primary role of any ABC's as a part of the numerical algorithm is to close the discrete (finite-difference) system solved inside the computational domain. Indeed, the stencil of the scheme used inside  $D_{in}$  cannot be applied to the nodes located at the external boundary or sufficiently close to it since the part of the stencil may merely "fall out" of the domain. Therefore, the finite-difference system appears subdefinite and requires that the missing relations between the values of the solution near the external boundary be provided by the ABC's. When the ABC's are constructed as a difference analogue to BEP (3.5), one can always think that the data on the internal layer(s) of the grid boundary  $\gamma$  are known and the data on the external layer(s) are to be determined using the BEP. The question of expressing the boundary values in terms of the data provided from inside  $D_{in}$  has been addressed in [46] along with the examples of implementation of the DPM-based ABC's with the different types of time-evolution procedures.



Ryaben'kii in work [46] provides a general framework of the algorithm for constructing the DPM-based ABC's. Any specific problem, of course, requires special treatment. As shown in Section 3, the nonlocal ABC's of type (1.6b) can be naturally and effectively incorporated into the structure of AP. For the steady-state diffraction problems governed by the Helmholtz equation, this effectiveness has been demonstrated by Ryaben'kii and Sofronov in work [28], see also work by Ryaben'kii [15]. Analogously to the Poisson equation AP of Section 3 (see also Section 1), the AP of [28] is formulated on a ball; the exact ABC's at the external spherical boundary are set using separation of variables for the Helmholtz equation in spherical coordinates. As mentioned in Section 2.1, the eigensolutions of the corresponding one-dimensional homogeneous equation in the transformed space, in particular, the function  $\hat{u}_l^{(1)}(r)$  that enters the determinant of (1.6a), are given by the modified Hankel functions in this case. In work [222], Sofronov had extended the methodology of [28] and proposed the approach to indirectly treat the continuous exact ABC's in the framework of the AP with the mesh not fitted to the boundary at which these ABC's are set; this approach has later been implemented by Weizman and Zinoviev in work [223] for solving the problems of acoustic diffraction.

Zueva, Mikhailova, and Ryaben'kii in work [224] and Brushlinskii, Ryaben'kii, and Tuzova in work [225] numerically study diffusion of the magnetic field in a finite conducting object surrounded by the vacuum; the formulation in both cases is two-dimensional. In work [224], the Cartesian coordinates are used to describe the magnetic field on the plane normal to a finite-cross-section conducting rod; in work [225], the conducting structure is a finite-cross-section axially symmetric annulus, and the magnetic field is studied in the meridian plane  $(r, z)$ . The diffusion process inside the conducting structure may be unsteady; however, the field in the surrounding vacuum is described by the Laplace equation. Therefore, the AP in work [224] and [225] is formulated and solved for the Laplace equation. Physical formulation of the problem generally assumes that the surrounding vacuum is infinite; in practice, however, the infinity is modeled by some remote boundary at which the magnetic field vanishes. Clearly, the size of the entire domain (including the vacuum area) must be chosen much larger than the cross section of the aforementioned rod or annulus; consequently, the ABC's that can exactly transfer the remote boundary condition to the surface of the conducting structure are as important in this model formulation as in the truly infinite one. In work [224, 225], the remote boundary condition is explicitly incorporated into the formulation of the AP. The exact ABC's are obtained in the form of the difference BEP; the latter is then solved for the reason of numerical convenience, which results in a matrix relation that connects the vectors of the unknowns on the internal and external layers of the two-layered grid boundary  $\gamma$ . The implementation of the ABC's of [224, 225] is algorithmically very simple: the matrix is calculated only once, and then it multiplies the corresponding vector at each time step. Actual numerical experiments in work [224, 225] have been carried out for the rectangular cross section of the conducting structure, although the DPM-based ABC's of [224, 225] can be obtained in exactly the same manner for any shape of the cross section. The experiments show that the computational cost of the algorithm that includes the DPM-based ABC's is much lower than the cost of the standard procedure that is based on actually solving the Laplace equation in a vacuum area at each time step. At the same time, the accuracy of the solution calculated inside the rod (annulus) with the ABC's of [224, 225] is the same as the accuracy obtained when the entire original problem is solved on an extended domain, which makes the DPM-based algorithm an attractive technique for solving the problems of this type.

In work [226], Mishkov and Ryaben'kii solve the inhomogeneous Helmholtz equation on a semi-plane. The RHS of the equation is compactly supported, it models the source of sound, e.g., a scatterer or an emitter. The surrounding semi-infinite sound-conducting medium is stratified; it consists of two layers with different speeds of sound, which makes the coefficient at the zeroth-order term of the Helmholtz equation piecewise constant. The radiation boundary conditions at infinity are formulated using the limitary absorption principle. The solution to this problem is to be calculated on some finite computational domain that entirely contains the aforementioned sound source. Because of the non-constancy of coefficients, there is apparently no means to analytically transfer the boundary conditions from infinity even to some regular finite boundary, as it has been done for the sphere in the case of the Laplace equation, see formula (1.6b). Therefore, the AP in work [226] is formulated for the perturbed Helmholtz equation (finite absorption is added) on a sufficiently large rectangle, and the homogeneous Dirichlet boundary conditions are set at the rectangle's sides. It is shown that as the size of this rectangle enlarges, the solution to the AP approaches the solution of the corresponding infinite-domain problem on any fixed neighborhood of the computational domain. It is also shown that as the absorption coefficient vanishes, the solution to the difference AP does approach the solution that corresponds to the true time-harmonic outgoing waves with no absorption. Moreover, a special technique proposed in [226] enables one to formally let the absorption coefficient be zero in the computations and still obtain the same solution. Therefore, the ABC's of [226] that are obtained in the form of the

difference BEP, can generally be constructed as close to the exact ABC's as desired. In other words, with these ABC's the truncated problem can be solved so that its solution differ from the corresponding fragment of the original solution within the initially prescribed accuracy. Of course, achieving the high accuracy may require a large domain for the AP. However, the ABC's of [226] become most effective when they are used to solve a series of similar problems, e.g., to calculate sound fields from different sources in the same medium, which is frequently the case important for applications.

Tsynkov in work [227, 228] and Sofronov and Tsynkov in work [229] consider a finite body (airfoil) immersed into an infinite flow of inviscid compressible fluid. The flow is governed by the Euler equations and assumed subsonic at infinity; the equations are discretized on the finite-difference O-type grid generated around the body. The computational domain in [227–229] is formed by this grid; the shape of its external boundary is completely determined by the grid as well, and no special assumptions regarding this shape are made. Outside the computational domain, the Euler equations are linearized against the constant free-stream background. Moreover, under the assumption that the velocity potential exists in the far field, the linearized system is split into the elliptic (velocity) and advection (entropy) parts. After the term associated with the circulation of flow around the airfoil has been subtracted, the regular part of the potential of velocity perturbations satisfies the Prandtl-Glauert equation

$$(4.1a) \quad (1 - M_0^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

with the zero boundary condition at infinity:

$$(4.1b) \quad \phi(x, y) \longrightarrow 0, \quad \text{as } x^2 + y^2 \longrightarrow \infty.$$

As mentioned in Section 2.2, the two-dimensional equation (4.1a) may produce inaccurate results when the free-stream Mach number  $M_0$  approaches one too closely. We, therefore, do not attempt to use the boundary conditions obtained on the basis of problem (4.1) (see [227–229]) for near-sonic flows.

Equation (4.1a) can be easily reduced to the Laplace equation by means of an affine coordinate transform; in so doing, boundary condition (4.1b) obviously does not change. To obtain the ABC's for velocity components, we represent the solution to (4.1) in the form of a generalized potential and then construct the corresponding BEP. We use the AP formulated on an annular domain  $\{R_1 \leq r \leq R_0\}$ ; the external circle  $\{r = R_0\}$  encompasses the artificial boundary and the internal circle  $\{r = R_1\}$  lies inside the computational domain. At  $r = R_1$ , we specify the homogeneous Dirichlet boundary conditions; at  $r = R_0$ , we specify the boundary conditions analogous to (1.6b) but obtained for the two-dimensional Laplace equation. It is possible to make sure that this AP is uniquely solvable and well-posed for any RHS concentrated inside the annulus. Numerically, the AP is easy to solve by means of the discrete Fourier transform in polar coordinates.

To close the discrete system that is solved inside the computational domain for the case of a second-order scheme used in the interior, we can always assume that the velocity components on the entire O-type grid including the artificial boundary  $\Gamma$  are known, whereas the corresponding values at the row of ghost nodes  $\Gamma_1$  are to be determined using the ABC's. ( $\Gamma$  and  $\Gamma_1$  can also be considered the penultimate and outermost coordinate lines of the grid, respectively.) The coordinate system and the grid for the AP are polar; the Laplacian is discretized with the second order of accuracy; the grid boundary  $\gamma$  and the BEP are constructed in accordance with the description in Section 3. Then, the BEP is solved so that the velocity on  $\Gamma$  provided from inside the computational domain coincides with the gradient of the potential (4.1) in a certain generalized sense. This gives the discrete density of the generalized potential for any data (velocity components) specified on  $\Gamma$ . Further, we calculate the generalized potential itself and find the trace of its gradient on the outermost coordinate line  $\Gamma_1$  by means of interpolation; this procedure yields the ABC's for the velocities. Finally, the ABC's for the thermodynamic parameters are obtained using local relations, specifically, the Bernoulli equation and the entropy advection equation.

The ABC's technique of [227–229] was combined with the iterative Euler solver by Sofronov [230] and several subsonic and transonic steady-state airfoil flows have been numerically studied. (For the transonic case, the near-surface local supercritical regions relevant have always been kept inside the computational domain.) The results of the numerical experiments presented in work [229] clearly demonstrate the superiority of the nonlocal DPM-based ABC's over the standard local techniques based on quasi-one-dimensional characteristic analysis (see Section 2.2). For a fixed computational domain, the nonlocal ABC's of [227–229] provide for better accuracy and faster convergence to the steady state than the local techniques do. Moreover, for some computational variants the algorithm with local boundary conditions simply fails to converge,

which never happens for the DPM-based ABC's of [227–229] and which indicates that the new boundary conditions increase robustness of the entire numerical procedure. Additionally, when the artificial boundary approaches the airfoil, the solution obtained with the technique of [227–229] appears essentially less influenced by the shrinkage of the computational domain than the solution obtained using local radiation boundary conditions. In other words, the ABC's of [227–229] allow one to maintain high accuracy for much smaller computational domains than the standard boundary conditions do. These results, along with the geometric universality of the DPM-based ABC's of [227–229], make this approach useful for calculating the external Euler flows.

The foundations of the DPM-based methodology for setting highly accurate nonlocal ABC's in computation of viscous flows have been proposed by Ryaben'kii and Tsynkov in work [231]; this work addresses the two-dimensional steady-state algorithm for Navier-Stokes' equations. In work [232], Tsynkov implements the approach of [231] along with the code FLOMG (see [60–63]) and presents some numerical results for subsonic and transonic laminar flows over single-element airfoils. In work [233], Tsynkov, Turkel, and Abarbanel show the results of subsequent numerical experiments and propose an approximate treatment of turbulence in the far field. This treatment is based on the concept of effective turbulent viscosity by Boussinesq (see, e.g., [234]). Work by Ryaben'kii and Tsynkov [235] delineates the algorithm for solving one-dimensional systems of ordinary difference equations that arise after the separation of variables in the AP. Some issues related to the steady-state two-dimensional algorithm are also discussed in work by Tsynkov [236, 237]. In work [238], Tsynkov extends the area of applications for the DPM-based ABC's by analyzing the two-dimensional flows that oscillate in time. In practice, this formulation may originate from the well-known problem of pitching airfoil or from some modern approaches to active flow control, e.g., the one by Seifert et al. [239]. Work by Tsynkov [238] also addresses some solvability issues for the linearized thin-layer equations used in formulation of the AP. In work [240], Ryaben'kii and Tsynkov present a general survey of the DPM-based methodology as applied to solving external problems in CFD, including parallel implementation of the algorithm, combined implementation of nonlocal ABC's with multigrid, and entry-wise interpolation of the matrices of boundary operators with respect to the Mach number and the angle of attack. Additionally, in [240] one can find some new theoretical results on the computation of generalized potentials, the construction of ABC's based on the direct implementation of boundary projections (thin-layer equations), and some numerical results for various airfoil flows: laminar and turbulent, transonic and subsonic, including very low Mach numbers.

The next natural objective after constructing the two-dimensional algorithm is the analysis of three-dimensional steady-state flows. This case is undoubtedly the one most demanded by the current practice in CFD. In work [241, 242], Tsynkov outlines the basic elements of the DPM-based ABC's for steady-state viscous flows around the wing-shaped configurations and shows some preliminary numerical results for the subsonic regime. The numerical results of work [242] have been obtained with the NASA-developed production code TLNS3D by Vatsa, et al. [64, 65]. In work [243], Tsynkov and Vatsa further develop the three-dimensional DPM-based algorithm and present the computational results for transonic flows. Finally, in work [244] Tsynkov systematically describes the DPM-based ABC's for calculation of viscous flows about three-dimensional wings. It turns out that in all three-dimensional cases that have been studied numerically (see [242–244]), the DPM-based ABC's allow one to greatly reduce the size of the computational domain (compared to the standard local boundary conditions) while still maintaining high accuracy of the numerical solution. This actually means the overall increase of accuracy due to the improved treatment of the artificial boundary; it also implies the substantial economy of the computer resources. Moreover, the DPM-based ABC's may provide for a noticeable speedup (up to a factor of three) of the convergence of multigrid iterations. The issue of the combined implementation of global boundary conditions with multigrid and the resulting speedup of multigrid convergence have been specifically addressed and emphasized by Tsynkov in [245].

The rest of this paper will primarily be devoted to surveying the DPM-based methodology as applied to setting the ABC's for calculating the external viscous flows. The construction of the ABC's itself will be described for the three-dimensional case; as concerns actual computations, both two-dimensional and three-dimensional results will be presented.

We consider an unbounded steady-state flow of viscous fluid past a finite body, e.g., a three-dimensional wing (or an airfoil in two dimensions). The flow is uniform at infinity. We consider both the incompressible and compressible formulations, in the latter case we assume that the fluid (gas) is thermodynamically perfect and that the free stream is subsonic. Moreover, as the fluid is viscous and the size of the immersed body (wing) is finite, the flow limit at infinity is a free stream.

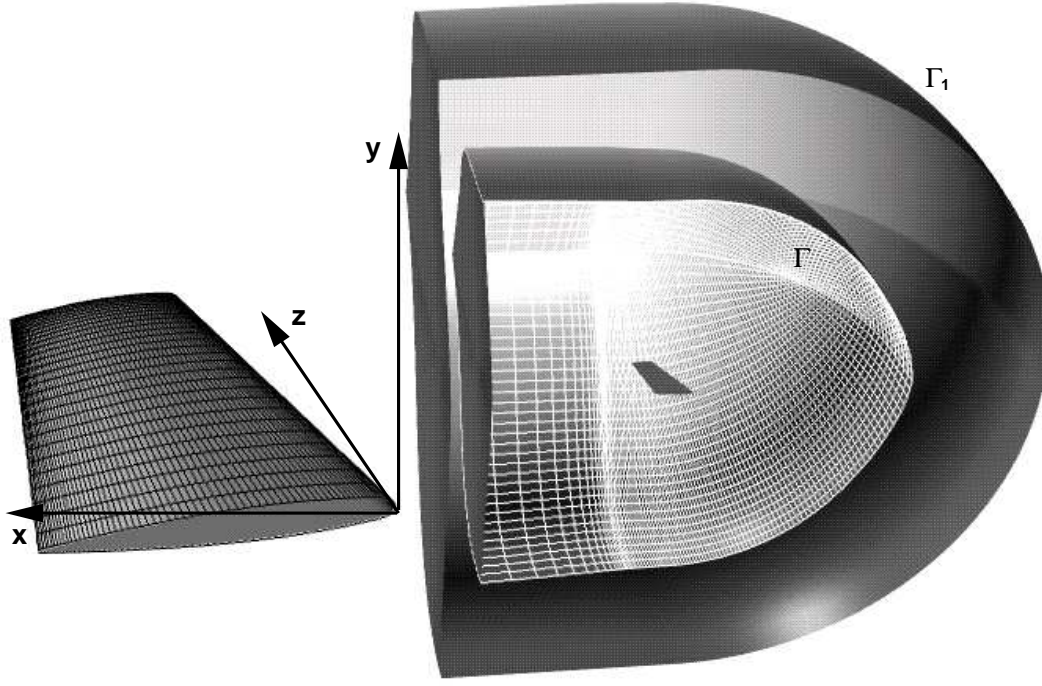


FIG. 4.1. *Schematic geometric setup for the three-dimensional case. The wing on the left is enlarged.*

Generally, the near-field flow is governed by the full Navier-Stokes equations. However, in many cases the full system can be simplified and reduced to the so-called thin-layer equations (see, e.g., [246]), which do not contain the streamwise viscous derivatives. In particular, this simplification is employed in the code TLNS3D (see [64, 65]) which has been used for our three-dimensional numerical tests; our far-field model is simplified accordingly as well.

The flow equations are integrated on the grid generated around the wing. This grid actually defines the finite computational domain  $D_{in}$ ; the ABC's that would close the truncated problem should be set at the external coordinate surface  $\Gamma = \partial D_{in}$  of the grid. For the one-block C-O-type curvilinear boundary-fitted grid around the ONERA M6 wing, the schematic geometric setup is shown in Figure 4.1. The flow is assumed symmetric with respect to the plane  $z = 0$ .

The coordinate surface designated  $\Gamma_1$  in Figure 4.1 represents the ghost nodes (or ghost cells for the finite-volume formulation) of the C-O grid. Clearly, when the stencil of the scheme used inside the computational domain is applied to any node from  $\Gamma$ , it generally requires some ghost cell data. (For a second-order scheme on the  $3 \times 3$  stencil inside  $D_{in}$ , it is basically sufficient to use only one row of ghost nodes, for larger stencils the entire construction would require minor changes.) Unless the required ghost cell data are provided, the finite-difference system solved inside the computational domain appears subdefinite (i.e., it has less equations than unknowns). If, for example, some iterative solver is employed to integrate the flow equations inside the computational domain, the values of the solution at the ghost cells should be prescribed at each iteration in order to be able to advance the next “time” step. Therefore, as has already been mentioned the closure of the discretized truncated problem in practical framework means specification of the solution at the ghost cells. This closure is provided by the DPM-based ABC's. In so doing, the resulting solution at the artificial boundary admits an exterior complement that solves the problem outside the computational domain. More precisely, outside the computational domain we consider the problem in small perturbations, and its approximate solution is obtained within some accuracy that can be controlled.

Let us now assume that the perturbations of flow against the constant free-stream background are small in  $D_{ex} = R^3 \setminus D_{in}$ , i.e., in the far field outside  $\Gamma$ . For the incompressible case, as well as for the small to moderate free-stream Mach numbers  $M_0$  (purely subsonic flows), one can retain in the governing equations only the first-order terms with respect to the small perturbations. In so doing, the original problem will

actually be linearized in the far field; the two-dimensional example for the compressible velocity potential is given by the Prandtl-Glauert equation (4.1a). In the transonic limit, however, equation (4.1a) may appear deficient (see comments in Section 2.2 and book [110] for details); therefore, the possibility of linearization in the far field as  $M_0 \rightarrow 1$  requires, generally speaking, some additional analysis. Below, we provide such an analysis for the model full-potential formulation and justify that the compressible three-dimensional far field remains linear for transonic flow regimes as well (see [244] for more details).

Formally (see [110]), in the transonic limit some quadratic terms with respect to the small perturbations should be retained in the governing equations along with the first-order terms. For the perturbation  $\phi$  of the full velocity potential  $\Phi$  in the compressible gas flow around a thin three-dimensional wing, this yields the Kármán–Guderley equation

$$(4.2) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial \hat{y}^2} + \frac{\partial^2 \phi}{\partial \hat{z}^2} = \frac{\kappa + 1}{K} \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2}.$$

Here

$$(4.3) \quad \begin{aligned} \frac{\Phi'_x}{u_0} &= 1 + \delta^{2/3} \phi'_x, & \frac{\Phi'_y}{u_0} &= \delta \phi'_y, & \frac{\Phi'_z}{u_0} &= \delta \phi'_z, \\ \tilde{y} &= \delta^{1/3} y, & \tilde{y} &= \delta^{1/3} y, & K &= \frac{1 - M_0^2}{\delta^{2/3}}, \\ \hat{y} &= \sqrt{K} \tilde{y}, & \hat{z} &= \sqrt{K} \tilde{z}, \end{aligned}$$

$\delta$  is the wing thickness ( $\delta \rightarrow +0$  along with  $M_0 \rightarrow 1$  in the transonic limit),  $K$  is the parameter of transonic similarity (the true linear theory corresponds to big values of  $K$ ),  $u_0$  is the flow speed at infinity,  $\kappa$  is the ratio of specific heats, and the positive  $x$  direction coincides with the free stream.

The common practice (see [110]) of developing the asymptotic expansions for solutions of the equations that involve transonic nonlinearities consists of substituting the leading linear term(s) into the nonlinear parts of the equation (right-hand side of (4.2)) and obtaining the corresponding corrections by solving the resulting non-homogeneous problem. In the specific case under study, the linear far-field expansion starts with the horseshoe vortex

$$(4.4a) \quad \phi_1 = \frac{\hat{y}}{\hat{y}^2 + \hat{z}^2} \left( 1 + \frac{x}{\hat{r}} \right), \quad \hat{r} \equiv \sqrt{x^2 + \hat{y}^2 + \hat{z}^2}.$$

Substituting the expression (4.4a) into the right-hand side of (4.2) and solving the resulting Poisson equation (this involves the Fourier transform in spherical functions) one obtains the nonlinear correction

$$(4.4b) \quad \phi_{1NL} \sim \hat{r}^{-3}$$

that decays at infinity two orders of magnitude faster than the source term (4.4a). Analogously, for the general doublet term  $\phi_2 \sim \hat{r}^{-2}$  the corresponding nonlinear correction can be shown to be  $\phi_{2NL} \sim \hat{r}^{-5}$ , which is the three orders of magnitude difference. We therefore conclude that the transonic nonlinear corrections can be neglected when analyzing the compressible far field in three space dimensions.

Of course, having established the fact of the far-field linearity, we cannot say in advance whether or not the linearization outside  $\Gamma$  is possible for every specific configuration of the domains. Clearly, for the very large computational domain one can linearize the flow outside  $\Gamma$ , and as we approach the wing (the source of perturbations), the validity of linearization can be verified a posteriori (see below, as well as our work [232, 233, 240, 242–244] for details).

The dimensionless governing equations linearized against the constant free-stream background in  $D_{ex}$  can be written as

$$(4.5a) \quad \mathbf{C} \frac{\partial \mathbf{u}}{\partial x} + \mathbf{D} \frac{\partial \mathbf{u}}{\partial y} + \mathbf{E} \frac{\partial \mathbf{u}}{\partial z} + \mathbf{F} \frac{\partial^2 \mathbf{u}}{\partial y^2} + \mathbf{H} \frac{\partial^2 \mathbf{u}}{\partial z^2} + \mathbf{J} \frac{\partial^2 \mathbf{u}}{\partial y \partial z} = \mathbf{0},$$

where  $(x, y, z)$  are the Cartesian coordinates,  $\mathbf{u} = [p, u, v, w]^t$  for the incompressible case, and  $\mathbf{u} = [p, u, v, w, \rho]^t$  for the compressible case;  $p, u, v, w$ , and  $\rho$  are the perturbations of pressure, velocity components, and density, respectively. The free stream is aligned with the positive  $x$  direction. Matrix coefficients  $\mathbf{C}$ ,  $\mathbf{D}$ , and  $\mathbf{E}$  correspond to the convection terms and matrix coefficients  $\mathbf{F}$ ,  $\mathbf{H}$ , and  $\mathbf{J}$  correspond to the viscous terms (the latter are proportional to the inverse Reynolds number  $Re^{-1}$ ); in both incompressible and compressible cases, the matrices can be considered symmetric, the explicit expressions can be found in our work [244]. For the two-dimensional case, the system can be written in a similar form, see our earlier work [238] and also [231].

System (4.5a) describes the flow in the far field. In both incompressible and compressible cases, it is supplemented by the boundary condition

$$(4.5b) \quad \mathbf{u} \longrightarrow \mathbf{0}, \quad \text{as} \quad r \equiv (x^2 + y^2 + z^2)^{1/2} \longrightarrow +\infty,$$

which means that all the perturbations vanish at infinity, or equivalently, the flow approaches the free stream limit.

From now on, instead of the original formulation we will be considering the new coupled problem; this problem is nonlinear in  $D_{in}$  (original thin-layer equations with no-slip boundary conditions at the body surface) and linear in  $D_{ex}$  (system (4.5a) with boundary condition (4.5b) at infinity). Since  $D_{ex}$  is still infinite, we cannot solve this coupled problem directly so we replace its entire linear portion by the ABC's at  $\Gamma$  using difference Calderon's projections and the DPM. To accomplish this step, we need to formulate the AP for system (4.5a), this AP should take into full account the structure of the solution from outside the computational domain. In the example of Section 3 (Poisson equation), we have been able to formulate such an AP on the ball  $\{r \leq R_0\}$ , and the boundary conditions could be analytically transferred from infinity to the surface of this ball, see formula (1.6b). The feasibility of this transfer is accounted for by the possibility to separate the variables for the Laplace equation in spherical coordinates.

However, existence of the special symmetries that allow, for example, to separate the variables for the Laplace equation in spherical coordinates, cannot be regarded as the general situation, it is rather an exception. For system (4.5a) such symmetries are apparently unavailable. Therefore, we first introduce a parallelepiped  $[0, X] \times [-Y/2, Y/2] \times [0, Z/2] \supset \Gamma_1$  and discretize system (4.5a) on the auxiliary Cartesian grid in this parallelepiped with the second order of accuracy. We use the first-order differences in  $x$  and the second-order differences in  $y$  and  $z$  (see our work [238, 240, 244] for details); in so doing, the residuals are evaluated and consequently, the RHS's for the AP are specified on the Cartesian grid that is shifted half-size in  $x$ . The DPM will provide us with the *complete boundary classification* (in terms of the appropriate traces) of all those and only those exterior grid vector-functions that solve the discrete counterpart to (4.5a) on  $D_{ex}$  and satisfy boundary condition (4.5b) in some approximate sense. This classification will be obtained as an image of the discrete counterpart to Calderon's boundary projection operator.

The finite-difference AP is formulated on the parallelepiped  $[0, X] \times [-Y/2, Y/2] \times [0, Z/2]$  for the inhomogeneous counterpart to (4.5a); its compactly supported RHS  $\mathbf{f}$ ,  $\text{supp } \mathbf{f} \subset D_{in}$ , is specified in accordance with the general description of Section 3 (our work [244] provides more detail). We also specify the periodicity boundary conditions in the  $y$  and  $z$  directions (with the periods  $Y$  and  $Z$ , respectively) and assume symmetry at  $z = 0$ . After the Fourier transform with respect to  $y$  and  $z$ , the discrete inhomogeneous counterpart to (4.5a) can be written as a family of one-dimensional difference equations

$$(4.6) \quad \begin{aligned} \mathbf{A}_k \hat{\mathbf{u}}_{m,k} + \mathbf{B}_k \hat{\mathbf{u}}_{m-1,k} &= \hat{\mathbf{f}}_{m-1/2,k}, \\ m &= 1, \dots, M, \quad \mathbf{k} \equiv (k_y, k_z), \\ k_y &= 0, \dots, J_y, \quad k_z = 0, \dots, J_z. \end{aligned}$$

The explicit expression for the matrices  $\mathbf{A}_k$  and  $\mathbf{B}_k$  can be found in [244]. In formula (4.6),  $M + 1$ ,  $2J_y + 1$ , and  $J_z + 1$  are the numbers of the Cartesian grid nodes in the  $x$ ,  $y$ , and  $z$  directions, respectively (symmetry has been taken into account, as well as the fact that both  $\mathbf{u}$  and  $\mathbf{f}$  are real-valued). Note, in our earlier two-dimensional work we have used the full Navier-Stokes rather than the thin-layer equations, in this case the reduction of the linearized problem to the family of one-dimensional first-order difference equations of type (4.6) requires introducing the additional variables, see [231].

Boundary conditions for the AP in the streamwise direction  $x$  (at  $x = 0$  and  $x = X$ ) are imposed on Fourier components of the solution separately for each pair of wavenumbers  $\mathbf{k}$ :

$$(4.7a) \quad \left[ \prod_{|\lambda(\mathbf{k})| > 1} (\mathbf{Q}_{\mathbf{k}} - \lambda(\mathbf{k})\mathbf{I}) \right] \hat{\mathbf{u}}_{0,\mathbf{k}} = \mathbf{0},$$

and

$$(4.7b) \quad \left[ \prod_{|\lambda(\mathbf{k})| \leq 1} (\mathbf{Q}_{\mathbf{k}} - \lambda(\mathbf{k})\mathbf{I}) \right] \hat{\mathbf{u}}_{M,\mathbf{k}} = \mathbf{0},$$

where  $\mathbf{Q}_{\mathbf{k}} = \mathbf{A}_{\mathbf{k}}^{-1}\mathbf{B}_{\mathbf{k}}$ ,  $\lambda(\mathbf{k})$  are the eigenvalues of  $\mathbf{Q}_{\mathbf{k}}$ , and  $\mathbf{I}$  is the identity matrix of the appropriate dimension.

The semi-analytic boundary conditions (4.7) are the principal part of the AP formulation. Since the RHS  $\mathbf{f}$  is compactly supported, we may formally consider (4.6) for  $m < 0$  and for  $m > M$  as a homogeneous system of ordinary difference equations. Then, in accordance with the requirement (4.5b), boundary condition (4.7a) prohibits at  $x = 0$  ( $m = 0$ ) all the modes (eigensolutions of the homogeneous system) that do not decrease to the left (inflow direction), and boundary condition (4.7b) prohibits at  $x = X$  ( $m = M$ ) all that modes that increase to the right (outflow direction). The eigenvalues  $\lambda(\mathbf{k})$  for (4.7) are calculated numerically. The role of boundary conditions (4.7) is, in fact, similar to the one of (1.2b) or (1.6b) (see Section 1.2) as the latter also explicitly prohibit the growing modes and allow only the decaying ones so that the desired behavior of the solution at infinity be achieved. The periods  $Y$  and  $Z$  should be chosen sufficiently large to ensure that the periodic solution considered near  $\Gamma$  and  $\Gamma_1$  is sufficiently close to the theoretical non-periodic solution; the latter can be thought of as a limit when the periods approach infinity. The approximation of the non-periodic solution by the periodic one on a finite fixed neighborhood of  $D_{in}$  as the period(s) increase(s), as well as the relations between the grid sizes and periods that are necessary for convergence (when the grid sizes shrink and the periods simultaneously increase), are discussed in detail in our work [231, 238, 240]. Moreover, recent work by Mishkov and Ryaben'kii [247, 248] delineates the issue of approximating the non-periodic solutions by the periodic ones and provides a rigorous justification of convergence for a model two-dimensional problem. In addition, in work [240, 244], we discuss the possibility to replace the Fourier transforms by the non-unitary transforms. The latter appear when the grid in  $y$  or  $z$  is stretched (which provides for a drastic cost reduction) and the corresponding eigenfunctions consequently form a skew basis.

The periodic AP with boundary conditions (4.7) has been constructed so that on any finite fixed neighborhood of  $D_{in}$  its solution can be made as close to the solution of the inhomogeneous problem (4.5) (with the same compactly supported RHS  $\mathbf{f}$ ,  $\text{supp} \mathbf{f} \subset D_{in}$ ) as initially prescribed. From the standpoint of constructing the DPM-based ABC's, the aforementioned type of convergence, i.e., the convergence considered only on some finite neighborhood of  $D_{in}$ , presents no loss of generality. Indeed, this neighborhood can always be chosen to entirely contain both  $\Gamma$  and  $\Gamma_1$ , and the difference projections operate in the space of functions defined on the grid boundary  $\gamma$ , which is located near  $\Gamma$  by definition. Therefore, all the operators can be constructed using this AP in the same way as described in Section 3; in so doing the discrepancy between the resulting ABC's and the ABC's that could have been obtained on the basis of the nonperiodic infinite-domain AP is controlled by the values of the periods  $Y$  and  $Z$  and can potentially be made as small as desired. The techniques for actually solving the foregoing difference AP on the computer are described in our work [235, 244].

The difference grid boundary  $\gamma$  for the second-order Cartesian discretization of system (4.5a) that we use is constructed in accordance with the algorithm of Section 3. The foregoing AP allows us to calculate the generalized difference potential  $\mathbf{P}_{ex}\mathbf{u}_{\gamma}$  for any grid density  $\mathbf{u}_{\gamma}$  specified on  $\gamma$  ( $\mathbf{u}_{\gamma}$  is the discrete analogue of  $\xi_{\Gamma}$ , see Section 3). The composition of the operators  $\mathbf{Tr}_{\gamma}$  and  $\mathbf{P}_{ex}$ , where  $\mathbf{Tr}_{\gamma}$  for this case is merely a contraction from the entire Cartesian grid of the AP to  $\gamma$ ,  $\mathbf{P}_{\gamma} \equiv \mathbf{Tr}_{\gamma}\mathbf{P}_{ex}$ , is a projection,  $\mathbf{P}_{\gamma}^2 = \mathbf{P}_{\gamma}$ , and it is a discrete counterpart of Calderon's boundary projection for the operator given by the left-hand side of (4.5a). The image of this projection,  $\text{Im} \mathbf{P}_{\gamma}$ , contains all those and only those  $\mathbf{u}_{\gamma}$ 's that are the traces of some exterior difference solution to (4.5a) that satisfies the boundary conditions of the AP (periodicity in

the  $y$  and  $z$  directions and boundary conditions (4.7) in the  $x$  direction). The latter boundary conditions, in turn, approximate (4.5b).

Having constructed the procedure for calculating the potentials and projections for the discrete version of (4.5a), we can now close the system inside the computational domain, i.e., obtain the ABC's. First, we take  $\mathbf{u}$  and  $\partial\mathbf{u}/\partial n$  on  $\Gamma$ ,  $n$  is the normal, (these data are available from inside the computational domain  $D_{in}$ ) and, using interpolation  $\mathbf{R}_\Gamma$  along  $\Gamma$  and the first two terms of the Taylor expansion (denoted  $\pi_\gamma$ ), obtain  $\mathbf{u}_\gamma$ :

$$(4.8) \quad \mathbf{u}_\gamma = \pi_\gamma \mathbf{R}_\Gamma \left( \mathbf{u}, \frac{\partial \mathbf{u}}{\partial n} \right) \Big|_\Gamma.$$

Then, we need to calculate the potential  $\mathbf{P}_{ex} \mathbf{v}_\gamma$  for the density  $\mathbf{v}_\gamma = \mathbf{P}_\gamma \mathbf{u}_\gamma$  and interpolate it to the nodes  $\Gamma_1$ :

$$(4.9) \quad \mathbf{u} \Big|_{\Gamma_1} = \mathbf{R}_{\Gamma_1} \mathbf{P}_{ex} \mathbf{v}_\gamma \equiv \mathbf{R}_{\Gamma_1} \mathbf{P}_{ex} \mathbf{u}_\gamma.$$

Finally, the ABC's are obtained in the operator form

$$(4.10) \quad \mathbf{u} \Big|_{\Gamma_1} = \mathbf{T} \left( \mathbf{u}, \frac{\partial \mathbf{u}}{\partial n} \right) \Big|_\Gamma,$$

where  $\mathbf{T}$  is composed of the operations (4.8) and (4.9). Boundary condition (4.10) is applied every time we need to update the ghost cell values in the course of the iteration process. The implementation of ABC's (4.10) can either be direct or involve preliminary calculation of the matrix  $\mathbf{T}$ . In the latter case, the runtime implementation of the ABC's (4.10) is reduced to a matrix-vector multiplication.

Let us also note, that in our earlier work (see [231–233]) we have implemented another approach for setting the DPM-based ABC's. For two-dimensional Navier-Stokes' equations, we were actually solving the BEP by means of a variational approach thus expressing the normal derivatives of the solution on  $\Gamma$  and therefore, the grid density of the generalized potential, in terms of only the Dirichlet-type boundary data on  $\Gamma$ . Then, we could obtain the data on  $\Gamma_1$  by calculating the generalized potential. From the standpoint of computing, the foregoing new approach with the direct implementation of boundary projections is less cumbersome and less expensive; at the same time it has been experimentally found at least as accurate and robust as the previous one. More comments on these two approaches can be found in our work [240].

Let us now demonstrate some numerical results obtained when computing the external viscous flows using DPM-based ABC's. In our work [231–233, 240], we have calculated some subsonic and transonic flows past single-element airfoils (NACA0012 and RAE2822). The computational domain is formed by the C-type curvilinear grid generated around the airfoil. On this grid, the Navier-Stokes equations are integrated using the code FLOMG, see [60–63]. The standard treatment of the external boundary in the code FLOMG is based on the locally one-dimensional characteristics analysis, which may or may not be supplemented by the point-vortex (p.-v.) correction, see [108].

In Table 4.1, we show the results of a particular computation (transonic turbulent flow) for the airfoil RAE2822 under the angle of attack  $\alpha = 2.79^\circ$ . Note, the grids for the smaller domains in Table 4.1 are the exact subsets of the biggest grid, and we therefore expect that the grid changes do not exert any direct influence on the solution in the near field.

Basic conclusions that could be drawn from the particular results presented in Table 4.1, as well as from our two-dimensional numerical experience in general (see [231–233, 240]), are the following. The DPM-based ABC's are geometrically universal, algorithmically simple and easy to implement along with the existing solver. For large computational domains (30–50 chords of the airfoil), the performance of the standard methods and the DPM-based ABC's is roughly the same. As, however, the artificial boundary approaches the airfoil the discrepancy between the corresponding solutions increases. The lift and drag coefficients obtained on the basis of the DPM-based boundary conditions deviate from their asymptotic (50 chords) values much slighter (within fractions of one percent) than the coefficients obtained using local ABC's do. In other words, the nonlocal DPM-based ABC's allow one to use much smaller computational domains (as small as 2–3 chords) than the standard boundary conditions do and still maintain high accuracy of



TABLE 4.1  
*RAE2822*:  $M_0 = 0.73$ ;  $Re_0 = 6.5 \cdot 10^6$ ;  $\alpha = 2.79^\circ$ .

| “Average radius” of $D_{in}$  | 3 chords         |        | 8 chords         |        | 50 chords        |        |
|-------------------------------|------------------|--------|------------------|--------|------------------|--------|
| Dimension of the grid         | $600 \times 104$ |        | $608 \times 112$ |        | $640 \times 128$ |        |
| Type of ABC’                  | p.-v.            | DPM    | p.- v.           | DPM    | p.-v.            | DPM    |
| Dynamic lift, $C_l$           | 0.8653           | 0.8591 | 0.8624           | 0.8589 | 0.8603           | 0.8593 |
| Relative error                | 0.58%            | 0.02%  | 0.24%            | 0.04%  | 0%               | 0%     |
| Dynamic drag, $C_d \times 10$ | 0.1203           | 0.1263 | 0.1209           | 0.1261 | 0.1255           | 0.1260 |
| Relative error                | 4.14%            | 0.24%  | 3.67%            | 0.08%  | 0%               | 0%     |
| Full drag, $C_D \times 10$    | 0.1755           | 0.1816 | 0.1762           | 0.1815 | 0.1810           | 0.1815 |
| Relative error                | 3.04%            | 0.05%  | 2.65%            | 0%     | 0%               | 0%     |

TABLE 4.2  
*ONERA M6*:  $M_0 = 0.84$ ;  $Re_0 = 11.7 \cdot 10^6$ ;  $\alpha = 3.06^\circ$ .

| “Average radius” of $D_{in}$ | 3 root chords             |        | 10 root chords            |        |
|------------------------------|---------------------------|--------|---------------------------|--------|
| Dimension of the grid        | $197 \times 49 \times 33$ |        | $209 \times 57 \times 33$ |        |
| Type of ABC’s                | standard                  | DPM    | standard                  | DPM    |
| Full lift, $C_L$             | $0.298 \pm 0.004$         | 0.2798 | 0.2805                    | 0.2786 |
| Relative error               | $6.24\% \pm 1.43\%$       | 0.43%  | 0%                        | 0%     |
| Full drag, $C_D \times 10$   | $0.168 \pm 0.008$         | 0.1537 | 0.1542                    | 0.1531 |
| Relative error               | $8.95\% \pm 5.19\%$       | 0.39%  | 0%                        | 0%     |

computations. Moreover, if we compare three models: DPM-based, point-vortex, and standard local, then it turns out that the DPM-based ABC’s display the best performance for small computational domains, the performance of the local characteristic boundary conditions for the small domains is very poor, and the point-vortex boundary conditions perform much better for the lift than they do for the drag coefficient. This behavior seems reasonable since the point-vortex model is a lift-based treatment. We also note that for certain variants of computation the DPM-based ABC’s may noticeably speed up (by up to a factor of three) the convergence of the multigrid iterations, see our work [231–233]. Some discussion on the combined implementation of the DPM-based ABC’s with multigrid can be found in work [240,245]. The phenomenon of multigrid convergence speedup caused by implementing the DPM-based ABC’s also occurs for the three-dimensional case, the corresponding results are presented below, see also [245].

In work [242–244], we have calculated several different flows around the ONERA M6 wing. An example of the grid for these computations is shown in Figure 4.1. The thin-layer equations are integrated on this grid by the code TLNS3D, see [64]. This code is based on the central-difference finite-volume discretization in space with the first- and third-order artificial dissipation. Transient iterations are used for obtaining the steady-state solution; the integration in time is done by the five-stage Runge-Kutta algorithm (with the Courant number calculated locally) supplemented by residual smoothing. For the purpose of accelerating the convergence, the multigrid methodology is implemented; in our computations we have used three subsequent grid levels with V cycles; the full multigrid methodology (FMG) could be employed as well. In addition, we use the preconditioning technique of [65] to improve the convergence to steady state. The details of implementation of the DPM-based ABC’s with the code TLNS3D are discussed in [243,244]. Unlike the two-dimensional case, the standard ABC’s in three dimensions are combined of the locally one-dimensional radiation boundary conditions for the inflow part of the boundary and extrapolation of all flow variables at the outflow part of the boundary (the point-vortex model is not applicable).

The flow regimes that we have studied numerically in our work [242–244] range from very low to transonic Mach numbers and include both attached and separated turbulent flows. (For the low speed flows ( $M_0 = 0.01$ ) the ABC’s were constructed on the basis of the linearized incompressible equations in the far field (see [244]).) In Table 4.2 we present the computational results for one standard transonic case. The smaller grid in Table 4.2 is again an exact subset of the bigger one. The results clearly demonstrate that for small computational domains the DPM-based ABC’s generate much more accurate solutions than standard boundary conditions do.

Besides the improvement of accuracy, the application of the DPM-based ABC’s to transonic flow com-

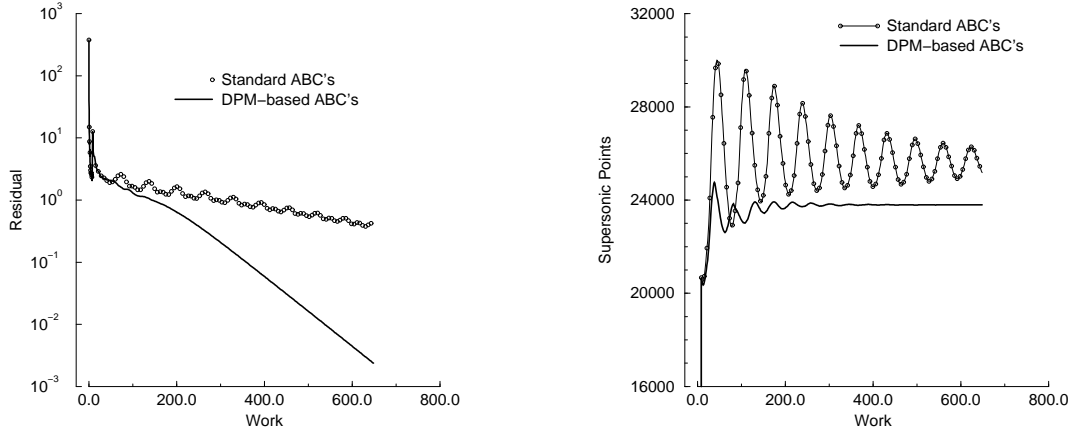


FIG. 4.2. *ONERA M6*:  $M_0 = 0.84$ ,  $Re_0 = 11.7 \cdot 10^6$ , angle of attack  $\alpha = 3.06^\circ$ . Convergence history for the residual of the continuity equation (left) and the number of supersonic nodes in the domain (right).

putations on the small (3 root chords) computational domain yields much higher multigrid convergence rate of the residual (continuity equation), as well as much faster convergence of other quantities, including those deemed as sensitive, e.g., the number of supersonic points in the domain. In Figure 4.2 we show the convergence history for this supersonic flow variant. One can see that the convergence for the standard boundary conditions is poor; therefore the corresponding force coefficients in Table 4.2 are given with the error bands indicated.

For the 10 chords domain, the DPM-based ABC's also provide for some convergence speedup, although the difference between the two ABC's techniques is less dramatic here. This is reasonable because one could generally expect that the bigger the computational domain, the smaller is the influence that the external boundary conditions exert on the numerical procedure.

From Figure 4.2 we conclude that the mere replacement of the standard local boundary conditions by the DPM-based nonlocal ABC's may produce as drastic change in the algorithm performance as a three-fold increase in the convergence rate. The effect becomes even more significant when we consider a separated turbulent flow around the ONERA M6 wing. This is a more complicated case, for which the multigrid algorithm with standard boundary conditions simply fails to converge on the small (3 chords) domain, whereas the nonlocal DPM-based ABC's still produce a good convergence. The corresponding results are presented in our work [244,245]. As mentioned previously, we have also been able to obtain the results of the similar kind for the two-dimensional flows.

The acceleration of multigrid convergence provided by the DPM-based ABC's is extremely important for applications; it shows that a combination of global boundary conditions with multigrid solvers has a solid potential for being widely used in modern CFD. On the other hand, we acknowledge that neither rigorous mathematical justification nor a conclusive experimental explanation of why and when the speedup occurs is available as of yet. For our two- and three-dimensional computations we have used different multigrid strategies (W- and V-cycles, respectively, with different number of levels). Moreover, the laminar airfoil flows (see [232,233,240]), for which the most noticeable convergence acceleration occurs in two dimensions, were computed on relatively coarse grids with Courant numbers determined globally, whereas the transonic turbulent flows around the ONERA M6 wing, see Figure 4.2, for which the most noticeable convergence acceleration occurs in three dimensions, were calculated on strongly stretched grids with the local time step. We have run some two-dimensional transonic turbulent computations as well (RAE2822 airfoil, see above and [233,240] for details) but have not been able to obtain the convergence acceleration as drastic as the one shown in Figure 4.2. As concerns laminar computations, the results are available for two dimensions only. For three dimensions, we have run only turbulent cases and in doing so found that for the purely subsonic flows the type of ABC's had a relatively small effect on the convergence rate of multigrid iterations (see [244]), and for the transonic flows (see above) this effect could be very significant. Finally, the influence that the ABC's may exert on the multigrid convergence rate (as well as on the numerical accuracy, see above and [240,244]) strongly depends on the size of the computational domain (compared to the size of the wing/airfoil) and apparently, the dimension/stretching rate of the grid, see [245]. Basically, the overall impact of the ABC's on the computational algorithm decreases as the domain enlarges. In particular, for

bigger domains the convergence speedup becomes less dramatic (see [244, 245]).

Besides our own work, there is other evidence of the improved convergence to steady state when highly accurate nonlocal ABC's are combined with multigrid. For example, the results of this type have been obtained by Ferm in work [54, 56, 57, 59] (see Section 2.2). Moreover, Bayliss, Goldstein, and Turkel in [249] have also noticed the improved multigrid convergence when the interior solver was coupled with global boundary conditions; the specific case studied in [249] was the Helmholtz equation in a waveguide. However, neither our own work nor the work of other authors can provide for a conclusive explanation of what causes the foregoing improvement in performance and for what combinations of parameters one can expect to obtain the maximal gain. At the same time, the results obtained up to date prove that the combination of nonlocal boundary conditions with multigrid has a very substantial promise for reducing the overall cost of computations. Therefore, a detailed study of interaction between the nonlocal boundary conditions and multigrid iterations becomes an important task for the future research.

Finally, we emphasize that the DPM-based may also improve the robustness of the entire computational procedure. Indeed, for certain computational variants, the new boundary conditions can ensure good convergence to steady state whereas the standard procedure would merely fail to converge. We observed this phenomenon for both two-dimensional and three-dimensional viscous computations, as well as for two-dimensional inviscid flows, see our work [229, 232, 233, 240, 244, 245].

The computational cost of the DPM-based ABC's in two space dimensions is modest; the ABC's may add about 10% to the cost of the original integration procedure. In three space dimensions, the relative cost of the DPM-based algorithm has so far been higher; the ABC's typically add about 20–25% of extra CPU time to the cost of the same procedure with the standard (characteristics-based) boundary conditions. Basically, this extra expense is quite acceptable taking into account the improvement of accuracy; moreover, it can often be compensated for and even noticeably prevailed over by the convergence acceleration and the reduction of the domain size. Besides, to explicitly decrease the computational cost associated with the DPM-based ABC's we plan on the future use of the entry-wise interpolation of boundary operators (see [240, 243]) and/or multiresolution based methodologies (see [242, 243]). We expect that the latter can also be employed when implementing the DPM-based ABC's for multi-block grids. The applicability of multiresolution based methodologies (see, e.g., work by Harten and Yad-Shalom [250]) is accounted for by the special structure that is typically relevant to the matrices  $\mathbf{T}$ . If the grid nodes on both  $\Gamma$  and  $\Gamma_1$  are enumerated consistently, then this structure may resemble the block-wise diagonal dominance because even though the boundary conditions are global and the matrix  $\mathbf{T}$  is therefore dense, the mutual influence between the close nodes is obviously stronger than between the remote ones. More details on this subject, as well as the specific model example for the Yukawa equation, can be found in our work [242, 243].

Summarizing our experience in developing and implementing the nonlocal DPM-based ABC's for steady-state flow computations, we can say that these boundary conditions are capable of greatly reducing the size of the computational domain (compared to the standard methods) while still maintaining high accuracy of the numerical solution. This size reduction amounts to either the possibility of refining the grid in the near field, which potentially leads to the improvement in accuracy, or usage of the smaller-dimension grids without compromising the accuracy. Moreover, the DPM-based ABC's may noticeably speed up the convergence of multigrid iterations and improve robustness of the entire numerical procedure. Finally, the new boundary conditions appear geometrically universal and easy to incorporate in the structure of the existing flow solvers. Altogether, these features constitute a foundation for this methodology to become an effective and widely used tool in modern CFD.

Extending the DPM-based approach to time-dependent formulations is a challenging problem for the future research. Our work [238] addresses a particular class of such formulations, namely, the oscillating flows, which, in a certain sense, occupy an intermediate position between the steady-state and truly unsteady problems. As concerns the latter, carrying out the general analysis of [46, 221] to particular applied problems that do not necessarily have regular geometry is likely to require a separate substantial effort especially as it implies paying particular attention to restricting the nonlocality of the boundary conditions in time.

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