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FIRST-PASSAGE DENSITIES OF A TWO-DIMENSIONAL PROCESS*

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Abstract. The two-dimensional stochastic process $(x(t), y(t))$, where $y(t) = dx(t)/dt$ is a Wiener process (Brownian motion) is considered. The value of $x(t)$ when $y(t)$ first hits a line in the plane is of interest. The joint moment generating function (m.g.f.) of the first hitting time and place, as well as the probability density function (p.d.f.) of the first hitting place, in a special case, are obtained by solving the appropriate Kolmogorov backward equation. In the last section of the paper, the joint m.g.f. of the first hitting time and place is used to obtain the optimal control of the process in the first quadrant.

Key words. integrated Brownian motion, first-passage density, optimal stochastic control

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1. Introduction. First-passage densities of stochastic processes are needed in many applications, but explicit results for processes of dimension $n \geqq 2$ are few: for example, in the case of the two-dimensional Brownian motion, see Buckholtz and Wasan [2] and Iyengar [7]; Wendel [11] has considered the n-dimensional Brownian motion inside, outside, and between spheres.

In this note, we consider the two-dimensional stochastic process defined by

$$dx(t) = y(t)\, dt, \qquad dy(t) = dW(t), \tag{1.1}$$

where $W(t)$ is a one-dimensional Wiener process (or Brownian motion) with zero mean and, for simplicity, variance parameter $\sigma^2 = 2$. That is, $x(t)$ is the integrated Wiener process. This process has already been studied by McKean [10], Goldman [5], and Gor'kov [6].

Suppose that the process starts at $(x(0), y(0)) = (0, 1)$ and let

$$t_1 = \min\{t\colon t > 0, x(t) = 0\}. \tag{1.2}$$

McKean has obtained the joint probability density function (p.d.f.) of t_1 and $y(t_1)$, and the marginal density of $y(t_1)$. In [5], Goldman has derived an expression, in terms of the joint p.d.f. of the half winding time t_1 and the hitting place $y(t_1)$, for the first passage of $x(t)$ to $x > 0$ when $x(0) = 0$ and $y(0) < 0$. Finally, Gor'kov has obtained a formula for the p.d.f. of $y(t_2)$, where

$$t_2 = \min\{t\colon x(t) = 0, y(t) \geqq 0\}, \tag{1.3}$$

assuming that $x(0) < 0$ and $-\infty < y(0) < \infty$.

Here we suppose that the process defined by (1.1) starts at (x, y), where $-\infty < x < \infty$ and $y \geqq \eta$, and we evaluate the joint moment generating function (m.g.f.) of $\tau(\eta)$ and $x[\tau(\eta)]$, where

$$\tau(\eta) = \inf\{t\colon y(t) = \eta\}. \tag{1.4}$$

We also obtain the p.d.f. of $x[\tau(0)]$.

Note that, in our case, τ is the time it takes the Wiener process to hit the line $y = \eta$, whereas in the papers cited above, t_1 (or t_2) is the time it takes the *integrated* Wiener process to hit the barrier.

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In § 2, we consider the case when $\eta \geqq 0$; the case when η is negative is treated in § 3. In the last section of the paper, we use the joint m.g.f. of $\tau(\eta)$ and $x[\tau(\eta)]$ to evaluate the optimal control of the process in the first quadrant. Our aim is then to minimize the expected value of $\tau(\eta)$ and $x[\tau(\eta)]$, without using too much control. That is, we want the Wiener process $y(t)$ to hit the line $y=\eta$ as soon as possible and with $x(t)$ as small as possible. This is an example of a type of problem that Whittle [12] has termed LQG homing.

2. The case when $\eta \geqq 0$. We consider the two-dimensional stochastic process defined by (1.1). Suppose that $x(0)=x$ is any real number, but that $y(0)=y \geqq \eta$. In this section, we assume that $\eta \geqq 0$. Let

$$\tau(=\tau(\eta)) = \inf\{t: y(t)=\eta\} \tag{2.1}$$

and write

$$P_{(x,y)}[\tau \in dt, x(\tau) \in d\xi] = \rho(x, y; \xi, t)\, dt\, d\xi. \tag{2.2}$$

That is, ρ is the joint probability density function of τ and $x(\tau)$.

Now, since $dx(t)/dt = y(t)$ and $y(0)=y \geqq \eta \geqq 0$, we see that we must have

$$x(\tau) > x \quad \text{if } \tau > 0. \tag{2.3}$$

Moreover, it is easy to show that the p.d.f. of $x(\tau)$ depends on the difference $\xi - x$; hence, we may write

$$\rho(x, y; \xi, t) = \rho(0, y; \xi - x, t) = \rho(x-\xi, y; 0, t). \tag{2.4}$$

The function ρ satisfies the Kolmogorov backward equation (see McKean [10], for this particular process, or Cox and Miller [3] for the general case)

$$\rho_{yy} + y\rho_x = \rho_t \qquad (x<\xi, y>\eta, t>0). \tag{2.5}$$

The boundary and initial conditions are

$$\begin{aligned} &\rho(\xi, y; \xi, t) = \delta(y-\xi, t) \\ &\rho(x, \eta; \xi, t) = \delta(x-\xi, t) \\ &\rho(x, y; \xi, 0) = \delta(x-\xi, y-\eta) \end{aligned} \tag{2.6}$$

where δ is the Dirac delta function. The first condition in (2.6) is a consequence of (2.3), while the second is the appropriate boundary condition in the case of an absorbing barrier at $y=\eta$ (see Cox and Miller [3], for instance). Indeed, if $y=\eta$, then $\tau=0$ and we must have $x(\tau)=x$. Finally, the last condition is the obvious initial condition that ρ must satisfy.

Also, using the fact that

$$0 \leqq \rho(x, y; \xi, t) \leqq p(x, y; \xi, \eta), \tag{2.7}$$

where

$$\begin{aligned} p(x, y; \xi, \eta)\, d\xi\, d\eta &= P_{(x,y)}[x(t) \in d\xi, y(t) \in d\eta] \\ &= 3^{1/2}/(2\pi t^2) \exp\{-3(\xi - x - ty)^2 t^{-3} \\ &\quad + 3(\eta - y)(\xi - x - ty)t^{-2} - (\eta - y)^2 t^{-1}\}\, d\xi\, d\eta \end{aligned} \tag{2.8}$$

(see McKean [10], or Goldman [5], but here σ^2 is equal to 2, rather than 1), we deduce that

$$\lim_{t\to\infty}\rho=\lim_{x\to-\infty}\rho=\lim_{y\to\infty}\rho=0. \tag{2.9}$$

Next, define

$$\Psi(x,y;\xi,\alpha)=\int_0^\infty e^{-\alpha t}\rho\,dt, \tag{2.10}$$

where α is a nonnegative real parameter. From (2.5) and (2.9), we find that the function Ψ satisfies the partial differential equation

$$\Psi_{yy}+y\Psi_x=\alpha\Psi \qquad (x<\xi,\ y>\eta). \tag{2.11}$$

Finally, let us write

$$\Phi(\beta,y;\xi,\alpha)=\int_{-\infty}^{\xi} e^{\beta x}\Psi\,dx \tag{2.12}$$

where β is a positive real parameter. Note that, because of (2.4), we may write

$$\begin{aligned}\Phi(\beta,y;\xi,\alpha)&=\int_{-\infty}^{\xi}\int_0^\infty e^{\beta x-\alpha t}\rho(x,y;\xi,t)\,dt\,dx\\&=e^{\beta\xi}\int_0^\infty\int_0^\infty e^{-\beta u-\alpha t}\rho(0,y;u,t)\,dt\,du,\end{aligned}$$

so that (2.12) is really just a Laplace transform of Ψ, multiplied by an exponential.

Using (2.11), (2.6), and (2.9), we deduce that Φ satisfies the ordinary differential equation

$$\Phi_{yy}-\beta y\Phi=\alpha\Phi \qquad (y>\eta), \tag{2.13}$$

subject to the boundary condition

$$\Phi(\beta,\eta;\xi,\alpha)=e^{\beta\xi}. \tag{2.14}$$

Let

$$w=\beta^{-2/3}(\alpha+\beta y). \tag{2.15}$$

Then, the solution of (2.13) is given by

$$\Phi(\beta,y;\xi,\alpha)=k_1\,\mathrm{Ai}(w)+k_2\,\mathrm{Bi}\,(w) \tag{2.16}$$

where k_1 and k_2 are independent of y, and where the Airy functions Ai (w) and Bi (w) are defined, in terms of Bessel functions, by [1]

$$\mathrm{Ai}\,(w)=\frac{1}{3}w^{1/2}[I_{-1/3}(\zeta)-I_{1/3}(\zeta)]=\pi^{-1}\left(\frac{w}{3}\right)^{1/2}K_{1/3}(\zeta), \tag{2.17}$$

$$\mathrm{Bi}\,(w)=\left(\frac{w}{3}\right)^{1/2}[I_{-1/3}(\zeta)+I_{1/3}(\zeta)], \tag{2.18}$$

with

$$\zeta=\tfrac{2}{3}w^{3/2}. \tag{2.19}$$

Finally, from (2.9) we can state that

$$\lim_{y\to\infty}\Phi(\beta,y;\xi,\alpha)=0. \tag{2.20}$$

Since Bi (w) diverges as y tends to $+\infty$, we must set k_2 equal to zero in (2.16), and we deduce the following from the boundary condition (2.14):

$$\Phi(\beta, y; \xi, \alpha) = e^{\beta\xi} \operatorname{Ai}[\beta^{-2/3}(\alpha+\beta y)]/\operatorname{Ai}[\beta^{-2/3}(\alpha+\beta\eta)]. \tag{2.21}$$

THEOREM 1. *Suppose that* $(x(0), y(0)) = (x, y)$, *where* $-\infty < x < \infty$ *and* $y \geqq \eta \geqq 0$. *Let* $\tau(\eta) = \inf\{t: y(t) = \eta\}$. *Then, the joint moment generating function of* $\tau(\eta)$ *and* $x[\tau(\eta)]$ *is given by the formula*

$$E_{(x,y)}\{\exp[-\alpha\tau(\eta) - \gamma x(\tau(\eta))]\} = e^{-\gamma\xi} \operatorname{Ai}[\gamma^{-2/3}(\alpha+\gamma y)]/\operatorname{Ai}[\gamma^{-2/3}(\alpha+\gamma\eta)] \tag{2.22}$$

where $\alpha \geqq 0$ and $\gamma > 0$.

Proof. From (2.21), we have

$$\begin{aligned}\int_{-\infty}^{\xi}\int_0^{\infty} e^{\beta x-\alpha t}\rho(x, y; \xi, t)\,dt\,dx \\ = e^{\beta x} \operatorname{Ai}[\beta^{-2/3}(\alpha+\beta y)]/\operatorname{Ai}[\beta^{-2/3}(\alpha+\beta\eta)].\end{aligned} \tag{2.23}$$

Now, using (2.4) we may write

$$\begin{aligned}\int_{-\infty}^{\xi}\int_0^{\infty} e^{\beta x-\alpha t}\rho(x, y; \xi, t)\,dt\,dx &= e^{\beta\xi}\int_0^{\infty}\int_0^{\infty} e^{-\beta u-\alpha t}\rho(0, y; u, t)\,dt\,du \\ &= e^{\beta(\xi+x)}\int_x^{\infty}\int_0^{\infty} e^{-\beta\xi-\alpha t}\rho(x, y; \xi, t)\,dt\,d\xi.\end{aligned}$$

Hence, the theorem follows. □

Note that (2.22) makes sense, since Ai (x) is positive if x is nonnegative [1]. If we let γ decrease to zero in (2.22), we obtain the well-known formula

$$E\{\exp[-\alpha\tau(\eta)] \mid y(0) = y\} = \exp[\alpha^{1/2}(\eta - y)]. \tag{2.24}$$

We also have

$$E_{(x,y)}\{\exp[-\gamma x(\tau(\eta))]\} = e^{-\gamma x} \operatorname{Ai}(\gamma^{1/3} y)/\operatorname{Ai}(\gamma^{1/3}\eta). \tag{2.25}$$

Using the definition of Ai (x) ([1]), it is easy to show that $x[\tau(\eta)]$, like $\tau(\eta)$, has no finite moments.

Next, from the asymptotic representation of Ai (x) (see [8]), we obtain the following approximate formula, valid for η large:

$$\begin{aligned}&E_{(x,y)}\{\exp[-\alpha\tau(\eta) - \gamma x(\tau(\eta))]\} \\ &\quad\simeq e^{-\gamma x}[(\alpha+\gamma\eta)/(\alpha+\gamma y)]^{1/4} \exp\{[2/(3\gamma)][(\alpha+\gamma\eta)^{3/2} - (\alpha+\gamma y)^{3/2}]\},\end{aligned} \tag{2.26}$$

so that

$$E_{(x,y)}\{\exp[-\gamma x(\tau(\eta))]\} \simeq e^{-\gamma x}(\eta/y)^{1/4} \exp\{(2/3)\gamma^{1/2}[\eta^{3/2} - y^{3/2}]\}. \tag{2.27}$$

Finally, we will use (2.22) in § 4 to obtain the optimal control of the process in the first quadrant.

THEOREM 2. *The probability density function of* $x[\tau(0)]$ *is given by*

$$\begin{aligned}&P_{(x,y)}[x(\tau(0)) \in d\xi] \\ &\quad= (2\pi)^{-1}3^{-1/6}\Gamma(2/3)[y(\xi-x)^{-4/3}] \exp[-\tfrac{1}{9}y^3(\xi-x)^{-1}]\,d\xi,\end{aligned} \tag{2.28}$$

where $x < \xi$ and $y > 0$.

Proof. Since $y(t)$ is a Wiener process, so that $P[\tau(n)<\infty]=1$, setting α and η equal to zero in (2.23) we obtain

$$\int_{-\infty}^{\xi} e^{\beta x}P_{(x,y)}[x(\tau(0))\in d\xi]\,dx = d\xi\, e^{\beta x}\,\mathrm{Ai}\,(\beta^{1/3}y)/c_1, \tag{2.29}$$

where

$$c_1=\mathrm{Ai}\,(0)=[3^{2/3}\Gamma(2/3)]^{-1}. \tag{2.30}$$

Hence, we may write

$$\int_{0}^{\infty} e^{-\beta u}P_{(\xi-u,y)}[x(\tau(0))\in d\xi]\,du = d\xi\,\mathrm{Ai}\,(\beta^{1/3}y)/c_1. \tag{2.31}$$

Using the relation

$$\mathrm{Ai}\,(\beta^{1/3}y)=\pi^{-1}(\beta^{1/3}y/3)^{1/2}K_{1/3}[\tfrac{2}{3}\beta^{1/2}y^{3/2}] \tag{2.32}$$

and the formula [4]

$$\int_{0}^{\infty} e^{-pt}t^{\nu-1}\,e^{-a/(4t)}\,dt=2[a/(4p)]^{\nu/2}K_\nu(a^{1/2}p^{1/2}) \tag{2.33}$$

(and the fact that $K_{-\nu}=K_\nu$) we obtain

$$\begin{aligned} &P_{(\xi-u,y)}[x(\tau(0))\in d\xi] \\ &\quad=(2\pi)^{-1}3^{-1/6}\Gamma(2/3)(y/u^{4/3})\exp[-y^3/(9u)]\,d\xi, \end{aligned} \tag{2.34}$$

from which we deduce formula (2.28). □

The p.d.f. of $x[\tau(0)]$ is very simple; however, $\eta=0$ is the only case for which we can obtain an explicit expression for the p.d.f. of $x[\tau(\eta)]$. Indeed, if η is positive we have

$$\int_{-\infty}^{\xi} e^{\beta x}P_{(x,y)}[x(\tau(\eta))\in d\xi]\,dx = d\xi\,e^{\beta x}\,\mathrm{Ai}\,(\beta^{1/3}y)/\mathrm{Ai}\,(\beta^{1/3}\eta), \tag{2.35}$$

and, due to the ratio of Airy functions, it does not seem possible to invert (2.35) to get the p.d.f. of $x[\tau(\eta)]$.

3. The case when $\eta<0$. In this section, we consider the same problem as in § 2, but now we assume that η is negative. Then, relation (2.3) is no longer valid. Indeed, in this case $x(\tau)$ can take any real value. Therefore, the joint p.d.f. $\rho(x, y;\, \xi, t)$ (defined in (2.2)) satisfies the Kolmogorov backward equation

$$\rho_{yy}+y\rho_x=\rho_t \qquad (-\infty<x<\infty,\ y>\eta,\ t>0). \tag{3.1}$$

Relation (2.4) still holds, even if $x\geqq\xi$. Furthermore, the boundary and initial conditions are

$$\rho(x,\eta;\,\xi,t)=\delta(x-\xi,t) \qquad \rho(x,y;\,\xi,0)=\delta(x-\xi,y-\eta), \tag{3.2}$$

and we also have

$$\lim_{t\to\infty}\rho=\lim_{x\to\pm\infty}\rho=\lim_{y\to\infty}\rho=0. \tag{3.3}$$

That is, the first condition in (2.6) is now replaced by $\rho(\infty, y;\, \xi, t)=0$, which follows from (2.7) and (2.8).

Next, we have

$$\Psi_{yy} + y\Psi_x = \alpha\Psi \qquad (-\infty < x < \infty, y > \eta) \tag{3.4}$$

where $\Psi(x, y; \xi, \alpha)$ is defined in (2.10). Now, let

$$\Phi(\beta, y; \xi, \alpha) = \int_{-\infty}^{\infty} e^{i\beta x}\Psi \, dx \tag{3.5}$$

where β is any real parameter. Then, using (3.2) and (3.3), we find that Φ satisfies the ordinary differential equation

$$\Phi_{yy} - i\beta y\Phi = \alpha\Phi \qquad (y > \eta). \tag{3.6}$$

The boundary condition is

$$\Phi(\beta, \eta; \xi, \alpha) = e^{i\beta\xi}. \tag{3.7}$$

Equation (3.6) can be written

$$\Phi_{zz} - z\Phi = 0, \tag{3.8}$$

where

$$z = (\alpha + i\beta y)\, e^{-\pi i/3}\beta^{-2/3}, \tag{3.9}$$

for any β different from zero. Thus, we deduce that

$$\Phi(\beta, y; \xi, \alpha) = k_1 \operatorname{Ai}(z) + k_2 \operatorname{Bi}(z). \tag{3.10}$$

Note that for $\beta = 0$, we deduce at once from (3.6) and (3.7) that

$$\Phi(0, y; \xi, \alpha) = E\{\exp[-\alpha\tau(\eta)] \,|\, y(0) = y\} = \exp[\alpha^{1/2}(n - y)], \tag{3.11}$$

which is the well-known formula already mentioned in § 2.

Now, suppose first that β is positive in (3.10). Then, when $y \to \infty$ we have $\arg(z) \to \pi/6$ and $|z| \to \infty$. Therefore, we may use the asymptotic representations of $\operatorname{Ai}(w)$ and $\operatorname{Bi}(w)$ for large $|w|$ [8]:

$$\begin{bmatrix} \operatorname{Ai}(w) \simeq (\pi^{-1/2}/2) w^{-1/4} \exp[-\tfrac{2}{3} w^{3/2}] & |\arg(w)| < 2\pi/3 \\ \operatorname{Bi}(w) \simeq \pi^{-1/2} w^{-1/4} \exp[\tfrac{2}{3} w^{3/2}] & |\arg(w)| < \pi/3. \end{bmatrix} \tag{3.12}$$

Choosing the appropriate complex roots, we find that

$$\lim_{y \to \infty} \operatorname{Re}\{\pm z^{3/2}\} = \pm\infty. \tag{3.13}$$

But the norm of Φ must obviously be finite. Indeed, we may write

$$\begin{aligned} 0 \leqq |\Phi| &\leqq \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\alpha t} \rho(x, y; \xi, \tau) \, dt \, dx \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\alpha t} \rho(0, y; \xi - x, \tau) \, dt \, dx \\ &\leqq E_{(x,y)}[\exp(-\alpha\tau)] \leqq 1. \end{aligned}$$

Thus, from (3.12) we conclude that we must eliminate the solution $\operatorname{Bi}(z)$ and, using the boundary condition (3.7), we obtain

$$\Phi(\beta, y; \xi, \alpha) = e^{i\beta\xi} \operatorname{Ai}[(\alpha + i\beta y)\, e^{-\pi i/3}\beta^{-2/3}] / \operatorname{Ai}[(\alpha + i\beta\eta)\, e^{-\pi i/3}\beta^{-2/3}] \tag{3.14}$$

for β positive.

Finally, by definition of the function $\Phi(\beta, y; \xi, \alpha)$, we may write

$$\Phi(-\beta, y; \xi, \alpha) = \overline{\Phi(\beta, y; \xi, \alpha)} \tag{3.15}$$

where the overbar denotes the complex conjugate of the expression. Hence, from (3.14) we obtain

$$\Phi(-\beta, y; \xi, \alpha) = e^{-i\beta\xi}\,\mathrm{Ai}\,[(\alpha - i\beta y)\, e^{\pi i/3}\beta^{-2/3}]/\mathrm{Ai}\,[(\alpha - i\beta\eta)\, e^{\pi i/3}\beta^{-2/3}], \tag{3.16}$$

which is also valid for β positive, i.e., for $-\beta$ negative.

Note that, η being a negative parameter in this section, it is easy to see that $(\alpha + i\beta\eta)\, e^{-\pi i/3}\beta^{-2/3}$ and $(\alpha - i\beta\eta)\, e^{\pi i/3}\beta^{-2/3}$ never take real values if β is positive. Therefore, since the function $\mathrm{Ai}\,(w)$ has zeros on the negative real axis only [1], the right-hand members in formulas (3.14) and (3.16) are well-defined. Note also that the right-hand side of (3.16) is indeed a solution of (3.6). This follows from [1]

$$\mathrm{Ai}\,(w\, e^{2\pi i/3}) = (\tfrac{1}{2})\, e^{\pi i/3}[\mathrm{Ai}\,(w) - i\,\mathrm{Bi}\,(w)]. \tag{3.17}$$

THEOREM 3. *Suppose* $(x(0), y(0)) = (x, y)$, *where* $-\infty < x < \infty$ *and* $y \geqq \eta$, *with* $\eta < 0$. *Let* $\tau(\eta) = \inf\{t: y(t) = \eta\}$. *Then we have*

$$E_{(x,y)}\{\exp[-\alpha\tau(\eta) - i\gamma x(\tau(\eta))]\} = \begin{cases} e^{-i\gamma x}\dfrac{\mathrm{Ai}\,[(\alpha + i\gamma y)\, e^{-\pi i/3}\gamma^{-2/3}]}{\mathrm{Ai}\,[(\alpha + i\gamma\eta)\, e^{-\pi i/3}\gamma^{-2/3}]} & (\gamma > 0), \\ \exp[\alpha^{1/2}(\eta - y)] \quad (\gamma = 0) & \\ e^{-i\gamma x}\dfrac{\mathrm{Ai}\,[(\alpha + i\gamma y)\, e^{\pi i/3}\gamma^{-2/3}]}{\mathrm{Ai}\,[(\alpha + i\gamma\eta)\, e^{\pi i/3}\gamma^{-2/3}]} & (\gamma < 0) \end{cases} \tag{3.18}$$

where $\alpha \geqq 0$.

Proof. Using (2.4), we may write

$$\begin{aligned} E_{(x,y)}\{\exp[-\alpha\tau(\eta) - i\gamma x(\tau(\eta))]\} &= \int_{-\infty}^{\infty}\int_0^{\infty} e^{-\alpha t - i\gamma\xi}\rho(x, y; \xi, t)\, dt\, d\xi \\ &= e^{-i\gamma x}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\alpha t - i\gamma u}\rho(0, y; u, t)\, dt\, du. \end{aligned}$$

Now, we have

$$\begin{aligned} \Phi(\beta, y; \xi, \alpha) &= \int_{-\infty}^{\infty}\int_0^{\infty} e^{-\alpha t + i\beta x}\rho(x, y; \xi, t)\, dt\, dx \\ &= e^{i\beta\xi}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\alpha t - i\beta u}\rho(0, y; u, t)\, dt\, du. \end{aligned}$$

Formula (3.18) then follows from (3.14) and (3.16). $\square$

By setting α equal to zero in (3.18), we obtain the following corollary.

COROLLARY 1. *The characteristic function of* $x[\tau(\eta)]$ *is given by*

$$E_{(x,y)}\{\exp[-i\gamma x(\tau(\eta))]\} = \begin{cases} e^{-i\gamma x}\dfrac{\mathrm{Ai}\,[\gamma^{1/3}\, e^{\pi i/6} y]}{\mathrm{Ai}\,[\gamma^{1/3}\, e^{\pi i/6}\eta]} & (\gamma > 0) \\ 1 \quad (\gamma = 0) & \\ e^{-i\gamma x}\dfrac{\mathrm{Ai}\,[\gamma^{1/3}\, e^{5\pi i/6} y]}{\mathrm{Ai}\,[\gamma^{1/3}\, e^{5\pi i/6}\eta]} & (\gamma < 0). \end{cases} \tag{3.19}$$

4. Optimal control in the first quadrant. Consider now the controlled process

$$dx(t) = y(t)\, dt, \qquad dy(t) = u\, dt + dW(t) \tag{4.1}$$

where $W(t)$ is still a one-dimensional Wiener process with zero mean and variance parameter $\sigma^2 = 2$. Suppose that the process starts at $(x(0), y(0)) = (x, y)$, where $x \geqq 0$

and $y \geqq \eta \geqq 0$, and that we want to minimize the expected value of the cost function

$$J(x, y) = \int_0^\tau \frac{qu^2}{2}\, dt + K[\tau, x(\tau)] \tag{4.2}$$

where

$$\tau = \inf\{t: y(t) = \eta\} \tag{4.3}$$

and q is a positive constant. This type of problem has been studied by Whittle [12], who termed it LQG homing, even if the problem does not have to be LQG; in particular, the final cost, K, can be almost any function of τ and $x(\tau)$ and is therefore not necessarily quadratic in x. Note also that the termination time τ is a random variable, rather than being fixed. Whittle has shown that, under certain conditions, the optimal control can be obtained from a mathematical expectation based on the *uncontrolled* process.

Let us take

$$K[\tau, x(\tau)] = a\tau + bx(\tau), \tag{4.4}$$

with a, $b \geqq 0$, so that our aim is really to hit the boundary $y = \eta$ as soon as possible and with $x(\tau)$ as small as possible, but without using too much control u. In [9], Lefebvre and Whittle have used (4.1) as a rudimentary model for the flight of an aircraft: $x(t)$ and $y(t)$ representing height above ground and vertical velocity, respectively. Then, if η is small, using symmetry our problem could become an optimal landing problem: we would want the aircraft to go from a negative vertical velocity to a near-zero vertical velocity as soon as possible, and we would also want the aircraft to be as close to the ground as possible when it reaches this small negative vertical velocity value. In fact, in this case, $-x(t)$, rather than $x(t)$, would represent height above ground.

THEOREM 4. *The value of the control u that minimizes the expected value of the cost function* (4.2) *is*

$$u^* = -(2/q)^{1/2}(a+by)^{1/2}K_{2/3}(\omega)/K_{1/3}(\omega) \tag{4.5}$$

where

$$\omega = [2/(3b)](a+by)^{3/2}/(2q)^{1/2}. \tag{4.6}$$

Proof. Since the uncontrolled process defined by (1.1) is certain to hit the boundary $y = \eta$, we can state that the optimal control is given by [12]

$$u^* = -F_y/q \tag{4.7}$$

where F is obtained from

$$\exp[-F(x, y)/c] = E_{(x,y)}\{\exp[-K(\tau, x(\tau))/c]\}, \tag{4.8}$$

τ now being the first-passage time of the *uncontrolled* Wiener process $y(t)$ to the line $y = \eta$.

In (4.8), the scalar c must be chosen so that the relation

$$\sigma^2 = c/q \tag{4.9}$$

holds. That is, we must take

$$c = 2q. \tag{4.10}$$

Now, using (2.22) with $\alpha = -a/(2q)$ and $\gamma = -b/(2q)$, we find that

$$\begin{aligned} &E_{(x,y)}\{\exp[-K(\tau, x(\tau))/c]\} \\ &\quad = \exp[-bx/(2q)]\,\mathrm{Ai}\,[(a+by)(2qb^2)^{-1/3}]/\mathrm{Ai}\,[(a+b\eta)(2qb^2)^{-1/3}]. \end{aligned} \tag{4.11}$$

Finally, in terms of Bessel functions we have [1]

$$\mathrm{Ai}\,(z)=\pi^{-1}(z/3)^{1/2}K_{1/3}[\tfrac{2}{3}z^{3/2}] \tag{4.12}$$

and

$$\mathrm{Ai}'\,(z)=-\pi^{-1}(z/3^{1/2})K_{2/3}[\tfrac{2}{3}z^{3/2}]. \tag{4.13}$$

Hence, (4.5) follows from (4.7) and (4.8). □

When $|z|$ is large, we may write [1]

$$K_\nu(z)\cong[\pi/(2z)]^{1/2}\,e^{-z}\{1+(\mu-1)/(8z)+\cdots\} \tag{4.14}$$

for $|\arg(z)|<3\pi/2$, where

$$\mu=4\nu^2. \tag{4.15}$$

Therefore, we obtain the approximate formula

$$u^*\simeq-(2/q)^{1/2}(a+by)^{1/2} \tag{4.16}$$

for large y.

Finally, if we set $a=0$ and $\eta=0$, we deduce from the formula

$$K_\nu(z)\simeq\frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \tag{4.17}$$

valid for small z, that

$$u^*\simeq-\left(\frac{12b}{q}\right)^{1/3}\Gamma(2/3)/\Gamma(1/3) \tag{4.18}$$

near the x-axis.

Formulas such as (4.5), giving the optimal control in terms of special functions, are not really useful. Formula (4.16) is much more interesting, since it gives an approximate optimal control that is very easy to implement. Note that this approximate formula will always be valid if η is large. Note also that when a is equal to zero, the approximate optimal control is simply proportional to the square root of y. Furthermore, if b is equal to zero, then we can show that the exact u^* is just a constant ($-(2a/q)^{1/2}$, actually). Finally, as we would expect, we deduce from (4.16) that, in absolute value, the approximate optimal control is decreasing in q and increasing in a and b.

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