J. Appl. Prob. **10**, 54–62 (1973)
Printed in Israel

A GAUSSIAN MARKOVIAN PROCESS ON A SQUARE LATTICE

P. A. P. MORAN, *The Australian National University*

Abstract

A definition of the Markovian property is given for a lattice process and a Gaussian Markovian lattice process is constructed on a torus lattice. From this a Gaussian Markovian process is constructed for a lattice in the plane and its properties are studied.

MARKOVIAN PROCESSES; LATTICE PROCESSES

1. A Markov process on a torus

In this paper we construct a stationary process on the square lattice formed by all the pairs of positive and negative integers, in which the random variables, X_{mn} say, are identically normally distributed and satisfy a Markovian property. We do this by first constructing a similar such process on a lattice torus and letting the size of the latter tend to infinity.

Let the torus be defined by all pairs of integers $(j,k), j = 0,\cdots, p-1$, $k=0,\cdots, q-1$ where p and q are identified with zero. There are thus pq random variables X_{jk} at the points (j,k) and we suppose these to have the joint normal distribution with probability density

$$(1) \qquad p = K\exp - \tfrac{1}{2}\{\Sigma X_{jk}^2 + a\Sigma X_{jk}(X_{j+1,k} + X_{j-1,k} + X_{j,k+1} + X_{j,k-1})\}$$

where $|a| < \frac{1}{4}$, and K is a constant to be determined. This is a proper statistical distribution because it can be shown that the quadratic form inside the brackets is positive definite. To see this we observe that any cross product is similar to $X_{jk}\, X_{j+1,k}$ and occurs twice in the sum, and that

$$(2) \qquad |2aX_{jk}X_{j+1,k}| < \tfrac{1}{4}(X_{jk}^2 + X_{j+1,k}^2).$$

Thus the quadratic form is positive definite.

We first prove that this possesses the natural generalisation of the Markovian property.

Let L be any finite set of n_1 points on the lattice and (changing the notation) let $X_1,\cdots,X_{n_1}$ be the corresponding set of random variables. Write ∂L for the boundary of the set L, i.e., the set of all lattice points not in L which are joined to

Received 19 January 1972.

L by a unit line on the lattice. Let $X'_1, \cdots, X'_{n2}$ be the random variables associated with the points of ∂L. Let M be any finite set of points outside $L + \partial L$, and $X''_1, \cdots, X''_{n_3}$ be the corresponding set of random variables. Then we say that the process satisfies the Markovian property if the distribution of $(X_1, \cdots, X_{n_1})$ conditional on $(X_1, \cdots X'_{n_2})$ is independent of the distribution of $(X''_1, \cdots, X''_{n_3})$ conditional on the same values of $(X_1', \cdots, X'_{n_2})$. Since the distribution (1) is multivariate normal this is a property depending only on the correlation matrix of the whole set of variables.

From (1) we see that because (1) can be written in the form

$$g_1(X_i, X'_j) g_2(X'_j) g_3(X'_j, X''_k) \tag{3}$$

the joint probability density of $(X_1, \cdots, X_{n_1})$, is dependent only on their values and the values of $(X'_1, \cdots, X'_{n_2})$, and similarly for $(X''_1, \cdots, X''_{n_3})$. Thus the process is certainly Markovian. We can also define the Markovian property in another way. Let l be a closed non-intersecting line on the lattice (i.e., consisting of segments joining neighbouring points) with an interior and an exterior. Define L to be all sets of points interior to l, replace ∂L by l, and M by any finite set of points outside l. l will include ∂L and may have other points as well. Then the above argument goes through in exactly the same way and the distribution of the variates corresponding to L, conditional on the values of those on l, will be independent of the distribution of the variates corresponding to M conditional on the same values of the variates on l.

We now find the variance-covariance matrix of the distribution of the X's by inverting the matrix of the coefficients of the quadratic form in (1). We write the pq random variables as a vector $\boldsymbol{X}$ equal to $(X_{00}, X_{01}, \cdots, X_{0q-1}, X_{11}, \cdots, X_{1q-1}, \cdots, X_{p-1,q-1})$. We write the matrix $\boldsymbol{Q}$ as $(C_{u_1-u_2, v_1-v_2})$ where $C_{u_1-u_2, v_1-v_2}$ is the element in the row $u_1 q + v_1$ and column $u_2 q + v_2$ where $u_j = 0, 1, \cdots, p-1$ and $v_j = 0, 1, \cdots, q-1$ and their sums and differences are taken modulo (p) and (q) respectively.

We construct the matrix $\boldsymbol{Q}$ as follows. Consider the $p \times p$ circulant matrix

$$\begin{bmatrix} x & y & 0 & \cdot & \cdot & \cdot & y \\ y & x & y & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y & 0 & 0 & \cdot & \cdot & y & x \end{bmatrix} \tag{4}$$

and replace the x, y, and zeros by $\boldsymbol{A}, \boldsymbol{B}$, and zero $q \times q$ matrices respectively where $\boldsymbol{A}$ is the $p \times p$ matrix

(5)
$$\begin{bmatrix} 1 & a & 0 & \cdot & \cdot & \cdot & a \\ a & 1 & a & 0 & \cdot & \cdot & 0 \\ 0 & a & 1 & a & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & 0 & \cdot & \cdot & 0 & a & 1 \end{bmatrix}$$

and $\boldsymbol{B}$ is a $p \times p$ diagonal matrix with a's down the main diagonal.

We now show that $\boldsymbol{Q}$ can be diagonalised by a unitary matrix $\boldsymbol{U}$ so that $\boldsymbol{U}^*\boldsymbol{Q}\,\boldsymbol{U}$ is a diagonal matrix. For $u_i = 0, 1, \cdots, p-1$, $v_j = 0, 1, \cdots, q-1$ we define the element in the row $u_1 q + v_1$ and column $u_2 q + v_2$ to be

(6)
$$(pq)^{-\frac{1}{2}} \exp i\{2\pi u_1 u_2 p^{-1} + 2\pi v_1 v_2 q^{-1}\}.$$

It is then easy to verify that $\boldsymbol{U}^*\boldsymbol{U} = \mathbf{1}$.

We now write $\boldsymbol{Q}$ generally as the matrix

(7)
$$(C_{u_1-u_2,\, v_1-v_2})$$

where $C_{u_1-u_2,\, v_1-v_2}$ is the element in the row $u_1 q + v_1$ and column $u_2 q + v_2$ where $u_i = 0, 1, \cdots, p-1$, $v_i = 0, 1, \cdots, q-1$, and the differences in (7) are taken modulo (p) and (q) respectively. This represents $\boldsymbol{Q}$ as a matrix depending on pq quantities which in fact only take the three values 1, a, 0 and

(8)
$$\begin{aligned} C_{kl} &= 1, \text{ if } k = 0,\ l = 0, \\ &= a, \text{ if } (k = 0,\ l = 1),\ (k = 0,\ l = q-1),\ (k = 1,\ l = 0), \text{ or} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (k = p-1,\ l = 0), \\ &= 0, \text{ otherwise.} \end{aligned}$$

Evaluating $\boldsymbol{U}^*\boldsymbol{Q}\,\boldsymbol{U}$ this equals $\boldsymbol{\Lambda}$, a diagonal $pq \times pq$ matrix with diagonal elements λ_{tt}, where $t = u_1 q + v_1$, given by

(9)
$$\lambda_{tt} = \sum_{k,l} C_{kl} \exp 2\pi i \{k u_1 p^{-1} + l v_1 q^{-1}\}.$$

In fact $\boldsymbol{Q}$ contains in each row and each column a single element unity, four elements equal to a, and the rest zeros. On inserting the values (8) we get

(10)
$$\begin{aligned} \lambda_{tt} &= 1 + a\{\exp(2\pi i u_1 q^{-1}) + \exp(-2\pi i u_1 q^{-1}) + \exp(2\pi i v_1 p^{-1}) + \exp(-2\pi i v_1 p^{-1})\} \\ &= 1 + 2a(\cos 2\pi u_1 q^{-1} + \cos 2\pi v_1 p^{-1}). \end{aligned}$$

Write $Q^{-1} = (b_{st})$ where $s = u_1 q + v_1$, $t = u_2 p + v_2$. Then

$$Q^{-1} = U\Lambda^{-1}U^*, \tag{11}$$

and writing $m = u_3 q + v_3$ we have

$$\begin{aligned} b_{st} &= \sum_m u_{sm}\lambda_{mm}^{-1}\bar{u}_{mt} \\ &= \sum_m (pq)^{-1}\exp 2\pi i\{u_1u_3p^{-1} + v_1v_3q^{-1}\} \qquad (12) \\ &\quad \cdot \{1 + 2a\cos 2\pi v_3 q^{-1} + 2a\cos 2\pi u_3 q^{-1}\}^{-1} \\ &\quad \cdot \exp - 2\pi i\{u_2u_3p^{-1} + v_2v_3q^{-1}\}. \qquad (13) \end{aligned}$$

This is real and is the variance-covariance matrix of the random variables on the torus. The coefficient K in (1) is the reciprocal of the square root of the product of the λ_{tt}. Notice that (13) is a function only of the differences $u_1 - u_2$ and $v_1 - v_2$ and is a symmetric function of each. Thus the process is invariant on the torus under rotations of angles $2\pi p^{-1}$ and $2\pi q^{-1}$, and symmetric under "reflection".

2. A Markov process on a plane lattice

Our aim is to construct a stationary Markovian Gaussian process on the square lattice formed by all positive and negative integers (m, n). To do this it is well known that it is sufficient to define such a process for all finite sets of such points provided the resulting distribution is invariant under translation. It is therefore sufficient to consider any fixed finite set of points on the torus lattice and then let p and q tend to infinity. Writing $\exp 2\pi i u_3 p^{-1} = z_1$ and $\exp 2\pi i u_3 q^{-1} = z_2$ and letting p, q tend to infinity, we find that the covariance between $X_{m,n}$ and $X_{m+s,n+t}$ is

$$-\frac{1}{4\pi^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\frac{z_1^{s-1}z_2^{t-1}}{[1 + a(z_1 + z_1^{-1}) + a(z_2 + z_2^{-1})]}\,dz_1dz_2, \tag{14}$$

the convergence being uniform on the finite set of points.

Thus, using a non-standard notation, the covariance $V_{s,t}$ between $X_{m,n}$ and $X_{m+s,n+t}$ is generated by the generating function

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} z_1^s z_2^t V_{s,t} = [1 + a(z_1 + z_1^{-1}) + a(z_2 + z_2^{-1})]^{-1}, \tag{15}$$

and we see that the process is not only stationary but also reflection invariant. We define the correlations by $r_{st} = V_{st}V_{00}^{-1}$. Furthermore it is Markovian, for if we take any finite sets of points (as before) L, ∂L, and M, their joint distribution is the limit of the distributions on the torus and their dependence (since the variates are Gaussian) is dependent only on their variance-covariance matrix which, being the limit of the one for the torus, is such that the distributions of the variates in L and M, conditional on the values of ∂L, are independent. Thus the process is Markovian, and also Markovian in the second definition above.

The result (15) also shows that we can now explicitly generate the process on the plane lattice in a direct way. Let ε_{st} be independent Gaussian variates with zero means and (say) unit variance. Define the X_{mn} by the series

$$X_{mn} = \sum \alpha_{m+s,n+t}\,\varepsilon_{st}, \tag{16}$$

where α_{st} is the coefficient of $z_1^s z_2^t$ in the expansion of

$$[1 + a(z_1 + z_1^{-1}) + a(z_2 + z_2^{-1})]^{-\frac{1}{2}}\,. \tag{17}$$

Because $|a| < \frac{1}{4}$ this expansion is convergent as a Laurent series in a ring $1 - \delta < z_1,\ z_2 < 1 + \delta$ with $\delta > 0$. Furthermore the coefficients α_{st} converge to zero geometrically so that (16) converges with probability one.

Lévy [5] has also constructed a Gaussian process on a lattice which is more general than the one considered here in that he includes in the quadratic form (1) also terms of the form $X_{jk}\, X_{j-1,k-1}$ etc. This is Markovian in the second sense defined above but not in general in the first. Moreover he does not show strictly that the process exists in that he starts from a formula like (1) with an infinite number of lattice points which does not therefore define a distribution, and furthermore does not observe that if we consider the joint distribution of the X_{jk} over a finite set of lattice points, such as on a rectangle, Formula (1) does not give the correct distribution since, for example, the X_{jk} would no longer all have the same variance. It is for this reason that we begin by defining the process on a torus, for which (1) holds, and then proceed to the limit obtaining the explicit representation (16). The correct quadratic form to use for a finite lattice in Lévy's case would have to be obtained by the very awkward inversion of the matrix of covariances defined by a generalisation of (18).

We can obtain the covariances explicitly from the integrals

$$V_{s,t} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos s\theta_1 \cos t\theta_2}{[1 + 2a\cos\theta_1 + 2a\cos\theta_2]}\, d\theta_1 d\theta_2, \tag{18}$$

(which is Bochner's spectral representation), and the coefficients X_{st} from

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos s\theta_1 \cos t\theta_2}{[1 + 2a\cos\theta_1 + 2a\cos\theta_2]^{-\frac{1}{2}}}\, d\theta_1 d\theta_2, \tag{19}$$

by expanding the denominators using the binomial theorem with a negative index, and integrating term by term using the standard results for the integral

$$\int_0^{2\pi} \cos m\theta (\cos\theta)^n d\theta. \tag{20}$$

The resulting series are complicated but convergent and computable. The problem of estimating a from a finite realisation of such a process is much more complicated and it is hoped to discuss this at a later date.

Notice that the process we have defined above is quite different from the nearest neighbour process defined by

$$X_{st} - a(X_{s+1,t} + X_{s-1,t} + X_{s,t+1} + X_{s,t-1}) = \varepsilon_{st}$$

discussed by Whittle [9]. This is not Markovian.

Notice also that we could have slightly generalised the whole of the above theory by replacing the a multiplying the last two products in (1) by a different constant b, and replacing the condition $|a| < \frac{1}{4}$ by $|a| + |b| < \frac{1}{2}$.

V_{00} and $V_{10} = V_{-10} = V_{01} = V_{0-1}$ are however simpler. Consider first V_{00}. This is given by, ($|a| < \frac{1}{4}$),

$$\begin{aligned} V_{00} &= (4\pi^2)^{-1} \int_0^{2\pi} \int_0^{2\pi} \sum_{r=0}^{\infty} (-2a)^r (\cos\theta_1 + \cos\theta_2)^r d\theta_1 d\theta_2 \\ &= (4\pi^2)^{-1} \sum_{r=0}^{\infty} (-2a)^r \sum_{m=0}^{r} \binom{r}{m} \int_0^{2\pi} (\cos\theta_1)^m d\theta_1 \int_0^{2\pi} (\cos\theta_2)^{r-m} d\theta_2. \end{aligned} \tag{21}$$

The integrals are zero unless m and $r - m$ are even so we rewrite this as

$$\sum_{r=0}^{\infty} a^{2r} \sum_{m=0}^{2r} \binom{2r}{2m} \binom{2m}{m} \binom{2r-2m}{r-m}$$

using standard results. This is equal to

$$\begin{aligned} \sum_{r=0}^{\infty} (a^2)^r \sum_{m=0}^{2r} \binom{2r}{r} \binom{r}{m}^2 &= \sum_{r=0}^{\infty} (a^2)^r \binom{2r}{r}^2 \\ &= \sum_{r=0}^{\infty} (16a^2)^r \left(\frac{1.3.5\cdots 2r-1}{2.4.6\cdots 2r}\right)^2 \\ &= 2\pi^{-1} \int_0^{\frac{1}{2}\pi} (1 - 16a^2 \sin^2\theta)^{-\frac{1}{2}} d\theta \to \infty, \text{ as } a \to \tfrac{1}{4}, \end{aligned}$$

which is $2\pi^{-1}$ times the complete elliptic integral of the first kind, tabulated in Abramowitz and Stegum [1].

As is well known, as $a \to \frac{1}{4}$ the above elliptic integral is asymptotically equal to

$$\tfrac{1}{2} \ln 16/(1 - 16a^2), \tag{23}$$

which gives the asymptotic behaviour of V_{00}. Thus $V_{00} \to \infty$ as $a \to -\frac{1}{4}$.

To obtain the value of V_{10} we use (15) which gives

$$V_{00} + a(V_{10} + V_{-10} + V_{01} + V_{0-1}) = 1,$$

so that

$$V_{10} = V_{-10} = V_{01} = V_{0-1} = \tfrac{1}{4}a^{-1}(1 - V_{00}),$$

and

$$r_{10} = \tfrac{1}{4}a^{-1}(V_{00}^{-1} - 1), \tag{24}$$

which converges to unity as $a \to -\frac{1}{4}$.

The evaluation of the integral (18), which is known as the lattice Green's function, for other values of m and n is much more complicated. It occurs in the theory of random walks on a lattice, in a number of applications in physics, and has (with various generalisations) a large literature for which the reader may consult Montroll [6], and Katsura and Inawashiro [4] who give extensive references.

3. Limiting behaviour on a continuous plane

It is also interesting to see if we can obtain, heuristically, a Markov process in the continuous plane by letting $a \to -\frac{1}{4}$ (to ensure continuity), and rescaling the lattice coordinates s and t to obtain continuous coordinates. We shall show that the present process then degenerates, whatever the scaling, to a process in which all the cross correlations are zero. Lévy [5] also considers this.

To do this we have to consider the limiting behaviour of the integral (18) when $a \to -\frac{1}{4}$, and s, t are rescaled by a factor $N(-a)$ so that $s = N(-a)x$, $t = N(-a)y$ where $N(-a)$ increases to infinity as $a \to -\frac{1}{4}$. The correlation between the random variable at $(0,0)$ and the random variable (x, y) will then be given by the asymptotic ratio of (18) to V_{00}. Write $a = -\alpha < 0$, and observe that since $V_{00} \to \infty$ we need to extend the double integral in (18) only over the range $-\frac{1}{2}\pi \leqq \theta_1$, $\theta_2 \leqq \frac{1}{2}\pi$, since the rest of the integral does not exceed $\frac{3}{4}$.

We take $\frac{1}{8} \leqq \alpha \leqq \frac{1}{4}$ and consider the integrals

$$C = \frac{1}{4\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos s\theta_1 \cos t\theta_2}{[1-2\alpha\cos\theta_1 - 2\alpha\cos\theta_2]}\,d\theta_1 d\theta_2, \tag{25}$$

and

$$C_1 = \frac{1}{4\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos s\theta_1 \cos t\theta_2}{[1 - 4\alpha + \alpha(\theta_1^2 + \theta_2^2)]}\,d\theta_1 d\theta_2. \tag{26}$$

Since for $0 \leqq \theta \leqq \frac{1}{2}\pi$, we have

$$1 - \cos\theta > \tfrac{1}{2}\theta^2,$$

and

$$\left|1 - \tfrac{1}{2}\theta^2 - \cos\theta\right| < \tfrac{1}{8}\,\theta^4,$$

we have

$$(27)\qquad \begin{aligned} |C - C_1| &\leqq \frac{1}{4\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\theta_1^4+\theta_2^4}{\alpha(\theta_1^2+\theta_2^2)^2}d\theta_1 d\theta_2 \\ &\leqq \tfrac{1}{4}\alpha^{-1}. \end{aligned}$$

It is therefore sufficient to consider the asymptotic behaviour of C_1. Rescaling and putting $(1-4\alpha)\ N(\alpha)^2 = k^2 > 0$ we get for x, y fixed,

$$(28)\qquad C_1 = \frac{1}{4\pi^2}\int_{-\frac{1}{2}\pi N(\alpha)}^{\frac{1}{2}\pi N(\alpha)}\int_{-\frac{1}{2}\pi N(\alpha)}^{\frac{1}{2}\pi N(\alpha)}\frac{\cos x\phi_1 \cos y\phi_2}{[k^2+\alpha(\phi_1^2+\phi_2^2)]}d\phi_1 d\phi_2$$

$$(29)\qquad = \frac{1}{4\pi^2}\int_{-\frac{1}{2}\pi N(\alpha)}^{\frac{1}{2}\pi N(\alpha)}\int_{-\frac{1}{2}\pi N(\alpha)}^{\frac{1}{2}\pi N(\alpha)}\frac{\exp\{ix\phi_1+iy\phi_2\}}{[k^2+\alpha(\phi_1^2+\phi_2^2)]}d\phi_1 d\phi_2.$$

First consider k^2 fixed. Then $8k^2 \leqq k^2\alpha^{-1} \leqq 4k^2$. Integrating by parts we check that so long as $x \neq 0$, $y \neq 0$ the integral converges as $N(\alpha) \to \infty$ uniformly for (say) $\frac{1}{8} \leqq \alpha \leqq 1$, $1 \leqq k \leqq \infty$. Furthermore using a formula of Titchmarsh ([8], p. 201) we see that, for k fixed

$$\lim_{\alpha \to \frac{1}{4}} C_1 = 4K_0(2kr),$$

where $K_0(x)$ is a Bessel function and $r^2 = x^2 + y^2$. This is finite and thus all cross correlations tend to zero as $\alpha \to \frac{1}{4}$. $K_0(x)$ is known to be a decreasing function of x which tends to zero as $x \to \infty$.

If $k \to 0$, we split the region of integration into a circle round the origin of radius R, where $R < \max(\frac{1}{4}\pi x^{-1}, \frac{1}{4}\pi y^{-1})$, and the rest of the region. The integral over the rest of the region is bounded and the integral over the circle is asymptotically a constant times $-\ln k^2$. Since V_{00} is asymptotically a constant times $-\ln(1-4\alpha)$ and $N(\alpha) \to \infty$ the correlation for fixed x, y again tends to zero. On the other hand suppose $k \to \infty$, then this implies that $N(\alpha)^2$ is increasing faster than $(1-4\alpha)^{-1}$. We have seen that if $\alpha > \frac{1}{8}$ say, $k \geqq 1$ say, and $N(\alpha)$ is otherwise arbitrary but tends to infinity, the integral is convergent as $N(\alpha)$, independent of α and k, tends to infinity, uniformly for $x \neq 0$, $y \neq 0$ fixed, $\frac{1}{8} < \alpha \leqq \frac{1}{4}$, $1 \leqq k \leqq \infty$. Thus using Titchmarsh's formula the integral C_1 is asymptotically equal to $4K_0(2kr)$. Thus there is no scaling function $N(\alpha)$ which will ensure that $C_1 V_{00}^{-1}$ will not tend to zero when x and y are fixed and non-zero. The process cannot converge to a continuous process in the plane and for the construction of processes in the plane which are continuous and Markovian quite other methods are necessary.

Gaussian Markovian processes in the plane have been constructed by Wong [10].

Much work has recently been put into the construction of Markov processes whose variates are 0 and 1 and defined on both regular and irregular lattices (see Spitzer [7] who gives extensive reference to Dobrushin's recent work, Bartlett [2] and Hammersley and Clifford [3]). This is a much more difficult problem and the resulting processes show very complicated behaviour related to the Ising problem and other critical phenomena in physics. However, the procedure of constructing the present process can be extended, with a little complication, to a three-dimensional lattice.

4. Acknowledgements

I am indebted to Professor Hannan for advice on the algebra, and to Dr J. M. Hammersley for arguments which led to the undertaking of this work.

Note added February 1972. Rosanov [11], using a quite different method, has also constructed a Gaussian Markov field on a lattice.

References

[1] Abramowitz, M. and Stegum, I. A. (1965) *Handbook of Mathematical Functions.* Dover, New York.

[2] Bartlett, M. S. (1971) Physical nearest neighbour systems and non-linear time series. *J. Appl. Prob.* **8,** 222–232.

[3] Hammersley, J. M. and Clifford, P. Markov fields on finite graphs and lattices. To appear.

[4] Katsura, S. and Inawashiro, S. (1971) Lattice Green's function for the rectangular and square lattices at arbitrary points. *J. Math. and Phys.* **12,** 1622–1630.

[5] Lévy, P. (1948) *Processus Stochastiques et Movement Brownien.* Gautier Villars, Paris.

[6] Montroll, E. W. (1964) Random walks on lattices. *Proceedings of Symposia in Applied Mathematics* **16,** 193–220.

[7] Spitzer, F. (1971) Markov random fields and Gibbs ensembles. *Amer. Math. Monthly* **78,** 142–154.

[8] Titchmarsh, E. C. (1937) *Introduction to the Theory of Fourier Integrals.* Oxford.

[9] Whittle, P. (1954) On stationary processes in the plane. *Biometrika* **41,** 434–449.

[10] Wong, E. (1969) Homogeneous Gauss-Markov random fields. *Ann. Math. Statist.* **40,** 1625–1634.

[11] Rosanov, Yu. A. (1967) On the Gaussian homogeneous fields with given conditional distributions. *Theor. Probability Appl.* **12,** 381–391.