PERGAMON Computers and Mathematics with Applications 44 (2002) 655–666

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Exact and Approximating Boundary Conditions for the Parabolic Problems on Unbounded Domains

HOUDE HAN AND ZHONGYI HUANG*
Department of Mathematical Sciences, Tsinghua University
Beijing 100084, P.R. China
hhan@math.tsinghua.edu.cn

(Received and accepted July 2001)

Abstract—In this paper, the numerical solutions of the problems of heat equation in two dimensions on unbounded domains are considered. For a given problem, we introduce an artificial boundary Γ to finite the computational domain. On the artificial boundary Γ, we propose an exact boundary condition to reduce the given problem to an initial-boundary problem of heat equation on the finite computational domain, which is equivalent to the original problem. Furthermore, a series of approximating artificial boundary conditions is given. Then the finite difference method and finite element method are used to solve the reduced problem on the finite computational domain. Finally, the numerical examples show the feasibility and effectiveness of the method given in this paper. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Heat equation, Artificial boundary, Exact artificial boundary conditions.

1. INTRODUCTION

The focus of this paper is heat problems on unbounded domains in two space dimensions. These kinds of problems come from heat transfer, fluid dynamics, finance, or other areas of applied mathematics. The integral equation method is one of the approaches to get the numerical solutions of these problems. Greengard and Lin [1] developed a new algorithm for solving the heat problems on unbounded domains several years ago. This algorithm is based on the evolution of the continuous spectrum of the solution. The new algorithm highly reduced the cost of evaluating heat potentials. We think that the appearance of this new algorithm will promote the spread of the integral equation method for solving the heat problems on unbounded domain. We also refer the reader to the work of Strain [2], which related to the fast Gauss transform. On the other hand, the finite element method or finite difference method can be used to get the numerical solutions of these problems. In this case, we need to introduce an artificial boundary to finite the computational domain. On the artificial boundary, the appropriate artificial boundary condition

*Current address: 205 Fine Hall, Department of Mathematics, Princeton University, NJ 08544, U.S.A. Email: zhongyih@princeton.edu.
This work was supported in part by the National Natural Science Foundation of China, the Climbing Program of National Key Project of Foundation of China.

PII: S0898-1221(02)00180-3

is needed. Then the original problem is reduced to an initial-boundary problem of the heat equation on the finite computational domain. For the elliptic problems on unbounded domain, the methods of constructing the artificial boundary condition on a given artificial boundary have often been studied by many authors [3–7], but for the parabolic problems on unbounded domain there are very few results on the artificial boundary condition. Givoli [8] studied the heat problems on unbounded domains, in which the author tries to get the DtN artificial boundary condition on the given artificial boundary and the numerical example was given only in the steady state case. Han and Huang [9] gave the exact artificial boundary conditions for the heat problems on unbounded domains in the one-dimensional case.

In this paper, we considered the problems of heat equations on unbounded domains in two space dimensions. We derived the exact artificial boundary condition on the given artificial boundary Γ. Namely, the relationship between $(u, \frac{\partial u}{\partial n})|_\Gamma$ and $\frac{\partial u}{\partial t}|_\Gamma$ is given. We can choose the first few items of the series of the artificial boundary condition, and therefore, we get a series of approximate problems of the original problem. Finally, three numerical examples show the feasibility and effectiveness of the given method.

2. THE ARTIFICIAL BOUNDARY CONDITION

Let $D \subset \mathbb{R}^2$ denote a bounded domain, namely $D \subset B(O,a) = \{x \in \mathbb{R}^2 \mid \|x\| < a\}$. Suppose

$$D^c = \mathbb{R}^2 \setminus \bar{D}, \qquad \Omega_c^T = D^c \times (0,T], \qquad \Gamma_0 = \partial D \times (0,T].$$

Consider the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} - \Delta u = f(x,t), \qquad (x,t) \in \Omega_c^T, \tag{2.1}$$

$$u|_{\Gamma_0} = g(x,t), \qquad (x,t) \in \Gamma_0, \tag{2.2}$$

$$u|_{t=0} = u_0(x), \qquad x \in D^c, \tag{2.3}$$

$$u \text{ is bounded}, \qquad \text{when } \|x\| \to +\infty, \tag{2.4}$$

where $f(x,t)$, $g(x,t)$, and $u_0(x)$ are given smooth functions and $f(x,t)$, $u_0(x)$ vanish outside a disc with the radius a, namely,

$$f(x,t) = 0, \quad u_0(x) = 0, \qquad \text{for } \|x\| \geq a. \tag{2.5}$$

We introduce an artificial boundary $\Gamma = \{(x,t) \mid \|x\| = b,\ 0 < t \leq T\}$ with $b > a$. Γ divides the domain Ω_c^T into two parts,

$$\Omega_i^T = \{(x,t) \mid x \in D^c \text{ and } \|x\| < b,\ 0 < t \leq T\},$$
$$\Omega_e^T = \{(x,t) \mid \|x\| > b,\ 0 < t \leq T\}.$$

If we can seek a boundary condition on Γ, we can reduce problem (2.1)–(2.4) to the bounded computational domain Ω_i^T. We consider the restriction of the solution $u(x,t)$ of problem (2.1)–(2.4) on the unbounded domain Ω_e^T in the polar coordinate $u(r,\theta,t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad (r,\theta,t) \in \Omega_e^T, \tag{2.6}$$

$$u|_\Gamma = u(b,\theta,t), \tag{2.7}$$

$$u|_{t=0} = 0, \tag{2.8}$$

$$u \text{ is bounded}, \qquad \text{when } r \to +\infty. \tag{2.9}$$

Since $u(b,\theta,t)$ is an unknown function, problem (2.6)–(2.9) is an incompletely posed problem. It cannot be solved independently. If the function $u(b,\theta,t)$ is given, we try to find the solution $u(r,\theta,t)$ of problem (2.6)–(2.9). Let

$$u(b,\theta,t) = a_0(t) + \sum_{n=1}^{\infty}(a_n(t)\cos n\theta + b_n(t)\sin n\theta), \tag{2.10}$$

$$a_0(t) = \frac{1}{2\pi}\int_0^{2\pi} u(b,\theta,t)\,d\theta, \tag{2.11}$$

$$\left.\begin{aligned} a_n(t) &= \frac{1}{\pi}\int_0^{2\pi} u(b,\theta,t)\cos n\theta\,d\theta,\\ b_n(t) &= \frac{1}{\pi}\int_0^{2\pi} u(b,\theta,t)\sin n\theta\,d\theta, \end{aligned}\right\} \quad n = 1,2,\ldots, \tag{2.12}$$

and

$$u(r,\theta,t) = u_0(r,t) + \sum_{n=1}^{\infty}\{u_n(r,t)\cos n\theta + v_n(r,t)\sin n\theta\}. \tag{2.13}$$

Substituting (2.13) into (2.6), we obtain

$$\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial r^2} - \frac{1}{r}\frac{\partial u_0}{\partial r} + \sum_{n=1}^{\infty}\left\{\left[\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial r^2} - \frac{1}{r}\frac{\partial u_n}{\partial r} + \frac{n^2}{r^2}u_n\right]\cos n\theta \right.$$
$$\left. + \left[\frac{\partial v_n}{\partial t} - \frac{\partial^2 v_n}{\partial r^2} - \frac{1}{r}\frac{\partial v_n}{\partial r} + \frac{n^2}{r^2}v_n\right]\sin n\theta\right\} = 0.$$

Hence, we have the following.

(i) $u_0(r,t)$ satisfies

$$\frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r}, \qquad r > b, \quad 0 < t \le T, \tag{2.14}$$

$$u_0|_{r=b} = a_0(t), \tag{2.15}$$

$$u_0|_{t=0} = 0, \tag{2.16}$$

$$u_0 \text{ is bounded}, \qquad \text{when } r \to +\infty. \tag{2.17}$$

(ii) $u_n(r,t)$ satisfies

$$\frac{\partial u_n}{\partial t} = \frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r} - \frac{n^2}{r^2}u_n, \qquad r > b, \quad 0 < t \le T, \tag{2.18}$$

$$u_n|_{r=b} = a_n(t), \tag{2.19}$$

$$u_n|_{t=0} = 0, \tag{2.20}$$

$$u_n \text{ is bounded}, \qquad \text{when } r \to +\infty. \tag{2.21}$$

(iii) $v_n(r,t)$ satisfies

$$\frac{\partial v_n}{\partial t} = \frac{\partial^2 v_n}{\partial r^2} + \frac{1}{r}\frac{\partial v_n}{\partial r} - \frac{n^2}{r^2}v_n, \qquad r > b, \quad 0 < t \le T, \tag{2.22}$$

$$v_n|_{r=b} = b_n(t), \tag{2.23}$$

$$v_n|_{t=0} = 0, \tag{2.24}$$

$$v_n \text{ is bounded}, \qquad \text{when } r \to +\infty. \tag{2.25}$$

If we obtained the $u_0(r,t)$, $u_n(r,t)$, and $v_n(r,t)$, then the function $u(r,\theta,t)$ given by (2.13) is the solution of (2.6)–(2.9).

In [10], the solution of problem (2.14)–(2.17) is given by

$$u_0(r,t) = \int_0^t \frac{da_0(\lambda)}{d\lambda} G(r,t-\lambda)\,d\lambda, \tag{2.26}$$

with

$$G(r,t) = 1 + \frac{2}{\pi}\int_0^\infty e^{-\mu^2 t}\frac{J_0(\mu r)Y_0(\mu b) - Y_0(\mu r)J_0(\mu b)}{J_0^2(\mu b) + Y_0^2(\mu b)}\frac{d\mu}{\mu}$$

and

$$\left.\frac{\partial u_0}{\partial r}(r,t)\right|_{r=b} = -\frac{1}{b\sqrt{\pi}}\int_0^t \frac{da_0(\lambda)}{d\lambda}\frac{H_0(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda, \tag{2.27}$$

with

$$H_0(t) = \frac{4\sqrt{t}}{\sqrt{\pi^3}}\int_0^\infty \frac{e^{-\mu^2 t}}{\mu\left\{J_0^2(b\mu) + Y_0^2(b\mu)\right\}}\,d\mu. \tag{2.28}$$

Then we will consider the initial-boundary value problem (2.18)–(2.21). First we consider the following simple problem:

$$\frac{\partial G_n}{\partial t} = \frac{\partial^2 G_n}{\partial r^2} + \frac{1}{r}\frac{\partial G_n}{\partial r} - \frac{n^2}{r^2}G_n, \qquad r > b, \quad 0 < t \le T, \tag{2.29}$$

$$G_n|_{r=b} = 1, \tag{2.30}$$

$$G_n|_{t=0} = 0, \tag{2.31}$$

$$G_n \text{ is bounded}, \qquad \text{when } r \to +\infty. \tag{2.32}$$

Let

$$G_n(r,t) = e^{-\mu^2 t}w(r). \tag{2.33}$$

Substituting (2.33) into (2.29), we obtain that $w(r)$ satisfies

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \left(\mu^2 - \frac{n^2}{r^2}\right)w = 0. \tag{2.34}$$

This is the Bessel equation of order n; there are two independent solutions $J_n(\mu r)$ and $Y_n(\mu r)$. Hence,

$$e^{-\mu^2 t}\frac{J_n(\mu r)Y_n(\mu b) - Y_n(\mu r)J_n(\mu b)}{J_n^2(\mu b) + Y_n^2(\mu b)}$$

is a solution of equation (2.29) for any $\mu > 0$. Let

$$G_*(r,t) = \frac{2}{\pi}\int_0^\infty e^{-\mu^2 t}\frac{J_n(\mu r)Y_n(\mu b) - Y_n(\mu r)J_n(\mu b)}{J_n^2(\mu b) + Y_n^2(\mu b)}\frac{d\mu}{\mu}. \tag{2.35}$$

$G_*(r,t)$ is a solution of equation (2.29) and

$$\begin{aligned}
&G_*(r,t)|_{r=b} = 0,\\
&G_*(r,t)|_{t=0} = \lim_{t\to+0} G_*(r,t) = \frac{2}{\pi}\int_0^\infty \frac{J_n(\mu r)Y_n(\mu b) - Y_n(\mu r)J_n(\mu b)}{J_n^2(\mu b) + Y_n^2(\mu b)}\frac{d\mu}{\mu}\\
&\qquad = -\left(\frac{b}{r}\right)^n, \qquad r > b.
\end{aligned}$$

The last equality is given in [11, p. 679]. Let

$$G_n(r,t) = \left(\frac{b}{r}\right)^n + G_*(r,t); \tag{2.36}$$

then $G_n(r,t)$ is the solution of problem (2.29)–(2.32). By Duhamel's theorem in [10, p. 30], we obtain

$$\begin{aligned}
u_n(r,t) &= \int_0^t a_n(\lambda)\,\frac{\partial}{\partial t}\,G_n(r,t-\lambda)\,d\lambda \\
&= -\int_0^t a_n(\lambda)\,\frac{\partial}{\partial \lambda}\,G_n(r,t-\lambda)\,d\lambda \\
&= -a_n(\lambda)G_n(r,t-\lambda)|_{\lambda=0}^{\lambda=t} + \int_0^t \frac{da_n(\lambda)}{d\lambda}\,G_n(r,t-\lambda)\,d\lambda \\
&= \int_0^t \frac{da_n(\lambda)}{d\lambda}\,G_n(r,t-\lambda)\,d\lambda,
\end{aligned}$$

where $u_n(r,t)$ is the solution of problem (2.18)–(2.21). Furthermore, we have

$$\left.\frac{\partial u_n(r,t)}{\partial r}\right|_{r=b} = \int_0^t \frac{da_n(\lambda)}{d\lambda}\,\frac{\partial G_n(r,t-\lambda)}{\partial r}\,d\lambda\Big|_{r=b}. \tag{2.37}$$

On the other hand,

$$\begin{aligned}
\left.\frac{\partial G_n(r,t)}{\partial r}\right|_{r=b} &= -\frac{n}{b} + \frac{2}{\pi}\int_0^\infty e^{-\mu^2 t}\,\frac{J_n'(\mu b)Y_n(\mu b) - Y_n'(\mu b)J_n(\mu b)}{J_n^2(\mu b) + Y_n^2(\mu b)}\,\frac{d\mu}{\mu}, \\
\text{(by the Wronskian relation)} &= -\frac{n}{b} - \frac{4}{\pi^2 b}\int_0^\infty \frac{e^{-\mu^2 t}}{J_n^2(\mu b) + Y_n^2(\mu b)}\,\frac{d\mu}{\mu}.
\end{aligned}$$

Let

$$H_n(t) = \frac{4\sqrt{t}}{\sqrt{\pi^3}}\int_0^\infty \frac{e^{-\mu^2 t}}{J_n^2(\mu b) + Y_n^2(\mu b)}\,\frac{d\mu}{\mu};$$

then

$$\left.\frac{\partial G_n(r,t)}{\partial r}\right|_{r=b} = -\frac{n}{b} - \frac{H_n(t)}{b\sqrt{\pi t}}. \tag{2.38}$$

Combining (2.37) and (2.38), we have

$$\begin{aligned}
\left.\frac{\partial u_n}{\partial r}\right|_{r=b} &= -\int_0^t \frac{da_n(\lambda)}{d\lambda}\,\frac{n}{b}\,d\lambda - \frac{1}{b\sqrt{\pi}}\int_0^t \frac{da_n(\lambda)}{d\lambda}\,\frac{H_n(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda \\
&= -\frac{n}{b}u_n\Big|_{r=b} - \frac{1}{b\sqrt{\pi}}\int_0^t \frac{\partial u_n(b,\lambda)}{\partial \lambda}\,\frac{H_n(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda.
\end{aligned} \tag{2.39}$$

Similarly, we obtain

$$\left.\frac{\partial v_n}{\partial r}\right|_{r=b} = -\frac{n}{b}v_n\Big|_{r=b} - \frac{1}{b\sqrt{\pi}}\int_0^t \frac{\partial v_n(b,\lambda)}{\partial \lambda}\,\frac{H_n(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda, \tag{2.40}$$

where $v_n(r,t)$ is the solution of problem (2.22)–(2.25)

$$v_n(r,t) = \int_0^t \frac{db_n(\lambda)}{d\lambda}\,G_n(r,t-\lambda)\,d\lambda. \tag{2.41}$$

By equality (2.13) with

$$\begin{aligned} u_0(r,t) &= \frac{1}{2\pi}\int_0^{2\pi} u(r,\phi,t)\,d\phi, \\ u_n(r,t) &= \frac{1}{\pi}\int_0^{2\pi} u(r,\phi,t)\cos n\phi\,d\phi, \qquad\qquad n = 1,2,\dots, \\ v_n(r,t) &= \frac{1}{\pi}\int_0^{2\pi} u(r,\phi,t)\sin n\phi\,d\phi, \end{aligned} \tag{2.42}$$

$$\begin{aligned} \left.\frac{\partial u}{\partial r}\right|_{r=b} &= \left.\frac{\partial u_0}{\partial r}\right|_{r=b} + \sum_{n=1}^{\infty}\left(\left.\frac{\partial u_n}{\partial r}\right|_{r=b}\cos n\theta + \left.\frac{\partial v_n}{\partial r}\right|_{r=b}\sin n\theta\right) \\ &= -\sum_{n=1}^{\infty}\frac{n}{b}\left(u_n|_{r=b}\cos n\theta + v_n|_{r=b}\sin n\theta\right) \\ &\quad - \frac{1}{b\sqrt{\pi}}\sum_{n=0}^{\infty}\int_0^t\left[\frac{\partial u_n(b,\lambda)}{\partial\lambda}\cos n\theta + \frac{\partial v_n(b,\lambda)}{\partial\lambda}\sin n\theta\right]\frac{H_n(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda. \end{aligned} \tag{2.43}$$

Finally, we obtain the exact boundary condition on the artificial boundary Γ:

$$\begin{aligned} \left.\frac{\partial u}{\partial r}(r,\theta,t)\right|_{r=b} &= -\frac{1}{2b\sqrt{\pi^3}}\int_0^t\int_0^{2\pi}\frac{\partial u(b,\phi,\lambda)}{\partial\lambda}\,d\phi\,\frac{H_0(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda \\ &\quad - \frac{1}{b\sqrt{\pi^3}}\int_0^t\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial u(b,\phi,\lambda)}{\partial\lambda}\cos n(\phi-\theta)\,d\phi\,\frac{H_n(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda \\ &\quad - \frac{1}{b\pi}\sum_{n=1}^{\infty} n\int_0^{2\pi} u(b,\phi,t)\cos n(\phi-\theta)\,d\phi \\ &\equiv \mathcal{K}_\infty\left(u(b,\cdot,\cdot),\frac{\partial u}{\partial t}(b,\cdot,\cdot)\right)(\theta,t). \end{aligned} \tag{2.44}$$

Furthermore, we take the first few terms of the above summation; we obtain a series of approximating artificial boundary conditions on Γ:

$$\begin{aligned} \left.\frac{\partial u}{\partial r}(r,\theta,t)\right|_{r=b} &= -\frac{1}{2b\sqrt{\pi^3}}\int_0^t\int_0^{2\pi}\frac{\partial u(b,\phi,\lambda)}{\partial\lambda}\,d\phi\,\frac{H_0(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda \\ &\quad - \frac{1}{b\sqrt{\pi^3}}\int_0^t\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial u(b,\phi,\lambda)}{\partial\lambda}\cos n(\phi-\theta)\,d\phi\,\frac{H_n(t-\lambda)}{\sqrt{t-\lambda}}\,d\lambda \\ &\quad - \frac{1}{b\pi}\sum_{n=1}^{N} n\int_0^{2\pi} u(b,\phi,t)\cos n(\phi-\theta)\,d\phi \\ &\equiv \mathcal{K}_N\left(u(b,\cdot,\cdot),\frac{\partial u}{\partial t}(b,\cdot,\cdot)\right)(\theta,t), \qquad \text{for } N = 0,1,2,\dots. \end{aligned} \tag{2.45}$$

By the boundary conditions (2.45), the original problem (2.1)–(2.4) is reduced to the following approximation problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + f(r,\theta,t), \qquad (r,\theta,t)\in\Omega_i^T, \tag{2.46}$$

$$u|_{\Gamma_0} = g(\theta,t), \tag{2.47}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=b} = \mathcal{K}_N\left(u(b,\cdot,\cdot),\frac{\partial u}{\partial t}(b,\cdot,\cdot)\right), \tag{2.48}$$

$$u|_{t=0} = u_0(x). \tag{2.49}$$

3. NUMERICAL EXAMPLES

In order to demonstrate the effectiveness of the artificial boundary conditions given in this paper, three numerical examples are discussed. We use two kinds of numerical methods to solve these examples: FDM (finite difference method) and FEM (finite element method).

EXAMPLE 1. Let us consider an initial-boundary problem on the domain out of a disc:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad (r,\theta,t) \in \Omega^T = \{r > a,\ \theta \in [0,2\pi],\ t \in [0,T]\}, \tag{3.1}$$

$$u|_{r=a} = g(\theta,t), \tag{3.2}$$

$$u|_{t=0} = 0, \tag{3.3}$$

$$u \text{ is bounded}, \qquad \text{when } r \to +\infty, \tag{3.4}$$

where $a = 2$, $T = 1$, $g(\theta,t) = (1/t)\exp(-a^2/4t)$. The exact solution of problem (3.1)–(3.4) is

$$u(r,\theta,t) = \frac{1}{t}\exp\left(-\frac{r^2}{4t}\right). \tag{3.5}$$

After we introduce the artificial boundary Γ_b to bound the domain $\Omega_0^T = \{(r,\theta,t) \mid a < r < b,\ 0 < t < T\}$, we need only to solve problem (2.46)–(2.49) with $f(x,t) \equiv 0$ and $u_0(x) \equiv 0$.

First, we use the Crank-Nicholson difference scheme to solve problem (3.1)–(3.4). We divided the interval $[0,T]$ into K equal parts:

$$0 = t_0 < t_1 < \cdots < t_K = T. \tag{3.6}$$

Here we let $b = 3$. Then we divided the interval $[a,b]$ of r-axis and the interval of $[0,2\pi]$ into I and J equal parts, respectively,

$$a = r_0 < r_1 < \cdots < r_I = b, \qquad 0 = \theta_0 < \theta_1 < \cdots < \theta_J = 2\pi. \tag{3.7}$$

Let $I = J/6$, $\tau = T/K$, $h = (b-a)/I$, and $\delta = 2\pi/J$. In this example, we let $N = 0$ and $\tau = h$. Now we have the following formulae by Crank-Nicholson scheme:

$$\begin{gathered}\frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k + u_{i+1,j}^{k-1} - 2u_{i,j}^{k-1} + u_{i-1,j}^{k-1}}{2h^2} + \frac{u_{i+1,j}^k - u_{i-1,j}^k + u_{i+1,j}^{k-1} - u_{i-1,j}^{k-1}}{4hr_i} \\ + \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k + u_{i,j+1}^{k-1} - 2u_{i,j}^{k-1} + u_{i,j-1}^{k-1}}{2r_i^2\delta^2} - \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\tau} = 0, \\ 1 \le i \le I, \qquad 1 \le j \le J, \qquad 1 \le k \le K,\end{gathered} \tag{3.8}$$

$$u_{i,j}^0 = 0, \qquad 0 \le i \le I+1, \quad 1 \le j \le J, \tag{3.9}$$

$$u_{0,j}^k = g(\theta_j, t_k), \qquad 1 \le j \le J, \qquad 1 \le k \le K, \tag{3.10}$$

$$\frac{u_{I+1,j}^k - u_{I-1,j}^k}{2h} = \mathcal{K}_0\left(u(r_I,\cdot,\cdot), \frac{\partial u}{\partial t}(r_I,\cdot,\cdot)\right)(\theta_j,t_k), \qquad 1 \le j \le J, \qquad 1 \le k \le K. \tag{3.11}$$

Due to the operator $\mathcal{K}_N$, at each time step we need to compute these integrals

$$\int_0^{t_k}\int_0^{2\pi} \frac{\partial u(b,\phi,\lambda)}{\partial \lambda} \cos n(\phi - \theta_j)\, d\phi\, \frac{H_n(t_k - \lambda)}{\sqrt{t_k - \lambda}}\, d\lambda, \qquad n = 0,\dots,N; \tag{3.12}$$

$$\int_0^{2\pi} u(b,\phi,t_k)\cos n(\phi - \theta_j)\, d\phi, \qquad n = 1,\dots,N. \tag{3.13}$$

It is easy to calculate the integrals in (3.13),

$$\int_0^{2\pi} u(b,\phi,t_k)\cos n(\phi-\theta_j)\,d\phi = \sum_{s=0}^{J-1}\int_{\theta_s}^{\theta_{s+1}}\left[u_{I,s}^k + \frac{(u_{I,s+1}-u_{I,s})(\phi-\theta_s)}{\delta}\right]\cos n(\phi-\theta_j)\,d\phi$$
$$= \frac{1}{n^2\delta}\sum_{s=1}^{J} u_{I,s}^k\left[2\cos(\theta_s-\theta_j)-\cos(\theta_{s+1}-\theta_j)-\cos(\theta_{s-1}-\theta_j)\right].$$

We can approximate the integrals in (3.12) as follows:

$$\begin{aligned}
&\int_0^{t_k}\int_0^{2\pi}\frac{\partial u(b,\phi,\lambda)}{\partial\lambda}\cos n(\phi-\theta_j)\,d\phi\,\frac{H_n(t_k-\lambda)}{\sqrt{t_k-\lambda}}\,d\lambda \\
&= \sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}}\sum_{s=0}^{J-1}\int_{\theta_s}^{\theta_{s+1}}\left\{\frac{u_{I,s}^{l+1}-u_{I,s}^l}{\tau}+\frac{u_{I,s+1}^{l+1}-u_{I,s+1}^l-u_{I,s}^{l+1}+u_{I,s}^l}{\tau}\cdot\frac{\phi-\theta_s}{\delta}\right\} \\
&\quad\cdot\cos n(\phi-\theta_j)\,d\phi\,\frac{H_n(t_k-\lambda)}{\sqrt{t_k-\lambda}}\,d\lambda \\
&= \frac{\delta}{\tau}\sum_{l=0}^{k-1}\sum_{s=0}^{J-1}\left(u_{I,s}^{l+1}-u_{I,s}^l\right)\left[2\cos(\theta_s-\theta_j)-\cos(\theta_{s+1}-\theta_j)-\cos(\theta_{s-1}-\theta_j)\right]\int_{t_l}^{t_{l+1}}\frac{H_n(t_k-\lambda)}{\sqrt{t_k-\lambda}}\,d\lambda.
\end{aligned} \tag{3.14}$$

Furthermore, we know that

$$\frac{H_0(t)}{\sqrt{t}}\sim\frac{1}{\log t},\qquad \text{as } t\to\infty, \tag{3.15}$$
$$\frac{H_n(t)}{\sqrt{t}}\sim\frac{1}{t^n},\qquad \text{as } t\to\infty,\quad n\ge 1. \tag{3.16}$$

Therefore, we can approximate the integrals in (3.12) by

$$\begin{aligned}
&\frac{\delta}{\tau}\sum_{l=k_0}^{k-1}\sum_{s=0}^{J-1}\left(u_{I,s}^{l+1}-u_{I,s}^l\right) \\
&\times\left[2\cos(\theta_s-\theta_j)-\cos(\theta_{s+1}-\theta_j)-\cos(\theta_{s-1}-\theta_j)\right]\int_{t_l}^{t_{l+1}}\frac{H_n(t_k-\lambda)}{\sqrt{t_k-\lambda}}\,d\lambda,
\end{aligned} \tag{3.17}$$

when t_k is very large. We can make some tables for functions $H_n(t)/\sqrt{t}$ ($n = 0,1,\dots,N$) first. Thus, it is easy to get the integrals in (3.17) by numerical integration. Certainly, it is a disadvantage of our method in this paper that the integral kernels do not decay fast. In our following paper, we will give another method to overcome this disadvantage.

For different I, we have the results in the second column of Table 1.

Table 1. The results of Example 1.

	$\frac{\|u-u_h\|_{1,\Omega}}{\|u\|_{1,\Omega}}$	
I	FDM	FEM
2	4.1585e−1	4.0883e−1
4	2.0973e−1	2.1113e−1
8	1.0344e−1	1.0328e−1
16	4.1552e−2	4.1534e−2

Furthermore, we use the finite element method to solve the approximation problem (2.46)–(2.49). To do so, we should give the variational form of problem (2.46)–(2.49):

$$\text{Find } u \in U, \text{ such that } \frac{d}{dt}(u,v)_{\Omega_i^T} + a(u,v) = 0, \qquad \forall v \in V, \tag{3.18}$$

where

$$\Omega_i^T = \{(r,\theta,t) \mid a < r < b,\ 0 \le \theta \le 2\pi,\ 0 < t < T\}, \tag{3.19}$$

$$(u,v)_{\Omega_i^T} = \iint_{\Omega_i^T} uv\,dr\,d\theta, \tag{3.20}$$

$$a(u,v) = \iint_{\Omega_i^T} \left\{ \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial r}v + \frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial v}{\partial \theta} \right\} dr - \int_0^{2\pi} \left\{ \mathcal{K}_N\left(u, \frac{\partial u}{\partial t}\right) v \right\}\Big|_{r=b} d\theta, \tag{3.21}$$

$$V = \left\{ v(r,\theta) \in H^1\left(\Omega_i^T\right) \mid v(a,\theta) = 0,\ v(r,0) = v(r,2\pi) \right\}, \tag{3.22}$$

$$U = \Big\{ w(r,\theta,t) \mid \text{ for fixed } t \in [0,T],\ w(\cdot,\cdot,t),\ \frac{\partial w}{\partial t}(\cdot,\cdot,t) \in H^1\left(\Omega_i^T\right), \text{ and}$$
$$w(r,\theta,0) = 0,\ w(r,0,t) = w(r,2\pi,t),\ w(a,\theta,t) = g(\theta,t) \Big\}. \tag{3.23}$$

If we give a partition $\mathcal{T}_h$ of $[a,b] \times [0,2\pi]$ such as (3.7), we can construct the finite dimension subspace V_h of V by using piecewise bilinear functions

$$V_h = \left\{ p_h(r,\theta) \in C^0([a,b] \times [0,2\pi]) \mid p_h|_{[r_{i-1},r_i]\times[\theta_{j-1},\theta_j]} \in P_{11}(r,\theta),\ 1 \le i \le I,\ 1 \le j \le J \right\},$$

where $P_{11}(r,\theta)$ is the space of bilinear functions with $p_h(r,0) = p_h(r,2\pi)$. Let $\{N_1(r,\theta), N_2(r,\theta), \dots, N_{\mathcal{I}}(r,\theta)\}$ be a basis of space V_h with $\mathcal{I} = \dim(V_h)$, and

$$V_h^0 = \{v_h(r,\theta) \in V_h \mid v_h(a,\theta) = 0\},$$
$$U_h = \left\{ w_h(r,\theta,t) \in U \mid w_h = \sum_{i=1}^{\mathcal{I}} \alpha_i(t) N_i(r,\theta) \text{ and } \alpha_i \in H_0^1([0,T]) \right\}.$$

Then we get the following approximation problem of (3.18):

$$\text{Find } u_h \in U_h, \text{ such that } \frac{d}{dt}(u_h,v_h)_{\Omega_i^T} + a(u_h,v_h) = 0, \qquad \forall v_h \in V_h^0. \tag{3.24}$$

Letting $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_{\mathcal{I}}(t))^T$, we can rewrite problem (3.24) as the following initial value problem of ordinary differential equations:

$$\begin{aligned} A\boldsymbol{\alpha}' + B\boldsymbol{\alpha} + C(\boldsymbol{\alpha}, \boldsymbol{\alpha}') &= 0, \\ \boldsymbol{\alpha}(0) &= 0, \end{aligned} \tag{3.25}$$

where $A = (a_{ij})_{\mathcal{I}\times\mathcal{I}}$ and $B = (b_{ij})_{\mathcal{I}\times\mathcal{I}}$ are two constant matrices with

$$a_{ij} = \iint_{\Omega_i^T} r N_i(r,\theta) N_j(r,\theta)\,dr\,d\theta,$$
$$b_{ij} = \iint_{\Omega_i^T} \left\{ r\frac{\partial N_i}{\partial r}(r,\theta)\frac{\partial N_j}{\partial r}(r,\theta) + \frac{1}{r}\frac{\partial N_i}{\partial \theta}(r,\theta)\frac{\partial N_j}{\partial \theta}(r,\theta) \right\} dr\,d\theta,$$

and

$$C(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = -\int_0^{2\pi} \mathcal{K}_N\left(\Phi(\phi,\theta)\boldsymbol{\alpha}, \Phi(\phi,\theta)\boldsymbol{\alpha}'\right) d\theta,$$
$$\Phi(\phi,\theta) = (N_1(b,\phi), \dots, N_{\mathcal{I}}(b,\phi))^T \cdot (N_1(b,\theta), \dots, N_{\mathcal{I}}(b,\theta)).$$

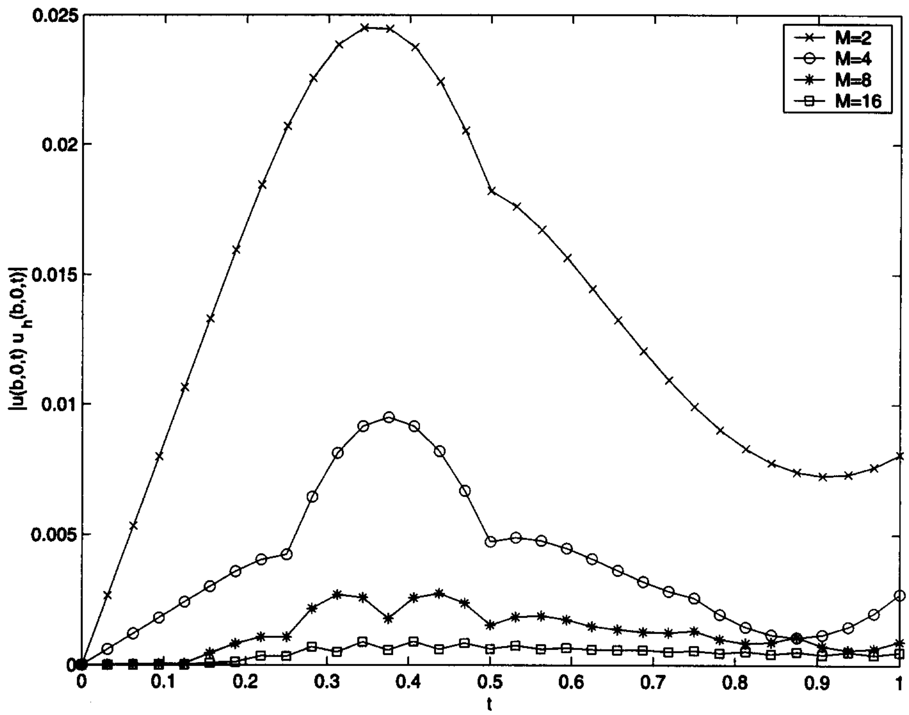

Figure 1. By FDM.

There is the similar problem as in FDM. We need to truncate those integrals in $C(\alpha, \alpha')$ when t is very large. After we use a finite difference method to solve problem (3.25), then we can get the results shown in the third column of Table 1. In Figure 1, we give the error function $|u(b,\theta,t) - u_h(b,\theta,t)|$ for $t \in [0,T]$ and $\theta = 0$.

EXAMPLE 2. Consider the same equations in Example 1, but now we give another boundary condition: $g(\theta,t) = (1/t)\exp(-a^2/4t) + G_1(b,t)\cos\theta$ where $G_1(r,t)$ is given in (2.36). Now the exact solution of problem (3.1)–(3.4) is

$$u(r,\theta,t) = \frac{1}{t}\exp\left(-\frac{r^2}{4t}\right) + G_1(r,t)\cos\theta. \tag{3.26}$$

After we introduce an artificial boundary $\Gamma_b = \{(x,t) \mid x = b,\ t \in [0,T]\}$ and let $b = 3$, we use a similar procedure to that in Example 1 for different choices of N—the order of the artificial boundary condition in (2.45)—then we can get the results shown in Table 2. In Figure 2, we give the error function $|u(b,\theta,t) - u_h(b,\theta,t)|$ for $t \in [0,T]$ and $\theta = 0$.

Table 2. The results of Example 2.

	$\frac{\|u-u_h\|_{1,\Omega}}{\|u\|_{1,\Omega}}$			
	FDM		FEM	
I	$N=0$	$N=1$	$N=0$	$N=1$
2	3.9610e−1	3.9512e−1	4.0725e−1	3.7474e−1
4	3.4304e−1	2.2206e−1	2.4875e−1	2.1911e−1
8	3.0578e−1	1.1544e−1	1.7837e−1	1.1429e−1
16	2.9220e−1	6.5095e−2	1.6107e−1	6.4684e−2

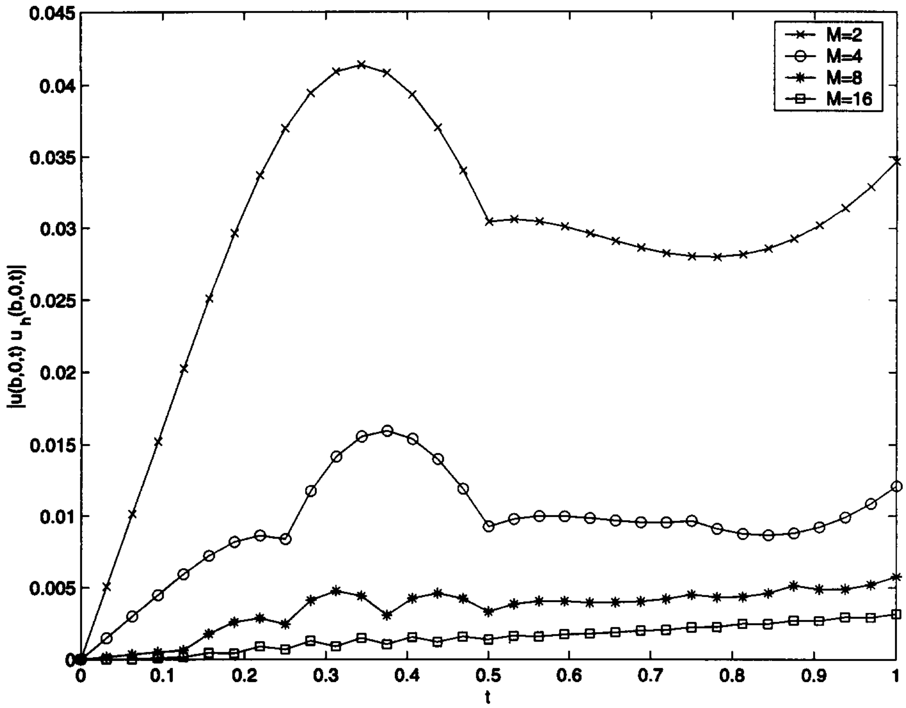

Figure 2. By FDM.

EXAMPLE 3. Consider the same equations in Example 1, but now we give another boundary condition

$$g(\theta,t)=\frac{1}{t}\exp\left(-\frac{(a\cos\theta-x_0)^2+(a\sin\theta-y_0)^2}{4t}\right),\qquad \text{with } (x_0,y_0)=(0.5,0.5).$$

Now the exact solution of problem (3.1)–(3.4) is

$$u(r,\theta,t)=\frac{1}{t}\exp\left(-\frac{(r\cos\theta-x_0)^2+(r\sin\theta-y_0)^2}{4t}\right). \tag{3.27}$$

We also introduce an artificial boundary $\Gamma_b=\{(x,t)\mid x=b,\ t\in[0,T]\}$ and let $b=3$; we use a similar procedure to that in Example 1 for different choices of N—the order of the artificial boundary condition in (2.45). Then we can get the results shown in Table 3. In Figure 3, we give the error function $|u(b,\theta,t)-u_h(b,\theta,t)|$ for $t\in[0,T]$ and $\theta=0$.

Table 3. The results of Example 3.

	$\frac{\|u-u_h\|_{1,\Omega}}{\|u\|_{1,\Omega}}$					
	FDM			FEM		
I	$N=0$	$N=1$	$N=2$	$N=0$	$N=1$	$N=2$
2	4.9818e−1	4.7656e−1	4.7509e−1	5.1129e−1	4.8706e−1	4.8549e−1
4	3.9483e−1	3.6034e−1	3.5612e−1	3.9678e−1	3.6089e−1	3.5654e−1
8	2.6815e−1	2.0596e−1	1.9747e−1	2.6935e−1	2.0663e−1	1.9810e−1
16	2.0418e−1	1.0468e−1	8.5249e−2	2.0447e−1	1.0475e−1	8.5308e−2

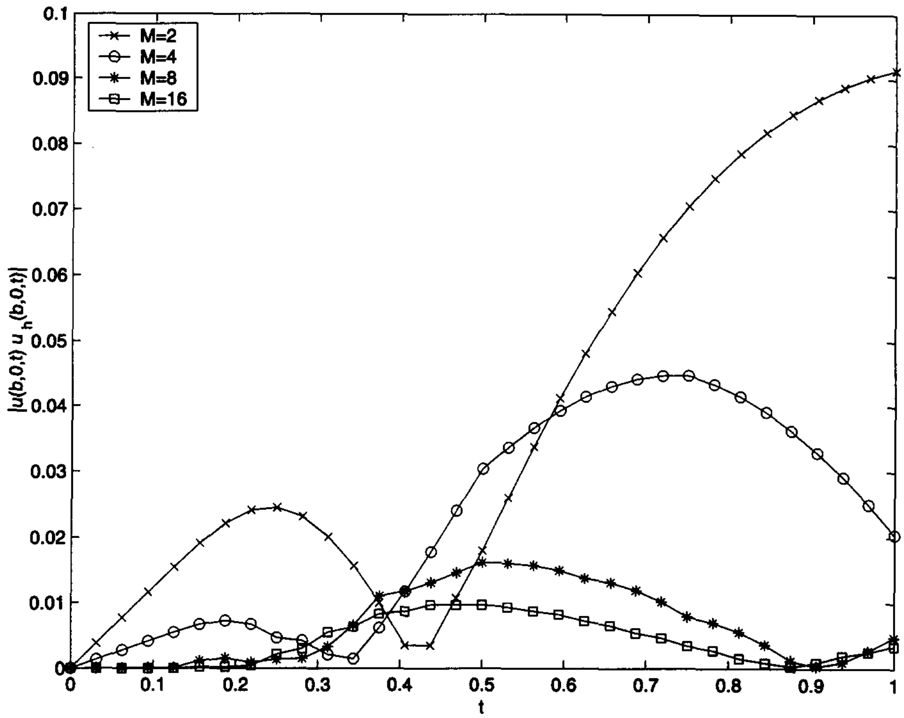

Figure 3. By FDM.

4. CONCLUSION

Here we provide a kind of exact artificial boundary condition for the numerical solution of heat problems on an unbounded domain in two space dimensions. After we introduce an artificial boundary, we get a initial-boundary problem on a finite domain enclosed by the artificial boundary which is equivalent to the original problem. In addition, we get the exact artificial boundary condition and a series of approximation artificial boundary conditions. From the numerical examples, we find that we can get good numerical solutions using our exact artificial boundary conditions whenever we use FEM or FDM.

REFERENCES

1. L. Greengard and P. Lin, On the numerical solution of the heat equation in unbounded domains (Part I), *SIAM Review.*
2. J. Strain, Fast adaptive methods for the free-space heat equation, *SIAM J. Sci. Comput.* **15**, 185–206, (1992).
3. W. Bao and H. Han, Nonlocal artificial boundary conditions for the incompressible viscous flows in a channel using spectral techniques, *J. Comp. Phys.* **126**, 52, (1996).
4. B. Engquist and A. Majda, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comp.* **31**, 629, (1977).
5. K. Feng, Asymptotic radiation conditions for reduced wave equations, *J. Comp. Math.* **2**, 130, (1984).
6. D. Givoli, *Numerical Methods for Problems in Infinite Domains*, Elsevier, Amsterdam, (1992).
7. H. Han and X. Wu, The approximation of the exact boundary condition at an artificial boundary for linear elastic equations and its application, *Math. Comp.* **59**, 21, (1992).
8. D. Givoli, Finite element analysis of heat problems in unbounded domains, In *Numerical Methods in Thermal Problems, Volume VI, Part 2,* (Edited by R.W. Lewis and K. Morgen), pp. 1094–1104, Pineridge Press, Swansea, U.K., (1989).
9. H. Han and Z. Huang, A class of artificial boundary conditions for heat equation in unbounded domains, *Computers Math. Applic.* **43** (6/7), 889–900, (2002).
10. H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, Second edition, Oxford University Press, New York, (1959).
11. I.S. Gradshteyn and M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, (1980).