# Differential Operators and their Adjoint Operators

**Differential Operators** Linear functions from $E^n$ to $E^m$ may be described, once bases have been selected in both spaces (ordinarily one uses the standard unit vectors), by means of $n \times n$ matrices $\mathbf{A}$; we write $Y = \mathbf{A}\,X$. This operation is linear because

$$\mathbf{A}(c_1\,X_1 + c_2\,X_2) = c_1\,\mathbf{A}\,X_1 + c_2\,\mathbf{A}\,X_2.$$

When $m = n$ we often use the term *linear transformation* for this operation because it then transforms $E^n$ into itself.

We will use the symbol $PC_2[a,b]$, or $PC_2^0[a,b]$ to stand for the set of piecewise continuous functions on the interval $[a,b]$; supplied with the inner product and norm

$$\langle f,\, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx; \;\; \|f\| = \sqrt{\int_a^b |f(x)|^2\,dx}.$$

This inner product and norm is ordinarily associated with the larger space $L^2[a,b]$, consisting of square Lebesgue integrable functions on the interval $[a,b]$, but we don't want to get into the details of that at the present. Since for any two functions $f$ and $g$ in $PC_2[a,b]$ it is easily verified that a linear combination $c_1\,f + c_2\,g$ continues to belong to that set, it follows that $PC_2[a,b]$ is a linear vector space; supplied with the inner product and norm we have just indicated it becomes an *inner product space*. Unlike the larger space $L^2[a,b]$ it is not *complete*; sequences of functions $f_k(x) \in PC_2[a,b]$ which have the *Cauchy property* $\lim_{j,k\to\infty}\|f_j - f_k\| = 0$ do not necessarily converge to a limit in $PC_2[a,b]$; the limit does exist in the larger, complete space $L^[a,b]$ (which, being a complete inner product space is a *Hilbert space*) but it may not be piecewise continuous.

A linear transformation on $PC_2[a,b]$ is simply a linear function from that space into itself. It is very easy to cite examples. One may take a fixed function, $h(x)$, piecewise continuous on $[a,b]$ and define

$$H : PC_2[a,b] \to PC_2[a,b]; \quad H(f)(x) = h(x)f(x).$$

It is straightforward to see that for each $f \in PC_2[a,b]$ this "operation" $H$ gives another function in that space and it is easy to verify the linearity of the operation since $h(x)(c_1 f(x) + c_2 g(x)) = c_1 h(x)f(x) + c_2 h(x)g(x)$. Another example of a linear transformation on $PC_2[a,b]$ is given by

$$K(f)(x) = \int_a^b k(x,y)f(y)\,dy, \;\; x \in [a,b],$$

where the "kernel" $k(x,y)$ satisfies the conditions indicated in the section on linear transforms (in particular, $k(x,y)$ continuous on $[a,b] \times [a,b]$ would suffice).

There is little precision in the terminology for these linear transformations; the terms *linear function*, *linear transformation*, *linear operator* are all used with no clear distinction between them. In these notes we will reserve the word "operator" for use in the term *linear differential operator* (the term linear differential operator is standard but "operator" by itself is often used in other contexts in the general literature). For our purposes here an $n$ *-th order linear differential operator* on $[a,b]$ takes the form

$$\mathcal{D} = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{d}{dx} + a_n(x),$$

where each $a_{n-k}(x)$ is a piecewise continuous function on $[a,b]$ (we often add further requirements). The "operation" of $\mathcal{D}$ on a piecewise $n$ times continuously differentiable function $y(x)$ defined for $x \in [a,b]$ is simply

$$\mathcal{D}(y)(x) = a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y(x).$$

Most, but not all, of the examples we will cite correspond to $a_k(x) \equiv a_k$, constant. As $x$ is allowed to vary in $[a,b]$ this process gives another function, $\mathcal{D}(y)(x)$ and it is a simple matter to verify that the process is a linear one.

It is clear that the operator $\mathcal{D}$ cannot be applied to an arbitrary function $y(x) \in PC_2[a,b]$ because, in general, functions in that space may not have the required derivatives. Unlike linear transformations on finite dimensional spaces, we here encounter a transformation, associated with the operator $\mathcal{D}$, for which we have to specify a *domain subspace*; a subspace of the basic space in which we are operating, in this case $PC_2[a,b] = PC_2^0[a,b]$, consisting of those elements of the basic space to which the operation is to be applied. For the differential operator $\mathcal{D}$ as just defined, one begins by restricting the operation of $\mathcal{D}$ to the subspace $PC_2^n[a,b]$, which consists of functions $y(x)$ for which the $n$-th derivative $y^{(n)}(x)$ is defined and piecewise continuous on $[a,b]$. This implies that $y(x)$ and the lower order derivatives $y^{(k)}(x)$, $k = 1,2,...,n-1$ are all continuous and, in fact, $y^{(k)}(x)$ must have derivatives up to order $n-k$.

In addition, the domain subspace for $\mathcal{D}$, which we will call $\mathcal{S}$, needs to be further restricted by imposing $n$ boundary conditions at the endpoints, $a$ and $b$, of the interval in question. In many cases $n = 2m$ is even and $m$ boundary conditions are applied at each of these endpoints; exactly which boundary conditions depends on the specific operator and will become clearer as we discuss some of the examples to follow.

Our main interest lies in *eigenvalues* and *eigenfunctions* of differential operators. A function $\phi(x)$, not equivalent to the zero function, in the domain subspace $\mathcal{S}$ of the differential operator $\mathcal{D}$ is an *eigenfunction* of $\mathcal{D}$ corresponding to the (possibly complex) *eigenvalue* $\lambda$ of $\mathcal{D}$ if

$$(\mathcal{D}\phi)(x) - \lambda\,\phi(x) \equiv 0, \;\; x \in [a,b].$$

This means that $\phi(x)$ satisfies, for $x \in [a,b]$, the differential equation

$$a_0(x)\,\frac{d^n\phi}{dx^n} + a_1(x)\,\frac{d^{n-1}\phi}{dx^{n-1}} + \ldots + a_{n-1}\,\frac{d\phi}{dx} + a_n(x)\,\phi(x) - \lambda\,\phi(x) \equiv 0.$$

If we assume the eigenvalue $\lambda$ is known, the general solution of this differential equation will involve $n$ arbitrary constants. The purpose of the boundary conditions completing the definition of $\mathcal{S}$, which we described earlier, is to determine these arbitrary constants. In general this specification of boundary conditions cannot be done in an entirely arbitrary way; it has to be done with due regard to the structure of the operator $\mathcal{D}$.

**Definition 1** *If $\mathcal{D}^*$ is a differential operator with domain subspace (we will normally just say "domain" from this point on) $\mathcal{S}^*$ and for all $y(x) \in \mathcal{S}$ and all $z(x) \in \mathcal{S}^*$ we have*

$$\langle \mathcal{D}y,\, z\rangle = \langle y,\, \mathcal{D}^* z\rangle\,,$$

*then $\mathcal{D}^*$ is the* adjoint operator *for $\mathcal{D}$. If $\mathcal{D} = \mathcal{D}^*$ and $\mathcal{S} = \mathcal{S}^*$ we say that $\mathcal{D}$ is* self-adjoint.

**Remark** Strictly speaking we should say that $\mathcal{D}^*$ is *an* adjoint operator for $\mathcal{D}$ but it turns out that in virtually all cases of practical interest the adjoint operator $\mathcal{D}^*$ is unique if the domains $\mathcal{S}$ and $\mathcal{S}^*$ are chosen correctly.

**Example 1** Consider the differential operator defined by

$$(\mathcal{D}y)(x) = y'(x),\;\; x \in [0,1];\;\; \mathcal{S} = \left\{y(x) \in PC_2^1[a,b]\; |y(0) = 0\right\}.$$

For $z(x)$ also in $PC_2^1[0,1]$ we compute

$$\langle \mathcal{D}y,\, z\rangle = \int_0^1 y'(x)\overline{z(x)}\,dx = y(x)\overline{z(x)}|_0^1 - \int_0^1 y(x)\overline{z'(x)}\,dx$$

$$= y(1)\overline{z(1)} - \int_0^1 y(x)\overline{z'(x)}\,dx.$$

Clearly, then, if we take $(\mathcal{D}^* z)(x) = -z'(x)$ and take $\mathcal{S}^*$ to consist of functions $z(x) \in PC_2^1[0,1]$ such that $z(1) = 0$ we will have

$$\langle \mathcal{D}y,\, z\rangle = \int_0^1 y'(x)\overline{z(x)}\,dx = -\int_0^1 y(x)\overline{z'(x)}\,dx = \langle y,\, \mathcal{D}^* z\rangle$$

and we see that $\mathcal{D}^*$ as defined on $\mathcal{S}^*$ is the adjoint operator for $\mathcal{D}$.

**Example 2** Let $\mathcal{D}$ be the (first order) differentiation operator of Example 1 but, in specifying $\mathcal{S}$, we replace the boundary condition used there by the periodic condition $y(0) = y(1)$. Essentially the same calculations as carried out in Example 1 then show that $\mathcal{D}^* = -\mathcal{D}$ with $\mathcal{S}^* = \mathcal{S}$. (In this situation, where the adjoint of $\mathcal{D}$ is $-\mathcal{D}$ with the same domain, we say that $\mathcal{D}$ is *skew-adjoint* or *anti-hermitian.*) The eigenvalue - eigenfunction equation is

$$\frac{d\phi}{dx} = \lambda\,\phi$$

for which the general solution is $\phi(x,\lambda) = c\,e^{\lambda x}$, $c$ being an arbitrary constant. Using the periodic boundary condition which is part of the definition of $\mathcal{S}^* = \mathcal{S}$ we have

$$c\,e^{\lambda\cdot 0} = c = e^{\lambda\cdot 1} = c\,e^{\lambda}.$$

This is satisfied for $c \neq 0$ just in case $\lambda = 2k\pi i$ for some integer $k$, $-\infty < k < \infty$. These eigenvalues are clearly complex, as are the eigenfunctions, which we may take to be $\phi_k(x) = e^{2k\pi i x}$, $-\infty < k < \infty$.

**Example 3** We consider the constant coefficient, second order, differential operator

$$(\mathcal{D}\,y)\,(x) = \frac{\partial^2 y}{\partial x^2} - c\,y(x), \quad c \text{ real},$$

defined on the domain

$$\mathcal{S} = \left\{ y(x) \in PC_2^2\left([0,L]\right) \left| a_0\, y(0) + b_0\, y^{'}(0) = 0,\ a_L\, y(L) + b_L\, y^{'}(L) = 0 \right. \right\}$$

with $a_0$, $b_0$, $a_L$, $b_L$ all real, $a_0$, $b_0$ not both zero, $a_L$, $b_L$ not both zero.

We claim that $\mathcal{D}$ is self-adjoint. This assertion rests on the calculation, for $y,\ z\ \in\ \mathcal{S}$,

$$\langle \mathcal{D}y,\, z\rangle \;=\; \int_0^L \left(y^{''}(x) - c\, y(x)\right) \overline{z(x)}\, dx$$

$$= y^{'}(x)\overline{z(x)}\Big|_0^L \;-\; \int_0^L \left(y^{'}(x)\overline{z^{'}(x)} \;+\; c\, y(x)\overline{z(x)}\right) dx$$

$$= \left(y^{'}(x)\overline{z(x)} \;-\; y(x)\overline{z^{'}(x)}\right)\Big|_0^L \;+\; \int_0^L y(x)\overline{\left(z^{''}(x) \;-\; c\, z(x)\right)}\, dx$$

$$= \left(y^{'}(x)\overline{z(x)} \;-\; y(x)\overline{z^{'}(x)}\right)\Big|_0^L \;+\; \langle y,\, \mathcal{D}z\rangle\,.$$

Clearly we have the self-adjointness property just in case

$$\left(y^{'}(x)\overline{z(x)} \;-\; y(x)\overline{z^{'}(x)}\right)\Big|_0^L \;=\; 0.$$

Suppose, first of all, that $a_L \neq 0$. Then, still for $y,\ z\ \in\ \mathcal{S}$, we compute

$$y^{'}(L)\overline{z(L)} \;-\; y(L)\overline{z^{'}(L)} \;=\; y^{'}(L)\left(\overline{z(L)} + \frac{b_L}{a_L}\overline{z^{'}(L)}\right)$$

$$-\;\left(\frac{b_L}{a_L}y^{'}(L) + y(L)\right)\overline{z^{'}(L)} \;=\; 0.$$

A similar result is obtained at $x = 0$ if $a_0 \neq 0$. If $a_L = 0$ then $b_L \neq 0$ and the boundary condition at $x = L$ gives $y^{'}(L) = 0,\ z^{'}(L) = 0$, in which case we again have $y^{'}(L)\overline{z(L)} - y(L)\overline{z^{'}(L)} = 0$. A similar result is obtained at $x = 0$ if $a_0 = 0$. The self-adjointness of $\mathcal{D}$ is proved.

Our final example for this section shows how we deal with a second order differential operator with coefficients depending on $x$.

**Example 4** Let $\mathcal{D}$ be the differential operator defined by

$$\mathcal{D}y(x) = c_0(x)\,\frac{d^2y}{dx^2} + c_1(x)\,\frac{dy}{dx} + c_2(x)\,y(x),$$

where the real valued functions $c_k(x) \in PC^{2-k}$, $k = 0,1,2$. We will assume $c_0(x) \neq 0$, $x \in [a,b]$. We take the domain of $\mathcal{D}$ to be the subspace of $PC_2[a,b]$ consisting of functions $y(x)$ which also lie in $PC^2[a,b]$ and satisfy boundary conditions, with real coefficients,

$$d_a\,y^{'}(a) + e_a\,y(a) = 0, \;\; d_b\,y^{'}(b) + e_b\,y(b) = 0.$$

We assume that at least one of $d_a$, $e_a$ and at least one of $d_b$, $e_b$ are different from zero. Assuming that $z(x) \in PC^2[a,b]$, we use integration by parts to compute

$$\langle \mathcal{D}y,\, z\rangle = \int_a^b \left(c_0(x)\,\frac{d^2y}{dx^2} + c_1(x)\,\frac{dy}{dx} + c_2(x)\,y(x)\right)\overline{z(x)}\,dx$$

$$= \left(c_0(x)\,y^{'}(x)\overline{z(x)} - y(x)\overline{\left(c_0(x)z^{'}(x) + \left(c_1(x) - c_0^{'}(x)\right)z(x)\right)}\right)\Big|_a^b +$$

$$\int_a^b y(x)\overline{\left(c_0(x)\,z''(x) + \left(2c_0^{'}(x) - c_1(x)\right)z^{'}(x) + \left(c_0^{''}(x) - c_1^{'}(x) + c_2(x)\right)z(x)\right)}\,dx.$$

The last integral shows that the adjoint differential operator should be

$$\mathcal{D}^*z(x) = c_0(x)\,\frac{d^2z}{dx^2}(x) + \left(2c_0^{'}(x) - c_1(x)\right)\frac{dz}{dx}(x) + \left(c_0^{''}(x) - c_1^{'}(x) + c_2(x)\right)z(x).$$

If $d_b \neq 0$ the boundary terms at $x = b$ can be rewritten as

$$c_0(b)\left(y^{'}(b) + \frac{e_b}{d_b}y(b)\right)\overline{z(b)} - \frac{c_0(b)e_b}{d_b}y(b)\overline{z(b)}$$
$$- y(b)\overline{\left(c_0(b)z^{'}(b) + \left(c_1(b) - c_0^{'}(b)\right)z(b)\right)}.$$

Applying the boundary condition satisfied by $y(x)$ at $x = b$, the boundary term at $x = b$ vanishes just in case

$$0 = -\frac{c_0(b)e_b}{d_b}y(b)\overline{z(b)} - y(b)\overline{\left(c_0(b)z^{'}(b) + \left(c_1(b) - c_0^{'}(b)\right)z(b)\right)}$$

$$= -y(b)\overline{\left(c_0(b)z'(b) + \left(c_1(b) - c_0'(b) + \frac{c_0(b)e_b}{d_b}\right) z(b)\right)},$$

which is true if $z(x)$ satisfies the boundary condition

$$c_0(b)z^{'}(b) + \left(c_1(b) - c_0^{'}(b) + \frac{c_0(b)e_b}{d_b}\right) z(b) = 0.$$

If we assume $d_a \neq 0$ we similarly obtain

$$c_0(a)z^{'}(a) + \left(c_1(a) - c_0^{'}(a) + \frac{c_0(a)e_a}{d_a}\right) z(a) = 0.$$

The adjoint domain $\mathcal{D}^*$ consists of functions in $PC^2[a, b]$ satisfying these boundary conditions. The cases wherein $d_a = 0$ and/or $d_b = 0$ (and hence $e_a \neq 0$ and/or $e_b \neq 0$ are left as exercises.

**QuickCheck Exercises**

**1.** Find the adjoint operator for $\mathcal{D}y(x) = \frac{d}{dx}\left((1+x)\frac{dy}{dx}\right)$, the domain of $\mathcal{D}$ consisting of functions in $PC^2[0, 1]$ for which $y(0) = y(1) = 0$.

**2.** Find the adjoint operator for $\mathcal{D}y(x) = y^{'''}(x)$, the domain consisting of functions in $PC^3[0, 1]$ for which $y(0) = y^{''}(0) = 0,\ y^{'}(1) = 0$.

**3.** Show that $-c$ is an eigenvalue of $\mathcal{D}$ as defined in Example 3 if $a_0 = a_L = 0$.

**4.** Compute the eigenvalues of $\mathcal{D}$ in Example 3 if $b_0 = b_L = 0$. What are the corresponding eigenfunctions?

**5.** How are the boundary conditions characterizing the domain of $\mathcal{D}^*$ modified in the case of the operator $\mathcal{D}$ of Example 4 if one or both of $d_a$, $d_b$ is equal to zero?