First Passage Time Problem: A Fokker-Planck Approach

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This chapter reviews the first passage time problem for one-dimensional stochastic processes and presents closed-form solutions for the underlying distribution function. Using the Fokker-Planck approach the case of Brownian motion with drift is solved in the diffusive limit. This technique is then generalized to obtain exact solutions in the case of anomalous diffusion, corresponding to a continuous time random walk.

1 Introduction

Let $X(t)$ be an one dimensional stochastic process. Consider the time when the process first crosses a threshold. This time T is obviously a random variable and is called the First Passage Time (FPT) [1]. An important problem is to find the probability density function (pdf) of T. This is known as the first passage time problem and has a long history [2–9].

The FPT problem finds applications in many areas of science and engineering [10–13]. A sampling of these applications is listed below:

- probability theory (study of Wiener process, fractional Brownian motion etc.)
- statistical physics (study of anomalous diffusion)
- neuroscience (analysis of neuron firing models)
- civil and mechanical engineering (analysis of structural failure)
- chemical physics (study of noise assisted potential barrier crossings)
- hydrology (optimal design of dams)
- financial mathematics (analysis of circuit breakers)
- imaging (study of image blurring due to hand jitter)

We now consider one application of the FPT problem in some detail. Consider a neuron which receives an input current $x(t)$. We wish to study the output spike train generated by the neuron. For simplicity, we restrict ourselves to the simplest model of neuron firing – the perfect or nonleaky Integrate and Fire (IF) model first introduced by Lapicque [12]. In this model, the membrane potential $V(t)$ of the neuron is obtained by integrating the input current. When $V(t)$ reaches a threshold value V_0, an action potential is generated and $V(t)$ is reset instantaneously to its resting value (assumed

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here to be zero). This discrete generation of the action potential leads to an output spike train. The time interval between two successive spikes is called the interspike interval (ISI). Since these intervals are known to codify information, it is important to characterize their distribution.

For most neurons, the output ISI is found to be a random variable [14] and hence we need to find its pdf. To find this, we first need a mechanism to generate stochastic spike trains. They can be generated by making the input current $x(t)$ a stochastic process. Therefore $V(t)$ which is obtained by integrating this input current is now a random walk. Assume that the neuron has generated an output spike at time t_i. It generates the next spike after a time interval T when the random walk $\{V(t), t \geq t_i\}$ starting from $V(t) = 0$ at $t = t_i$ first crosses the threshold value V_0. Therefore it is clear that T is a first passage time and a random variable. Its pdf is nothing but the FPT distribution. Thus the pdf of the output ISI's is nothing but the distribution of FPT's obtained as above.

Given the ubiquitous role played by the FPT distribution in various applications, it is natural to derive the FPT distribution for different types of stochastic processes. In particular, we will first investigate Brownian motion with drift in the diffusive limit. Then we consider continuous time random walks which give rise to anomalous diffusions where mean squared displacement varies as t^γ, $\gamma \neq 1$ for large t. We consider both subdiffusive processes ($0 < \gamma < 1$) and superdiffusive processes ($1 < \gamma < 2$). We employ the Fokker-Planck approach [10] (and its generalizations) in this review. It should be noted that in some cases, probabilistic arguments can be used to derive the FPT distribution in a much simpler fashion. But this approach is not always applicable. For pedagogical reasons and for the sake of consistency we will use the Fokker-Planck approach even in cases where simpler probabilistic derivations are possible.

2 FPT Distribution for Brownian Motion

We first study the first passage time problem for the simplest case of Brownian motion (Wiener process). This will illustrate the manner in which the Fokker-Planck equation enters the picture and how it facilitates the solution of the FPT problem.

We start with a simple random walk. A step Y is taken for every τ units of time with the following probabilities:

$$Y = \begin{cases} l, & \text{with probability } 0.5; \\ -l, & \text{with probability } 0.5. \end{cases} \tag{1}$$

Consider a new random variable X_n defined as

$$X_n = \sum_{i=1}^{n} Y_i, \tag{2}$$

with $X_0 = 0$. It is obvious that X_n gives the position of the random walker at time $t = n\tau$. Assume that the steps Y_i are mutually independent. Then the mean squared displacement $\langle X_n^2 \rangle$ is given by

$$\langle X_n^2 \rangle = \sum_{i=1}^{n} \langle Y_i^2 \rangle + \sum_{i \neq j} \langle Y_i Y_j \rangle. \tag{3}$$

The second term is zero since the Y_i's are mutually independent. Hence

$$\langle X_n^2 \rangle = \sum_{i=1}^{n} l^2 = nl^2. \tag{4}$$

The above equation can be written as

$$\langle X_n^2 \rangle = nl^2 = \frac{l^2}{\tau}(n\tau) = \frac{l^2}{\tau} t. \tag{5}$$

The random walk described above has discontinuous jumps. It is often easier to deal with continuous quantities. We can make the sample paths continuous by taking the so-called diffusive limit: $\tau \to 0$, $l \to 0$ such that

$$\frac{l^2}{\tau} = 2D. \tag{6}$$

Here D is called the diffusion constant. In this diffusive limit, we have

$$\langle X^2(t) \rangle = 2Dt. \tag{7}$$

The reader will recognize this as the equation that characterizes Brownian motion (or Wiener process). More formally, $X(t)$ is a Wiener process if

- $X(t)$ is a Gaussian process,
- $X(0) = 0$, $\langle X(t) \rangle = 0$,
- $\langle (X(t) - X(s))^2 \rangle = 2D|t - s| \quad \forall t, s$.

Further, $X_\mu(t)$ is a Wiener process (or Brownian motion) with drift μ if

$$X_\mu(t) = \mu t + X(t). \tag{8}$$

Brownian motion is an example of a regular or ordinary diffusion. All processes that belong to this class are characterized by the following relation:

$$\langle X^2(t) \rangle \sim t \text{ for large } t. \tag{9}$$

Thus in regular diffusive processes other than Brownian motion, the proportionality of the mean squared displacement with time is satisfied only for large times.

In the analysis above, we have directly dealt with the stochastic process X_n (or $X(t)$ in the diffusive limit). It turns out that it is easier to deal with the pdf for the stochastic process than the process itself for many applications (including the calculation of the FPT distribution). As we show below, the pdf for the simple random walk described above satisfies a partial differential equation known as the Fokker-Planck equation in the diffusive limit.

Let $W(ml,(n+1)\tau)$ denote the probability that the random walker is at position ml at time $(n+1)\tau$ (i.e. after $n+1$ time steps). But the walker could have reached this position only by either being at position $(m-1)l$ at time $n\tau$ and jumping right (with probability 0.5) or being at position $(m+1)l$ at time $n\tau$ and jumping left (again with probability 0.5). Hence we have from simple probability considerations:

$$W(ml,(n+1)\tau) = \frac{1}{2}[W((m+1)l,n\tau) + W((m-1)l,n\tau)]. \tag{10}$$

If τ is small, we have

$$W(ml,(n+1)\tau) \approx W(ml,t) + \tau\frac{\partial}{\partial t}W(ml,t), \tag{11}$$

where $t = n\tau$. Hence, we get (for τ small)

$$\tau\frac{\partial}{\partial t}W(ml,t) = \frac{1}{2}[W((m+1)l,t) + W((m-1)l,t) - 2W(ml,t)]. \tag{12}$$

But for l small, we have

$$[W((m+1)l,t) + W((m-1)l,t) - 2W(ml,t)] \approx l^2\frac{\partial^2}{\partial x^2}W(x,t), \tag{13}$$

where $x = ml$. Thus, we have

$$\frac{\partial}{\partial t}W(x,t) \approx \frac{l^2}{2\tau}\frac{\partial^2}{\partial x^2}W(x,t). \tag{14}$$

Taking the diffusive limit described earlier, we finally get

$$\frac{\partial}{\partial t}W(x,t) = D\frac{\partial^2}{\partial x^2}W(x,t). \tag{15}$$

This is nothing but the diffusion equation for $W(x,t)$ and is known as the Fokker-Planck Equation (FPE). The natural boundary conditions for this equation are that $W(x,t) = 0$ at $x = \pm\infty$. Since the random walker is assumed to start at the origin, we also have the initial condition $W(x,0) = \delta(x)$.

We can easily verify that the above Fokker-Planck equation describes the evolution of the pdf for Brownian motion. First we compute $\langle X\rangle$. Multiplying the FPE given in Eq. (15) by x and integrating from $x=-\infty$ to $x=\infty$ we get

$$\frac{\partial}{\partial t}\langle X\rangle = -D\int_{-\infty}^{\infty} dx\,\frac{\partial W(x,t)}{\partial x}, \tag{16}$$

where we have performed integration by parts and assumed that $x\frac{\partial W(x,t)}{\partial x}=0$ at $x=\pm\infty$. Evaluating the remaining integral using the natural boundary conditions, we get

$$\frac{\partial}{\partial t}\langle X\rangle = 0. \tag{17}$$

This gives $\langle X\rangle = 0$ which is the expected result.

Next we compute $\langle X^2\rangle$. Multiplying the FPE in Eq. (15) by x^2 and integrating we get

$$\frac{\partial}{\partial t}\langle X^2\rangle = -2D\int_{-\infty}^{\infty} dx\, x\frac{\partial W(x,t)}{\partial x}, \tag{18}$$

where we have again used integration by parts and assumed that $x^2\frac{\partial W(x,t)}{\partial x} = 0$ at $x=\pm\infty$. Evaluating the remaining integral using the the boundary conditions and the normalization condition $\int_{-\infty}^{\infty} dx\, W(x,t)=1$ we obtain

$$\langle X^2\rangle = 2Dt. \tag{19}$$

This shows that the FPE in Eq. (15) does describe the usual Brownian motion.

For Brownian motion with drift, the corresponding FPE is:

$$\frac{\partial}{\partial t}W(x,t) = \mu\frac{\partial}{\partial x}W(x,t) + D\frac{\partial^2}{\partial x^2}W(x,t). \tag{20}$$

Next we obtain the FPT distribution for Brownian motion with drift starting from the Fokker-Planck equation. Consider a stochastic process $X(t)$ with $X(0)=0$. The first passage time (FPT) T to the point $X=a>0$ is defined as [1]

$$T = \inf\{t : X(t) = a\}. \tag{21}$$

We would like to obtain the probability density function for T for Brownian motion.

Since Brownian motion (in the diffusive limit) is described by Fokker-Planck equations, the problem of obtaining the FPT density function can be recast as a boundary value problem with absorbing boundaries [10]. In our case, to obtain the FPT density function, we first need to solve Eq. (20) with absorbing boundaries at $x=-\infty$ and $x=a$, where a is the predetermined level of crossing, with the initial condition $W(x,0)=\delta(x)$ [10]. An equivalent formulation, due to symmetry, is to solve Eq. (20) with the following boundary and initial conditions:

$$W(0,t)=0, \quad W(\infty,t)=0, \quad W(x,0)=\delta(x-a), \tag{22}$$

where $x=a$ is the new starting point of the Brownian motion, containing the initial concentration of the distribution. The equivalence is easily seen by making the change of variables $x \to a - x$ in Eq. (20). This latter formulation makes the subsequent derivation less cumbersome.

Once we solve for $W(x,t)$, the first passage time density $f(t)$ can be determined as follows. From simple probability considerations, the probability $P(T > t)$ that the first passage time exceeds a given time t is nothing but

$$P(T > t) = \int_0^\infty dx\, W(x,t). \tag{23}$$

Therefore,

$$P(T \leq t) = 1 - \int_0^\infty dx\, W(x,t). \tag{24}$$

Hence the FPT density $f(t)$ is given by

$$f(t) = \frac{d}{dt} P(T \leq t) = -\frac{d}{dt} \int_0^\infty dx\, W(x,t). \tag{25}$$

First we solve for $W(x,t)$ using the given boundary and initial conditions. We solve the FPE using the method of separation of variables [15]. Let $W(x,t) = X(x)T(t)$. Substituting in Eq. (20) we obtain

$$X(x)\frac{dT(t)}{dt} = T(t)\left[\mu X'(x) + DX''(x)\right], \tag{26}$$

where the primes denote the derivatives with respect to x. Separating out the variables and introducing the separation constant λ we get

$$DX''(x) + \mu X'(x) = -\lambda X(x), \tag{27}$$

and

$$\frac{dT(t)}{dt} = -\lambda T(t). \tag{28}$$

First we solve the simple Eq. (28). We obtain

$$T(t) = \exp\left[-\lambda t\right]. \tag{29}$$

Next consider Eq. (27). The solution of this equation satisfying the boundary conditions is given by

$$X(x) = \exp[-\mu(x-a)/2D]\frac{\sin[x\sqrt{\lambda/D - \mu^2/4D^2}]}{2\sqrt{D\lambda - \mu^2/4}}, \quad \lambda \geq \mu^2/4D. \tag{30}$$

Thus we have a continuous spectrum for λ. Combining the solutions for $X(x)$ and $T(t)$, $W(x,t)$ is given by the following integral over λ:

$$W(x,t) = \frac{2}{\pi}\int_{\mu^2/4D}^\infty d\lambda A(\lambda)\exp[-\mu(x-a)/2D]\frac{\sin[x\sqrt{\lambda/D - \mu^2/4D^2}]}{2\sqrt{D\lambda - \mu^2/4}} \times \exp\left[-\lambda t\right]. \tag{31}$$

The coefficient $A(\lambda)$ is fixed by the initial condition ($W(x,0) = \delta(x-a)$) and we get

$$W(x,t) = \frac{2}{\pi}\int_{\mu^2/4D}^{\infty} d\lambda \exp[-\mu(x-a)/2D]\sin[a\sqrt{\lambda/D-\mu^2/4D^2}] \\ \times\frac{\sin[x\sqrt{\lambda/D-\mu^2/4D^2}]}{2\sqrt{D\lambda-\mu^2/4}}\exp\left[-\lambda t\right]. \tag{32}$$

Letting $\lambda' = \sqrt{\lambda/D-\mu^2/4D^2}$ we obtain

$$W(x,t) = \frac{2}{\pi}\int_{0}^{\infty} d\lambda' \sin\lambda' a \sin\lambda' x \exp[-\mu(x-a)/2D] \\ \times\exp\left[-(D\lambda'^2+\mu^2/4D)t\right]. \tag{33}$$

Using standard trigonometric identities and dropping the primes, the above equation can be rewritten as

$$W(x,t) = \frac{1}{\pi}\int_{0}^{\infty} d\lambda \exp[-\mu(x-a)/2D]\exp\left[-(D\lambda^2+\mu^2/4D)t\right] \\ \times[\cos\lambda(x-a)-\cos\lambda(x+a)]. \tag{34}$$

Taking the Laplace transform with respect to time we get

$$q(x,s) = \frac{1}{\pi}\int_{0}^{\infty} d\lambda \exp[-\mu(x-a)/2D]\frac{1}{s+\mu^2/4D+D\lambda^2} \\ \times[\cos\lambda(x-a)-\cos\lambda(x+a)], \tag{35}$$

where $q(x,s)$ is the Laplace transform of $W(x,t)$. Here we have used the fact that the Laplace transform of $\exp(-Bt)$ is $1/(s+B)$ [16].

We can now perform the integration over λ by using the following result [17]

$$\int_{0}^{\infty} d\lambda\frac{\cos\lambda x}{\alpha^2+\lambda^2} = \frac{\pi}{2\alpha}e^{-\alpha|x|}. \tag{36}$$

Thus we obtain

$$q(x,s) = \frac{1}{2\sqrt{D}}\frac{\exp[-\mu(x-a)/2D]}{\sqrt{s+\mu^2/4D}}\exp[-\sqrt{s+\mu^2/4D}\;|x-a|/\sqrt{D}] \\ - \frac{1}{2\sqrt{D}}\frac{\exp[-\mu(x-a)/2D]}{\sqrt{s+\mu^2/4D}}\exp[-\sqrt{s+\mu^2/4D}\;(x+a)/\sqrt{D}]. \tag{37}$$

To obtain the Laplace transform $F(s)$ of the FPT density function $f(t)$, we take the Laplace transform of Eq. (25) to get

$$F(s) = -s\int_{0}^{\infty} dx\, q(x,s) + \int_{0}^{\infty} dx\, W(x,0). \tag{38}$$

Here we have used the fact that Laplace transform of $dW(x,t)/dt$ is given by [17] $sq(x,s) - W(x,0)$. Since $W(x,0) = \delta(x-a)$, we obtain

$$F(s) = 1 - s \int_0^\infty dx\, q(x,s). \tag{39}$$

Substituting for $q(x,s)$ from Eq. (37), we get

$$\begin{aligned} F(s) = 1 &- \frac{1}{2\sqrt{D}} \frac{1}{\sqrt{s+\mu^2/4D}} \\ &\times \int_0^\infty dx\ \exp[-\mu(x-a)/2D] \exp[-\sqrt{s+\mu^2/4D}\ |x-a|/\sqrt{D}] \\ &+ \frac{1}{2\sqrt{D}} \frac{1}{\sqrt{s+\mu^2/4D}} \\ &\times \int_0^\infty dx\ \exp[-\mu(x-a)/2D] \exp[-\sqrt{s+\mu^2/4D}\ (x+a)/\sqrt{D}]. \end{aligned} \tag{40}$$

Upon evaluating the integrals we obtain

$$F(s) = \exp\left[-a\left(-\mu + \sqrt{\mu^2 + 4Ds}\right)/2D\right]. \tag{41}$$

Performing the inverse Laplace transform [17] we finally get the desired FPT distribution for a Brownian motion with drift:

$$f(t) = \frac{a}{\sqrt{4\pi D t^3}} \exp\left[-\frac{(a-\mu t)^2}{4Dt}\right], \quad a > 0, \quad t > 0. \tag{42}$$

This is the famous inverse Gaussian distribution. This distribution when $\mu = 0$ was first derived by Bachelier [2] and the $\mu \neq 0$ case was first derived by Schrödinger and Smoluchowski [3,4].

3 FPT Distribution for Continuous Time Random Walks

Consider a one dimensional continuous time random walk [18,19] described by the following Langevin equation:

$$\frac{dX}{dt} = \sum_{i=1}^\infty Y_i\ \delta(t - t_i). \tag{43}$$

Here the random walker starts at $x = 0$ at time $t_0 = 0$. Subsequently, the random walker waits at a given location x_i for time $t_i - t_{i-1}$ before taking a jump Y_i which could depend on the waiting time. The waiting time $u > 0$ and the jump size y ($-\infty < y < \infty$) are drawn from the joint probability density function $\phi(y,u)$. The waiting time distribution $\psi(u)$ is given by

$$\psi(u) = \int_{-\infty}^\infty dy \phi(y,u). \tag{44}$$

The process is non Markovian if $\psi(u)$ is a non-exponential distribution since the probability for the next jump to occur depends on how long the the random walker has been waiting since the previous jump. But CTRW is non Markovian in a special way since it does not depend on the history of the process prior to the previous jump.

We now relate [18] the probability distribution $W(x,t)$ for the CTRW to $\phi(y,u)$ and $\psi(u)$. The probability density $\eta(x,t)$ of the random walker just arriving at x in the time interval t to $t+dt$ is

$$\eta(x,t) = \int_{-\infty}^{\infty} dx' \int_0^t \eta(x',\tau)\phi(x-x',t-\tau)d\tau + \delta(t)\delta(x). \tag{45}$$

Thus $\eta(x,t)$ is obtained by summing over all x' and τ the probability $\eta(x',\tau)$ of being exactly at x' at time τ multiplied by the probability $\phi(x-x',t-\tau)$ of jumping a distance $x-x'$ in time $t-\tau$ to exactly arrive at x at time t. The second term $\delta(x)\delta(t)$ is just the initial condition. At $x=0$, $t=0$, we have $\eta(0,0)=1$ i.e. the walker starts at $x=0$ at time $t=0$ with probability 1.

The probability $W(x,t)$ of the random walker being at x at time t is obtained by summing over all τ' the probability $\eta(x,t-\tau')$ of exactly arriving at x at time $t-\tau'$ and then waiting without jumping up to time t. The probability of not jumping from time $t-\tau'$ to t is given by $\xi(\tau')$ where $\xi(t)$ is the so-called survival probability:

$$\xi(t) = 1 - \int_0^t \psi(\tau)d\tau. \tag{46}$$

Hence we have

$$W(x,t) = \int_0^t \eta(x,t-\tau')\xi(\tau')d\tau'. \tag{47}$$

From the equation for $\eta(x,t)$ we have

$$\begin{aligned}\eta(x,t-\tau') = & \int_{-\infty}^{\infty} dx' \int_0^{t-\tau'} \eta(x',\tau)\phi(x-x',t-\tau'-\tau)d\tau \\ & +\delta(t-\tau')\delta(x).\end{aligned} \tag{48}$$

Substituting this expression in $W(x,t)$ we get

$$\begin{aligned}W(x,t) = & \int_{-\infty}^{\infty} dx' \int_0^t d\tau' \int_0^{t-\tau'} \eta(x',\tau)\phi(x-x',t-\tau'-\tau)\xi(\tau')d\tau \\ & +\xi(t)\delta(x).\end{aligned} \tag{49}$$

Letting $\tau'' = \tau + \tau'$ we obtain

$$\begin{aligned}W(x,t) = & \int_{-\infty}^{\infty} dx' \int_0^t d\tau' \int_{\tau'}^t \eta(x',\tau''-\tau')\phi(x-x',t-\tau'')\xi(\tau')d\tau'' \\ & +\xi(t)\delta(x).\end{aligned} \tag{50}$$

Changing the order of integration, the above expression can be rewritten as

$$W(x,t) = \int_{-\infty}^{\infty} dx' \int_0^t d\tau'' \left[\int_0^{\tau''} \eta(x',\tau''-\tau')\xi(\tau')d\tau'\right] \phi(x-x',t-\tau'')$$
$$+\xi(t)\delta(x). \tag{51}$$

But the term within brackets is $W(x',\tau'')$. Thus

$$W(x,t) = \int_{-\infty}^{\infty} dx' \int_0^t d\tau'' W(x',\tau'')\phi(x-x',t-\tau'') + \xi(t)\delta(x). \tag{52}$$

Taking the Fourier-Laplace transform and denoting the transforms of $W(x,t)$ and $\phi(y,u)$ by $\tilde{W}(k,s)$ and $\tilde{\phi}(k,s)$ respectively (where k is the Fourier transform of the space variable and s the Laplace transform of the time variable) we get [18]

$$\tilde{W}(k,s) = \tilde{W}(k,s)\tilde{\phi}(k,s) + \tilde{\xi}(s). \tag{53}$$

Here $\tilde{\xi}(s)$ is the Laplace transform of $\xi(t)$. It can be related to the Laplace transform $\tilde{\psi}(s)$ of the waiting time distribution $\psi(u)$ as follows [cf. Eq. (46)]:

$$\tilde{\xi}(s) = \frac{1}{s} - \frac{\tilde{\psi}(s)}{s}. \tag{54}$$

Here we have used the fact that the Laplace transforms of 1 and $\int_0^t \psi(u)du$ are $1/s$ and $\tilde{\psi}(s)/s$ respectively. Substituting the above expression in the equation for $\tilde{W}(k,s)$ and solving for $\tilde{W}(k,s)$ we finally obtain:

$$\tilde{W}(k,s) = \frac{1}{s}\frac{1-\tilde{\psi}(s)}{1-\tilde{\phi}(k,s)}. \tag{55}$$

It can be further shown that, depending on the specific form of $\phi(y,u)$, the CTRW can produce anomalous diffusion as well as ordinary diffusion [18,20]. For example, consider

$$\phi(y,u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-y^2/2\sigma^2] \frac{(\alpha-1)/\tau}{(1+u/\tau)^{\alpha}}, \tag{56}$$

where y and u are decoupled with y being a Gaussian variable with zero mean. Here the parameters σ and τ can be thought of as giving characteristic step size and waiting time for the random walk. For $1 < \alpha < 2$, the corresponding CTRW gives subdiffusion where $\langle X^2(t)\rangle \sim t^{\gamma}$ for large t with $\gamma = \alpha - 1$ between 0 and 1 [38]. We call this Lévy type anomalous diffusion since the waiting time distribution $\psi(u)$ given by [cf. Eqs. (56) and (44)]

$$\psi(u) = \frac{(\alpha-1)/\tau}{(1+u/\tau)^{\alpha}} \tag{57}$$

is a Lévy type distribution [38]. For $\alpha \geq 2$, one gets ordinary diffusion with $\gamma = 1$. Similarly, for a CTRW characterized by [18]

$$\phi(y,u) = \frac{1}{2}\delta(u/\tau - |y|/\sigma)\frac{(\beta-1)/\tau}{(1+u/\tau)^{\beta}}, \tag{58}$$

where $2 < \beta < 3$ and $\delta(\cdot)$ is the Dirac delta function, we obtain a Lévy type superdiffusive process with $\gamma = \beta - 1$.

Above we had described CTRW processes that gives rise to Lévy type anomalous diffusion. However, it is difficult to derive analytical results directly from the process. As mentioned earlier, it is more convenient to work in the general framework of Fokker-Planck equations [10]. One can go from the CTRW process to a fractional Fokker-Planck equation (FFPE) [22] by taking the generalized diffusive limit. This limit is analogous to the regular diffusive limit that we considered earlier to derive the regular Fokker-Planck equation from a random walk with $\langle X^2(t)\rangle \sim t$ for large t. In the regular diffusive limit, we took $\sigma, \tau \to 0$ such that σ^2/τ is maintained a constant. For a CTRW where $\langle X^2(t)\rangle \sim t^{\gamma}$ $(0 < \gamma < 2)$, the generalized diffusive limit is obtained by taking the limit $\sigma, \tau \to 0$ such that σ^2/τ^{γ} is maintained a constant.

Thus to obtain a FFPE from the CTRW process, we need to take the limit $\sigma, \tau \to 0$. We will take this limit for the Fourier-Laplace transform $\tilde{W}(k,s)$ of $W(x,t)$ and then invert the transform to obtain the FFPE. First consider the subdiffusive CTRW characterized by Eq. (56). The Laplace transform $\tilde{\psi}(s)$ of $\psi(u)$ given in Eq. (57) is [17]

$$\tilde{\psi}(s) = (\alpha-1)(\tau s)^{\alpha-1}\Gamma(1-\alpha,\tau s)e^{\tau s}. \tag{59}$$

The Fourier-Laplace transform $\tilde{\phi}(k,s)$ of $\phi(y,u)$ in Eq. (56) is given by [17]

$$\tilde{\phi}(k,s) = \exp(-\sigma^2 k^2/2)\tilde{\psi}(s). \tag{60}$$

To obtain an expression for $W(k,s)$ in the limit $\tau \to 0$, we first consider $\tilde{\psi}(s)$. We have [23]

$$\Gamma(1-\alpha,\tau s) = \Gamma(1-\alpha) - \sum_{n=0}^{\infty}\frac{(-1)^n(\tau s)^{1-\alpha+n}}{n!(1-\alpha+n)}. \tag{61}$$

As $\tau \to 0$,

$$\Gamma(1-\alpha,\tau s) \approx -\frac{\Gamma(2-\alpha)}{\alpha-1} + \frac{(\tau s)^{1-\alpha}}{\alpha-1} + \frac{(\tau s)^{2-\alpha}}{2-\alpha}. \tag{62}$$

Therefore $\tilde{\psi}(s)$ as $\tau \to 0$ is given by

$$\tilde{\psi}(s) \approx 1 - \Gamma(2-\alpha)(\tau s)^{\alpha-1}, \quad 1 < \alpha < 2, \tag{63}$$

$$\approx 1 - (2\alpha-3)\tau s/(\alpha-2), \quad \alpha > 2. \tag{64}$$

Substituting this in the Eq. (55) we have [cf. Eq. (60)]

$$\tilde{W}(k,s) \approx \frac{1}{s}\frac{\Gamma(1-\gamma)(\tau s)^\gamma}{1-[1-\Gamma(1-\gamma)(\tau s)^\gamma]\exp(-\sigma^2 k^2/2)}. \tag{65}$$

Here we have used $\gamma = \alpha - 1$ ($0 < \gamma < 1$) instead of α since it is the physically relevant quantity. Now we take the further limit $\sigma \to 0$ such that $\sigma^2/2\Gamma(1-\gamma)\tau^\gamma = K$ is a constant. K is called the generalized diffusion constant. We then obtain

$$\tilde{W}(k,s) = \frac{1}{s + Kk^2 s^{1-\gamma}}. \tag{66}$$

This can be rewritten as

$$\tilde{W}(k,s) - \frac{1}{s} = -Kk^2 s^{-\gamma}\tilde{W}(k,s). \tag{67}$$

To take the inverse Fourier-Laplace transform of the above equation, we need the inverse Laplace transform of $s^{-\gamma}\tilde{W}(k,s)$. This is given by the Riemann-Liouville fractional integral ${}_0D_t^{-\gamma}W(k,t)$ which is defined as [24,25]

$${}_0D_t^{-\gamma}W(k,t) = \frac{1}{\Gamma(\gamma)}\int_0^t dt'\ (t-t')^{\gamma-1}W(k,t'), \quad \gamma > 0. \tag{68}$$

This result enables us to take the inverse Fourier-Laplace transform of Eq. (67) giving [22]

$$W(x,t) - W(x,0) = K\ {}_0D_t^{-\gamma}\frac{\partial^2}{\partial x^2}W(x,t), \quad 0 < \gamma < 1. \tag{69}$$

Here we have incorporated the initial condition $W(x,0) = \delta(x)$. One obtains the same equation for a subdiffusive process but with a different K.

To obtain the FPT distribution for the above process, we follow a procedure identical to that we used for analyzing the regular Fokker-Planck equation in the previous section. We first solve for $W(x,t)$ (the boundary and initial conditions are the same as for the FPE case). Then we use Eq. (25) to obtain the FPT distribution.

Let $W(x,t) = X(x)T(t)$. Substituting in Eq. (69) we obtain

$$X(x)T(t) - X(x) = \left[\ {}_0D_t^{-\gamma}T(t)\right][KX''(x)], \tag{70}$$

where the primes denote the derivatives with respect to x. Separating out the variables and introducing the separation constant λ we get

$$KX''(x) = -\lambda X(x), \tag{71}$$

and

$$T(t) - 1 = -\lambda\ {}_0D_t^{-\gamma}T(t). \tag{72}$$

First we solve Eq. (72). Taking its Laplace transform we obtain

$$T(s) - \frac{1}{s} = \frac{\lambda}{s^\gamma} T(s). \tag{73}$$

Here we have used the fact that the Laplace transform of ${}_0D_t^{-\gamma}T(t)$ is given by $T(s)/s^\gamma$. Solving for $T(s)$ we get

$$T(s) = \frac{1}{s - \lambda s^{1-\gamma}}. \tag{74}$$

Taking the inverse Laplace transform [16] we finally obtain

$$T(t) = E_\gamma\left[-\lambda t^\gamma\right], \tag{75}$$

where $E_\gamma(z)$ is the Mittag-Leffler function [16] with the following power series expansion:

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\gamma n)}. \tag{76}$$

Note that $E_\gamma(z)$ reduces to the regular exponential function when $\gamma = 1$.

Next consider Eq. (71). The solution of this equation satisfying the boundary conditions is given by

$$X(x) = \frac{\sin[x\sqrt{\lambda/K}]}{2\sqrt{K\lambda}}, \quad \lambda \geq 0. \tag{77}$$

Thus we have a continuous spectrum for λ. Combining the solutions for $X(x)$ and $T(t)$, $W(x,t)$ is given by the following integral over λ:

$$W(x,t) = \frac{2}{\pi}\int_0^\infty d\lambda A(\lambda)\frac{\sin[x\sqrt{\lambda/K}]}{2\sqrt{K\lambda}} E_\gamma\left[-\lambda t^\gamma\right]. \tag{78}$$

The coefficient $A(\lambda)$ is fixed by the initial condition ($W(x,0) = \delta(x-a)$) and we get

$$W(x,t) = \frac{2}{\pi}\int_0^\infty d\lambda \sin[a\sqrt{\lambda/K}]\frac{\sin[x\sqrt{\lambda/K}]}{2\sqrt{K\lambda}} E_\gamma\left[-\lambda t^\gamma\right]. \tag{79}$$

Letting $k = \sqrt{\lambda/K}$ we obtain

$$W(x,t) = \frac{2}{\pi}\int_0^\infty dk \sin[ka]\sin[kx] E_\gamma\left[-k^2 K t^\gamma\right]. \tag{80}$$

Taking the Laplace transform and performing the integral over k we get

$$q(x,s) = \frac{s^{\gamma/2-1}}{2\sqrt{K}}\left[\exp(-s^{\gamma/2}|x-a|/\sqrt{K}) - \exp(-s^{\gamma/2}(x+a)/\sqrt{K})\right].$$

The inverse Laplace transform of $s^{\gamma/2-1}\exp(-|x|s^{\gamma/2})$ is known [26]. We get

$$p(x,t) = \frac{1}{2(Kt^\gamma)^{1/2}} H^{1,0}_{1,1}\left(\frac{|x-a|}{(Kt^\gamma)^{1/2}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right) - \frac{1}{2(Kt^\gamma)^{1/2}} H^{1,0}_{1,1}\left(\frac{x+a}{(Kt^\gamma)^{1/2}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right). \tag{81}$$

Here, the Fox or H-function [27,28] has the following alternating power series expansion:

$$H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(a_j,A_j)_{j=1,\ldots,p}\\(b_j,B_j)_{j=1,\ldots,q}\end{matrix}\right) = \sum_{l=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^k z^{s_{lk}}}{k!B_l} \times \frac{\prod_{j=1,j\neq l}^{m}\Gamma(b_j - B_j s_{lk})\prod_{r=1}^{n}\Gamma(1-a_r+A_r s_{lk})}{\prod_{u=m+1}^{q}\Gamma(1-b_u+B_u s_{lk})\prod_{v=n+1}^{p}\Gamma(a_v - A_v s_{lk})}, \tag{82}$$

where $s_{lk} = (b_l+k)/B_l$ and an empty product is interpreted as unity. Further, m,n,p,q are nonnegative integers such that $0 \le n \le p$, $1 \le m \le q$; A_j, B_j are positive numbers; a_j, b_j can be complex numbers. For further discussions of the H-function, see Mathai [28].

Substituting Eq. (81) into Eq. (25) we have

$$f(t) = -\frac{d}{dt}\left[\frac{1}{2(Kt^\gamma)^{1/2}}\int_0^\infty dx\, H^{1,0}_{1,1}\left(\frac{|x-a|}{(Kt^\gamma)^{1/2}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right)\right] + \frac{d}{dt}\left[\frac{1}{2(Kt^\gamma)^{1/2}}\int_0^\infty dx\, H^{1,0}_{1,1}\left(\frac{x+a}{(Kt^\gamma)^{1/2}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right)\right].$$

Defining $z = (x-a)/(Kt^\gamma)^{1/2}$, $z' = (x+a)/(Kt^\gamma)^{1/2}$, we obtain

$$\begin{aligned} f(t) &= -\frac{d}{dt}\int_{-a/(Kt^\gamma)^{1/2}}^{\infty} dz\, H^{1,0}_{1,1}\left(|z|\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right) + \frac{d}{dt}\int_{a/(Kt^\gamma)^{1/2}}^{\infty} dz'\, H^{1,0}_{1,1}\left(z'\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right) \\ &= \frac{a\gamma}{2K^{1/2}t^{(2+\gamma)/2}} H^{1,0}_{1,1}\left(\frac{a}{(Kt^\gamma)^{1/2}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right). \end{aligned}$$

Thus using the fractional Fokker-Planck approach and H-functions, we are able to obtain an exact form for the FPT distribution for Lévy-type anomalous diffusion with zero drift. When $\gamma = 1$, the above expression reduces to the inverse Gaussian distribution with $\mu = 0$. Thus FPT distribution for Brownian motion is contained as a special case of this more general result. Expressions for the FPT distribution (or its Laplace transform) have been derived [38,29,30] in other cases also.

4 Summary

In this paper, we highlighted the important role played by the first passage time distribution in different areas. We explained in detail how the Fokker-Planck approach can be used to find the FPT distribution for ordinary Brownian motion. We then extended the approach using fractional Fokker-Planck equation to find the FPT distribution for anomalous diffusion.

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