1 Introduction

- "Because two-locus and other types of model cannot describe inheritance and selection for continuously varying traits (such as ornaments and preferences), these models are fundamentally disconnected from the empirical study of sexual selection," [7]
- "The male traits most likely to become exaggerated by such mechanisms are those under weak natural selection and subject to relatively large variance in female sexual preferences, such as some behavioral and morphological elements of courtship and mating," [6]
- quantitative genetics model / like mine: "Variation among females in their preferences selects for increased variance in the male trait; we assume that this effect is weak by requiring that $\sigma_{\mathrm{z}}^2\sigma_{\mathrm{y}}^2 << 1$," [4] WHY?!?!
- two-allele model: "Neither copying model supports variation at the male trait locus, and copying makes it more difficult for a novel male trait phenotype to spread," [3]. "Since Lill's study, anecdotal evidence for copying has also been found in other species of birds, mammals, and fishes (reviewed by Gibson and Hoglund 1992; Pruett-Jones 1992)," [3]. "Bradbury and Gibson (1983) and Bradbury et al. (1985) found that the large variance of male mating success observed in sage grouse could not be plausibly attributed to independent female choice, and suggested that copying might be responsible," [3]. "Y-linkage is of interest because many of the loci that affect sexually-selected male coloration in guppies are Y-linked (Winge 1927; Haskins et al. 1961)," [3]. "Copying itself does not support polymorphism in the male trait, at least in haploid and Y-linked systems," [3]. "Guppies are a perplexing example in this context. The species is famous for highly polymorphic male coloration (Endler 1983) which is known to be under sexual selection (Endler 1980, 1983; Breden and Stoner 1987; Houde 1987, 1988; Stoner and Breden 1988; Houde and Endler 1990) and is at least in part caused by Y-linked loci (Winge 1927; Haskins et al 1961)," [3].
- "Mate preferences can be genetically determined (Saether et al. 2007), but they can also be learned (Virzijden et al. 2012). For example, in sticklebacks females learn to prefer mates with phenotypes similar to their fathers (Kozak et al. 2011) and in some cichlids females learn to prefer mates with phenotypes similar to their mothers (Verzijden et al. 2008). Recent work suggests that learned preferences for parental phenotypes can promote speciation (Verzijden et al. 2005; Servedio and Dukas 2013)," [1].

2 Question

1. H1 Song learning can decelerate speciation by allowing for genetically diverse birds to mate with each other, maintaining gene flow between subpopulations that might either begin to diverge
2. H2 Song learning can accelerate speciation by increasing standing genetic variation, which would allow for quicker divergence once new selection pressures arise [5]
3. H3 Song learning can accelerate speciation because culturally inherited traits can evolve more quickly than genetically inherited ones [2]

3 Goals

4 Model

4.1 Continuous traits Each male has a song and each female has preference for a particular song. She will mate with males with songs other than her preferred song, but the probability of her doing so decreases as the potential mate's song gets less similar to her preferred song. Each female mates once and chooses a male according to her preferences and the distribution of songs present in the population. Each male, therefore, may breed multiple times or not at all. We assume that generations are non-overlapping, so once the adults breed they die and we can shift our focus to the new generation. To begin with, we assume that males acquire their songs from their fathers at birth and females acquire their preferences from their mothers at birth. We further assume an unbiased sex ratio. Before the new generation mates, each male has a small probability of "mutating" its song. This can be interpreted as a learning error or as innovation. After mutation, the new males and females mate.

Mathematically, each male has a song $x \in \mathbb{R}$ and each female has a preference $y \in \mathbb{R}$. The probability density of the male songs will be written $P_{\mathrm{m}}(x)$ and the probability density of female preferences will be written $P_{\mathrm{f}}(y)$. The preference of a female with preference y mates for a male with song x is

$$f_y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right),$$

Table 1: Summary of choices made in previous models

	Lachlan and Servedio [5]
♂trait(s)	allele A / a: song predisposition
	song
How ♂ trait is learned	
Inherited	
♀ trait(s)	
How ♀ trait is learned	
Inherited	
Errors	
Population structure	
Mating structure	
♀ preference	
Selection	

Table 2: Parameters

σ^2	width of female preference function
σ_{f}^2	variance of female preferences within each population
σ_{m}^2	variance of male traits within each population

which is maximal when $x = y$ and decreases as $|x - y|$ increases. The probability that a female with preference y choose a male with song x is

$$\frac{P_{\mathrm{m}}(x) f_y(x)}{\int_{\mathbb{R}} P_{\mathrm{m}}(x') f_y(x') dx'},$$

so the probability of a (x, y) pair mating is then

$$P_{\mathrm{mate}}(x,y) = \frac{P_{\mathrm{f}}(y) P_{\mathrm{m}}(x) f_y(x)}{\int_{\mathbb{R}} P_{\mathrm{m}}(x') f_y(x') dx'} = \frac{P_{\mathrm{f}}(y) P_{\mathrm{m}}(x) f_y(x)}{Z_y},$$

where Z_y is the normalizing factor $\int_{\mathbb{R}} P_{\mathrm{m}}(x') f_y(x') dx'$. Each such pair will produce a male with song x and a female with preference y. Before mating, the male's song changes to $x - \delta_{\mathrm{mut}}$ with probability $p_{\mathrm{mut}}/2$ and to $x + \delta_{\mathrm{mut}}$ with probability $p_{\mathrm{mut}}/2$. Under these assumptions, the probability density of female preferences does not change over time:

$$P_{\mathrm{f}}(y, t+1) = P_{\mathrm{f}}(y, t).$$

The probability density of male songs in the next generation, before mutating and after mutating, follows

$$P_{\mathrm{m}}(x, t+1/2) = \int_{\mathbb{R}} P_{\mathrm{mate}}(x, y, t) dy$$
$$P_{\mathrm{m}}(x, t+1) = (1 - p_{\mathrm{mut}}) P_{\mathrm{m}}(x, t+1/2) + p_{\mathrm{mut}}/2 P_{\mathrm{m}}(x - \delta_{\mathrm{mut}}, t+1/2) + p_{\mathrm{mut}}/2 P_{\mathrm{m}}(x + \delta_{\mathrm{mut}}, t+1/2)$$

(Population dynamics are modeled in Code **??**. Global and / or needed parameters are given in Code 2.)

We start with mixtures of normal distributions of both male and female:

$$P_{\mathrm{f}}(y, 0) = p_{\mathrm{f}} \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}}^2}} \exp\left(-\frac{(y+1)^2}{2\sigma_{\mathrm{f}}^2}\right) + (1 - p_{\mathrm{f}}) \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}}^2}} \exp\left(-\frac{(y-1)^2}{2\sigma_{\mathrm{f}}^2}\right)$$
$$P_{\mathrm{m}}(x, 0) = p_{\mathrm{m}} \frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}}^2}} \exp\left(-\frac{(x+1)^2}{2\sigma_{\mathrm{m}}^2}\right) + (1 - p_{\mathrm{m}}) \frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}}^2}} \exp\left(-\frac{(x-1)^2}{2\sigma_{\mathrm{m}}^2}\right)$$

Even with a continuous distribution of female preferences, we find that after several generations, the male song distribution is concentrated in several discrete peaks (Figure **??**). (Parameters and initial conditions given in Code **??**.)

There are three critical parameters:

Mutations give rise to a small number of males with song traits that are slightly higher (lower) than the rest of the population. The "edge" females that most prefer these extreme songs have limited options because they do not like the songs most males are singing. They therefore mate with the new "edge" males with very high probability, which leads to a very fast increase of these songs. As mutations continue to generate new songs, extreme songs will be selected for at the expense of similar songs closer to the middle of the song range. Ultimately, this results in a number of discrete songs at high density.

Ignoring mutations for the moment, the rate of change of the probability density of a particular song is given by

$$\frac{P_{\mathrm{m}}(x,t+1)}{P_{\mathrm{m}}(x,t)} = \int_{\mathbb{R}} \frac{P_{\mathrm{f}}(y) f_y(x)}{Z_y} dy \tag{1}$$

This shows that the normalizing factor Z_y affects how the density of male songs changes from one generation to the next: if a male is recognized by females with very few options (small Z_y), then the density of that song will increase more than the song a male recognized by females with many options (large Z_y) (Figure **??**).

4.2 Analysis

Claim 4.1 $\exp\left(-\frac{(x-\mu_{\mathrm{a}})^2}{2\sigma_{\mathrm{a}}^2}\right) \times \exp\left(-\frac{(x-\mu_{\mathrm{b}})^2}{2\sigma_{\mathrm{b}}^2}\right) = \exp\left(-\frac{(x-\mu_{\mathrm{ab}})^2}{2\sigma_{\mathrm{ab}}^2}\right) \times \exp\left(-\frac{(\mu_{\mathrm{a}}-\mu_{\mathrm{b}})^2}{2(\sigma_{\mathrm{a}}^2+\sigma_{\mathrm{b}}^2)}\right)$ where $\mu_{\mathrm{ab}} = \frac{\mu_{\mathrm{a}}\sigma_{\mathrm{b}}^2+\mu_{\mathrm{b}}\sigma_{\mathrm{a}}^2}{\sigma_{\mathrm{a}}^2+\sigma_b^2}$ and $\sigma_{\mathrm{ab}}^2 = \frac{\sigma_{\mathrm{a}}^2\sigma_{\mathrm{b}}^2}{\sigma_{\mathrm{a}}^2+\sigma_{\mathrm{b}}^2}$

$$
\begin{aligned}
P_{\mathrm{m}}(x,y) &\sim N\left(\mu_{\mathrm{m}}, \Sigma_{\mathrm{m}}\right) \text{ where } \quad \mu_{\mathrm{m}} = \begin{pmatrix} \mu_{\mathrm{m}x} \\ \mu_{\mathrm{m}y} \end{pmatrix} \quad \Sigma_{\mathrm{m}} = \begin{pmatrix} \sigma_{\mathrm{m}x}^2 & \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \sigma_{\mathrm{m}y}^2 \end{pmatrix} \\
f_y(x) &\sim N\left(y, \sigma^2\right) \\
\Rightarrow P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) &= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \bar{\mu}_{\mathrm{m}})^T \bar{\Sigma}_{\mathrm{m}}^{-1} (\vec{x}_{\mathrm{m}} - \bar{\mu}_{\mathrm{m}})\right) \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_{\mathrm{m}x}^2)}} \exp\left(-\frac{(y_2 - \mu_{\mathrm{m}x})^2}{2(\sigma^2 + \sigma_{\mathrm{m}x}^2)}\right) \\
\text{where } \bar{\mu}_{\mathrm{m}} &= \begin{pmatrix} \frac{\sigma^2 \mu_{\mathrm{m}x}}{\sigma^2 + \sigma_{\mathrm{m}x}^2} + \frac{\sigma_{\mathrm{m}x}^2 y_2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} \\ \mu_{\mathrm{m}y} + \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}}{\sigma^2 + \sigma_{\mathrm{m}x}^2}(y_2 - \mu_{\mathrm{m}x}) \end{pmatrix} = \begin{pmatrix} \frac{\sigma^2 \mu_{\mathrm{m}x}}{\sigma^2 + \sigma_{\mathrm{m}x}^2} + \frac{\sigma_{\mathrm{m}x}^2 y_2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} \\ \mu_{\mathrm{m}y} + \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}} \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2}(y_2 - \mu_{\mathrm{m}x}) \end{pmatrix} \\
\text{and } \bar{\Sigma}_{\mathrm{m}} &= \begin{pmatrix} \frac{\sigma^2 \sigma_{\mathrm{m}x}^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} & \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} \\ \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} & \left(\frac{\sigma^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} + \frac{\sigma_{\mathrm{m}x}^2(1-\rho_{\mathrm{m}}^2)}{\sigma^2 + \sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{m}y}^2 \end{pmatrix} = \frac{\sigma^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2}\Sigma_{\mathrm{m}} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2 + \sigma_{\mathrm{m}x}^2} \begin{pmatrix} 0 & 0 \\ 0 & (1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2 \end{pmatrix} \\
\Rightarrow \int\int P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) dx_1 dy_1 &= \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_{\mathrm{m}x}^2)}} \exp\left(-\frac{(y_2 - \mu_{\mathrm{m}x})^2}{2(\sigma^2 + \sigma_{\mathrm{m}x}^2)}\right) \\
\Rightarrow \frac{P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) dx_1 dy_1} &= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \bar{\mu}_{\mathrm{m}})^T \bar{\Sigma}_{\mathrm{m}}^{-1} (\vec{x}_{\mathrm{m}} - \bar{\mu}_{\mathrm{m}})\right) \\
\Rightarrow \frac{P_{\mathrm{f}}(x_2,y_2) P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) dx_1 dy_1} &= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}} - \mu_{\mathrm{f}})^T \Sigma_{\mathrm{f}}^{-1} (\vec{x}_{\mathrm{f}} - \mu_{\mathrm{f}})\right) \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \bar{\mu}_{\mathrm{m}})^T \bar{\Sigma}_{\mathrm{m}}^{-1} (\vec{x}_{\mathrm{m}} - \bar{\mu}_{\mathrm{m}})\right)
\end{aligned}
$$

CAN INTEGRATE x_1, y_1, x_2

$$\text{OR } \frac{P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}}\exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)}$$

$$= P_{\mathrm{m}}(x_1,y_1)\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}\exp\left(-\frac{(y_2-x_1)^2}{2\sigma^2}\right)\exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2(-\sigma^2-\sigma_{\mathrm{m}x}^2)}\right)$$

$$= P_{\mathrm{m}}(x_1,y_1)\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}\exp\left(-\frac{\left(y_2-\left(\frac{(-\sigma^2-\sigma_{\mathrm{m}x}^2)x_1+\mu_{\mathrm{m}x}\sigma^2}{-\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(-\sigma^2-\sigma_{\mathrm{m}x}^2)}{-\sigma_{\mathrm{m}x}^2}}\right)\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2(-\sigma_{\mathrm{m}x}^2)}\right)$$

$$= \frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}\exp\left(-\frac{\left(y_2-\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right)$$

$$\Rightarrow \frac{P_{\mathrm{f}}(x_2,y_2)P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{\left(y_2-\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}$$

$$= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}$$

$$\text{where } \bar{\mu}_{\mathrm{f}} = \begin{pmatrix} \mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right) \\ \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}+\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2} \end{pmatrix}$$

$$\text{and } \bar{\Sigma}_f = \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\Sigma_{\mathrm{f}}+\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^{+}\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\begin{pmatrix}(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}x}^2 & 0 \\ 0 & 0\end{pmatrix}$$

$$|\bar{\Sigma}_{\mathrm{f}}| = |\Sigma_{\mathrm{f}}|\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}$$

$$\Rightarrow \frac{P_{\mathrm{f}}(x_2,y_2)P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)\times$$

$$\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma^2}}\sqrt{\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

$$= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

$$= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

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$$
P_{\mathrm{mate}}(x_1,y_1,x_2,y_2) = \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right) \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right) \frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma^2_{\mathrm{m}x}}\right)} \sqrt{\frac{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}}
$$

$$
\Rightarrow \int P_{\mathrm{mate}}dy_1 = \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right) \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right) \frac{\frac{1}{\sqrt{2\pi\sigma^2_{\mathrm{m}x}}}\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma^2_{\mathrm{m}x}}\right)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma^2_{\mathrm{m}x}}\right)} \sqrt{\frac{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}}
$$

$$
= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right) \frac{1}{\sqrt{2\pi\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}} \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)
$$

$$
\Rightarrow \int\int P_{\mathrm{mate}}dy_1dy_2 = \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}(1-\rho_{\mathrm{f}}^2)}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}\right)\sigma^2_{\mathrm{f}x}}} \exp\left(-\frac{\left(x_2-\mu_{\mathrm{f}x}-\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}\left(\frac{(\sigma^2+\sigma^2_{\mathrm{m}x})x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma^2_{\mathrm{m}x}}-\mu_{\mathrm{f}y}\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}(1-\rho_{\mathrm{f}}^2)}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}\right)\sigma^2_{\mathrm{f}x}}\right) \times
$$

$$
\frac{1}{\sqrt{2\pi\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}} \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)
$$

$$
\Rightarrow \int\int P_{\mathrm{mate}}dy_1dy_2 = \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}(1-\rho_{\mathrm{f}}^2)}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}\right)\sigma^2_{\mathrm{f}x}}} \exp\left(-\frac{\left(x_2-\mu_{\mathrm{f}x}-\sigma_{\mathrm{f}x}\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}}\frac{\sigma^2+\sigma^2_{\mathrm{m}x}}{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}}\left(x_1-\frac{\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{m}x}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)\right)^2}{2\left(1-\frac{\rho_{\mathrm{f}}^2\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}\right)\sigma^2_{\mathrm{f}x}}\right) \times
$$

$$
\frac{1}{\sqrt{2\pi\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}} \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)
$$

$$\Rightarrow P_{\mathrm{mate}}(x_1,x_2) \sim N\left(\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{f}x} \end{pmatrix}, \begin{pmatrix} \sigma_{\mathrm{m}x}'^2 & \rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}' \\ \rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}' & \sigma_{\mathrm{f}x}'^2 \end{pmatrix}\right)$$

$$\text{where } \sigma_{\mathrm{m}x}'^2 = \frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$\text{and } \sigma_{\mathrm{f}x}'^2 = \sigma_{\mathrm{f}x}^2$$

$$\text{and } \rho_{x_1,x_2} = \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

$$\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dy_1 = \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dy_1dx_2 = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{\left(y_2-\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x})}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}+\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\right)\times$$

$$\frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{\left(y_2-\left(\mu_{\mathrm{f}y}+\frac{\sigma_{\mathrm{f}y}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2\mu_{\mathrm{f}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\right)\right)\right)^2}{2\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\right)\times$$

$$\frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)$$

$$\Rightarrow P_{\mathrm{mate}}(x_1,y_2) \sim N\left(\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{f}y} \end{pmatrix}, \begin{pmatrix} \sigma_{\mathrm{m}x}'^2 & \rho_{x_1,y_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}y}' \\ \rho_{x_1,y_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}y}' & \sigma_{\mathrm{f}y}'^2 \end{pmatrix}\right)$$

$$\text{where } \sigma_{\mathrm{m}x}'^2 = \frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$\text{and } \sigma_{\mathrm{f}y}'^2 = \sigma_{\mathrm{f}y}^2$$

$$\text{and } \rho_{x_1,y_2} = \frac{\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

$$
\begin{aligned}
\int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_2dy_2 &= \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma^2_{\mathrm{m}x}}\right)}\sqrt{\frac{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}} \\
&= \frac{1}{\sqrt{2\pi(1-\rho^2_{\mathrm{m}})\sigma^2_{\mathrm{m}y}}}\frac{1}{\sqrt{2\pi\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)\times \\
&\exp\left(-\frac{\left(y_1-\left(\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(x_1-\mu_{\mathrm{m}x})\right)\right)^2}{2(1-\rho^2_{\mathrm{m}})\sigma^2_{\mathrm{m}y}}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma'^2_{\mathrm{m}x}}}\exp\left(-\frac{(x_1-\mu'_{\mathrm{m}x})^2}{2\sigma'^2_{\mathrm{m}x}}\right)\times \\
&\frac{1}{\sqrt{2\pi(1-\rho^2_{x_1,y_2})\sigma'^2_{\mathrm{m}y}}}\exp\left(-\frac{(y_1-(\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(\mu'_{\mathrm{m}x}-\mu_{\mathrm{m}x})+\frac{\rho_{x_1,y_1}\sigma'_{\mathrm{m}y}}{\sigma'_{\mathrm{m}x}}(x_1-\mu'_{\mathrm{m}x})))^2}{2(1-\rho^2_{x_1,y_1})\sigma'^2_{\mathrm{m}y}}\right) \\
\text{where } \rho_{x_1,y_1} &= \rho_{\mathrm{m}}\sqrt{\frac{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+(1-\rho^2_{\mathrm{m}})\sigma^2_{\mathrm{m}x}(\sigma^2+\sigma^2_{\mathrm{m}x})+\rho^2_{\mathrm{m}}\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y}}} \\
&= \frac{1}{\sqrt{2\pi\sigma'^2_{\mathrm{m}x}}}\exp\left(-\frac{(x_1-\mu'_{\mathrm{m}x})^2}{2\sigma'^2_{\mathrm{m}x}}\right)\frac{1}{\sqrt{2\pi(1-\rho^2_{x_1,y_2})\sigma'^2_{\mathrm{m}y}}}\exp\left(-\frac{(y_1-(\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})+\frac{\rho_{x_1,y_1}\sigma'_{\mathrm{m}y}}{\sigma'_{\mathrm{m}x}}(x_1-\mu'_{\mathrm{m}x})))^2}{2(1-\rho^2_{x_1,y_2})\sigma'^2_{\mathrm{m}y}}\right) \\
\Rightarrow P_{\mathrm{mate}}(x_1,y_1) &\sim N\left(\begin{pmatrix}\frac{\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{m}x}+\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\end{pmatrix},\begin{pmatrix}\sigma'^2_{\mathrm{m}x} & \rho_{x_1,y_1}\sigma'_{\mathrm{m}x}\sigma'_{\mathrm{m}y} \\ \rho_{x_1,y_1}\sigma'_{\mathrm{m}x}\sigma'_{\mathrm{m}y} & \sigma'^2_{\mathrm{m}y}\end{pmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dy_2 &= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}}\exp\left(-\frac{\left(x_2-\left(\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
&\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \\
&= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}}\exp\left(-\frac{\left(x_2-\left(\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
&\frac{1}{2\pi\sqrt{\sigma_{\mathrm{m}x}'^2}}\exp\left(-\frac{\left(x_1-\left(\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(x_1-\mu_{\mathrm{m}x})\right)^2}{2(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}\right) \\
&= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}}\exp\left(-\frac{\left(x_2-\left(\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
&\frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}x}'^2}}\exp\left(-\frac{\left(x_1-\left(\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{1}{2\pi\sqrt{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{(x_1-(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})))^2}{2\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}\right)\exp\left(-\frac{(y_1-\mu_{\mathrm{m}y})^2}{2\sigma_{\mathrm{m}y}^2}\right) \\
&= \frac{1}{2\pi\sqrt{|\Sigma_{x_1,x_2}|}}\exp\left(-(v-\mu_{x_1,x_2})^T\Sigma_{x_1,x_2}^{-1}(v-\mu_{x_1,x_2})\right)\times \\
&\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{(x_1-(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})))^2}{2\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}\right) \\
\text{where } v = (x_1,x_2)^T,\ \mu_{x_1,x_2} &= \begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{f}x} \end{pmatrix} \\
\text{and } \Sigma_{x_1,x_2} &= \begin{pmatrix} \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\sigma_{\mathrm{m}x}^2 & \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}y} \\ \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}y} & \sigma_{\mathrm{f}x}^2 \end{pmatrix} \\
&= \frac{1}{2\pi\sqrt{|\Sigma_{x_1,x_2}|}}\exp\left(-(v-\nu)^TM^{-1}(v-\nu)\right)\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})-\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}+\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\sigma_{\mathrm{m}x}^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\sigma_{\mathrm{m}x}^2+\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 &= \sigma_{\mathrm{m}x}^2\left(\frac{\rho_{\mathrm{m}}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)+(\sigma^2+\sigma_{\mathrm{m}x}^2)^2(1-\rho_{\mathrm{m}}^2)}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\\
&= \sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\\
&= \frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}'^2}{\sigma_{\mathrm{m}y}^2\rho_{\mathrm{m}}^2}\\
\Rightarrow \nu &= \begin{pmatrix}
\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\\
\mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})-\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)
\end{pmatrix}\\
&= \begin{pmatrix}
\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\\
\mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)
\end{pmatrix}\\
&= \begin{pmatrix}
\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\\
\mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}\sigma_{\mathrm{m}x}}{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)
\end{pmatrix}\\
&= \begin{pmatrix}
\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\\
\mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}y}\sigma_{\mathrm{m}x}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)
\end{pmatrix}\\
\rho_{x_1,x_2}\sigma_{\mathrm{m}x}' &= \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}x}=\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\\
\Rightarrow \nu_2 &= \mu_{\mathrm{f}x}+\sigma_{\mathrm{f}x}\frac{\rho_{\mathrm{m}}\rho_{\mathrm{f}}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\right)}\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)\\
&= \mu_{\mathrm{f}x}+\frac{\sigma_{\mathrm{f}x}\rho_{\mathrm{m}}\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}'\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}(y_1-\mu_{\mathrm{m}y}')\\
&= \mu_{\mathrm{f}x}+\frac{\rho_{y_1,x_2}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}y}'}(y_1-\mu_{\mathrm{m}y}')\\
\text{where }\rho_{y_1,x_2} &= \frac{\rho_{\mathrm{m}}\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}
\end{aligned}
$$

$$\begin{aligned}
\text{and } M &= \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2}\Sigma_{x_1,x_2} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{(1-\rho_{x_1,x_2}^2)\sigma_{\mathrm{m}x}'^2\sigma_{\mathrm{f}y}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2} \end{pmatrix} \\
&= \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2}\Sigma_{x_1,x_2} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{|\Sigma_{x_1,x_2}|}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2} \end{pmatrix} \\
\Rightarrow |M| &= \left(\frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2}\right)^2 |\Sigma_{x_1,x_2}| + \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}x}'^2}{\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2\right)^2}|\Sigma_{x_1,x_2}| \\
&= \frac{\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2\right)\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2\right)}{\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2\right)^2}|\Sigma_{x_1,x_2}| \\
&= \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2 + \sigma_{\mathrm{m}x}'^2}|\Sigma_{x_1,x_2}| \\
&= \frac{(1-\rho_{\mathrm{m}}^2)(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}|\Sigma_{x_1,x_2}| \\
&= \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}{\sigma_{\mathrm{m}y}'^2}|\Sigma_{x_1,x_2}| \\
\text{and } \Rightarrow M_{2,2} &= \left(1 - \frac{\sigma_{\mathrm{m}x}'^2\rho_{x_1,x_2}^2}{\sigma_{\mathrm{m}x}^2\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\sigma_{\mathrm{f}y}^2 \\
&= \left(1 - \frac{\rho_{\mathrm{f}}^2\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}y}^2 \\
&= \left(1-\rho_{y_1,x_2}^2\right)\sigma_{\mathrm{f}y}^2
\end{aligned}$$

$$\Rightarrow \int P_{\mathrm{mate}} y_2 = \frac{1}{2\pi\sqrt{|M|\left(\frac{\sigma_{\mathrm{m}y}'^2}{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}\right)}}\exp\left(-(v-\nu)^T M^{-1}(v-\nu)\right)\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{(y_1-\mu_{\mathrm{m}y}')^2}{2\frac{\rho_{\mathrm{m}}^2\sigma_{\mathrm{m}y}^2}{\sigma_{\mathrm{m}x}^2}\frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}'^2}{\sigma_{\mathrm{m}y}^2\rho_{\mathrm{m}}^2}}\right)$$

$$\Rightarrow \int P_{\mathrm{mate}} dx_1 dy_2 = \frac{1}{\sqrt{2\pi(1-\rho_{y_1,x_2}^2)\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{\left(x_2-\mu_{\mathrm{f}x}-\frac{\rho_{y_1,x_2}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}y}'}(y_1-\mu_{\mathrm{m}y}')\right)^2}{2(1-\rho_{y_2,x_1}^2)\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{(y_1-\mu_{\mathrm{m}y}')^2}{2\sigma_{\mathrm{m}y}'^2}\right)$$

$$\Rightarrow \int P_{\mathrm{mate}} dx_1 dy_2 \sim N\left(\begin{pmatrix}\mu_{\mathrm{m}y}' \\ \mu_{\mathrm{f}x}\end{pmatrix}, \begin{pmatrix}\sigma_{\mathrm{m}y}'^2 & \rho_{y_1,x_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}x} \\ \rho_{y_1,x_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}x} & \sigma_{\mathrm{f}x}^2\end{pmatrix}\right)$$

$$\frac{P_{\mathrm{f}}(x_2,y_2)P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}}\exp\left(-\frac{1}{2}(\vec{x}_m-\bar{\mu}_{\mathrm{m}})^T\bar{\Sigma}_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}}-\bar{\mu}_{\mathrm{m}})\right)$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_1dx_2 = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(1-\rho_{\mathrm{m}})^2\right)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\frac{1}{\sigma^2+\sigma_{\mathrm{m}x}^2}(y_2-\mu_{\mathrm{m}x})\right)^2}{2\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(1-\rho_{\mathrm{m}}^2)\right)\sigma_{\mathrm{m}y}^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\times$$

$$\frac{1}{\sqrt{2\pi\left(1-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}^2\right)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{1}{\sigma_{\mathrm{f}y}}\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}}\frac{\sigma_{\mathrm{m}y}\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}(y_2-\mu_{\mathrm{m}x})\right)^2}{2\left(1-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}^2\right)\sigma_{\mathrm{m}y}^2}\right)$$

$$\left(1-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}^2\right)\sigma_{\mathrm{m}y}^2 = (\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+\sigma_{\mathrm{m}x}^2-\rho_{\mathrm{m}}^2\sigma_{\mathrm{m}x}^2)\frac{\sigma_{\mathrm{m}y}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$= \frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}y}^2((\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$= \left(1-\frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}\right)\frac{\sigma_{\mathrm{m}y}^2((\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$\text{If } \sigma_{\mathrm{m}y}'^2 = \frac{\sigma_{\mathrm{m}y}^2((\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\ ,\ \sigma_{\mathrm{f}y}'^2=\sigma_{\mathrm{f}y}^2\ ,\ \text{and } \rho_{y_1,y_2} = \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}}$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_1dx_2 = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(y_2-\mu_{\mathrm{m}x})\right)^2}{2(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(y_2-\mu_{\mathrm{f}y})\right)^2}{2(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(y_2-\mu_{\mathrm{f}y})\right)^2}{2(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}\right)$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_1dx_2 \sim N\left(\begin{pmatrix}\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\\ \mu_{\mathrm{f}y}\end{pmatrix}, \begin{pmatrix}\sigma_{\mathrm{m}y}'^2 & \rho_{y_1,y_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}y}'\\ \rho_{y_1,y_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}y}' & \sigma_{\mathrm{f}y}'^2\end{pmatrix}\right)$$

$$
\begin{aligned}
P_{\mathrm{m}}(x_1, y_1) f_{y_2}(x_1) &= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \mu_{\mathrm{m}})^T \Sigma_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}} - \mu_{\mathrm{m}})\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_2 - x_2)^2}{2\sigma^2}\right) \\
&= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \mu_{\mathrm{m}})^T \Sigma_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}} - \mu_{\mathrm{m}})\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}} - \nu)^T B^{-1}(\vec{x}_{\mathrm{m}} - \nu)\right) \\
\text{where } \nu &= \begin{pmatrix} y_2 \\ 0 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} \\
\mu) + (\vec{x} - \nu)^T B^{-1}(\vec{x} - \nu) &= \vec{x}^T(\Sigma^{-1} + B^{-1})\vec{x} - 2\vec{x}^T\Sigma^{-1}\mu - 2\vec{x}^T B^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= \vec{x}^T(\Sigma^{-1} + B^{-1})\vec{x} + -2\vec{x}^T(\Sigma^{-1} + B^{-1})(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= \vec{x}^T(\Sigma^{-1} + B^{-1})\vec{x} - 2\vec{x}^T(\Sigma^{-1} + B^{-1})(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1} + B^{-1})(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) \\
&- (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= (\vec{x} - (\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu))^T(\Sigma^{-1} + B^{-1})(\vec{x} - (\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu)) \\
&- (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= (\vec{x} - c)^T C^{-1}(\vec{x} - c) \\
&- (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&\text{where } c = (\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) \text{ and } C = (\Sigma^{-1} + B^{-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
|\Sigma| &= (1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2 \\
\Rightarrow \Sigma^{-1} &= \frac{1}{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2}\begin{pmatrix} \sigma_{\mathrm{m}y}^2 & -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \sigma_{\mathrm{m}x}^2 \end{pmatrix} = \frac{1}{1-\rho_{\mathrm{m}}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
\Rightarrow \Sigma^{-1}+B^{-1} &= \begin{pmatrix} \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}x}^2}+\frac{1}{\sigma^2} & -\frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} = \frac{1}{1-\rho_{\mathrm{m}}^2}\begin{pmatrix} \frac{\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
\Rightarrow |\Sigma^{-1}+B^{-1}| &= \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2} + \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma^2}\frac{1}{\sigma_{\mathrm{m}y}^2} = \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}y}^2}\left(\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma^2\sigma_{\mathrm{m}x}^2}\right) \\
\Rightarrow (\Sigma^{-1}+B^{-1})^{-1} &= \frac{\sigma^2\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}y}^2} & \frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ \frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2\sigma_{\mathrm{m}x}^2} \end{pmatrix} = \begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}x}^2 & \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \frac{\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}y}^2 \end{pmatrix} = \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\Sigma + \begin{pmatrix} 0 & 0 \\ 0 & \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}y}^2 \end{pmatrix} \\
\text{Let } s &= \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \text{ and } D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}y}^2 \end{pmatrix} \text{ so } (\Sigma^{-1}+B^{-1})^{-1} = s\Sigma + D \\
B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu &= -(\Sigma^{-1}\mu+B^{-1}\nu)(s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu \\
&= -s\mu^T\Sigma^{-1}\mu - s\mu^TB^{-1}\nu - \mu^T\Sigma^{-1}D\Sigma^{-1}\mu - s\nu^TB^{-1}\mu - s\nu^TB^{-1}\Sigma B^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^TB^{-1}\nu \\
\Sigma^{-1}D\Sigma^{-1} &= \frac{1}{(1-\rho_{\mathrm{m}}^2)^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
&= \frac{1}{(1-\rho_{\mathrm{m}}^2)^2}\begin{pmatrix} 0 & -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ 0 & \sigma_{\mathrm{m}x}^2\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
&= \frac{1}{(1-\rho_{\mathrm{m}}^2)^2}\begin{pmatrix} \rho_{\mathrm{m}}^2\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ -\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{m}y}^2}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \end{pmatrix} \\
&= \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{\rho_{\mathrm{m}}^2}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
&= \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} + \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{\rho_{\mathrm{m}}^2-1}{\sigma_{\mathrm{m}x}^2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= (1-s)\Sigma^{-1} - sB^{-1} \\
\text{and } sB^{-1}\Sigma B^{-1} &= s\begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{\mathrm{m}x}^2 & \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \sigma_{\mathrm{m}y}^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= s\begin{pmatrix} \frac{1}{\sigma^2}\sigma_{\mathrm{m}x}^2 & \frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= s\begin{pmatrix} \frac{1}{\sigma^4}\sigma_{\mathrm{m}x}^2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= (1-s)B^{-1} \\
B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu &= -s\mu^T\Sigma^{-1}\mu - s\mu^TB^{-1}\nu - \mu^T\left((1-s)\Sigma^{-1}-sB^{-1}\right)\mu - s\nu^TB^{-1}\mu - (1-s)\nu^TB^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^TB^{-1}\nu \\
&= s\mu^TB^{-1}\mu - 2s\nu^TB^{-1}\mu + s\nu^TB^{-1}\nu \\
&= s(\nu-\mu)^TB^{-1}(\nu-\mu)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (\vec{x}-\mu)^T\Sigma^{-1}(\vec{x}-\mu)+(\vec{x}-\nu)^T B^{-1}(\vec{x}-\nu) &= (\vec{x}-c)^T C^{-1}(\vec{x}-c)+s(\nu-\mu)^T B^{-1}(\nu-\mu)\\
\text{where } C &= (\Sigma^{-1}+B^{-1})^{-1} = s\Sigma + D,\ \ c = C(\Sigma^{-1}\mu + B^{-1}\nu)\\
\Rightarrow c &= (s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)\\
&= s\mu + s\Sigma B^{-1}\nu + D\Sigma^{-1}\mu\\
&= s\mu + s\begin{pmatrix} \frac{1}{\sigma^2}\sigma_{\mathrm{m}x}^2 & 0\\ \frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & 0\end{pmatrix}\nu + (1-s)\begin{pmatrix} 0 & 0\\ -\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}} & 1\end{pmatrix}\mu\\
&= \begin{pmatrix} s\mu_{\mathrm{m}x} + s\frac{1}{\sigma^2}\sigma_{\mathrm{m}x}^2 y_2\\ s\mu_{\mathrm{m}y} + s\frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}y_2 + (1-s)(-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\mu_{\mathrm{m}x}+\mu_{\mathrm{m}y})\end{pmatrix}\\
&= \begin{pmatrix} s\mu_{\mathrm{m}x} + (1-s)y_2\\ \mu_{\mathrm{m}y} + s\frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}y_2 + (1-s)(-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\mu_{\mathrm{m}x})\end{pmatrix}\\
&= \begin{pmatrix} s\mu_{\mathrm{m}x} + (1-s)y_2\\ \mu_{\mathrm{m}y} + (1-s)\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(y_2-\mu_{\mathrm{m}x})\end{pmatrix}\\
|C| &= |(\Sigma^{-1}+B^{-1})^{-1}| = |\Sigma^{-1}+B^{-1}|^{-1}\\
&= \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} = |\Sigma|\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\\
\Rightarrow P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1) &= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-\mu_{\mathrm{m}})^T\Sigma_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}}-\mu_{\mathrm{m}})\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2\sigma^2}\right)\\
&= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-c)^T C^{-1}(\vec{x}_{\mathrm{m}}-c)\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\mu_{\mathrm{m}x}-y_2)^2}{2\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma^2}\sigma^2}\right)\\
&= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-c)^T C^{-1}(\vec{x}_{\mathrm{m}}-c)\right)\frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}}\exp\left(-\frac{(\mu_{\mathrm{m}x}-y_2)^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)\\
&= \frac{1}{2\pi\sqrt{|C|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-c)^T C^{-1}(\vec{x}_{\mathrm{m}}-c)\right)\frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}}\exp\left(-\frac{(\mu_{\mathrm{m}x}-y_2)^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)
\end{aligned}
$$

$$-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\Big)\exp\left(-\frac{\left(y_2-\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right)=\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\nu)^TB^{-1}(\vec{x}_{\mathrm{f}}-\nu)\right)$$

$$\text{where } \nu=\begin{pmatrix}0\\ \frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\end{pmatrix}\text{ and } B^{-1}=\begin{pmatrix}0 & 0\\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}$$

$$(\vec{x}-\mu)^T\Sigma^{-1}(\vec{x}-\mu)+(\vec{x}-\nu)^TB^{-1}(\vec{x}-\nu)=(\vec{x}-c)^TC^{-1}(\vec{x}-c)-(\Sigma^{-1}\mu+B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu$$

$$\text{where } c=(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)\text{ and } C=(\Sigma^{-1}+B^{-1})^{-1}$$

$$\Sigma^{-1}+B^{-1}=\frac{1}{1-\rho_{\mathrm{f}}^2}\begin{pmatrix}\frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}}\\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2}\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}=\begin{pmatrix}\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}}\\ -\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{f}y}^2\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}$$

$$\Rightarrow|\Sigma^{-1}+B^{-1}|=\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{1}{\sigma_{\mathrm{f}x}^2}\frac{1}{\sigma_{\mathrm{f}y}^2}+\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{f}x}^2\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}=\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{1}{\sigma_{\mathrm{f}x}^2\sigma_{\mathrm{f}y}^2}\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}$$

$$\text{Let } s=\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}\text{ and } D=\begin{pmatrix}(1-\rho_{\mathrm{f}}^2)(1-s)\sigma_{\mathrm{f}x}^2 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow C=(\Sigma^{-1}+B^{-1})^{-1}=\begin{pmatrix}\sigma_{\mathrm{f}x}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2} & -\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}s\\ -\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}s & \sigma_{\mathrm{f}y}^2s\end{pmatrix}=s\Sigma+D$$

$$\Rightarrow c=(s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)=s\mu+s\Sigma B^{-1}\nu+D\Sigma^{-1}\mu$$

$$=s\mu+s\begin{pmatrix}0 & \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\\ 0 & \sigma_{\mathrm{f}y}^2\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}\nu+\begin{pmatrix}(1-s) & -\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-s)\\ 0 & 0\end{pmatrix}\mu$$

$$=\begin{pmatrix}\mu_{\mathrm{f}x}+\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}s\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-s)\mu_{\mathrm{f}y}\\ s\mu_{\mathrm{f}y}+s\sigma_{\mathrm{f}y}^2\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\end{pmatrix}$$

$$=\begin{pmatrix}\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-s)\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\\ s\mu_{\mathrm{f}y}+(1-s)\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\end{pmatrix}$$

$$\Sigma^{-1}\mu+B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu=-(\Sigma^{-1}\mu+B^{-1}\nu)(s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu$$

$$=-s\mu^T\Sigma^{-1}\mu-s\mu^TB^{-1}\nu-\mu^T\Sigma^{-1}D\Sigma^{-1}\mu-s\nu^TB^{-1}\mu-s\nu^TB^{-1}\Sigma B^{-1}\nu+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu$$

$$
\begin{aligned}
\Sigma^{-1} D \Sigma^{-1} &= \frac{1}{(1-\rho_{\mathrm{f}}^2)^2} \begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \begin{pmatrix} (1-\rho_{\mathrm{f}}^2)(1-s)\sigma_{\mathrm{f}x}^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \\
&= \frac{1}{(1-\rho_{\mathrm{f}}^2)^2} \begin{pmatrix} (1-\rho_{\mathrm{f}}^2)(1-s) & 0 \\ -\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-\rho_{\mathrm{f}}^2)(1-s) & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \\
&= \frac{1-s}{1-\rho_{\mathrm{f}}^2} \begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{\rho_{\mathrm{f}}^2}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \\
&= (1-s)\Sigma^{-1} - sB^{-1} \\
sB^{-1}\Sigma B^{-1} &= s \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} \begin{pmatrix} \sigma_{\mathrm{f}x}^2 & \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} \\ \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} & \sigma_{\mathrm{f}y}^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} \\
&= s \begin{pmatrix} 0 & 0 \\ \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} & \sigma_{\mathrm{f}y}^2\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} \\
&= s \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{\mathrm{f}y}^2\left(\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{m}x}}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} = (1-s)B^{-1} \\
\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu &= -s\mu^T\Sigma^{-1}\mu - s\mu^T B^{-1}\nu - \mu^T\left((1-s)\Sigma^{-1} - sB^{-1}\right)\mu - s\nu^T B^{-1}\mu - (1-s)\nu^T B^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= s\nu^T B^{-1}\nu - 2s\mu^T B^{-1}\nu + \mu^T sB^{-1}\mu \\
&= (\mu-\nu)^T sB^{-1}(\mu-\nu) \\
&= \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\left(\mu_{\mathrm{f}y} - \frac{(\sigma^2+\sigma_{\mathrm{m}x})^2 x_1 - \mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)^2 \\
&= \frac{\left(\mu_{\mathrm{f}y} - \frac{(\sigma^2+\sigma_{\mathrm{m}x})^2 x_1 - \mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)^2}{\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}
\end{aligned}
$$

$$
\Rightarrow \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{\left(y_2 - \left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1 - \mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right) = \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-c)^T C^{-1}(\vec{x}_{\mathrm{f}}-c)\right)\exp\left(-\frac{\left(\mu_{\mathrm{f}y} - \frac{(\sigma^2+\sigma_{\mathrm{m}x})^2 x_1 - \mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)
$$

$$P_{\mathrm{mate}}(x_1, y_1, x_2, y_2) \sim N\left(\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{m}y} + \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y} - \mu_{\mathrm{m}x}) \\ \mu_{\mathrm{f}x} \\ \mu_{\mathrm{f}y} \end{pmatrix}, \Sigma'\right)$$

$$\frac{\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \quad \frac{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}y}$$

$$\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \quad \frac{\sigma_{\mathrm{m}y}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}y}$$

$$\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} \quad \sigma_{\mathrm{f}y}^2$$

$$\cdots+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2 \qquad \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \quad \frac{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}x}$$

$$\frac{\rho_{\mathrm{m}}\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \quad \frac{\sigma_{\mathrm{m}y}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}x}$$

$$\sigma_{\mathrm{f}x}^2$$

$$\Sigma' = \begin{pmatrix} \sigma_{\mathrm{m}x}^2 \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_\mathrm{m}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\left(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2\right)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_\mathrm{f}\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ & \sigma_{\mathrm{m}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_\mathrm{m}\rho_\mathrm{f}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\rho_\mathrm{m}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma_{\mathrm{f}y}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ & & \sigma_{\mathrm{f}x}^2 & \rho_\mathrm{f}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} \\ & & & \sigma_{\mathrm{f}y}^2 \end{pmatrix}$$

	song from father	song obliquely learned	song genetic
pref genetic	$\lim_{t\to\infty}\sigma_y^2 = \lim_{t\to\infty}\sigma_x^2 = 0$	$\lim_{t\to\infty}\sigma_x^2 = \sigma_x(0)^2, \lim_{t\to\infty}\sigma_y^2 = 0$	
pref imprinted from father	$\lim_{t\to\infty}\sigma_y^2 = \lim_{t\to\infty}\sigma_x^2 = 0$	$\lim_{t\to\infty}\sigma_x^2 = \sigma_x^2(0), \lim_{t\to\infty}\sigma_y^2 = \frac{\sigma_x^2(\sigma^2+\sigma_x^2)}{2\sigma_x^2+\sigma^2}$	
pref learned from mother / obliquely	$\lim_{t\to\infty}\sigma_y^2 = \sigma_y^2(0), \lim_{t\to\infty}\sigma_x^2 = \sigma_y^2 - \sigma^2$	$\lim_{t\to\infty}\sigma_y^2 = \sigma_y^2(0), \lim_{t\to\infty}\sigma_x^2 = \sigma_x^2(0)$	

5 Equilibrium for various modes of inheritance

5.1 Song from father, pref genetic

$$\sigma_x(t+1)^2 = \sigma_x^2\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2}$$

$$\sigma_y(t+1)^2 = \frac{1}{4}\sigma_y^2\frac{\sigma^2(\sigma^2+\sigma_x^2)+(1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2)+\rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\rho\sigma_x\sigma_y^3}{\sigma^2+\sigma_x^2} + \frac{1}{4}\sigma_y^2$$

$$= \frac{\sigma_y^2}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + (1-\rho^2)\sigma_x^2\frac{\sigma^2+\sigma_x^2-\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right)$$

$$\mathrm{Cov}_{x,y}(t+1) = \frac{1}{2}\frac{\rho\sigma_x\sigma_y\left(\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2\right)}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2}$$

$$\text{equilibrium } \Rightarrow \sigma^2+\sigma_x^2 = \sigma_y^2 \Rightarrow \sigma_x < \sigma_y$$

$$\Rightarrow \sigma_y(t+1)^2 = \frac{\sigma_y^2}{4}\left(1+\frac{2\rho\sigma_x}{\sigma_y}+1\right)$$

$$\Rightarrow \sigma_y(t+1)^2 < \sigma_y(t)^2$$

$$\Rightarrow \lim_{t\to\infty}\sigma_x(t)^2 = \lim_{t\to\infty}\sigma_y(t)^2 = 0$$

$$
\begin{aligned}
\text{If } \sigma^2 = 0,\ \sigma_x(t+1)^2 &= \sigma_y^2 \\
\sigma_y(t+1)^2 &= \frac{\sigma_y^2}{4}\left(\frac{\sigma_y^2}{\sigma_x^2} + (1-\rho^2)\frac{\sigma_x^2 - \sigma_y^2}{\sigma_x^2} + \frac{2\rho\sigma_y}{\sigma_x} + 1\right) \\
\mathrm{Cov}(t+1) &= \frac{1}{2}\frac{\rho\sigma_x\sigma_y\sigma_y^2}{\sigma_x^2} + \frac{1}{2}\sigma_y^2 \\
\text{equilibrium } &\Rightarrow \sigma_x^2 = \sigma_y^2 \\
&\Rightarrow 2 + 2\rho = 4 \\
&\Rightarrow \rho = 1 \\
\text{YOU'RE HERE!!!}
\end{aligned}
$$

5.2 Song learned from father, pref imprinted from father

$$
\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2\frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} \\
\sigma_y(t+1)^2 &= \sigma_x^2\frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2}
\end{aligned}
$$

Cannot happen that both $\sigma_x^2 = \sigma_y^2 - \sigma^2$ and $\sigma_y^2 = \sigma_x^2$ unless $\sigma^2 = 0$, in which case $\lim_{t\to\infty} \sigma_x(t)^2 = \lim_{t\to\infty} \sigma_y(t)^2 = \sigma_y(0)^2$.

5.3 Song learned from father, pref learned from mother

$$
\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2\frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} \\
\sigma_x(t+1)^2 &= \sigma_x^2 \Rightarrow \sigma^2 + \sigma_x^2 = \sigma_y^2
\end{aligned}
$$

5.4 Song learned obliquely, pref genetic

$$
\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2 \\
\sigma_y(t+1)^2 &= \frac{1}{4}\sigma_y^2\frac{\sigma^2(\sigma^2+\sigma_x^2) + (1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2) + \rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\rho\sigma_x\sigma_y^3}{\sigma^2+\sigma_x^2} + \frac{1}{4}\sigma_y^2 \\
&= \frac{1}{4}\sigma_y^2\frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_x^2(\sigma^2+\sigma_x^2)}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{4}\sigma_y^2 \text{ since in this case } \rho = 0 \\
&= \frac{1}{2}\sigma_y^2 \\
\Rightarrow \lim_{t\to\infty} \sigma_y(t)^2 &= 0
\end{aligned}
$$

5.5 Song learned obliquely, pref imprinted from father

$$
\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2 \\
\sigma_y(t+1)^2 &= \sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} \\
\sigma_y(t+1)^2 \geq \sigma_y^2 &\Rightarrow \sigma_x^2(\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2) \geq \sigma_y^2(\sigma^2+\sigma_x^2)^2 \\
\Rightarrow \left(\sigma^4 + 2\sigma^2\sigma_x^2 + \sigma_x^4 - \sigma_x^4\right)\sigma_y^2 &\leq \sigma_x^2\sigma^2(\sigma^2+\sigma_x^2) \\
\Rightarrow \sigma^2(\sigma^2+2\sigma_x^2)\sigma_y^2 &\leq \sigma^2\sigma_x^2(\sigma^2+\sigma_x^2) \\
\Rightarrow \sigma_y^2 &\leq \frac{\sigma_x^2(\sigma^2+\sigma_x^2)}{\sigma^2+2\sigma_x^2}
\end{aligned}
$$

Stable equilibrium with $\sigma_x^2 = \sigma_x(0)^2$ and $\sigma_y^2 = \frac{\sigma_x^2(\sigma^2+\sigma_x^2)}{2\sigma_x^2+\sigma^2}$.

5.6 Song obliquely, pref obliquely Neither distribution changes.

5.7 Song genetic, pref genetic

$$
\begin{aligned}
\sigma_x(t+1)^2 &= \frac{1}{4}\sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\rho\sigma_x^3\sigma_y}{\sigma^2+\sigma_x^2} + \frac{1}{4}\sigma_x^2 \\
&= \frac{\sigma_x^2}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) \\
\sigma_y(t+1)^2 &= \frac{1}{4}\sigma_y^2\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+(1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2)+\rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) \\
&= \frac{1}{4}\sigma_y^2\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2(\sigma^2+\sigma_x^2-\sigma_y^2)}{(\sigma^2+\sigma_x^2)^2} \\
\mathrm{Cov}(t+1) &= \frac{1}{4}\left(\rho\sigma_x\sigma_y\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} + \frac{\rho^2\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} + \rho\sigma_x\sigma_y\right) \\
&= \frac{\mathrm{Cov}(t)}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2}
\end{aligned}
$$

$$\text{Let } Q = \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1$$

$$\Rightarrow \sigma_x(t+1)^2 = \frac{Q}{4}\sigma_x^2$$

$$\sigma_y(t+1)^2 = \frac{Q}{4}\sigma_y^2 + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2(\sigma^2+\sigma_x^2-\sigma_y^2)}{(\sigma^2+\sigma_x^2)^2}$$

$$\mathrm{Cov}(t+1) = \frac{Q}{4}\mathrm{Cov}(t) + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2}$$

$$\text{equilibrium } \Rightarrow Q = 4$$

$$\Rightarrow \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} = 0$$

$$\Rightarrow \rho = 1 \text{ or } \sigma_x^2 = 0 \text{ or } \sigma_y^2 = 0$$

$$\sigma_x^2 = 0 \Rightarrow Q = 2 \text{ and } \sigma_y^2 = 0 \Rightarrow Q = \frac{\sigma^2}{\sigma^2+\sigma_x^2} + 1 \leq 2$$

$$\text{so at equilibrium need both } \sigma_x^2 = 0, \sigma_y^2 = 0$$

$$\text{or } \sigma_x^2 > 0, \sigma_y^2 > 0, \text{ and } \rho = 1$$

$$Q = 4 \Rightarrow \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} = 3$$

$$\Rightarrow \sigma^2(\sigma^2+\sigma_x^2) + \sigma_x^2\sigma_y^2 + 2\rho\sigma_x\sigma_y(\sigma^2+\sigma_x^2) = 3(\sigma^2+\sigma_x^2)^2$$

$$\Rightarrow$$

This model was built to study whether song diversity can be maintained when two populations come back together. However, even within one population there are interesting dynamics.

1. There can be many discrete "popular" songs when σ_{m}^2 is small, σ_{f}^2 is high, and σ^2 is intermediate.

2. When the equilibrium population is unimodal, there is an interesting interaction between σ^2 and σ_{f}^2: At low σ^2 and low σ_{f}^2, the males are even more narrowly distributed than the females because the male trait preferred by the most females does way better than anyone else. At low σ^2 and high σ_{f}^2, the male distribution matches the female distribution. At high σ^2, σ_{f}^2 does not matter and the male distribution is moderately narrow because average males can mate with most females.

References

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