Contents

1 Introduction

- "Because two-locus and other types of model cannot describe inheritance and selection for continuously varying traits (such as ornaments and preferences), these models are fundamentally disconnected from the empirical study of sexual selection," [7]

- "The male traits most likely to become exaggerated by such mechanisms are those under weak natural selection and subject to relatively large variance in female sexual preferences, such as some behavioral and morphological elements of courtship and mating," [6]

- quantitative genetics model / like mine: "Variation among females in their preferences selects for increased variance in the male trait; we assume that this effect is weak by requiring that $\sigma_z^2 \sigma_y^2 << 1$," [4] WHY?!?!

- two-allele model: "Neither copying model supports variation at the male trait locus, and copying makes it more difficult for a novel male trait phenotype to spread," [3]. "Since Lill's study, anecdotal evidence for copying has also been found in other species of birds, mammals, and fishes (reviewed by Gibson and Hoglund 1992; Pruett-Jones 1992)," [3]. "Bradbury and Gibson (1983) and Bradbury et al. (1985) found that the large variance of male mating success observed in sage grouse could not be plausibly attributed to independent female choice, and suggested that copying might be responsible," [3]. "Y-linkage is of interest because many of the loci that affect sexually-selected male coloration in guppies are Y-linked (Winge 1927; Haskins et al. 1961)," [3]. "Copying itself does not support polymorphism in the male trait, at least in haploid and Y-linked systems," [3]. "Guppies are a perplexing example in this context. The species is famous for highly polymorphic male coloration (Endler 1983) which is known to be under sexual selection (Endler 1980, 1983; Breden and Stoner 1987; Houde 1987, 1988; Stoner and Breden 1988; Houde and Endler 1990) and is at least in part caused by Y-linked loci (Winge 1927; Haskins et al 1961)," [3].

- "Mate preferences can be genetically determined (Saether et al. 2007), but they can also be learned (Virzijden et al. 2012). For example, in sticklebacks females learn to prefer mates with phenotypes similar to their fathers (Kozak et al. 2011) and in some cichlids females learn to prefer mates with phenotypes similar to their mothers (Verzijden et al. 2008). Recent work suggests that learned preferences for parental phenotypes can promote speciation (Verzijden et al. 2005; Servedio and Dukas 2013)," [1].

1. H1 Song learning can decelerate speciation by allowing for genetically diverse birds to mate with each other, maintaining gene flow between subpopulations that might either begin to diverge

2. H2 Song learning can accelerate speciation by increasing standing genetic variation, which would allow for quicker divergence once new selection pressures arise [5]

Table 1: Summary of choices made in previous models

	Lachlan and Servedio [5]
♂trait(s)	allele A / a: song predisposition song
How ♂ trait is learned	
Inherited	
♀ trait(s)	
How ♀ trait is learned	
Inherited	
Errors	
Population structure	
Mating structure	
♀ preference	
Selection	

3. H3 Song learning can accelerate speciation because culturally inherited traits can evolve more quickly than genetically inherited ones [2]

2 Model

2.1 Continuous traits Each male has a song and each female has preference for a particular song. She will mate with males with songs other than her preferred song, but the probability of her doing so decreases as the potential mate's song gets less similar to her preferred song. Each female mates once and chooses a male according to her preferences and the distribution of songs present in the population. Each male, therefore, may breed multiple times or not at all. We assume that generations are non-overlapping, so once the adults breed they die and we can shift our focus to the new generation. To begin with, we assume that males acquire their songs from their fathers at birth and females acquire their preferences from their mothers at birth. We further assume an unbiased sex ratio. Before the new generation mates, each male has a small probability of "mutating" its song. This can be interpreted as a learning error or as innovation. After mutation, the new males and females mate.

Mathematically, each male has a song $x \in \mathbb{R}$ and each female has a preference $y \in \mathbb{R}$. The probability density of the male songs will be written $P_{\mathrm{m}}(x)$ and the probability density of female preferences will be written $P_{\mathrm{f}}(y)$. The preference of a female with preference y mates for a male with song x is

$$f_y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right),$$

which is maximal when $x = y$ and decreases as $|x - y|$ increases. The probability that a female with preference y choose a male with song x is

$$\frac{P_{\mathrm{m}}(x) f_y(x)}{\int_{\mathbb{R}} P_{\mathrm{m}}(x') f_y(x') dx'},$$

so the probability of a (x, y) pair mating is then

$$P_{\mathrm{mate}}(x, y) = \frac{P_{\mathrm{f}}(y) P_{\mathrm{m}}(x) f_y(x)}{\int_{\mathbb{R}} P_{\mathrm{m}}(x') f_y(x') dx'} = \frac{P_{\mathrm{f}}(y) P_{\mathrm{m}}(x) f_y(x)}{Z_y},$$

where Z_y is the normalizing factor $\int_{\mathbb{R}} P_{\mathrm{m}}(x') f_y(x') dx'$. Each such pair will produce a male with song x and a female with preference y. Before mating, the male's song changes to $x - \delta_{\mathrm{mut}}$ with probability $p_{\mathrm{mut}}/2$ and to $x + \delta_{\mathrm{mut}}$ with probability $p_{\mathrm{mut}}/2$. Under these assumptions, the probability density of female preferences does not change over time:

$$P_{\mathrm{f}}(y, t+1) = P_{\mathrm{f}}(y, t).$$

The probability density of male songs in the next generation, before mutating and after mutating, follows

$$P_{\mathrm{m}}(x, t+1/2) = \int_{\mathbb{R}} P_{\mathrm{mate}}(x, y, t) dy$$

$$P_{\mathrm{m}}(x, t+1) = (1 - p_{\mathrm{mut}}) P_{\mathrm{m}}(x, t+1/2) + p_{\mathrm{mut}}/2 P_{\mathrm{m}}(x - \delta_{\mathrm{mut}}, t+1/2) + p_{\mathrm{mut}}/2 P_{\mathrm{m}}(x + \delta_{\mathrm{mut}}, t+1/2)$$

Table 2: Parameters

σ^2	width of female preference function
σ_{f}^2	variance of female preferences within each population
σ_{m}^2	variance of male traits within each population

(Population dynamics are modeled in Code **??**. Global and / or needed parameters are given in Code 2.)

We start with mixtures of normal distributions of both male and female:

$$P_{\mathrm{f}}(y,0) = p_{\mathrm{f}} \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}}^2}} \exp\left(-\frac{(y+1)^2}{2\sigma_{\mathrm{f}}^2}\right) + (1-p_{\mathrm{f}}) \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}}^2}} \exp\left(-\frac{(y-1)^2}{2\sigma_{\mathrm{f}}^2}\right)$$
$$P_{\mathrm{m}}(x,0) = p_{\mathrm{m}} \frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}}^2}} \exp\left(-\frac{(x+1)^2}{2\sigma_{\mathrm{m}}^2}\right) + (1-p_{\mathrm{m}}) \frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}}^2}} \exp\left(-\frac{(x-1)^2}{2\sigma_{\mathrm{m}}^2}\right)$$

Even with a continuous distribution of female preferences, we find that after several generations, the male song distribution is concentrated in several discrete peaks (Figure **??**). (Parameters and initial conditions given in Code **??**.)

There are three critical parameters:

Mutations give rise to a small number of males with song traits that are slightly higher (lower) than the rest of the population. The "edge" females that most prefer these extreme songs have limited options because they do not like the songs most males are singing. They therefore mate with the new "edge" males with very high probability, which leads to a very fast increase of these songs. As mutations continue to generate new songs, extreme songs will be selected for at the expense of similar songs closer to the middle of the song range. Ultimately, this results in a number of discrete songs at high density.

Ignoring mutations for the moment, the rate of change of the probability density of a particular song is given by

$$\frac{P_{\mathrm{m}}(x,t+1)}{P_{\mathrm{m}}(x,t)} = \int_{\mathbb{R}} \frac{P_{\mathrm{f}}(y) f_y(x)}{Z_y} dy \tag{1}$$

This shows that the normalizing factor Z_y affects how the density of male songs changes from one generation to the next: if a male is recognized by females with very few options (small Z_y), then the density of that song will increase more than the song a male recognized by females with many options (large Z_y) (Figure **??**).

2.2 Analysis

Claim 2.1 $\exp\left(-\frac{(x-\mu_{\mathrm{a}})^2}{2\sigma_{\mathrm{a}}^2}\right) \times \exp\left(-\frac{(x-\mu_{\mathrm{b}})^2}{2\sigma_{\mathrm{b}}^2}\right) = \exp\left(-\frac{(x-\mu_{\mathrm{ab}})^2}{2\sigma_{\mathrm{ab}}^2}\right) \times \exp\left(-\frac{(\mu_{\mathrm{a}}-\mu_{\mathrm{b}})^2}{2(\sigma_{\mathrm{a}}^2+\sigma_{\mathrm{b}}^2)}\right)$ where $\mu_{\mathrm{ab}} = \frac{\mu_{\mathrm{a}}\sigma_{\mathrm{b}}^2+\mu_{\mathrm{b}}\sigma_{\mathrm{a}}^2}{\sigma_{\mathrm{a}}^2+\sigma_b^2}$ and $\sigma_{\mathrm{ab}}^2 = \frac{\sigma_{\mathrm{a}}^2\sigma_{\mathrm{b}}^2}{\sigma_{\mathrm{a}}^2+\sigma_{\mathrm{b}}^2}$

Formulas

$$P_{\mathrm{m}}(x,y) \sim N\left(\mu_{\mathrm{m}}, \Sigma_{\mathrm{m}}\right) \text{ where } \quad \mu_{\mathrm{m}} = \begin{pmatrix} \mu_{\mathrm{m}x} \\ \mu_{\mathrm{m}y} \end{pmatrix} \quad \Sigma_{\mathrm{m}} = \begin{pmatrix} \sigma_{\mathrm{m}x}^2 & \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \sigma_{\mathrm{m}y}^2 \end{pmatrix}$$

$$f_y(x) \sim N\left(y, \sigma^2\right)$$

$$\Rightarrow P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) = \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \bar{\mu}_{\mathrm{m}})^T \bar{\Sigma}_{\mathrm{m}}^{-1} (\vec{x}_{\mathrm{m}} - \bar{\mu}_{\mathrm{m}})\right) \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}} \exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)$$

$$\text{where } \bar{\mu}_{\mathrm{m}} = \begin{pmatrix} \frac{\sigma^2\mu_{\mathrm{m}x}}{\sigma^2+\sigma_{\mathrm{m}x}^2} + \frac{\sigma_{\mathrm{m}x}^2 y_2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ \mu_{\mathrm{m}y} + \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2}(y_2 - \mu_{\mathrm{m}x}) \end{pmatrix} = \begin{pmatrix} \frac{\sigma^2\mu_{\mathrm{m}x}}{\sigma^2+\sigma_{\mathrm{m}x}^2} + \frac{\sigma_{\mathrm{m}x}^2 y_2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ \mu_{\mathrm{m}y} + \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(y_2 - \mu_{\mathrm{m}x}) \end{pmatrix}$$

$$\text{and } \bar{\Sigma}_{\mathrm{m}} = \begin{pmatrix} \frac{\sigma^2\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} + \frac{\sigma_{\mathrm{m}x}^2(1-\rho_{\mathrm{m}}^2)}{\sigma^2+\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{m}y}^2 \end{pmatrix} = \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\Sigma_{\mathrm{m}} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} 0 & 0 \\ 0 & (1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2 \end{pmatrix}$$

$$\Rightarrow \int\int P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) dx_1 dy_1 = \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}} \exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)$$

$$\Rightarrow \frac{P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) dx_1 dy_1} = \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \bar{\mu}_{\mathrm{m}})^T \bar{\Sigma}_{\mathrm{m}}^{-1} (\vec{x}_{\mathrm{m}} - \bar{\mu}_{\mathrm{m}})\right)$$

$$\Rightarrow \frac{P_{\mathrm{f}}(x_2,y_2) P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1) f_{y_2}(x_1) dx_1 dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}} - \mu_{\mathrm{f}})^T \Sigma_{\mathrm{f}}^{-1} (\vec{x}_{\mathrm{f}} - \mu_{\mathrm{f}})\right) \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \bar{\mu}_{\mathrm{m}})^T \bar{\Sigma}_{\mathrm{m}}^{-1} (\vec{x}_{\mathrm{m}} - \bar{\mu}_{\mathrm{m}})\right)$$

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$$\text{OR}\ \frac{P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}}\exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)}$$

$$= P_{\mathrm{m}}(x_1,y_1)\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}\exp\left(-\frac{(y_2-x_1)^2}{2\sigma^2}\right)\exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2(-\sigma^2-\sigma_{\mathrm{m}x}^2)}\right)$$

$$= P_{\mathrm{m}}(x_1,y_1)\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}\exp\left(-\frac{\left(y_2-\left(\frac{(-\sigma^2-\sigma_{\mathrm{m}x}^2)x_1+\mu_{\mathrm{m}x}\sigma^2}{-\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(-\sigma^2-\sigma_{\mathrm{m}x}^2)}{-\sigma_{\mathrm{m}x}^2}}\right)\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2(-\sigma_{\mathrm{m}x}^2)}\right)$$

$$= \frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}\exp\left(-\frac{\left(y_2-\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right)$$

$$\Rightarrow \frac{P_{\mathrm{f}}(x_2,y_2)P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{\left(y_2-\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}$$

$$= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\frac{\sqrt{\sigma^2+\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2}}$$

$$\text{where } \bar{\mu}_{\mathrm{f}} = \begin{pmatrix} \mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right) \\ \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}+\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2} \end{pmatrix}$$

$$\text{and } \bar{\Sigma}_f = \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\Sigma_{\mathrm{f}}+\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\begin{pmatrix}(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}x}^2 & 0 \\ 0 & 0\end{pmatrix}$$

$$|\bar{\Sigma}_{\mathrm{f}}| = |\Sigma_{\mathrm{f}}|\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}$$

$$\Rightarrow \frac{P_{\mathrm{f}}(x_2,y_2)P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)\times$$

$$\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma^2}}\sqrt{\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

$$= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$$

CAN INTEGRATE y_1, x_2, y_2

3 Pairs

3.1 Integrate y_1 and y_2

$$
\begin{aligned}
P_{\mathrm{mate}}(x_1,y_1,x_2,y_2) &= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \\
\Rightarrow \int P_{\mathrm{mate}}dy_1 &= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{\frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}x}^2}}\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \\
&= \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right) \\
\Rightarrow \int\int P_{\mathrm{mate}}dy_1dy_2 &= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2(1-\rho_{\mathrm{f}}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}}\exp\left(-\frac{\left(x_2-\mu_{\mathrm{f}x}-\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2(1-\rho_{\mathrm{f}}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
&\quad \frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right) \\
\Rightarrow \int\int P_{\mathrm{mate}}dy_1dy_2 &= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2(1-\rho_{\mathrm{f}}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}}\exp\left(-\frac{\left(x_2-\mu_{\mathrm{f}x}-\sigma_{\mathrm{f}x}\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\left(x_1-\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\left(1-\frac{\rho_{\mathrm{f}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
&\quad \frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow P_{\mathrm{mate}}(x_1,x_2) &\sim N\left(\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{f}x} \end{pmatrix}, \begin{pmatrix} \sigma_{\mathrm{m}x}'^2 & \rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}' \\ \rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}' & \sigma_{\mathrm{f}x}'^2 \end{pmatrix}\right) \\
\text{where } \sigma_{\mathrm{m}x}'^2 &= \frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} \\
\text{and } \sigma_{\mathrm{f}x}'^2 &= \sigma_{\mathrm{f}x}^2 \\
\text{and } \rho_{x_1,x_2} &= \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}
\end{aligned}
$$

3.2 Integrate y_1 and x_2

$$
\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dy_1 = \frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})^T\bar{\Sigma}_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\bar{\mu}_{\mathrm{f}})\right)\frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)
$$

$$
\begin{aligned}
\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dy_1dx_2 &= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{\left(y_2-\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x})}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}+\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\right)\times \\
&\quad \frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{\left(y_2-\left(\mu_{\mathrm{f}y}+\frac{\sigma_{\mathrm{f}y}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2\mu_{\mathrm{f}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\right)\right)\right)^2}{2\sigma_{\mathrm{f}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\right)\times \\
&\quad \frac{1}{\sqrt{2\pi\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow P_{\mathrm{mate}}(x_1,y_2) &\sim N\left(\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{f}y} \end{pmatrix}, \begin{pmatrix} \sigma_{\mathrm{m}x}'^2 & \rho_{x_1,y_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}y}' \\ \rho_{x_1,y_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}y}' & \sigma_{\mathrm{f}y}'^2 \end{pmatrix}\right) \\
\text{where } \sigma_{\mathrm{m}x}'^2 &= \frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} \\
\text{and } \sigma_{\mathrm{f}y}'^2 &= \sigma_{\mathrm{f}y}^2 \\
\text{and } \rho_{x_1,y_2} &= \frac{\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}
\end{aligned}
$$

3.3 Integrate x_2 and y_2

$$\int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_2dy_2 = \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma^2_{\mathrm{m}x}}\right)}\sqrt{\frac{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2_{\mathrm{m}})\sigma^2_{\mathrm{m}y}}}\frac{1}{\sqrt{2\pi\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}}\exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}-\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma^2_{\mathrm{m}x}(\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y})}{(\sigma^2+\sigma^2_{\mathrm{m}x})^2}}\right)\times$$

$$\exp\left(-\frac{\left(y_1-\left(\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(x_1-\mu_{\mathrm{m}x})\right)\right)^2}{2(1-\rho^2_{\mathrm{m}})\sigma^2_{\mathrm{m}y}}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma'^2_{\mathrm{m}x}}}\exp\left(-\frac{(x_1-\mu'_{\mathrm{m}x})^2}{2\sigma'^2_{\mathrm{m}x}}\right)\times$$

$$\frac{1}{\sqrt{2\pi(1-\rho^2_{x_1,y_2})\sigma'^2_{\mathrm{m}y}}}\exp\left(-\frac{(y_1-(\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(\mu'_{\mathrm{m}x}-\mu_{\mathrm{m}x})+\frac{\rho_{x_1,y_1}\sigma'_{\mathrm{m}y}}{\sigma'_{\mathrm{m}x}}(x_1-\mu'_{\mathrm{m}x})))^2}{2(1-\rho^2_{x_1,y_1})\sigma'^2_{\mathrm{m}y}}\right)$$

$$\text{where } \rho_{x_1,y_1} = \rho_{\mathrm{m}}\sqrt{\frac{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+\sigma^2_{\mathrm{f}y}\sigma^2_{\mathrm{m}x}}{\sigma^2(\sigma^2+\sigma^2_{\mathrm{m}x})+(1-\rho^2_{\mathrm{m}})\sigma^2_{\mathrm{m}x}(\sigma^2+\sigma^2_{\mathrm{m}x})+\rho^2_{\mathrm{m}}\sigma^2_{\mathrm{m}x}\sigma^2_{\mathrm{f}y}}}$$

$$= \frac{1}{\sqrt{2\pi\sigma'^2_{\mathrm{m}x}}}\exp\left(-\frac{(x_1-\mu'_{\mathrm{m}x})^2}{2\sigma'^2_{\mathrm{m}x}}\right)\frac{1}{\sqrt{2\pi(1-\rho^2_{x_1,y_2})\sigma'^2_{\mathrm{m}y}}}\exp\left(-\frac{(y_1-(\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})+\frac{\rho_{x_1,y_1}\sigma'_{\mathrm{m}y}}{\sigma'_{\mathrm{m}x}}(x_1-\mu'_{\mathrm{m}x})))^2}{2(1-\rho^2_{x_1,y_2})\sigma'^2_{\mathrm{m}y}}\right)$$

$$\Rightarrow P_{\mathrm{mate}}(x_1,y_1) \sim N\left(\begin{pmatrix}\frac{\sigma^2}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{m}x}+\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}\mu_{\mathrm{f}y}\\ \mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma^2_{\mathrm{m}x}}{\sigma^2+\sigma^2_{\mathrm{m}x}}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\end{pmatrix}, \begin{pmatrix}\sigma'^2_{\mathrm{m}x} & \rho_{x_1,y_1}\sigma'_{\mathrm{m}x}\sigma'_{\mathrm{m}y}\\ \rho_{x_1,y_1}\sigma'_{\mathrm{m}x}\sigma'_{\mathrm{m}y} & \sigma'^2_{\mathrm{m}y}\end{pmatrix}\right)$$

3.4 Integrate y_2 and x_1

$$
\begin{aligned}
\int P_{\mathrm{mate}}(x_1, y_1, x_2, y_2)dy_2 = {} & \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}} \exp\left(-\frac{\left(x_2-\left(\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
& \exp\left(-\frac{\left(x_1-\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{P_{\mathrm{m}}(x_1,y_1)}{\exp\left(-\frac{(x_1-\mu_{\mathrm{m}x})^2}{2\sigma_{\mathrm{m}x}^2}\right)}\sqrt{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}} \\
= {} & \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}} \exp\left(-\frac{\left(x_2-\left(\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
& \frac{1}{2\pi\sqrt{\sigma_{\mathrm{m}x}'^2}}\exp\left(-\frac{\left(x_1-\left(\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(x_1-\mu_{\mathrm{m}x})\right)^2}{2(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}\right) \\
= {} & \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}} \exp\left(-\frac{\left(x_2-\left(\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right)\right)\right)^2}{2\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}x}^2}\right)\times \\
& \frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}x}'^2}}\exp\left(-\frac{\left(x_1-\left(\frac{\mu_{\mathrm{m}x}\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\frac{\sigma_{\mathrm{m}x}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\frac{1}{2\pi\sqrt{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{(x_1-(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})))^2}{2\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}\right){\color{red}\exp\left(-\frac{(y_1-\mu_{\mathrm{m}y})^2}{2\sigma_{\mathrm{m}y}^2}\right)} \\
= {} & \frac{1}{2\pi\sqrt{|\Sigma_{x_1,x_2}|}}\exp\left(-(v-\mu_{x_1,x_2})^T\Sigma_{x_1,x_2}^{-1}(v-\mu_{x_1,x_2})\right)\times\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{(x_1-(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})))^2}{2\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}\right)
\end{aligned}
$$

$$
\text{where } v = (x_1, x_2)^T,\ \mu_{x_1,x_2} = \begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{f}x} \end{pmatrix}
$$

$$
\text{and } \Sigma_{x_1,x_2} = \begin{pmatrix} \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\sigma_{\mathrm{m}x}^2 & \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}y}{\color{red}x?} \\ \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sigma_{\mathrm{m}x}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{f}y}{\color{red}x?} & \sigma_{\mathrm{f}x}^2 \end{pmatrix}
$$

$$
= \frac{1}{2\pi\sqrt{|\Sigma_{x_1,x_2}|}}\exp\left(-(v-\nu)^TM^{-1}(v-\nu)\right)\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_{\mathrm{m}}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})-\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)^2}{2\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}+\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\sigma_{\mathrm{m}x}^2}\right)
$$

$$
\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\sigma_{\mathrm{m}x}^2+\frac{1-\rho_\mathrm{m}^2}{\rho_\mathrm{m}^2}\sigma_{\mathrm{m}x}^2=\sigma_{\mathrm{m}x}^2\left(\frac{\rho_\mathrm{m}^2(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2)+(\sigma^2+\sigma_{\mathrm{m}x}^2)^2(1-\rho_\mathrm{m}^2)}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)
$$

$$
=\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)
$$

$$
=\frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}'^2}{\sigma_{\mathrm{m}y}^2\rho_\mathrm{m}^2}
$$

$$
\Rightarrow \nu=\begin{pmatrix}\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_\mathrm{m}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_\mathrm{m}^2}{\rho_\mathrm{m}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}} \\ \mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_\mathrm{m}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})-\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)\right)\end{pmatrix}
$$

$$
=\begin{pmatrix}\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_\mathrm{m}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_\mathrm{m}^2}{\rho_\mathrm{m}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}} \\ \mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(\frac{1}{\rho_\mathrm{m}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)\end{pmatrix}
$$

$$
=\begin{pmatrix}\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_\mathrm{m}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_\mathrm{m}^2}{\rho_\mathrm{m}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}} \\ \mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}\sigma_{\mathrm{m}x}}{\rho_\mathrm{m}\sigma_{\mathrm{m}y}\sigma_{\mathrm{m}x}^2\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_\mathrm{m}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)\end{pmatrix}
$$

$$
=\begin{pmatrix}\dfrac{\left(\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)\left(\mu_{\mathrm{m}x}+\frac{1}{\rho_\mathrm{m}}\frac{\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}(y_1-\mu_{\mathrm{m}y})\right)+\frac{1-\rho_\mathrm{m}^2}{\rho_\mathrm{m}^2}\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y}\right)}{\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}} \\ \mu_{\mathrm{f}x}+\dfrac{\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}y}\sigma_{\mathrm{m}x}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_\mathrm{m}(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}\right)}\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_\mathrm{m}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)\end{pmatrix}
$$

$$
\rho_{x_1,x_2}\sigma_{\mathrm{m}x}'=\frac{\rho_\mathrm{f}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\frac{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}x}=\frac{\rho_\mathrm{f}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}
$$

$$
\Rightarrow \nu_2=\mu_{\mathrm{f}x}+\sigma_{\mathrm{f}x}\frac{\rho_\mathrm{m}\rho_\mathrm{f}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sigma_{\mathrm{m}y}\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\right)}\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_\mathrm{m}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\right)
$$

$$
=\mu_{\mathrm{f}x}+\frac{\sigma_{\mathrm{f}x}\rho_\mathrm{m}\rho_\mathrm{f}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}'\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}(y_1-\mu_{\mathrm{m}y}')
$$

$$
=\mu_{\mathrm{f}x}+\frac{\rho_{y_1,x_2}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}y}'}(y_1-\mu_{\mathrm{m}y}')
$$

where $\rho_{y_1,x_2}=\dfrac{\rho_\mathrm{m}\rho_\mathrm{f}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}$

$$
\begin{aligned}
\text{and } M &= \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2}\Sigma_{x_1,x_2} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{(1-\rho_{x_1,x_2}^2)\sigma_{\mathrm{m}x}'^2\sigma_{\mathrm{f}y x?}{}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2} \end{pmatrix} \\
&= \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2}\Sigma_{x_1,x_2} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{|\Sigma_{x_1,x_2}|}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2} \end{pmatrix} \\
\Rightarrow |M| &= \left(\frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2}\right)^2 |\Sigma_{x_1,x_2}| + \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}x}'^2}{\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2\right)^2}|\Sigma_{x_1,x_2}| \\
&= \frac{\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2\right)\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2\right)}{\left(\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2\right)^2}|\Sigma_{x_1,x_2}| \\
&= \frac{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2}{\frac{1-\rho_{\mathrm{m}}^2}{\rho_{\mathrm{m}}^2}\sigma_{\mathrm{m}x}^2+\sigma_{\mathrm{m}x}'^2}|\Sigma_{x_1,x_2}| \\
&= \frac{(1-\rho_{\mathrm{m}}^2)(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}|\Sigma_{x_1,x_2}| \\
&= \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}{\sigma_{\mathrm{m}y}'^2}|\Sigma_{x_1,x_2}| \\
\text{and } \Rightarrow M_{2,2} &= \left(1-\frac{\sigma_{\mathrm{m}x}'^2\rho_{x_1,x_2}^2}{\sigma_{\mathrm{m}x}^2\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\rho_{\mathrm{m}}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}}\right)\sigma_{\mathrm{f}y x?}{}^2 \\
&= \left(1-\frac{\rho_{\mathrm{f}}^2\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}\right)\sigma_{\mathrm{f}y x?}{}^2 \\
&= \left(1-\rho_{y_1,x_2}^2\right)\sigma_{\mathrm{f}y x?}{}^2
\end{aligned}
$$

$$
\Rightarrow \int P_{\mathrm{mate}}dy_2 = \frac{1}{2\pi\sqrt{|M|\left(\frac{\sigma_{\mathrm{m}y}'^2}{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}\right)}}\exp\left(-(v-\nu)^T M^{-1}(v-\nu)\right)\frac{1}{\sqrt{2\pi(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{(y_1-\mu_{\mathrm{m}y}')^2}{2\frac{\rho_{\mathrm{m}}^2\sigma_{\mathrm{m}y}^2}{\sigma_{\mathrm{m}x}^2}\frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}'^2}{\sigma_{\mathrm{m}y}^2\rho_{\mathrm{m}}^2}}\right)
$$

$$
\Rightarrow \int\int P_{\mathrm{mate}}dx_1dy_2 = \frac{1}{\sqrt{2\pi(1-\rho_{y_1,x_2}^2)\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{\left(x_2-\mu_{\mathrm{f}x}-\frac{\rho_{y_1,x_2}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{m}y}'}(y_1-\mu_{\mathrm{m}y}')\right)^2}{2(1-\rho_{y_2,x_1}^2)\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{(y_1-\mu_{\mathrm{m}y}')^2}{2\sigma_{\mathrm{m}y}'^2}\right)
$$

$$
\Rightarrow \int\int P_{\mathrm{mate}}dx_1dy_2 \sim N\left(\begin{pmatrix}\mu_{\mathrm{m}y}' \\ \mu_{\mathrm{f}x}\end{pmatrix},\begin{pmatrix}\sigma_{\mathrm{m}y}'^2 & \rho_{y_1,x_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}x} \\ \rho_{y_1,x_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}x} & \sigma_{\mathrm{f}x}^2\end{pmatrix}\right)
$$

3.5 Integrate x_1 and x_2

$$\frac{P_{\mathrm{f}}(x_2,y_2)P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)}{\int\int P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1)dx_1dy_1} = \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{f}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\frac{1}{2\pi\sqrt{|\bar{\Sigma}_{\mathrm{m}}|}}\exp\left(-\frac{1}{2}(\vec{x}_m-\bar{\mu}_{\mathrm{m}})^T\bar{\Sigma}_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}}-\bar{\mu}_{\mathrm{m}})\right)$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_1dx_2 = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(1-\rho_{\mathrm{m}})^2\right)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\frac{1}{\sigma^2+\sigma_{\mathrm{m}x}^2}(y_2-\mu_{\mathrm{m}x})\right)^2}{2\left(\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}+\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(1-\rho_{\mathrm{m}}^2)\right)\sigma_{\mathrm{m}y}^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\times$$

$$\frac{1}{\sqrt{2\pi\left(1-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}^2\right)\sigma_{\mathrm{m}y}^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{1}{\sigma_{\mathrm{f}y}}\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}}\frac{\sigma_{\mathrm{m}y}\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}}{\sigma^2+\sigma_{\mathrm{m}x}^2}(y_2-\mu_{\mathrm{m}x})\right)^2}{2\left(1-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}^2\right)\sigma_{\mathrm{m}y}^2}\right)$$

$$\left(1-\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}^2\right)\sigma_{\mathrm{m}y}^2 = (\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+\sigma_{\mathrm{m}x}^2-\rho_{\mathrm{m}}^2\sigma_{\mathrm{m}x}^2)\frac{\sigma_{\mathrm{m}y}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$= \frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}y}^2((\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$= \left(1-\frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}\right)\frac{\sigma_{\mathrm{m}y}^2((\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2}$$

$$\text{If } \sigma_{\mathrm{m}y}'^2 = \frac{\sigma_{\mathrm{m}y}^2((\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2},\ \sigma_{\mathrm{f}y}'^2=\sigma_{\mathrm{f}y}^2,\ \text{and } \rho_{y_1,y_2} = \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sqrt{(\sigma^2+\sigma_{\mathrm{m}x}^2)(\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2\rho_{\mathrm{m}}^2}}$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_1dx_2 = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(y_2-\mu_{\mathrm{m}x})\right)^2}{2(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(y_2-\mu_{\mathrm{f}y})\right)^2}{2(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{f}y}^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{f}y})^2}{2\sigma_{\mathrm{f}y}^2}\right)\frac{1}{\sqrt{2\pi(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}}\exp\left(-\frac{\left(y_1-\mu_{\mathrm{m}y}-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})-\frac{\rho_{y_1,y_2}\sigma_{\mathrm{m}y}'}{\sigma_{\mathrm{f}y}}(y_2-\mu_{\mathrm{f}y})\right)^2}{2(1-\rho_{y_1,y_2}^2)\sigma_{\mathrm{m}y}'^2}\right)$$

$$\Rightarrow \int\int P_{\mathrm{mate}}(x_1,y_1,x_2,y_2)dx_1dx_2 \sim N\left(\begin{pmatrix}\mu_{\mathrm{m}y}+\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y}-\mu_{\mathrm{m}x})\\ \mu_{\mathrm{f}y}\end{pmatrix},\begin{pmatrix}\sigma_{\mathrm{m}y}'^2 & \rho_{y_1,y_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}y}'\\ \rho_{y_1,y_2}\sigma_{\mathrm{m}y}'\sigma_{\mathrm{f}y}' & \sigma_{\mathrm{f}y}'^2\end{pmatrix}\right)$$

$$
\begin{aligned}
P_{\mathrm{m}}(x_1, y_1) f_{y_2}(x_1) &= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \mu_{\mathrm{m}})^T \Sigma_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}} - \mu_{\mathrm{m}})\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_2 - x_2)^2}{2\sigma^2}\right) \\
&= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}} \exp\left(-\frac{1}{2}(\vec{x}_m - \mu_{\mathrm{m}})^T \Sigma_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}} - \mu_{\mathrm{m}})\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}} - \nu)^T B^{-1}(\vec{x}_{\mathrm{m}} - \nu)\right) \\
\text{where } \nu &= \begin{pmatrix} y_2 \\ 0 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} \\
(\vec{x} - \mu)^T \Sigma^{-1}(\vec{x} - \mu) + (\vec{x} - \nu)^T B^{-1}(\vec{x} - \nu) &= \vec{x}^T(\Sigma^{-1} + B^{-1})\vec{x} - 2\vec{x}^T\Sigma^{-1}\mu - 2\vec{x}^T B^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= \vec{x}^T(\Sigma^{-1} + B^{-1})\vec{x} + -2\vec{x}^T(\Sigma^{-1} + B^{-1})(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= \vec{x}^T(\Sigma^{-1} + B^{-1})\vec{x} - 2\vec{x}^T(\Sigma^{-1} + B^{-1})(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1} + B^{-1})(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) \\
&- (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= (\vec{x} - (\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu))^T(\Sigma^{-1} + B^{-1})(\vec{x} - (\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu)) \\
&- (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&= (\vec{x} - c)^T C^{-1}(\vec{x} - c) \\
&- (\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^T B^{-1}\nu \\
&\text{where } c = (\Sigma^{-1} + B^{-1})^{-1}(\Sigma^{-1}\mu + B^{-1}\nu) \text{ and } C = (\Sigma^{-1} + B^{-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
|\Sigma| &= (1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2 \\
\Rightarrow \Sigma^{-1} &= \frac{1}{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2}\begin{pmatrix} \sigma_{\mathrm{m}y}^2 & -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \sigma_{\mathrm{m}x}^2 \end{pmatrix} = \frac{1}{1-\rho_{\mathrm{m}}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
\Rightarrow \Sigma^{-1}+B^{-1} &= \begin{pmatrix} \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}x}^2}+\frac{1}{\sigma^2} & -\frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} = \frac{1}{1-\rho_{\mathrm{m}}^2}\begin{pmatrix} \frac{\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
\Rightarrow |\Sigma^{-1}+B^{-1}| &= \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2} + \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma^2}\frac{1}{\sigma_{\mathrm{m}y}^2} = \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{1}{\sigma_{\mathrm{m}y}^2}\left(\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma^2\sigma_{\mathrm{m}x}^2}\right) \\
\Rightarrow (\Sigma^{-1}+B^{-1})^{-1} &= \frac{\sigma^2\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}y}^2} & \frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ \frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2\sigma_{\mathrm{m}x}^2} \end{pmatrix} = \begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}x}^2 & \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \frac{\sigma^2+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}y}^2 \end{pmatrix} = \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\Sigma + \begin{pmatrix} 0 & 0 \\ 0 & \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}y}^2 \end{pmatrix} \\
\text{Let } s &= \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \text{ and } D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}y}^2 \end{pmatrix} \text{ so } (\Sigma^{-1}+B^{-1})^{-1} = s\Sigma + D \\
\Rightarrow -(\Sigma^{-1}\mu+B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu &= -(\Sigma^{-1}\mu+B^{-1}\nu)(s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu \\
&= -s\mu^T\Sigma^{-1}\mu - s\mu^TB^{-1}\nu - \mu^T\Sigma^{-1}D\Sigma^{-1}\mu - s\nu^TB^{-1}\mu - s\nu^TB^{-1}\Sigma B^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^TB^{-1}\nu \\
\Sigma^{-1}D\Sigma^{-1} &= \frac{1}{(1-\rho_{\mathrm{m}}^2)^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
&= \frac{1}{(1-\rho_{\mathrm{m}}^2)^2}\begin{pmatrix} 0 & -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ 0 & \sigma_{\mathrm{m}x}^2\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
&= \frac{1}{(1-\rho_{\mathrm{m}}^2)^2}\begin{pmatrix} \rho_{\mathrm{m}}^2\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ -\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}}{\sigma_{\mathrm{m}y}}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{m}y}^2}\frac{1-\rho_{\mathrm{m}}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \end{pmatrix} \\
&= \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{\rho_{\mathrm{m}}^2}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} \\
&= \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{m}x}^2} & -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} \\ -\frac{\rho_{\mathrm{m}}}{\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}} & \frac{1}{\sigma_{\mathrm{m}y}^2} \end{pmatrix} + \frac{1}{1-\rho_{\mathrm{m}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\begin{pmatrix} \frac{\rho_{\mathrm{m}}^2-1}{\sigma_{\mathrm{m}x}^2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= (1-s)\Sigma^{-1} - sB^{-1} \\
\text{and } sB^{-1}\Sigma B^{-1} &= s\begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{\mathrm{m}x}^2 & \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ \rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & \sigma_{\mathrm{m}y}^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= s\begin{pmatrix} \frac{1}{\sigma^2}\sigma_{\mathrm{m}x}^2 & \frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= s\begin{pmatrix} \frac{1}{\sigma^4}\sigma_{\mathrm{m}x}^2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= (1-s)B^{-1} \\
\Rightarrow -(\Sigma^{-1}\mu+B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^TB^{-1}\nu &= -s\mu^T\Sigma^{-1}\mu - s\mu^TB^{-1}\nu - \mu^T\left((1-s)\Sigma^{-1}-sB^{-1}\right)\mu - s\nu^TB^{-1}\mu - (1-s)\nu^TB^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^TB^{-1}\nu \\
&= s\mu^TB^{-1}\mu - 2s\nu^TB^{-1}\mu + s\nu^TB^{-1}\nu \\
&= s(\nu-\mu)^TB^{-1}(\nu-\mu)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (\vec{x}-\mu)^T\Sigma^{-1}(\vec{x}-\mu)+(\vec{x}-\nu)^T B^{-1}(\vec{x}-\nu) &= (\vec{x}-c)^T C^{-1}(\vec{x}-c)+s(\nu-\mu)^T B^{-1}(\nu-\mu)\\
\text{where } C &= (\Sigma^{-1}+B^{-1})^{-1} = s\Sigma + D,\ \ c = C(\Sigma^{-1}\mu + B^{-1}\nu)\\
\Rightarrow c &= (s\Sigma + D)(\Sigma^{-1}\mu + B^{-1}\nu)\\
&= s\mu + s\Sigma B^{-1}\nu + D\Sigma^{-1}\mu\\
&= s\mu + s\begin{pmatrix} \frac{1}{\sigma^2}\sigma_{\mathrm{m}x}^2 & 0 \\ \frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} & 0 \end{pmatrix}\nu + (1-s)\begin{pmatrix} 0 & 0 \\ -\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}} & 1 \end{pmatrix}\mu\\
&= \begin{pmatrix} s\mu_{\mathrm{m}x} + s\frac{1}{\sigma^2}\sigma_{\mathrm{m}x}^2 y_2 \\ s\mu_{\mathrm{m}y} + s\frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} y_2 + (1-s)(-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\mu_{\mathrm{m}x} + \mu_{\mathrm{m}y}) \end{pmatrix}\\
&= \begin{pmatrix} s\mu_{\mathrm{m}x} + (1-s)y_2 \\ \mu_{\mathrm{m}y} + s\frac{1}{\sigma^2}\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y} y_2 + (1-s)(-\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\mu_{\mathrm{m}x}) \end{pmatrix}\\
&= \begin{pmatrix} s\mu_{\mathrm{m}x} + (1-s)y_2 \\ \mu_{\mathrm{m}y} + (1-s)\frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}(y_2 - \mu_{\mathrm{m}x}) \end{pmatrix}\\
|C| &= |(\Sigma^{-1}+B^{-1})^{-1}| = |\Sigma^{-1}+B^{-1}|^{-1}\\
&= \frac{(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{m}y}^2\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} = |\Sigma|\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\\
\Rightarrow P_{\mathrm{m}}(x_1,y_1)f_{y_2}(x_1) &= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-\mu_{\mathrm{m}})^T\Sigma_{\mathrm{m}}^{-1}(\vec{x}_{\mathrm{m}}-\mu_{\mathrm{m}})\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y_2-\mu_{\mathrm{m}x})^2}{2\sigma^2}\right)\\
&= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-c)^T C^{-1}(\vec{x}_{\mathrm{m}}-c)\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\mu_{\mathrm{m}x}-y_2)^2}{2\frac{\sigma^2+\sigma_{\mathrm{m}x}^2}{\sigma^2}\sigma^2}\right)\\
&= \frac{1}{2\pi\sqrt{|\Sigma_{\mathrm{m}}|\frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-c)^T C^{-1}(\vec{x}_{\mathrm{m}}-c)\right)\frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}}\exp\left(-\frac{(\mu_{\mathrm{m}x}-y_2)^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)\\
&= \frac{1}{2\pi\sqrt{|C|}}\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{m}}-c)^T C^{-1}(\vec{x}_{\mathrm{m}}-c)\right)\frac{1}{\sqrt{2\pi(\sigma^2+\sigma_{\mathrm{m}x}^2)}}\exp\left(-\frac{(\mu_{\mathrm{m}x}-y_2)^2}{2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)
\end{aligned}
$$

$$\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{\left(y_2-\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right)=\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\nu)^T B^{-1}(\vec{x}_{\mathrm{f}}-\nu)\right)$$

$$\text{where } \nu=\begin{pmatrix} 0 \\ \frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\end{pmatrix} \text{ and } B^{-1}=\begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}$$

$$(\vec{x}-\mu)^T\Sigma^{-1}(\vec{x}-\mu)+(\vec{x}-\nu)^T B^{-1}(\vec{x}-\nu)=(\vec{x}-\mathrm{c})^T C^{-1}(\vec{x}-\mathrm{c})-(\Sigma^{-1}\mu+B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^T B^{-1}\nu$$

$$\text{where } c=(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu) \text{ and } C=(\Sigma^{-1}+B^{-1})^{-1}$$

$$\Sigma^{-1}+B^{-1}=\frac{1}{1-\rho_{\mathrm{f}}^2}\begin{pmatrix}\frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2}\end{pmatrix}+\begin{pmatrix}0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}=\begin{pmatrix}\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{f}y}^2\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}$$

$$\Rightarrow|\Sigma^{-1}+B^{-1}|=\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{1}{\sigma_{\mathrm{f}x}^2}\frac{1}{\sigma_{\mathrm{f}y}^2}+\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma_{\mathrm{f}x}^2\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}=\frac{1}{1-\rho_{\mathrm{f}}^2}\frac{1}{\sigma_{\mathrm{f}x}^2\sigma_{\mathrm{f}y}^2}\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}$$

$$\text{Let } s=\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2} \text{ and } D=\begin{pmatrix}(1-\rho_{\mathrm{f}}^2)(1-s)\sigma_{\mathrm{f}x}^2 & 0 \\ 0 & 0\end{pmatrix}$$

$$\Rightarrow C=(\Sigma^{-1}+B^{-1})^{-1}=\begin{pmatrix}\sigma_{\mathrm{f}x}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{f}}^2)\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2} & -\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}s \\ -\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}s & \sigma_{\mathrm{f}y}^2 s\end{pmatrix}=s\Sigma+D$$

$$\Rightarrow c=(s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)=s\mu+s\Sigma B^{-1}\nu+D\Sigma^{-1}\mu$$

$$=s\mu+s\begin{pmatrix}0 & \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \\ 0 & \sigma_{\mathrm{f}y}^2\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\end{pmatrix}\nu+\begin{pmatrix}(1-s) & -\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-s) \\ 0 & 0\end{pmatrix}\mu$$

$$=\begin{pmatrix}\mu_{\mathrm{f}x}+\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}s\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-s)\mu_{\mathrm{f}y} \\ s\mu_{\mathrm{f}y}+s\sigma_{\mathrm{f}y}^2\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\end{pmatrix}$$

$$=\begin{pmatrix}\mu_{\mathrm{f}x}+\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-s)\left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}-\mu_{\mathrm{f}y}\right) \\ s\mu_{\mathrm{f}y}+(1-s)\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\end{pmatrix}$$

$$-(\Sigma^{-1}\mu+B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^T B^{-1}\nu=-(\Sigma^{-1}\mu+B^{-1}\nu)(s\Sigma+D)(\Sigma^{-1}\mu+B^{-1}\nu)+\mu^T\Sigma^{-1}\mu+\nu^T B^{-1}\nu$$

$$=-s\mu^T\Sigma^{-1}\mu-s\mu^T B^{-1}\nu-\mu^T\Sigma^{-1}D\Sigma^{-1}\mu-s\nu^T B^{-1}\mu-s\nu^T B^{-1}\Sigma B^{-1}\nu+\mu^T\Sigma^{-1}\mu+\nu^T B^{-1}\nu$$

$$
\begin{aligned}
\Sigma^{-1}D\Sigma^{-1} &= \frac{1}{(1-\rho_{\mathrm{f}}^2)^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2} \end{pmatrix}\begin{pmatrix} (1-\rho_{\mathrm{f}}^2)(1-s)\sigma_{\mathrm{f}x}^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \\
&= \frac{1}{(1-\rho_{\mathrm{f}}^2)^2}\begin{pmatrix} (1-\rho_{\mathrm{f}}^2)(1-s) & 0 \\ -\frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}x}}{\sigma_{\mathrm{f}y}}(1-\rho_{\mathrm{f}}^2)(1-s) & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{1}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \\
&= \frac{1-s}{1-\rho_{\mathrm{f}}^2}\begin{pmatrix} \frac{1}{\sigma_{\mathrm{f}x}^2} & -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} \\ -\frac{\rho_{\mathrm{f}}}{\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}} & \frac{\rho_{\mathrm{f}}^2}{\sigma_{\mathrm{f}y}^2} \end{pmatrix} \\
&= (1-s)\Sigma^{-1} - sB^{-1} \\
sB^{-1}\Sigma B^{-1} &= s\begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix}\begin{pmatrix} \sigma_{\mathrm{f}x}^2 & \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} \\ \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} & \sigma_{\mathrm{f}y}^2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} \\
&= s\begin{pmatrix} 0 & 0 \\ \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} & \sigma_{\mathrm{f}y}^2\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} \\
&= s\begin{pmatrix} 0 & 0 \\ 0 & \sigma_{\mathrm{f}y}^2\left(\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}\frac{\sigma_{\mathrm{m}x}}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)} \end{pmatrix} = (1-s)B^{-1} \\
\Rightarrow -(\Sigma^{-1}\mu + B^{-1}\nu)(\Sigma^{-1}+B^{-1})^{-1}(\Sigma^{-1}\mu+B^{-1}\nu) + \mu^T\Sigma^{-1}\mu + \nu^TB^{-1}\nu &= -s\mu^T\Sigma^{-1}\mu - s\mu^TB^{-1}\nu - \mu^T\left((1-s)\Sigma^{-1} - sB^{-1}\right)\mu - s\nu^TB^{-1}\mu - (1-s)\nu^TB^{-1}\nu + \mu^T\Sigma^{-1}\mu + \nu^TB^{-1}\nu \\
&= s\nu^TB^{-1}\nu - 2s\mu^TB^{-1}\nu + \mu^TsB^{-1}\mu \\
&= (\mu-\nu)^TsB^{-1}(\mu-\nu) \\
&= \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}\left(\mu_{\mathrm{f}y} - \frac{(\sigma^2+\sigma_{\mathrm{m}x})^2x_1 - \mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)^2 \\
&= \frac{\left(\mu_{\mathrm{f}y} - \frac{(\sigma^2+\sigma_{\mathrm{m}x})^2x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)^2}{\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}
\end{aligned}
$$

$$
\Rightarrow \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})^T\Sigma_{\mathrm{f}}^{-1}(\vec{x}_{\mathrm{f}}-\mu_{\mathrm{f}})\right)\exp\left(-\frac{\left(y_2 - \left(\frac{(\sigma^2+\sigma_{\mathrm{m}x}^2)x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)}{\sigma_{\mathrm{m}x}^2}}\right) = \exp\left(-\frac{1}{2}(\vec{x}_{\mathrm{f}}-\mathrm{c})^TC^{-1}(\vec{x}_{\mathrm{f}}-\mathrm{c})\right)\exp\left(-\frac{\left(\mu_{\mathrm{f}y} - \frac{(\sigma^2+\sigma_{\mathrm{m}x})^2x_1-\mu_{\mathrm{m}x}\sigma^2}{\sigma_{\mathrm{m}x}^2}\right)^2}{2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma_{\mathrm{m}x}^2}}\right)
$$

$$P_{\mathrm{mate}}(x_1, y_1, x_2, y_2) \sim N\left(\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{m}x} + \frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}\mu_{\mathrm{f}y} \\ \mu_{\mathrm{m}y} + \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}y}}{\sigma_{\mathrm{m}x}}\frac{\sigma_{\mathrm{m}x}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2}(\mu_{\mathrm{f}y} - \mu_{\mathrm{m}x}) \\ \mu_{\mathrm{f}x} \\ \mu_{\mathrm{f}y} \end{pmatrix}, \Sigma'\right)$$

$$\text{where } \Sigma' = \begin{pmatrix} \sigma_{\mathrm{m}x}^2 \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_{\mathrm{m}}\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\sigma'_{\mathrm{m}x}\sigma'_{\mathrm{m}y} & \frac{\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\sigma'_{\mathrm{m}x}\sigma_{\mathrm{f}x} & \frac{\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\sigma'_{\mathrm{m}x}\sigma_{\mathrm{f}y} \\ & \sigma_{\mathrm{m}y}^2\frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_{\mathrm{m}}\rho_{\mathrm{f}}\sigma_{\mathrm{f}y}\sigma_{\mathrm{m}x}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\sigma'_{\mathrm{m}y}\sigma_{\mathrm{f}x} & \frac{\rho_{\mathrm{m}}\sigma_{\mathrm{m}x}\sigma_{\mathrm{f}y}}{\sqrt{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_{\mathrm{m}}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_{\mathrm{m}}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}}\sigma'_{\mathrm{m}y}\sigma_{\mathrm{f}y} \\ & & \sigma_{\mathrm{f}x}^2 & \rho_{\mathrm{f}}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} \\ & & & \sigma_{\mathrm{f}y}^2 \end{pmatrix}$$

$$\Sigma' = \begin{pmatrix} \sigma_{\mathrm{m}x}^2 \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_\mathrm{m}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\left(\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2\right)}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_\mathrm{f}\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\sigma_{\mathrm{m}x}^2\sigma_{\mathrm{f}y}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ & \sigma_{\mathrm{m}y}^2 \frac{\sigma^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+(1-\rho_\mathrm{m}^2)\sigma_{\mathrm{m}x}^2(\sigma^2+\sigma_{\mathrm{m}x}^2)+\rho_\mathrm{m}^2\sigma_{\mathrm{f}y}^2\sigma_{\mathrm{m}x}^2}{(\sigma^2+\sigma_{\mathrm{m}x}^2)^2} & \frac{\rho_\mathrm{m}\rho_\mathrm{f}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y}}{\sigma^2+\sigma_{\mathrm{m}x}^2} & \frac{\rho_\mathrm{m}\sigma_{\mathrm{m}x}\sigma_{\mathrm{m}y}\sigma_{\mathrm{f}y}^2}{\sigma^2+\sigma_{\mathrm{m}x}^2} \\ & & \sigma_{\mathrm{f}x}^2 & \rho_\mathrm{f}\sigma_{\mathrm{f}x}\sigma_{\mathrm{f}y} \\ & & & \sigma_{\mathrm{f}y}^2 \end{pmatrix}$$

	song from father	song obliquely learned	song genetic
pref genetic	$\lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \sigma_y^2 = 0$ unless $\sigma = 0$	$\lim_{t\to\infty} \sigma_x^2 = \sigma_x(0)^2, \lim_{t\to\infty} \sigma_y^2 = 0$	bistability: $\lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \sigma_y^2 = 0$ or $\lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \sigma_y^2 = \infty$
from father	$\lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \sigma_y^2 = 0$ unless $\sigma^2 = 0$	$\lim_{t\to\infty} \sigma_x^2 = \sigma_x^2(0), \lim_{t\to\infty} \sigma_y^2 = \frac{\sigma_x^2(\sigma^2+\sigma_x^2)}{2\sigma_x^2+\sigma^2}$ unless $\sigma^2 = 0$	$\lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \sigma_y^2 = 0$ unless $\sigma^2 = 0 \Rightarrow \lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \sigma_y^2 = \sigma_y^2(0)$
from moth	$\lim_{t\to\infty} \sigma_x^2 = \sigma_y^2 - \sigma^2, \lim_{t\to\infty} \sigma_y^2 = \sigma_y^2(0)$	$\lim_{t\to\infty} \sigma_x^2 = \sigma_x^2(0), \lim_{t\to\infty} \sigma_y^2 = \sigma_y^2(0)$	as long as $\sigma_y^2 > (\frac{30+\sqrt{864}}{18})\sigma^2$ bistability: $\lim_{t\to\infty} \sigma_x^2 = \lim_{t\to\infty} \mathrm{Cov} = 0$ or $\lim_{t\to\infty} \sigma_x^2 = \sigma_x^{2*} = \frac{3\sigma_y^2-5\sigma^2+\sqrt{\sigma^4+\sigma_y^2(9\sigma_y^2-30\sigma^2)}}{6}$ ($\lim_{\sigma_y\to\infty}$ = something like $\sigma_y^2 - \sigma^2$?) $\lim_{t\to\infty} \mathrm{Cov} = \frac{\sigma_x^{2*}\sigma_y^2}{\sigma^2+\sigma_x^{2*}}$

Learning of songs can either destabilize or stabilize male song distribution and / or female preference distribution, depending on the learning set / imprinting set (what's Servedio's term?!?). Learning of preferences can stabilize female preference distribution but doesn't affect male song distribution.

4 Equilibrium for various modes of inheritance

4.1 Song from father, pref genetic $\rho = \rho_{\mathrm{m}}$, $\rho_{\mathrm{f}} = 0$

$$
\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} \\
\sigma_y(t+1)^2 &= \frac{1}{4}\sigma_y^2 \frac{\sigma^2(\sigma^2+\sigma_x^2)+(1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2)+\rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\rho\sigma_x\sigma_y^3}{\sigma^2+\sigma_x^2} + \frac{1}{4}\sigma_y^2 \\
&= \frac{\sigma_y^2}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + (1-\rho^2)\sigma_x^2\frac{\sigma^2+\sigma_x^2-\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) \\
\mathrm{Cov}_{x,y}(t+1) &= \frac{1}{2}\frac{\rho\sigma_x\sigma_y\left(\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2\right)}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} \\
\text{equilibrium } &\Rightarrow \sigma^2+\sigma_x^2 = \sigma_y^2 \Rightarrow \sigma_x < \sigma_y \\
&\Rightarrow \sigma_y(t+1)^2 = \frac{\sigma_y^2}{4}\left(1+\frac{2\rho\sigma_x}{\sigma_y}+1\right) \\
&\Rightarrow \sigma_y(t+1)^2 < \sigma_y(t)^2 \\
\Rightarrow \lim_{t\to\infty}\sigma_x(t)^2 &= \lim_{t\to\infty}\sigma_y(t)^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{If } \sigma^2 = 0,\ \sigma_x(t+1)^2 &= \sigma_y^2 \\
\sigma_y(t+1)^2 &= \frac{\sigma_y^2}{4}\left(\frac{\sigma_y^2}{\sigma_x^2} + (1-\rho^2)\frac{\sigma_x^2-\sigma_y^2}{\sigma_x^2} + \frac{2\rho\sigma_y}{\sigma_x} + 1\right) \\
\mathrm{Cov}(t+1) &= \frac{1}{2}\frac{\rho\sigma_x\sigma_y\sigma_y^2}{\sigma_x^2} + \frac{1}{2}\sigma_y^2 \\
\text{equilibrium } &\Rightarrow \sigma_x^2 = \sigma_y^2 \\
&\Rightarrow 2+2\rho = 4 \\
&\Rightarrow \rho = 1
\end{aligned}
$$

I have not been able to find an analytic solution to the equilibrium variance or to show *why* there is non-zero equilibrium variance. But I can show in song_learned_pref_genetic.R that there is non-zero equilibrium for every initial ρ and ratio of σ_x to σ_y, which must be stable because I find it numerically by running the above recursive equations. (As σ^2 decreases, the rate at which variance disappears slows down. It looks like, regardless of σ^2, points always move toward the one-to-one line and as σ^2 decreases movement along the line decreases.)

Mathematical aside

There are three contributions to the distribution of preferences in the population:

1. from the male preferences $\frac{\sigma^2(\sigma^2+\sigma_x^2)+(1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2)+\rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} = 1 + \rho^2\sigma_x^2\frac{\sigma_y^2-(\sigma^2+\sigma_x^2)}{(\sigma^2+\sigma_x^2)^2}$: If $\sigma^2+\sigma_x^2 > \sigma_y^2$, this is a concave-down quadratic minimized at $\rho = 1$ (for $\rho \in [0,1]$) and maximized at $\rho = 0$. In other words, if the preference distribution is sufficiently narrow, this contribution is highest when there is no correlation between male songs and preferences, because the male song distribution is being collapsed to be closer to the narrow

preference distribution. If $\sigma^2 + \sigma_x^2 < \sigma_y^2$, this is is a concave-up quadratic, maximized at $\rho = 1$ (for $\rho \in [0,1]$) and minimized at $\rho = 0$. If the preference distribution is large, this contribution is highest when the male songs and preferences are perfectly correlated so the preference distribution can be swept along with the widening song distribution.

2. from covariance between male preferences and female preferences $\frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2}$: This increases as ρ increases because the correlation between male and female preferences only arises through the interaction between (a) the correlation between male song and female preferences and (b) the correlation between male song and male preferences (ρ). When $\sigma^2 + \sigma_x^2 < \sigma_y^2$, the first and second factors interact such that their sum is maximized at an intermediate ρ. Their sum represents a tradeoff between avoiding getting swept along with the male distribution and increasing the correlation between male and female preferences.

3. the female preference 1

4.2 Song learned from father, pref imprinted from father $\rho_{\mathrm{m}} = \rho_{\mathrm{f}} = 0$

$$\sigma_x(t+1)^2 = \sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2}$$
$$\sigma_y(t+1)^2 = \sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2}$$

Cannot happen that both $\sigma_x^2 = \sigma_y^2 - \sigma^2$ and $\sigma_y^2 = \sigma_x^2$ unless $\sigma^2 = 0$, in which case $\lim_{t\to\infty} \sigma_x(t)^2 = \lim_{t\to\infty} \sigma_y(t)^2 = \sigma_y(0)^2$.

4.3 Song learned from father, pref learned from mother $\rho_{\mathrm{m}} = \rho_{\mathrm{f}} = 0$

$$\sigma_x(t+1)^2 = \sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2}$$
$$\sigma_x(t+1)^2 = \sigma_x^2 \Rightarrow \sigma^2 + \sigma_x^2 = \sigma_y^2$$

4.4 Song learned obliquely, pref genetic $\rho = \rho_{\mathrm{m}}$, $\rho_{\mathrm{f}} = 0$

$$\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2 \\
\sigma_y(t+1)^2 &= \frac{1}{4}\sigma_y^2 \frac{\sigma^2(\sigma^2+\sigma_x^2) + (1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2) + \rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\rho\sigma_x\sigma_y^3}{\sigma^2+\sigma_x^2} + \frac{1}{4}\sigma_y^2 \\
&= \frac{1}{4}\sigma_y^2 \frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_x^2(\sigma^2+\sigma_x^2)}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{4}\sigma_y^2 \text{ since after one generation } \rho \text{ disappears since} \\
&\quad \text{there can be no correlation between parents' traits and song model} \\
&= \frac{1}{2}\sigma_y^2 \\
\Rightarrow \lim_{t\to\infty} \sigma_y(t)^2 &= 0
\end{aligned}$$

4.5 Song learned obliquely, pref imprinted from father $\rho_{\mathrm{m}} = \rho_{\mathrm{f}} = 0$

$$\begin{aligned}
\sigma_x(t+1)^2 &= \sigma_x^2 \\
\sigma_y(t+1)^2 &= \sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2) + \sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} \\
\sigma_y(t+1)^2 \geq \sigma_y^2 &\Rightarrow \sigma_x^2(\sigma^2(\sigma^2+\sigma_x^2) + \sigma_x^2\sigma_y^2) \geq \sigma_y^2(\sigma^2+\sigma_x^2)^2 \\
\Rightarrow \left(\sigma^4 + 2\sigma^2\sigma_x^2 + \sigma_x^4 - \sigma_x^4\right)\sigma_y^2 &\leq \sigma_x^2\sigma^2(\sigma^2+\sigma_x^2) \\
\Rightarrow \sigma^2(\sigma^2 + 2\sigma_x^2)\sigma_y^2 &\leq \sigma^2\sigma_x^2(\sigma^2+\sigma_x^2) \\
\Rightarrow \sigma_y^2 &\leq \frac{\sigma_x^2(\sigma^2+\sigma_x^2)}{\sigma^2+2\sigma_x^2}
\end{aligned}$$

Stable equilibrium with $\sigma_x^2 = \sigma_x(0)^2$ and $\sigma_y^2 = \frac{\sigma_x^2(\sigma^2+\sigma_x^2)}{2\sigma_x^2+\sigma^2}$.

4.6 Song obliquely, pref obliquely
Neither distribution changes.

4.7 Song genetic, pref genetic $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{f}}$

$$\begin{aligned}
\sigma_x(t+1)^2 &= \frac{1}{4}\sigma_x^2 \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{1}{2}\frac{\rho\sigma_x^3\sigma_y}{\sigma^2+\sigma_x^2} + \frac{1}{4}\sigma_x^2 \\
&= \frac{\sigma_x^2}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) \\
\sigma_y(t+1)^2 &= \frac{1}{4}\sigma_y^2\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+(1-\rho^2)\sigma_x^2(\sigma^2+\sigma_x^2)+\rho^2\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) \\
&= \frac{1}{4}\sigma_y^2\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2(\sigma^2+\sigma_x^2-\sigma_y^2)}{(\sigma^2+\sigma_x^2)^2} \\
\mathrm{Cov}(t+1) &= \frac{1}{4}\left(\rho\sigma_x\sigma_y\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} + \frac{\rho^2\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} + \rho\sigma_x\sigma_y\right) \\
&= \frac{\mathrm{Cov}(t)}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1\right) + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2}
\end{aligned}$$

$$\begin{aligned}
\text{Let } Q &= \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} + 1 \\
\Rightarrow \sigma_x(t+1)^2 &= \frac{Q}{4}\sigma_x^2 \\
\sigma_y(t+1)^2 &= \frac{Q}{4}\sigma_y^2 + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2(\sigma^2+\sigma_x^2-\sigma_y^2)}{(\sigma^2+\sigma_x^2)^2} \\
\mathrm{Cov}(t+1) &= \frac{Q}{4}\mathrm{Cov}(t) + \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2}
\end{aligned}$$

$$\begin{aligned}
&\text{equilibrium } \Rightarrow Q = 4 \\
&\Rightarrow \frac{1-\rho^2}{4}\frac{\sigma_x^2\sigma_y^2}{\sigma^2+\sigma_x^2} = 0 \\
&\qquad \Rightarrow \rho = 1 \text{ or } \sigma_x^2 = 0 \text{ or } \sigma_y^2 = 0 \\
&\qquad \sigma_x^2 = 0 \Rightarrow Q = 2 \text{ and } \sigma_y^2 = 0 \Rightarrow Q = \frac{\sigma^2}{\sigma^2+\sigma_x^2} + 1 \le 2 \\
&\text{so at equilibrium need both } \sigma_x^2 = 0, \sigma_y^2 = 0 \\
&\qquad \text{or } \sigma_x^2 > 0, \sigma_y^2 > 0, \text{ and } \rho = 1
\end{aligned}$$

$$\begin{aligned}
Q = 4 &\Rightarrow \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_y^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2} + \frac{2\rho\sigma_x\sigma_y}{\sigma^2+\sigma_x^2} = 3 \\
&\Rightarrow \sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2 + 2\rho\sigma_x\sigma_y(\sigma^2+\sigma_x^2) = 3(\sigma^2+\sigma_x^2)^2 \\
&\Rightarrow
\end{aligned}$$

Assuming $\rho = 1$, there's an unstable equilibrum: if σ_x, σ_y are above the equilibrium, $Q > 4$ and they'll increase to ∞. If σ_x, σ_y are below the equilibrium, $Q < 4$ and they'll decrease to 0. Figure 1. NEED TO SHOW THAT the slope of the $Q = 4$ contour (on the right side) is less than 1.

With ρ evolving as well, it is possible for σ_x, σ_y, ρ to change such that $Q = 4$ is reached with $\sigma_x > 0$, $\sigma_y > 0$, and $\rho = 1$, but this is a 1 dimensional subspace of (σ_x, σ_y) initial conditions, for given initial ρ.

Points above the curve move off to ∞ and points below the curve collapse to $0, 0$. If $\sigma^2 = 0$, the curve is the one-to-one line. As σ^2 increases, the curve moves up and becomes U-shaped.

4.8 Song genetic, pref imprinted from father $\rho = \rho_{\mathrm{m}}, \rho_{\mathrm{f}} = 0$

$$\begin{aligned}
\sigma_x(t+1)^2 &= \frac{\sigma_x^2}{4}\left(\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} + 1\right) \\
\sigma_y(t+1)^2 &= \sigma_x^2\frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2}
\end{aligned}$$

Let $F = \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2}$. So

$$\begin{aligned}\sigma_x(t+1)^2 &= \sigma_x^2\frac{F+1}{4}\\ \sigma_y(t+1)^2 &= F\sigma_x^2\end{aligned}$$

At equilibrium, need $F = 3$ and therefore $\sigma_y^2 = 3\sigma_x^2$.

$$\begin{aligned}F = 3 \text{ and } \sigma_y^2 = 3\sigma_x^2 \Rightarrow F &= \frac{\sigma^2(\sigma^2+\sigma_x^2)+3\sigma_x^4}{(\sigma^2+\sigma_x^2)^2} = 3\\ \Rightarrow \sigma^4+\sigma^2\sigma_x^2+3\sigma_x^4 &= 3\sigma_x^4+6\sigma^2\sigma_x^2+3\sigma^4\\ \Rightarrow 2\sigma^4+5\sigma^2\sigma_x^2 &= 0\\ \Rightarrow \sigma^2 = 0 \text{ or } 2\sigma^2+5\sigma_x^2 &= 0\end{aligned}$$

So no such equilibrium is possible unless $\sigma^2 = 0$. If $\sigma^2 = 0$, $F = \frac{\sigma_y^2}{\sigma_x^2}$, so $\sigma_y(t+1)^2 = \sigma_y(t)^2$. If $\sigma_x^2 < \sigma_y^2/3$, $F > 3$ and σ_x^2 will increase until $\sigma_x^2 = \sigma_y^2/3$. If $\sigma_x^2 > \sigma_y^2/3$, $F < 3$ and σ_x^2 will decrease until $\sigma_x^2 = \sigma_y^2/3$. If $\sigma^2 > 0$,

$$\begin{aligned}\sigma_y(t+1)^2 > \sigma_y(t)^2 &\Leftrightarrow F(t)\sigma_x(t)^2 > \sigma_y(t)^2\\ &\Rightarrow F(t) > \frac{\sigma_y(t)^2}{\sigma_x(t)^2}\end{aligned}$$

Suppose $\sigma_y(t)^2/\sigma_x(t)^2 = c$.

$$\begin{aligned}&F(t) > c\\ \Leftrightarrow\ &\frac{\sigma^2(\sigma^2+\sigma_x^2)+c\sigma_x(t)^4}{(\sigma^2+\sigma_x(t)^2)^2} > c\\ \Leftrightarrow\ &\sigma^2(\sigma^2+\sigma_x^2)+c\sigma_x^4 > c(\sigma^4+2\sigma^2\sigma_x^2+\sigma_x^4)\\ \Leftrightarrow\ &(2c-1)\sigma^2\sigma_x^2+(c-1)\sigma^4 < 0\\ \Leftrightarrow\ &(2c-1)\sigma_x^2+(c-1)\sigma^2 < 0\\ \Leftrightarrow\ &\sigma_x^2 < \frac{(1-c)\sigma^2}{2c-1}\\ \text{Note that } &\frac{(1-c)\sigma^2}{2c-1} > 0 \Leftrightarrow c \in \left(\frac{1}{2},1\right)\end{aligned}$$

So the only $\sigma_x(t)^2, \sigma_y(t)^2$ such that σ_y^2 are those such that $\sigma_y(t)^2/\sigma_x(t)^2 \in (1/2,1)$. In order for σ_x^2 to increase, F must be greater than 3.

$$\begin{aligned}F = c \Rightarrow \frac{\sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2}{(\sigma^2+\sigma_x^2)^2} &= c\\ \Rightarrow c &> 0\\ \Rightarrow \sigma^2(\sigma^2+\sigma_x^2)+\sigma_x^2\sigma_y^2 &= c\sigma^4+2c\sigma^2\sigma_x^2+c\sigma_x^4\\ 0 &\Rightarrow c\sigma_x^4+((2c-1)\sigma^2-\sigma_y^2)\sigma_x^4+(c-1)\sigma^4\\ \Rightarrow \sigma_x^2 &= \frac{\sigma_y^2-(2c-1)\sigma^2 \pm \sqrt{((2c-1)\sigma^2-\sigma_y^2)^2-4c(c-1)}}{2c}\end{aligned}$$

If $c = 1$, $\sigma_x^2 = \frac{\sigma_y^2-\sigma^2\pm|\sigma^2-\sigma_y^2|}{2}$. If $\sigma_y^2 < \sigma^2$, $\sigma_x^2 = 0$ is the only solution. If $\sigma_y > \sigma^2$, $\sigma_x^2 = 0, \sigma_y^2 - \sigma^2$. Suppose $c \neq 1$.

$$\operatorname{sign}(\sigma_x^2) = \operatorname{sign}\left(\sigma_y^2-(2c-1)\sigma^2 \pm \sqrt{((2c-1)\sigma^2-\sigma_y^2)^2-4c(c-1)}\right)$$

Let $D = ((2c-1)\sigma^2-\sigma_y^2)^2-4c(c-1)$. Suppose $c < 1$. Then $D \geq 0$. Further, $D > ((2c-1)\sigma^2-\sigma-y^2)^2$, so $|\sqrt{D}| > |\sigma_y^2-(2c-1)\sigma^2|$. Regardless of the sign of $\sigma_y^2-(2c-1)\sigma^2$, the only positive solution is $\sigma_x^2 = \sigma_y^2-\sigma^2(2c-1)+\sqrt{D}$.

Suppose $c > 1$. Then

$$\begin{aligned}D \geq 0 &\Leftrightarrow |(2c-1)\sigma^2-\sigma_y^2| > 2\sqrt{c(c-1)}\\ &\Leftrightarrow \sigma_y^2 < (2c-1)\sigma^2-2\sqrt{c(c-1)} \text{ or } \sigma_y^2 > (2c-1)\sigma^2+2\sqrt{c(c-1)}\end{aligned}$$

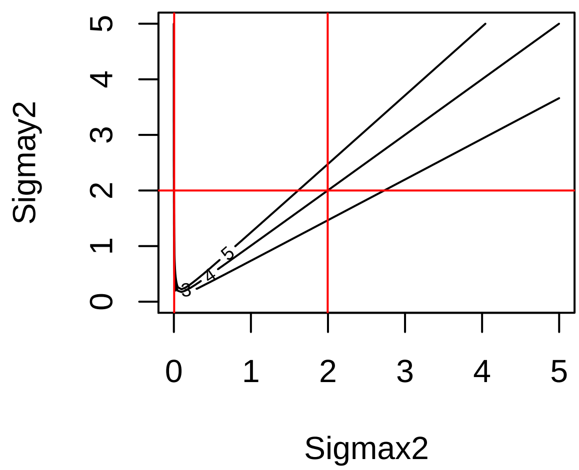

Figure 1: Contour of Q with $\rho = 1$.

If $D \geq 0$, $|\sqrt{D}| < |\sigma_y^2 - (2c-1)\sigma^2|$. If $c > 1$, $(2c-1)\sigma^2 + 2\sqrt{c(c-1)} > 0$, so if $\sigma_y^2 > (2c-1)\sigma^2 + 2\sqrt{c(c-1)}$, there are two positive solutions to σ_x^2. If $\sigma_y^2 < (2c-1)\sigma^2 - 2\sqrt{c(c-1)}$, there are two negative solutions to σ_x^2, which is not possible. So

- For $c < 1$, $\sigma_x^2 = \sigma_y^2 - \sigma^2(2c-1) + \sqrt{D}$
- For $c = 1$, $\sigma_x^2 = \sigma_y^2 - \sigma^2$
- For $c > 1$, $\sigma_x^2 = \frac{\sigma_y^2 - \sigma^2(2c-1) \pm \sqrt{D}}{2c}$ where

Therefore, there is an upward facing curve of σ_x^2, σ_y^2 such that $F = 3$, above the line $\sigma_y^2 = 3\sigma_x^2$. In this region, σ_x^2 increases. Not only is there no equilibrium in this mode, but there is no region in which both σ_x^2 and σ_y^2 increase. If they start in a region where one increases, it will eventually leave the region. All points eventually go to $(0, 0)$.

4.9 Song genetic, pref learned from mother

This model was built to study whether song diversity can be maintained when two populations come back together. However, even within one population there are interesting dynamics.

1. There can be many discrete "popular" songs when σ_{m}^2 is small, σ_{f}^2 is high, and σ^2 is intermediate.
2. When the equilibrium population is unimodal, there is an interesting interaction between σ^2 and σ_{f}^2: At low σ^2 and low σ_{f}^2, the males are even more narrowly distributed than the females because the male trait preferred by the most females does way better than anyone else. At low σ^2 and high σ_{f}^2, the male distribution matches the female distribution. At high σ^2, σ_{f}^2 does not matter and the male distribution is moderately narrow because average males can mate with most females.

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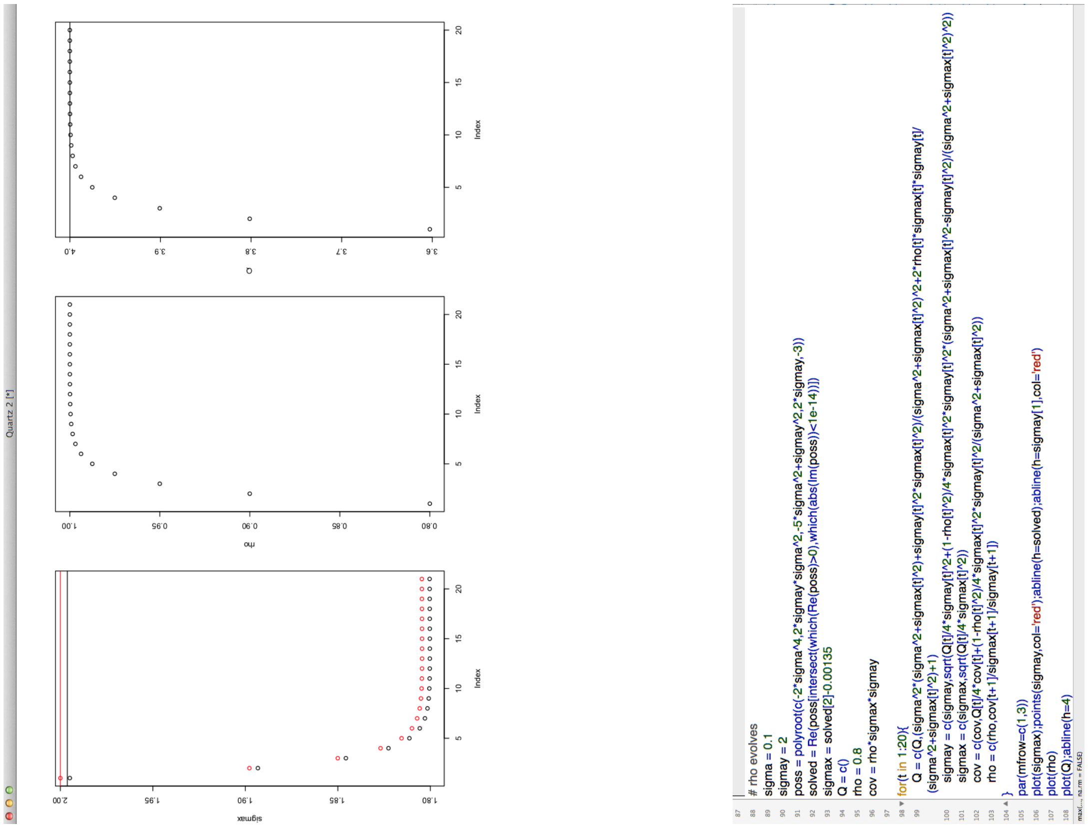

Figure 2: Dynamics of σ_x, σ_y, ρ, and Q.

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