Multigrid Solver for Laplacian Linear Systems on Power-Law Graphs

Eric Buras

Adviser: Dr. Matthew Knepley

Rice University
Department of Computational and Applied Mathematics

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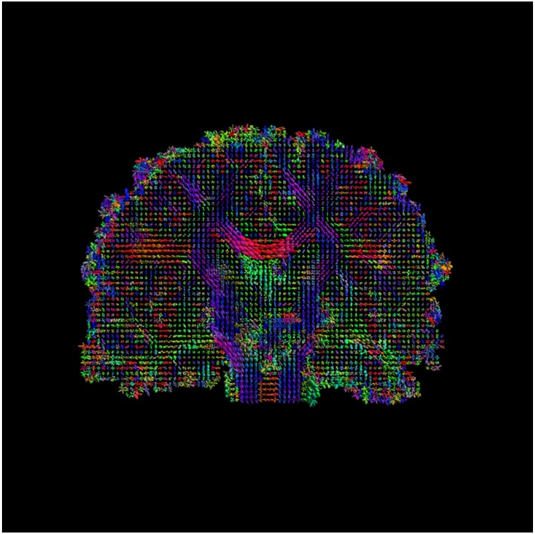
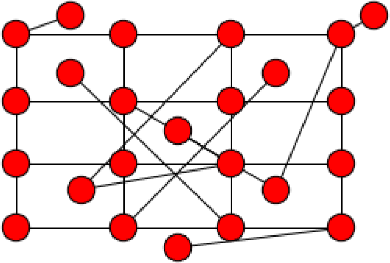


Human brain connectome; coronal view [5]

Foundation

Graph: $G = (N, E)$ where N is the set of nodes and E is the set of edges.

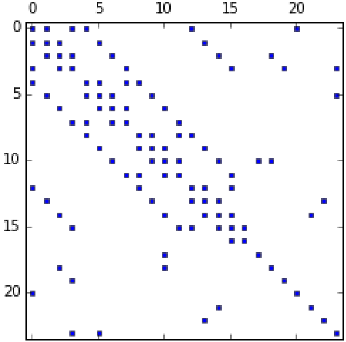


Foundation

Laplacian matrix of G:

$$L = D - A$$

D: diagonal matrix of node degrees, A: adjacency matrix of G



Why Do We Care About the Laplacian?

- Laplacian operator inverse is weighted average of paths through a network:

$$b + Lb + L^2 b + L^3 b + \ldots = \sum_{i=0}^{\infty} L^i b = (I - L)^{-1} b$$

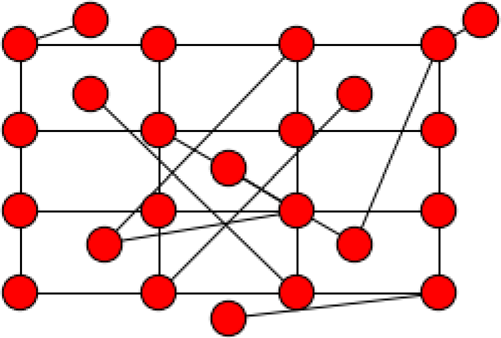
- PageRank linear system:

$$(I - \alpha L)x = (1 - \alpha)b$$

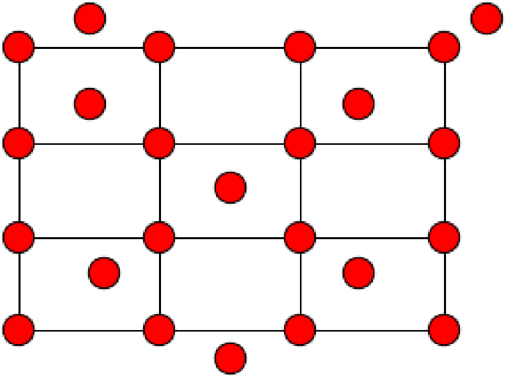
My Work

- Partition the graph into large planar subgraph and small teleportation subgraph
- Solve the large planar part with algebraic multigrid
- Solve the teleportation part directly
- Linear algebra to combine and solve the entire linear system

Graph Partitioning



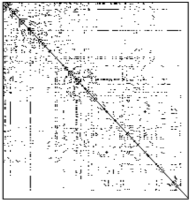
Graph Partitioning

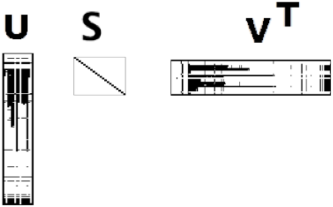


Max locally-connected subgraph using Chun-Lu [1]

Linear System Solve

$$L = P_L + T_L$$





Linear System Solve

Definition (Sherman-Woodberry-Morrison)

$$(P_L + USV^T)^{-1} = P_L^{-1} - P_L^{-1}U(S^{-1} + V^T P_L^{-1} U)^{-1} V^T P_L^{-1}$$

$$
\begin{aligned}
Lx &= b \\
x &= L^{-1} b \\
x &= (P_L + T_L)^{-1} b \\
x &= (P_L + USV)^{-1} b \\
x &= (P_L^{-1} - P_L^{-1}U(S^{-1} + VP_L^{-1}U)^{-1} VP_L^{-1}) b \\
x &= P_L^{-1} b - P_L^{-1}U(S^{-1} + VP_L^{-1}U)^{-1} VP_L^{-1} b
\end{aligned}
$$

Current Solution Approaches

Multigrid on entire graph: LAMG, CMG, Cascadic

- How do you coarsen the grid? Based on heuristics

Brute force factorization

- Slow and memory inefficient

Partitioning Complexity

- Graph partitioning complexity relies on the degree distribution of the graph
- Power law degree graphs with degree sequence:

$$P(d) \sim d^{-\gamma}$$

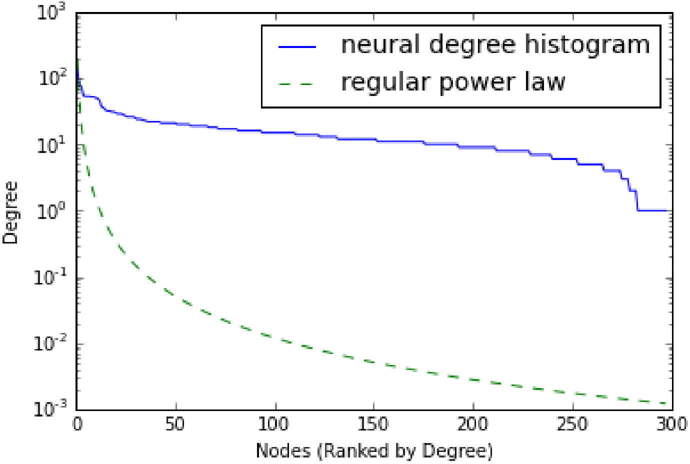
$$E[d] = \frac{\zeta(\gamma-1)}{\zeta(\gamma)}$$

- Partitioning is $O(iter. \times edges \times E[d]^3)$

Solve Complexity

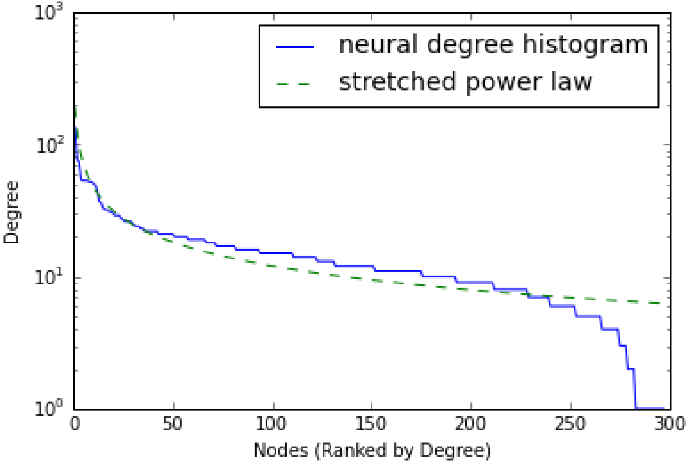
Operation	**O(FLOPs)**
$USV = T_L$	$O(n^3)$
S^{-1}	$O(r)$
$y = P_L^{-1} b$ (MG)	O(n)
$y_1 = Vy$	$O(rn)$
$Q = P_L^{-1} U$ (r MG solves)	$O(rn)$
$Q_1 = VQ$	$O(r^2 n)$
$Q_2 = S^{-1} + Q_1$	$O(r^2)$
$y_2 = Q_2^{-1} y_1$	$O(r^3)$
$y_3 = Uy_2$	$O(rn)$
$y_4 = P_L^{-1} y_3$ (MG)	$O(n)$
$x = y - y_4$	$O(n)$

Power Law vs. Neural Network Degree Sequence



$\gamma = 2.1$

Stretched Power Law



stretched degree is 0.6

Graphs and Results

Graph (nodes, edges)	Avg. Deg.	Rank T_L	Part. (s)	Iter.
Neural (297, 2148) [6, 7]	14.46	22	1.6	3
Metabolic (453, 2025) [2]	8.94	49	2.3	3
Protein (912, 22738) [4]	49.86	26	53	2
Facebook (4039, 88234) [3]	43.69	180	480	3
Power (4941, 6594) [6]	2.67	4284	.33	3

Solve Component Results

These are the limiting solve components according to theoretical complexity and timed results:

Operation	$O(r, n)$	**Neur.**	**Meta.**	**Prot.**	**FB**	**Pow.**
$USV = T_L$	$O(n^3)$	.0334	.0737	.4183	38.47	72.86
$Q = P_L^{-1}U$ ($r\times$MG)	$O(rn)$	.3292	.7360	.5190	23.11	106.9
$Q_1 = VQ$	$O(r^2n)$	.0006	.0036	.0013	.4124	285.1
$y_2 = Q_2^{-1}y_1$	$O(r^3)$	.0023	.0025	.0013	.0099	86.49
Total	$O(n^3)$	**.51**	**.966**	**1.44**	**64.46**	**560**

What Do These Solutions Mean?

C. Elegans data:

- Important regions in the worm nervous system
- Key metabolic processes
- Unobserved protein-phenotype relationships

Facebook friend networks:

- Origins of influence in a social group

Power Grid:

- Flow of electricity through urban infrastructure

Conclusions and Future Work

- Method to solve the graph laplacian linear system using partitioning and multigrid
- Optimize the solve operations and write faster, parallelizable code
- Find appropriate linear systems to solve the graphs I have
- Test larger graphs; more complex biological systems

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