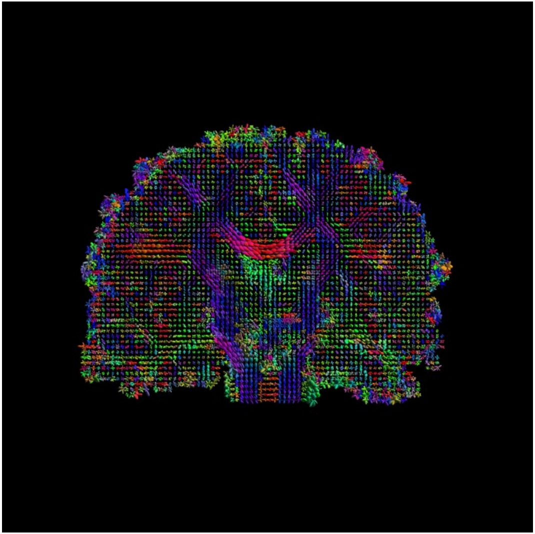
Multigrid Solver for Laplacian Linear Systems on Power-Law Graphs

Eric Buras

Adviser: Dr. Matthew Knepley

Rice University
Department of Computational and Applied Mathematics

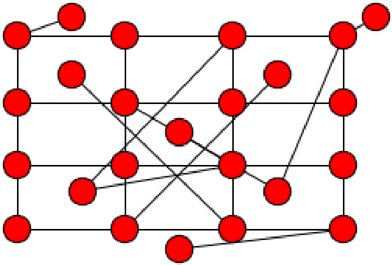
March 25, 2016



Human brain connectome; coronal view [5]

Foundation

Graph: $G = (N, E)$ where N is the set of nodes and E is the set of edges.

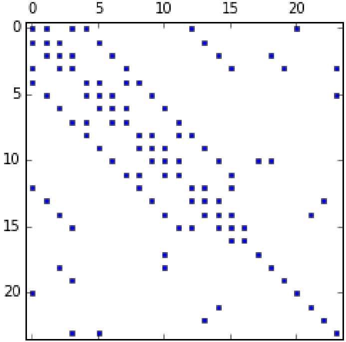


Foundation

Laplacian matrix of G:

$$L = D - A$$

D: diagonal matrix of node degrees, A: adjacency matrix of G



Why Do We Care About the Laplacian?

- Laplacian operator inverse is weighted average of paths through a network:

$$b + Lb + L^2 b + L^3 b + \ldots = \sum_{i=0}^{\infty} L^i b = (I - L)^{-1} b$$

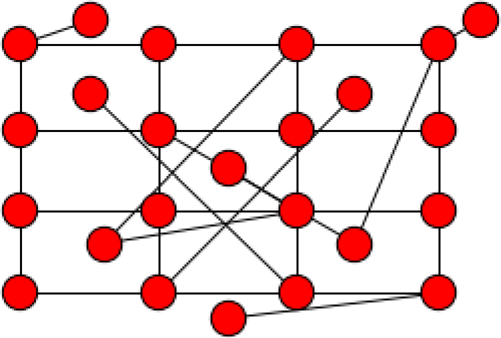
- PageRank linear system:

$$(I - \alpha L)x = (1 - \alpha)b$$

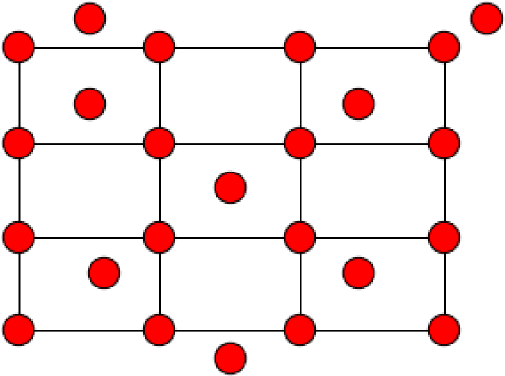
My Work

- Partition the graph into large planar subgraph and small teleportation subgraph
- Solve the large planar part with algebraic multigrid
- Solve the teleportation part directly
- Linear algebra to combine and solve the entire linear system

Graph Partitioning



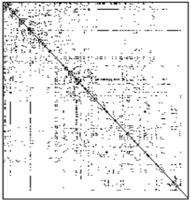
Graph Partitioning



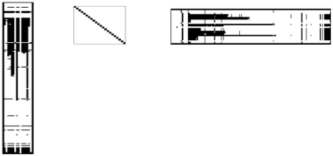
Max locally-connected subgraph using Chun-Lu [1]

Linear System Solve

$$L = P_L + T_L$$



$U \quad S \quad V^T$



Linear System Solve

Definition (Sherman-Woodberry-Morrison)

$$(P_L + USV^T)^{-1} = P_L^{-1} - P_L^{-1}U(S^{-1} + V^T P_L^{-1} U)^{-1} V^T P_L^{-1}$$

$$\begin{aligned}
Lx &= b \\
x &= L^{-1} b \\
x &= (P_L + T_L)^{-1} b \\
x &= (P_L + USV)^{-1} b \\
x &= (P_L^{-1} - P_L^{-1}U(S^{-1} + VP_L^{-1}U)^{-1} VP_L^{-1}) b \\
x &= P_L^{-1} b - P_L^{-1}U(S^{-1} + VP_L^{-1}U)^{-1} VP_L^{-1} b
\end{aligned}$$

Current Solution Approaches

Multigrid on entire graph: LAMG, CMG, Cascadic

- How do you coarsen the grid? Based on heuristics

Brute force factorization

- Slow and memory inefficient

Partitioning Complexity

- Graph partitioning complexity relies on the degree distribution of the graph
- Power law degree graphs with degree sequence:

$$P(d) \sim d^{-\gamma}$$

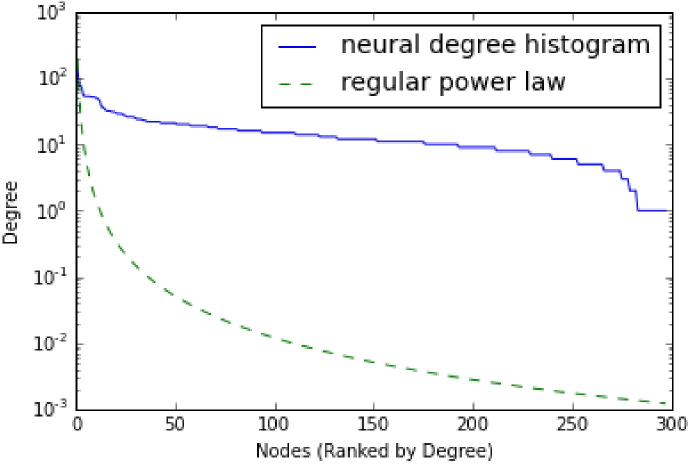
$$E[d] = \frac{\zeta(\gamma-1)}{\zeta(\gamma)}$$

- Partitioning is $O(iter. \times edges \times E[d]^3)$

Solve Complexity

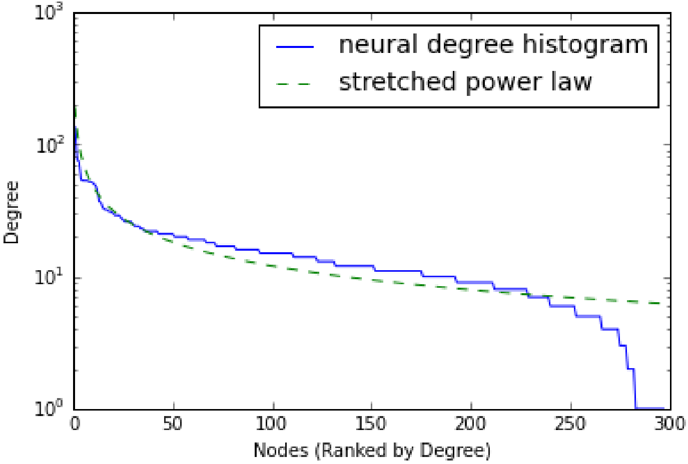
Operation	**O(FLOPs)**
$USV = T_L$	$O(n^3)$
S^{-1}	$O(r)$
$y = P_L^{-1}b$ (MG)	O(n)
$y_1 = Vy$	$O(rn)$
$Q = P_L^{-1}U$ (r MG solves)	$O(rn)$
$Q_1 = VQ$	$O(r^2n)$
$Q_2 = S^{-1} + Q_1$	$O(r^2)$
$y_2 = Q_2^{-1}y_1$	$O(r^3)$
$y_3 = Uy_2$	$O(rn)$
$y_4 = P_L^{-1}y_3$ (MG)	$O(n)$
$x = y - y_4$	$O(n)$

Power Law vs. Neural Network Degree Sequence



$\gamma = 2.1$

Stretched Power Law



stretched degree is 0.6

Graphs and Results

Graph (nodes, edges)	Avg. Deg.	Rank T_L	Part. (s)	Iter.
Neural (297, 2148) [6, 7]	14.46	22	1.6	3
Metabolic (453, 2025) [2]	8.94	49	2.3	3
Protein (912, 22738) [4]	49.86	26	53	2
Facebook (4039, 88234) [3]	43.69	180	480	3
Power (4941, 6594) [6]	2.67	4284	.33	3

Solve Component Results

These are the limiting solve components according to theoretical complexity and timed results:

Operation	$O(r, n)$	**Neur.**	**Meta.**	**Prot.**	**FB**	**Pow.**
$USV = T_L$	$O(n^3)$	.0334	.0737	.4183	38.47	72.86
$Q = P_L^{-1} U$ ($r \times$MG)	$O(rn)$	.3292	.7360	.5190	23.11	106.9
$Q_1 = VQ$	$O(r^2 n)$	.0006	.0036	.0013	.4124	285.1
$y_2 = Q_2^{-1} y_1$	$O(r^3)$	.0023	.0025	.0013	.0099	86.49
Total	$O(n^3)$	**.51**	**.966**	**1.44**	**64.46**	**560**

What Do These Solutions Mean?

C. Elegans data:

- Important regions in the worm nervous system
- Key metabolic processes
- Unobserved protein-phenotype relationships

Facebook friend networks:

- Origins of influence in a social group

Power Grid:

- Flow of electricity through urban infrastructure

Conclusions and Future Work

- Method to solve the graph laplacian linear system using partitioning and multigrid
- Optimize the solve operations and write faster, parallelizable code
- Find appropriate linear systems to solve the graphs I have
- Test larger graphs; more complex biological systems

References

Fan Chung and Linyuan Lu.
Complex Networks, chapter The Small World Phenomenon in Hybrid Power Law Graphs, pages 89–104.
Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

Jordi Duch and Alex Arenas.
Community detection in complex networks using extremal optimization.
Physical review E, 72(2):027104, 2005.

Julian J McAuley and Jure Leskovec.
Learning to discover social circles in ego networks.
In *NIPS*, volume 2012, pages 548–56, 2012.

Nicolas Simonis, Jean-François Rual, Anne-Ruxandra Carvunis, Murat Tasan, Irma Lemmens, Tomoko Hirozane-Kishikawa, Tong Hao, Julie M Sahalie, Kavitha Venkatesan, Fana Gebreab, et al.
Empirically controlled mapping of the caenorhabditis elegans protein-protein interactome network.
Nature methods, 6(1):47–54, 2009.

Arthur W Toga, Kristi A Clark, Paul M Thompson, David W Shattuck, and John Darrell Van Horn.
Mapping the human connectome.
Neurosurgery, 71(1):1, 2012.

Duncan J Watts and Steven H Strogatz.
Collective dynamics of 'small-world' networks.
nature, 393(6684):440–442, 1998.

JG White, E Southgate, JN Thomson, and S Brenner.
The structure of the nervous system of the nematode caenorhabditis elegans: the mind of a worm.
Phil. Trans. R. Soc. Lond, 314:1–340, 1986.