Homework Assignment #8
Introduction to Sequences.

Due: Friday, November 9, 2018, by 1pm.

Guidelines

Do the assigned reading. Details not covered in class can be found in the textbook.

You will only be submitting solutions to the "Submission Problems."

You should sincerely attempt to solve each of the Reading Problems that are also listed. Once you have made an attempt at answering an entire weekly batch of Reading Problems, feel free to discuss and share solutions with others.

Rewrites

Note: No rewrites.

Reading

- Read Rudin, chapter 3, pp.47–52 (sequences, subsequences).

Problems List

- Submission Problems. Submit solutions to Questions 1, 2, 3 below on Gradescope.
- Reading problems. Read, but do not submit: Rudin, chapter 3, problems 1 – 4.

Question 1 The terms *limit* and *limit point* mean different things, but the notions are easy to confuse. For example, the constant sequence $1, 1, \cdots, 1, \cdots$ is convergent with *limit* 1. As a subset of the real line, however, its values are just equal to the set $\{1\}$, which cannot have a limit point.

(a) Explain why the set $\{1\}$ in $\mathbb{R}$ has no limit points.

(b) To clarify the notions of limit and limit point, prove the following statement: If a convergent sequence in a metric space has infinitely many distinct points, then its limit is a limit point of the set of points of the sequence.

Question 2 Give an example of a sequence $\{x_n\}$ with values in $[0, 1]$ that has the following property: For every $x \in [0, 1]$, we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$.

Question 3 This exercise reveals additional interesting qualities of the Cantor set.

Let C be the Cantor set as defined in Rudin 2.44. Further, note that the sum $C + C = \{x + y : x, y \in C\}$. The goal of this exercise is to prove that $C + C = [0, 2]$. Since $C \subset [0, 1]$, it follows immediately that $C + C \subset [0, 2]$. Thus, it remains only to show the inclusion in the other direction, namely that $[0, 2] \in \{x + y : x, y \in C\}$. That is, given any $s \in [0, 2]$, we must show there exist numbers $x, y \in C$ that satisfy $x + y = s$. Do so through the following steps:

(a) Recall the notation from class in the construction of the Cantor set: $C = \cap_{n=0}^{\infty} K_n$ where K_n is the union of 2^n closed subintervals of $[0, 1]$ in which successive open middle third subintervals have been removed. Namely, $K_0 = [0, 1]$, $K_1 = [0, 1/3] \cup [2/3, 1]$, $K_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$, and so forth. Show that there exist $x_1, y_1 \in K_1$ for which $x_1 + y_1 = s$.

(b) Use induction to show in general that for an arbitrary $n \in \mathbb{N}$, we can always find $x_n, y_n \in K_n$ for which $x_n + y_n = s$.

(c) Show that we can find x and y in C satisfying $x + y = s$. (Note: The sequences $\{x_n\}$ and $\{y_n\}$ do not necessarily converge. Nonetheless, these can be used to produce the desired result.)