Student Name: ________________
Math 131
Homework 8
Due Date: November 9, 2018

Homework 8

Problem 1.

The terms *limit* and *limit point* mean different things, but the notions are easy to confuse. For example, the constant sequence $1, 1, \cdots, 1, \cdots$ is convergent with *limit* 1. As a subset of the real line, however, its values are just equal to the set $\{1\}$, which cannot have a limit point.

1. Explain why the set $\{1\}$ in $\mathbb{R}$ has no limit points.
2. To clarify the notions of limit and limit point, prove the following statement: If a convergent sequence in a metric space has infinitely many distinct points, then its limit is a limit point of the set of points of the sequence.

Solution.

Problem 2.

Give an example of a sequence $\{x_n\}$ with values in $[0, 1]$ that has the following property: For every $x \in [0, 1]$, we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$.

Solution.

Problem 3.
This exercise reveals additional interesting qualities of the Cantor set.

Let C be the Cantor set as defined in Rudin 2.44. Further, note that the sum $C + C = \{x + y : x, y \in C\}$. The goal of this exercise is to prove that $C + C = [0, 2]$. Since $C \subset [0, 1]$, it follows immediately that $C + C \subset [0, 2]$. Thus, it remains only to show the inclusion in the other direction, namely that $[0, 2] \in \{x + y : x, y \in C\}$. That is, given any $s \in [0, 2]$, we must show there exist numbers $x, y \in C$ that satisfy $x + y = s$. Do so through the following steps:

1. Recall the notation from class in the construction of the Cantor set: $C = \cap_{n=0}^{\infty} K_n$ where K_n is the union of 2^N closed subintervals of $[0, 1]$ in which successive open middle third subintervals have been removed. Namely, $K_0 = [0, 1]$, $K_1 = [0, 1/3] \cup [2/3, 1]$, $K_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$, and so forth. Show that there exist $x_1, y_1 \in K_1$ for which $x_1 + y_1 = s$.

2. Use induction to show in general that for an arbitrary $n \in N$, we can always find $x_n, y_n \in K_n$ for which $x_n + y_n = s$.

3. Show that we can find x and y in C satisfying $x + y = s$. (Note: The sequences $\{x_n\}$ and $\{y_n\}$ do not necessarily converge. Nonetheless, these can be used to produce the desired result.)

Solution.