Student Name ______________
Math 131
Homework 7
Due Date: November 2, 2018

Homework 7

Problem 1.
Let C be the Cantor set, as defined in 2.44 of Rudin.
Recall that the Schroder-Berstein theorem (sometimes known as the Cantor-Bernstein-Schroder theorem) gives another way to show the equality of the cardinality of two sets. The theorem states: Given two sets X and Y, $|X| = |Y|$ if and only if $|X| \leq |Y|$ and $|Y| \leq |X|$.

1. Find a function $f : C \to [0, 1]$ that is onto.
2. Prove that Cantor set C has the same cardinality as the closed interval $[0, 1] \in \mathbb{R}$ by using the function from part (a) and invoking the Schroder-Berstein theorem.

Solution.

Problem 2.
For this problem, it may help to review exercise 2.9 in Rudin. Let A be a subset of a metric space X with closure $\overline{A}$. Let A° be the *interior* of set A. That is, A° denotes the set of interior points of (recall definition 2.18(e) of Rudin). Let ∂A be the *boundary* of A. Thus,

$$A^\circ = \cup\{G \subset A : G \text{ is open}\}$$
$$\partial A = \overline{A} \backslash A^\circ$$

1. Prove or disprove: If set A is open, then it follows that $(\overline{A})^\circ = A$.
2. Prove or disprove: If set A is connected, $\overline{A}$ will also be connected.
3. Prove or disprove: If set A is connected, A° will also be connected.

Solution.

Problem 3.
Convince yourself of the following theorem (do not turn in this part): If $\{G_1, G_2, G_3, \cdots\}$ is a countable collection of dense, open sets, then the intersection $\cap_{n=1}^{\infty} G_n$ is a dense (and therefore non-empty) subset of $\mathbb{R}$

Note that a set G is dense in $\mathbb{R}$ if and only if $\overline{G} = \mathbb{R}$. So, for example, the set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$, but the set of integers $\mathbb{Z}$ is not.

We call a set E in $\mathbb{R}$ nowhere-dense if $\overline{E}$ contains no nonempty open intervals. Note that $\mathbb{Z}$ is nowhere-dense in $\mathbb{R}$ since $\mathbb{Z} = \overline{\mathbb{Z}}$ contains no nonempty open intervals.

1. Show that a set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of the closure, $(\overline{E})^c$, is dense in $\mathbb{R}$.

2. Prove that the set $\mathbb{R}$ cannot be written as a countable union of nowhere dense sets. Hint: Use a proof by contradiction.

Solution.