Testing Smooth Structural Changes in Predictive Regression Models

Daniel Gulti Kebede

August 07, 2025

What are Structural Breaks?

- Time series models estimate the relationship between variables that are observed over a period of time
- Many models assume that the relationship between these variables stays constant across the entire period
- There are cases where changes in factors outside of the model cause changes in the underlying relationship between the variables in the model
- Structural break models capture exactly these cases by incorporating sudden, permanent changes in the parameters of models

Why Should We Care About Structural Break?

"Structural change is pervasive in economic time series relationships, and it can be quite perilous to ignore. Inferences about economic relationships can go astray, forecasts can be inaccurate, and policy recommendations can be misleading or worse." – Bruce Hansen (2001)

Motivation

- Predictive relations are structurally unstable and change over time
- Time-varying parameter or structural break in the marginal distribution of the regressors
- Test for the presence of structural breaks and characterize the timing and nature of the breaks

Motivation

- Structural break represents an abrupt or smooth change in a time series
- Several factors to consider: nature of breaks? Known or unknown break dates? Number of breaks?
- LM test has best power against randomly and slowly evolving parameter changes (Georgiev et al (2018))
- SupF test displays a better power against alternatives where the parameters display a small number of breaks at deterministic points (Georgiev et al (2018))

Motivation

- Usually, no prior information about the structural change alternative is available in practice
- Another difficulty is controlling for the change in the marginal distribution of the regressor
- We propose a nonparametric test invariant to the nature of the alternative hypothesis
  - First to apply the test statistic to the structural break test with a non-stationary regressor
- We use Hansen's Fixed regressor bootstrap to control for the change in the marginal distribution of the regressor

Literature Review

- Chow (1960), Andrew, D.W.K. (1993), Nyblom, J. (1989), Bai and Perron (1998, 2003), Hansen, B.E. (1992a, 1992b, 2000)
- Chen and Hong (2012): Compared the parametric and non-parametric fitted values via a simple quadratic form.
- Georgiev et al (2018): used SupF and LM test statistics to test structural changes in parameters of a predictive regression model where the predictors display strong persistence.
- Cai et al (2015) propose a test allowing for time-varying coefficients in a predictive regression model with potentially non-stationary regressors.
- Bruce E. Hansen (2000): Derived the large sample distributions of the test statistics allowing for structural change in the marginal distribution of the regressors

Predictive Regression Model

We adopted a local-to-unit root framework to analyze the power and size of the test statistics when the regressors are close to exhibiting a unit root process.

$$y_t = \gamma_t + \phi_t x_{t-1} + \epsilon_t, \quad t = 1, ..., T \tag{1}$$

$$x_t = \mu + \rho_x x_{t-1} + \nu_t, \quad t = 0, ..., T \tag{2}$$

where $\rho_x := 1 - c_x T^{-1}$ with $c_x \geq 0$. Also, I let x_0 be an Op(1) variate. Assume

$$\boldsymbol{\theta}_t = \{\gamma_t, \phi_t\}$$

Predictive Regression Model

- To nest the constant parameter model within (2), we formulate the time varying intercept and slope coefficients as: $\gamma_t = \alpha + a\alpha_t$ and $\phi_t = \beta + b\beta_t$.
- We adopt a local-to-zero parameterization for β and α; where $\beta = gT^{-1}$ and $\alpha = gT^{-1}$. Also $a = gT^{-c}$ and $b = gT^{-c}$, for some constants g and c

Stochastic Coefficient Variation

This mechanism considers time variations in α_t and β_t to follow a (near) unit root processes.

$$\begin{bmatrix} \alpha_t \\ \beta_t \end{bmatrix} = \begin{bmatrix} \rho_\alpha & 0 \\ 0 & \rho_\beta \end{bmatrix} \begin{bmatrix} \alpha_{t-1} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{\alpha_t} \\ \epsilon_{\beta_t} \end{bmatrix} \tag{3}$$

where $\rho_\alpha = 1 - c_\alpha T^{-1}, \quad \rho_\beta = 1 - c_\beta T^{-1} \quad \textit{with} \quad c_\alpha \geq 0, \quad c_\beta \geq 0,$ which are local-to-unit root autoregressive processes. The coefficient processes are initialized at $\alpha_0 = \beta_0 = 0$.

Non-stochastic Coefficient Variation

Here we consider abrupt changes that occur at a fixed number of deterministic points in the sample.

$$\alpha_t = \beta_t = D_t(\lfloor \tau_0 T \rfloor) \tag{4}$$

where $D_t(\lfloor \tau T \rfloor) := 1(t \geq \lfloor \tau T \rfloor)$ with $\lfloor \tau T \rfloor$ representing a generic shift point with associated break fraction τ, $\lfloor \cdot \rfloor$ the integer part of the argument and $1(\cdot)$ the indicator function. We assume the true shift fraction τ_0 is unknown but $\tau_0 \in [\tau_L, \tau_U]$

Stochastic vs Non-stochastic coefficient variation

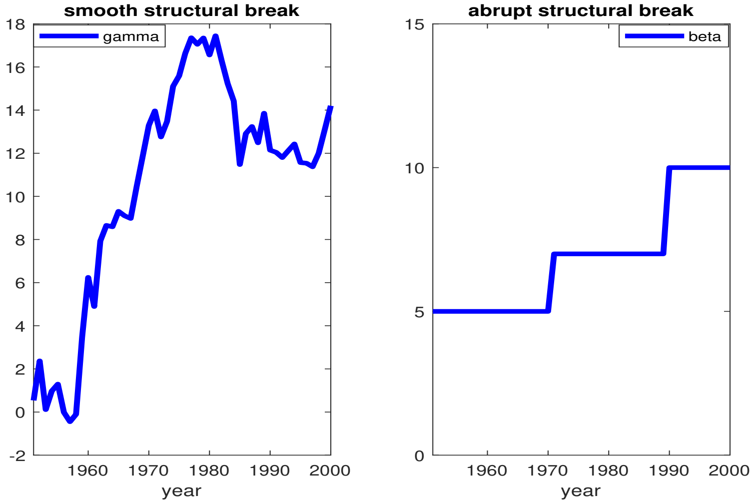

Figure 1: Smooth vs abrupt structural breaks

Test Statistics

- Under the null hypothesis, we have a constant parameter regression model, $\beta_t = \beta$; and β can be consistently estimated by OLS estimator, $\hat{\beta}$
- Under the alternative hypothesis, we assume a time varying parameter, $\beta = \beta(t/T)$
- Under the alternative hypothesis, the OLS estimator will not be consistent and the time-varying parameter, β_t, will be consistently estimated with a nonparametric estimator

Test Statistics

$$\hat{Q} = \frac{1}{T}\sum_{t=1}^{T}(X_t\hat{\boldsymbol{\theta}}_t - X_t\hat{\boldsymbol{\theta}})^2 \tag{5}$$

where $\boldsymbol{\theta}_t$ is estimated by a locally weighted least square estimator.

- Any significant departure of $\hat{Q}$ from 0 can be considered as an evidence of structural changes
- Standardizing the $\hat{Q}$ test will give us our Hausman test,

$$\hat{H}_{het} = (T\sqrt{h}\hat{Q} - \mu_H)/\sigma_H \tag{6}$$

Test Statistics

where μ_H and σ_H are approximately the mean and the variance of $T\sqrt{h}\hat{Q}$.

$$\mu_H = h^{-1/2} C_A trace(\hat{\omega}\hat{M}^{-1})$$
$$\sigma_H = \sqrt{4 C_B trace(\hat{M}^{-1}\hat{\omega}\hat{M}^{-1}\hat{\omega})}$$

where $\hat{M} = T^{-1}\sum_{t=1}^{T} X_t X_t'$, $\hat{\omega} = T^{-1}\sum_{t=1}^{T} \hat{\epsilon}_t X_t X_t'$,
$C_A = \int_{-1}^{1} k^2(s)ds + \mathrm{o}(1)$ and $C_B = \int_0^1 [\int_{-1}^{1} k(s)k(s+t)]^2 dt + \mathrm{o}(1)$

Heteroskedasticity and serial correlation

Let $\xi_t := [\epsilon_t, \nu_t, \epsilon_{\alpha_t}, \epsilon_{\beta_t}]'$,H and D_t are 4 x 4 non-stochastic matrices

$$H := \begin{vmatrix} 1 & 0 & 0 & 0 \\ h_{21} & 1 & 0 & 0 \\ h_{31} & h_{32} & 1 & 0 \\ h_{41} & h_{42} & h_{43} & 1 \end{vmatrix}, \quad D_t := \begin{vmatrix} d_{1t} & 0 & 0 & 0 \\ 0 & d_{2t} & 0 & 0 \\ 0 & 0 & d_{3t} & 0 \\ 0 & 0 & 0 & d_{4t} \end{vmatrix}$$

such that HH' is strictly positive definite. The volatility terms d_{it} satisfy $d_{it} = d_i(t/T)$, where $d_i \in D := D^1, \quad D^k := D_k[0,1]$ denoting the space of right continuous with left limit on [0,1] equipped with the Skorokhod topology.

Assumptions

- The innovation process ξ_t can be expressed as the product of non-stochastic matrices and a vector of martingale difference sequence: $\xi_t = HD_t e_t$. Where e_t is a $4x1$ vector of m.d.s.

$$\sigma_t := E(e_t e_t')$$

- The kernel function $K(.)$ is a symmetric and has a closed and bounded support. Since, $k : [-1, \quad 1] \rightarrow \Re^+$; we can see that the kernel function has a compact support $[-1, \quad 1]$

Assumptions

- $\hat{\boldsymbol{\theta}}$ is a parameter estimator such that $T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) = Op(1)$, where $\boldsymbol{\theta}^* = plim_{T\to\infty}\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^* = \boldsymbol{\theta}$ under the null hypothesis, where $\boldsymbol{\theta}$ is given in null hypothesis. Due to the high persistence of the regressors, the parameter estimate converges at a faster rate (converges at the rate of T, not $\sqrt{T}$)
- The bandwidth h satisfies that $h \to 0$, $Th \to \infty$ and $Th^4 \to 0$. We set $h = (1/\sqrt{12})T^{-1/20}$.

Asymptotic Distribution

- Unconditional heteroskedasticity present in ν_t and ϵ_t, and on the persistence parameter c_x
- We used Hansen's fixed regressor bootstrap to derive the asymptotic distribution.

Theorem 1

Consider a model given by 1-2, and the above Assumptions hold. Then, under the null hypothesis and the local alternatives discussed above, $\hat{H}_{het} \xrightarrow{d} N(0,1)$.

Finite Sample Performance

We further considered three cases for ϵ_t:

- $\epsilon_t \sim i.i.d.N(0,1)$;
- $\epsilon_t = \sqrt{h_t}u_t, \quad h_t = 0.2 + 0.5\epsilon_{t-1}^2, \quad u_t \sim i.i.d.N(0,1)$;
- $\epsilon_t = \sqrt{h_t}u_t, \quad h_t = 0.2 + 0.5X_t^2, \quad u_t \sim i.i.d.N(0,1)$

We used the following parameter values in the simulations: $\alpha = 1$, $\beta = 0.5$, $\mu = 0$, $c_x = [0, 10]$ and T=100. Also, we generate 500 data sets of random samples and used B=200 bootstrap iterations for each simulated data set. Size results for 5% level of significance

Finite Sample Performance: The Bootstrap corrects for size

Table 1: Rejection Rates Based on Asymptotic and Bootstrap Critical Values

	Asymptotic crit.		Bootstrap crit.	
	$c_x = 0$	$c_x = 10$	$c_x = 0$	$c_x = 10$
$\epsilon_t \sim i.i.d.N(0,1)$	0.0942	0.0784	0.066	0.068
$\epsilon_t \sim ARCH(1)$	0.094	0.08	0.064	0.07
$\epsilon_t \vert X_t \sim N(0, f(X_t))$	0.063	0.0434	0.064	0.062

‡ The level of significance is 5%. Any value above 5% will over-reject the null hypothesis.

Finite Sample Performance: Empirical Sizes

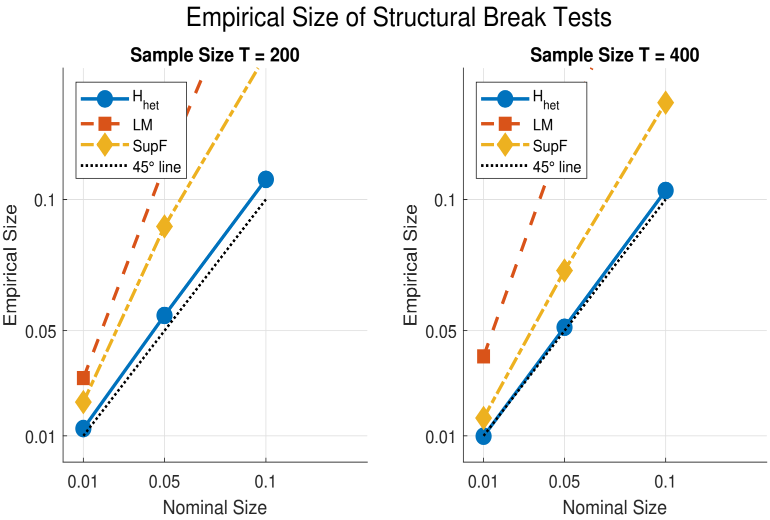

Figure 2: Empirical Sizes of the Structural Break Tests

The Nonparametric Test statistics have good power

$C_\alpha = C_\beta = 0$ is the case when the parameter variation is a unit root.

Table 2: Finite sample power of tests under H^S

τ_{0h}	σ	g	LM		SupF		$\hat{H}_{het}$	
			$C_\alpha = C_\beta = 0$		$C_\alpha = C_\beta = 0$		$C_\alpha = C_\beta = 0$	
			T=100	T=200	T=100	T=200	T=100	T=200
-	1	15	0.95	0.96	0.96	0.97	0.95	0.95
		35	0.99	1	0.99	1	1	1
1/2	4	15	0.78	0.82	0.77	0.81	0.69	0.75
		35	0.95	0.99	0.95	0.99	0.94	0.97
	1/4	15	0.97	0.97	0.94	0.93	0.9	0.92
		35	1	1	1	1	0.99	1
3/4	4	15	0.82	0.84	0.77	0.72	0.57	0.62
		35	0.96	0.99	0.95	0.98	0.89	0.93
	1/4	15	0.96	0.96	0.95	0.95	0.92	0.95
		35	1	1	1	1	1	1

The Nonparametric Test statistics have good power

$C_\alpha = C_\beta = 10$ is the case when the parameter variation is persistent.

Table 3: Finite sample power of tests under H^S

τ_{0h}	σ	g	LM		SupF		$\hat{H}_{het}$	
			C_α=C_β=10		C_α=C_β=10		C_α=C_β=10	
			T=100	T=200	T=100	T=200	T=100	T=200
-	1	15	0.65	0.65	0.63	0.72	0.71	0.8
		35	0.91	0.94	0.91	0.97	0.94	0.97
1/2	4	15	0.47	0.45	0.49	0.45	0.42	0.49
		35	0.78	0.82	0.77	0.85	0.72	0.78
	1/4	15	0.65	0.68	0.56	0.61	0.58	0.67
		35	0.91	0.97	0.89	0.96	0.89	0.95
3/4	4	15	0.43	0.39	0.37	0.36	0.27	0.32
		35	0.77	0.82	0.71	0.76	0.55	0.63
	1/4	15	0.66	0.72	0.62	0.71	0.67	0.77
		35	0.91	0.96	0.9	0.97	0.95	0.96

Abrupt Breaks: The Nonparametric Test Statistic Has Good Power

Table 4: Finite sample power of tests under H^N

τ_{0h}	σ	g	LM		SupF		$\hat{H}_{het}$	
			$\tau_0 = 1/2$		$\tau_0 = 1/2$		$\tau_0 = 1/2$	
			T=100	T=200	T=100	T=200	T=100	T=200
-	1	15	0.92	0.9	0.89	0.89	0.65	0.79
		35	1	1	1	1	0.98	0.99
1/2	4	15	0.28	0.33	0.23	0.26	0.31	0.41
		35	0.71	0.74	0.8	0.8	0.51	0.63
	1/4	15	0.83	0.83	0.71	0.7	0.52	0.63
		35	1	0.99	0.99	1	0.89	0.92
3/4	4	15	0.49	0.51	0.31	0.29	0.21	0.32
		35	0.92	0.93	0.88	0.88	0.52	0.63
	1/4	15	0.91	0.9	0.85	0.87	0.66	0.75
		35	1	1	1	1	0.97	0.98

Abrupt Breaks: The Nonparametric Test Statistic Has Good Power

Table 5: Finite sample power of tests under H^N

τ_{0h}	σ	g	LM		SupF		$\hat{H}_{het}$	
			$\tau_0 = 3/4$		$\tau_0 = 3/4$		$\tau_0 = 3/4$	
			T=100	T=200	T=100	T=200	T=100	T=200
-	1	15	0.76	0.76	0.79	0.8	0.63	0.76
		35	0.98	0.97	1	1	0.97	0.98
1/2	4	15	0.71	0.71	0.69	0.73	0.43	0.53
		35	0.99	0.99	1	1	0.84	0.88
	1/4	15	0.65	0.67	0.4	0.4	0.45	0.55
		35	0.94	0.96	0.94	0.94	0.85	0.88
3/4	4	15	0.6	0.6	0.47	0.43	0.25	0.33
		35	0.98	0.98	0.96	0.96	0.64	0.7
	1/4	15	0.58	0.6	0.52	0.52	0.54	0.86
		35	0.91	0.88	0.95	0.91	0.86	0.89

Finite Sample Power: Selected DGPs

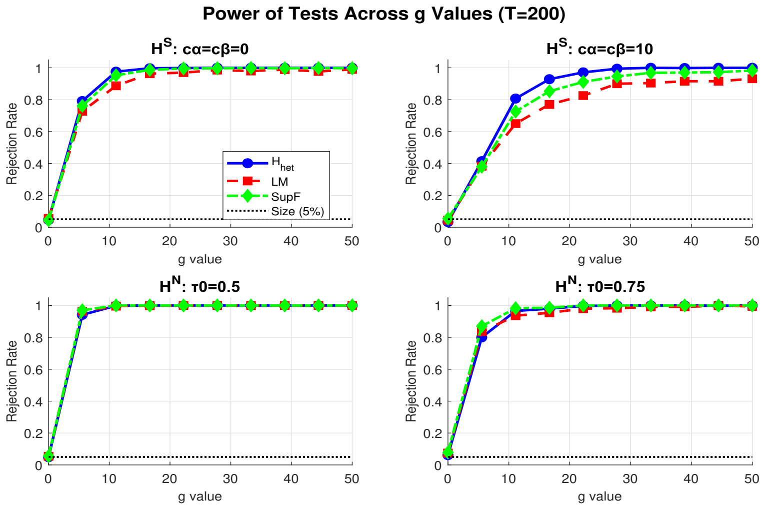

Figure 3: Empirical Sizes of the Structural Break Tests

The Nonparametric Test Performs Better in Application

Table 6: Application to Welch and Goyal (2008) data: multivariate regressions

y_t	x_{1t}	x_{2t}	$\hat{H}_{het}$	LM	SupF	$\hat{H}_{het}$	LM	SupF
Panel A. 1926-2015						Panel B. 1926-2007		
R_t	DY_t	DE_t	0	0.810	1	0	0.685	0.998
	DY_t	LTR_t	0	0.112	0.018	0	0.170	0.024
EP_t	DY_t	DE_t	0	0.441	0.908	0	0.319	0.898
	DY_t	LTR_t	0	0.102	0.028	0	0.114	0.034

The Nonparametric Test Performs Better in Application

Table 7: Application to Welch and Goyal (2008) data: using monthly and quarterly data

	Monthly Excess Return			Quarterly Excess Return		
	$\hat{H}_{het}$	$SupF$	LM	$\hat{H}_{het}$	$SupF$	LM
	Panel A: 1947 January - 2005 December			Panel B: 1947Q1-2005Q4		
D/P	0	0	0	0	0	0
DY	0	0.016	0	0	0.112	0.016
E/P	0.013	0	0	0.079	0.007	0
DE	0	0.076	0	0.113	0.427	0.137
SVAR	0	0.031	0	0	0.251	0.235
B/M	0	0	0	0	0.260	0
NTIS	0	0.110	0	0.020	0.398	0.002
TBL	0	0.158	0	0.002	0.596	0.005
LTY	0	0.172	0	0.036	0.591	0.006
LTR	0	0.071	0	0.083	0.350	0.007
TMS	0	0.189	0	0.009	0.373	0
DFY	0	0.011	0	0.001	0.260	0.003
DFR	0	0.110	0	0.006	0.384	0.018
INFL	0	0.151	0	0.259	0.589	0.156

Conclusion

- The performance of the nonparametric test is invariant to the alternative hypothesis
- The bootstrap procedure properly accounts for heteroskedasticity
- The nonparametric test demonstrates good size and power under the variety of alternative hypothesis

Other DGPs: The Nonparametric Test Statistic

Table 8: Empirical Powers of Tests under the other alternative mechanism

τ_{0h}	σ	DGP	LM		SupF		$\hat{H}_{het}$	
			T=100	T=200	T=100	T=200	T=100	T=200
-	1	P.1	0.76	0.98	0.66	0.99	0.69	0.87
		P.2	0.35	0.85	0.22	0.85	0.64	0.91
		P.3	0.43	0.83	0.25	0.87	0.71	0.92
		P.4	0.51	0.89	0.10	0.69	0.63	0.81
		P.5	0.98	1	0.84	0.99	0.99	1
1/2	4	P.1	0.58	0.88	0.46	0.95	0.54	0.93
		P.2	0.41	0.82	0.21	0.54	0.51	0.88
		P.3	0.67	0.94	0.67	0.96	0.83	0.99
		P.4	0.89	1	0.15	0.39	0.69	0.98
		P.5	0.96	1	0.65	0.78	0.98	1
	1/4	P.1	1	1	1	1	0.97	1
		P.2	0.87	1	0.83	1	0.93	1
		P.3	0.9	0.99	0.92	0.99	0.98	1
		P.4	0.92	1	0.6	0.9	0.86	0.99
		P.5	0.99	1	0.95	0.98	1	1
3/4	4	P.1	0.74	0.96	0.53	0.97	0.55	0.94
		P.2	0.33	0.71	0.08	0.34	0.47	0.85
		P.3	0.46	0.78	0.12	0.53	0.49	0.91
		P.4	0.76	0.97	0.07	0.15	0.49	0.93
		P.5	0.96	1	0.62	0.71	0.97	1
	1/4	P.1	1	1	1	1	0.98	1
		P.2	0.86	1	0.74	0.99	0.91	1
		P.3	0.9	1	0.85	0.99	0.97	1
		P.4	0.93	1	0.66	0.93	0.88	0.99
		P.5	0.99	1	0.97	0.99	1	1

Empirical Power of the Nonparametric Test Statistics

Table 9: Application to Welch and Goyal (2008) data: bivariate regressions

y_t	x_t	$\hat{H}_{het}$	LM	SupF	$\hat{H}_{het}$	LM	SupF
Panel A. 1926-2015					Panel B. 1926-2007		
R_t	DY	0.004	0.112	0.170	0.004	0.132	0.0902
	DE	0	0.042	0.008	0	0.014	0.004
	E/P	0	0.735	0.611	0.016	0.681	0.413
	D/P	0.134	0.782	0.707	0.261	0.872	0.810
	SVAR	0.233	0.946	0.968	0.315	0.908	0.840
	B/M	0.188	0.651	0.176	0.355	0.701	0.425
	NTIS	0.227	0.926	0.810	0.034	0.892	0.970
	TBL	0.247	0.840	0.832	0.263	0.721	0.832
	LTY	0.697	0.798	0.737	0.577	0.723	0.721
	LTR	0.186	0.409	0.164	0.705	0.479	0.200
	TMS	0.715	0.958	0.936	0.804	0.902	0.912
EP_t	DY	0.004	0.182	0.186	0.01	0.174	0.082
	DE	0	0.180	0.0180	0	0.092	0.002
	E/P	0	0.637	0.525	0.016	0.768	0.509
	D/P	0.076	0.649	0.633	0.176	0.727	0.824
	SVAR	0.180	0.794	0.894	0.188	0.625	0.930
	B/M	0.287	0.497	0.140	0.359	0.447	0.405
	NTIS	0.200	0.653	0.828	0.018	0.709	0.948
	TBL	0.235	0.872	0.828	0.281	0.752	0.842
	LTY	0.655	0.806	0.731	0.495	0.707	0.703
	LTR	0.190	0.413	0.226	0.557	0.309	0.242
	TMS	0.631	0.910	0.990	0.707	0.848	0.990

Fixed Regressor Wild Bootstrap Algorithm

- Use (Y_t, X_t') to estimate model via OLS and non-parametric regression. Then compute $\hat{H}$ statistic and the non-parametric residuals

$$\hat{\epsilon}_t = Y_t - X_t'\hat{\alpha}_t$$

- Obtain the wild bootstrap residual $\hat{\epsilon}_t^*$.

$$\bar{\epsilon}_t = \hat{\epsilon}_t - T^{-1}\sum_{t=1}^{T}\hat{\epsilon}_t$$

$$\hat{\epsilon}_t^* = \begin{cases} c\bar{\epsilon}_t & \text{with prob} \quad 1 - \frac{c}{\sqrt{5}} \\ (1-c)\bar{\epsilon}_t & \text{with prob} \quad \frac{c}{\sqrt{5}} \end{cases}$$

where $c = \frac{(1+\sqrt{5})}{2}$

Fixed Regressor Wild Bootstrap Algorithm

$$Y_t^* = X_t'\alpha + \hat{\epsilon}_t^*$$

- Compute the bootstrap statistic $\hat{H}^*$ with (Y_t^*, X_t')
- Repeat steps (bullet points) two and three B times

Variable Description

Variable	Description	Variable	Description
B/M	Book to market ratio	DFY	Default yield spread
E/P	Earnings Price Ratio	TBL	Treasury bill rate
D/Y	Dividend Yield	NTIS	Net equity expansion
D/E	Dividend Payout Ratio		
LTR	Long Term Return		
SVAR	Stock Variance		
D/P	Dividend Price Ratio		