Option Pricing Project – Monte Carlo Methods

1 Introduction

In this project, I used Monte Carlo methods to price different types of options. I started by pricing European options using both the Black-Scholes formula and Monte Carlo simulation. I then priced American options using the Longstaff-Schwartz algorithm, and then extended the approach to Asian and barrier options. To improve the efficiency of the Monte Carlo simulations, I applied variance reduction techniques such as antithetic variates, control variates and importance sampling. I also calculated the Greeks using finite differences and used them to simulate delta-hedging with transaction costs. The objective is to demonstrate both practical implementation and theoretical understanding of numerical option pricing methods commonly used in quantitative finance.

2 Pricing Models

2.1 Setup

I first imported one year of historical Apple stock price data to compute the annual volatility for the past year. This was done by computing the daily log returns $\log(C_t/C_{t-1})$ for $t = 1, ..., 249$, where C_t is the closing price at time t. I then computed the standard deviation of the daily log returns and multiplied by $\sqrt{250}$ to attain the annual volatility for the last year [1]. All simulations assume a constant risk-free rate of 4.09%, corresponding to the 1-Year U.S. Treasury Constant Maturity Rate for May 2025. A constant expiration of 1 year was also used. The latest Apple stock price from the imported data was used as the initial stock price.

2.2 Black-Scholes Formula

I then implemented the Black-Scholes formula to price European options. The Black-Scholes Model describes the evolution of the stock price through the equation $dS(t) = rS(t)dt + \sigma S(t)dW(t)$ [2, page 4] where $S(t)$ is the stock price at time t, $W(t)$ is a standard Brownian motion, σ is the volatility and r is the risk-free rate. Note that I have not included dividends. The solution to this equation leads to the following formulas:

- $d_1 = \frac{\log(S_0/K)+[r+\frac{1}{2}\sigma^2]T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$
- Call Option Price $= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$
- Put Option Price $= Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$

2.3 Monte Carlo Simulations

I then used Monte Carlo simulations to price the same option, this is done as follows [2, page 5]:

- $S_i(T) = S_i(0)\exp\left([r - \frac{1}{2}\sigma^2] + \sigma\sqrt{T}Z_i\right)$ for $Z_i \sim N(0, 1), i = 1, ..., n$ where n is the number of simulations,
- $C_i = e^{-rT}(S_i(T) - K)^+$,
- $\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i$ is the estimated European call price

With sufficient simulations, these methods result in the same prices, as seen below in figure 1.

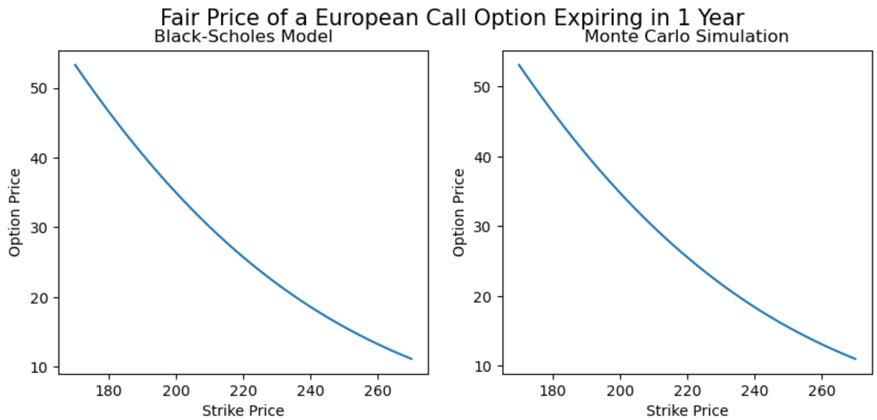

Figure 1: Black-Scholes Monte Carlo Comparison

2.4 Longstaff-Schwartz Algorithm

For American options, you can exercise at any time. The time to maturity is thus split into steps and the option is priced at each step. The price of an American call option at time t is $\max\left((S_t - K)^+, C_t\right)$ where C_t is the continuation value at time t. To compute the continuation value, least squares regression is used to estimate the expected cash flow from continuing the option's life conditional on the stock price at time t [3]. Let X be the stock prices at time t such that the option is in-the-money, and Y the corresponding discounted cash flows received at time $t+1$ if the option is not exercised at time t. Y is then regressed on X and X^2, leading to the continuation values $E\left[Y|X\right]$. It is then considered optimal to exercise if the exercise price $(S_T - K)^+$ is larger than the corresponding continuation value. Cash-flows are only adjusted on a path if it is optimal to exercise. This process is done backwards from the payoffs at expiration through each step to time 1. The results of this algorithm can be seen in figure 2 below.

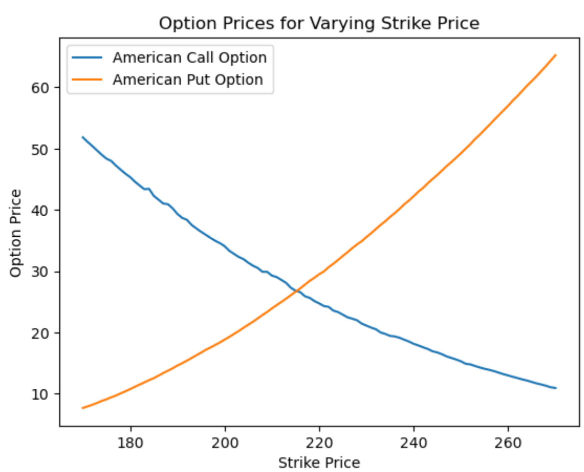

Figure 2: American Call and Put Option Prices

2.5 Exotic Options

For Asian call options, the payoff at expiration is $(S_{\text{avg}} - K)^+$ where S_{avg} is the mean of the stock price over time, excluding the price at time 0. For put options this is $(K - S_{\text{avg}})^+$. Since the average is required, the stock price needs to be simulated over steps, similarly to the American option. However, pricing an Asian option is computationally cheaper than an American option as the Longstaff-Schwartz algorithm is not needed.

For knock-in barrier options, the payoff is the same as a European option but it requires a barrier to have been met, i.e. the stock price must reach a certain barrier at some time $(0, T]$. For call options this is from below (price must hit or exceed barrier) and for put options this is from above (price must hit or fall below the barrier) [4]. Thus the stock price was again simulated over steps, so that the barrier conditions could be checked.

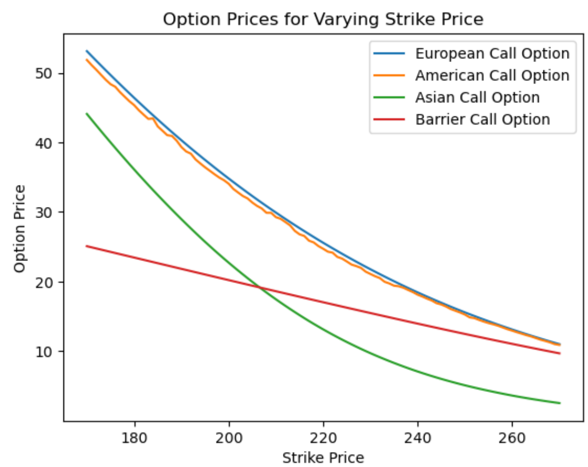

Figure 3: Call Option Prices for Different Options

3 Variance Reduction

3.1 Antithetic Variates

The method of antithetic variates involves pairing $Z_1, Z_2, ...$ of i.i.d. $N(0, 1)$ variables with $-Z_1, -Z_2, ...$ when simulating stock price paths [2, page 205]. When implemented into the pricing of a European call option, using 3000 Monte Carlo simulations, it gave the following results:

```
Price of Option without Variance Reduction: 25.52
Price of Option with Antithetic Variates: 25.06

Variance of Payoffs without Variance Reduction: 2162.77
Variance of Payoffs with Antithetic Variates: 2093.43
Variance reduction: 3.21%
```

Figure 4: Results from Antithetic Variates

This led to a modest variance reduction of 3.21%, this is likely not too effective as antithetic variates are most effective when the payoff function is linear in Z, which is not the case here.

3.2 Control Variates

The method of control variates in this case involves using the error from pricing a European option to adjust the price of an Asian option. Let Y_i be a payoff from pricing an Asian option using Monte Carlo, and X_i a payoff from pricing a European option using Monte Carlo, where both correspond to the same stock price path. Let $E(X)$ be the price given by the Black-Scholes formula as there is no error in the formula's output. Then the adjusted payoff is $Y_i(b) = Y_i - b(X_i - E(X))$ for some b [2, page 186]. The optimal b coefficient is $b^* = \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]}$ which can easily be estimated. When implemented into the pricing of an Asian call option, using 2000 Monte Carlo simulations, it gave the following results:

```
Price of Option without Variance Reduction: 13.08
Price of Option with Control Variates: 13.1

Variance of Payoffs without Variance Reduction: 596.41
Variance of Payoffs with Control Variates: 571.88
Variance reduction: 4.11%
```

Figure 5: Results from Control Variates

This led to a variance reduction of 4.11%, this is again quite modest, likely because the payoffs for the Asian and European call options are only somewhat correlated. Two more correlated payoffs might improve results.

3.3 Importance Sampling

Importance sampling was used to reduce the variance of barrier option payoffs. This involved shifting the mean of the $Z_1, Z_2, ...$ i.i.d. $N(0,1)$ variables to become i.i.d. $N(\mu,1)$ variables. The following likelihood ratio is then used to correct the payoffs (payoffs are multiplied by this factor) [2, page 255]:

$$\frac{L(0)}{L(\mu)} = \frac{\Pi_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}Z_i^2\right)\right)}{\Pi_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(Z_i-\mu)^2\right)\right)} = \exp\left(-\mu\sum_{i=1}^{n} Z_i + \frac{n}{2}\mu^2\right)$$

where n is the number of steps. This small shift results in more stock prices surpassing the barrier, i.e. paths that were originally quite rare become common. I used a shift of 0.3 and 2000 simulations, it gave the following results:

```
Price of Option without Variance Reduction: 8.39
Price of Option with Importance Sampling: 8.78

Variance of Payoffs without Variance Reduction: 1514.79
Variance of Payoffs with Importance Sampling: 128.14
Variance reduction: 91.54%
```

Figure 6: Results from Importance Sampling

This led to a variance reduction of 91.54%, which is extremely effective. The price of the option was very slightly higher when using importance sampling, this was likely due to there being only 2000 simulations. Due to the large variance reduction, computational costs will be significantly lower as less simulations are needed. This worked well for barrier options due to the large amount of zero payoffs from stock price paths not reaching the barrier. If this were to be used for a put barrier option, the shift would have to be negative.

4 Greeks

The Greeks are defined as follows:

- Delta: $\Delta = \frac{\partial V}{\partial S}$
- Gamma: $\Gamma = \frac{\partial^2 V}{\partial S^2}$
- Vega: $\text{Vega} = \frac{\partial V}{\partial \sigma}$
- Rho: $\rho = \frac{\partial V}{\partial r}$
- Theta: $\Theta = \frac{\partial V}{\partial t}$

Note that V is the option price, S is the stock price at time 0, σ is the volatility, r is the risk-free rate and t is the time until maturity. I used finite differences to compute these values, Black-Scholes could have been used for European options but to keep things simple I chose not to.

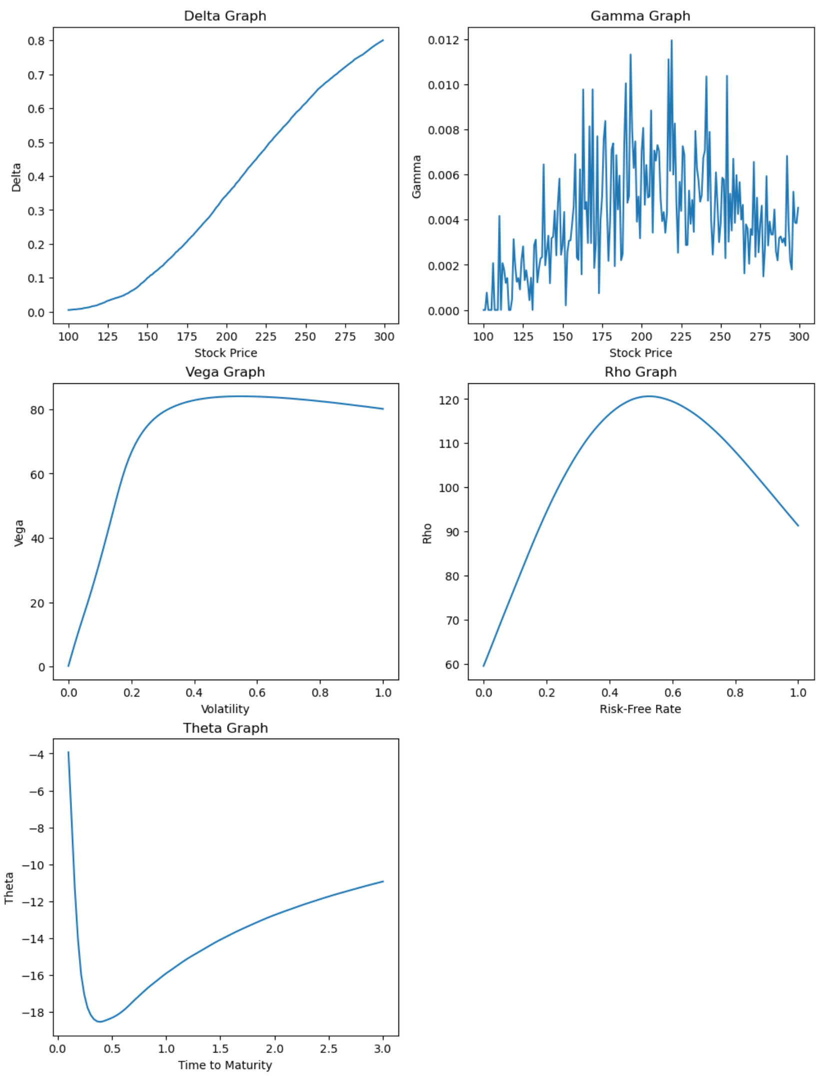

Figure 7: Greeks for a European Call Option

The Gamma graph shows significant noise, however the y-values do not exceed 0.012. The noise is likely due to the Gamma estimate not being accurate to this level.

5 Delta Hedging with Transactional Costs

Delta hedging a call option in this case involves being short the European call option and owning Δ shares. Each day is considered as a step, so the stock price is updated daily and if the delta value changes by at least the threshold ϵ then the amount of shares held is updated to the new delta value. Proportional transactional costs have been included to make the simulation more realistic. At expiration, the shares are all sold and the option exercised (assuming it's in-the-money). The profits and losses were tracked for each Monte Carlo stock price simulation, and the results are shown below. 100,000 simulations were performed for each ϵ value and proportional transactional costs were 1%, which is quite high.

```
Epsilon = 0.001
Mean P&L: -£9.5
Variance of P&Ls: 29.56
95% Confidence Interval for P&Ls: (-£22.1, -£2.09)

Epsilon = 0.05
Mean P&L: -£6.94
Variance of P&Ls: 26.8
95% Confidence Interval for P&Ls: (-£19.28, £0.19)

Epsilon = 0.1
Mean P&L: -£5.16
Variance of P&Ls: 30.75
95% Confidence Interval for P&Ls: (-£17.93, £3.63)

Epsilon = 0.2
Mean P&L: -£4.01
Variance of P&Ls: 53.85
95% Confidence Interval for P&Ls: (-£21.4, £8.57)

Epsilon = 0.5
Mean P&L: -£2.56
Variance of P&Ls: 284.75
95% Confidence Interval for P&Ls: (-£39.94, £24.61)

Epsilon = 1
Mean P&L: -£1.25
Variance of P&Ls: 488.71
95% Confidence Interval for P&Ls: (-£52.58, £25.25)
```

Figure 8: Profit and Loss for Varying Epsilon

As ϵ increases, the mean profit and loss gets closer to 0 but the variance increases significantly. This is expected as less trades are carried out for larger ϵ values, leading to more consistent results but a lower mean profit due to transaction costs. For smaller ϵ values, more trades are carried out so the mean loss is higher but results are significantly more consistent. Note that when $\epsilon = 1$, since $\Delta \in [0, 1]$, essentially no trades are carried out leading to the extremely high variance.

References

[1] Historical Volatility Calculation Macroption 2025 [Cited 2025 July 1]. Available from: https://www.macroption.com/historical-volatility-calculation/

[2] Glasserman, Paul Monte Carlo Methods in Financial Engineering Springer 2003 ProQuest Ebook Central https://ebookcentral.proquest.com/lib/imperial/detail.action?docID=4976106

[3] Longstaff, Francis A. ; Schwartz, Eduardo S Valuing American Options by Simulation: A Simple Least-Squares Approach The Review of Financial Studies Vol. 14, No. 1 Spring, 2001 [Cited 2025 July 1] Available from: https://people.math.ethz.ch/~hjfurrer/teaching/LongstaffSchwartzAmericanOptionsLeastSquareMonteCarlo.pdf

[4] Chen, James Investopedia 2024 [Cited 2025 July 1] Available from: https://www.investopedia.com/terms/b/barrieroption.asp