Homework Set #1

Due: Wednesday, March 2, 2022.

Please attach the MATLAB codes for the problems with the MATLAB parts.

1. *A Basic Probability Appetizer...*

   Show that two events A and B are independent if and only if $P(A|B) = P(A|B^C)$.

2. *Memoryless Property of the Exponential Distribution*

   Let X be exponentially distributed with parameter $\lambda > 0$, i.e., the pdf of X is given by

   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   Find the conditional pdf $f_X(x|X \geq a)$ for some $a > 0$. Draw both $f_X(x)$ and $f_X(x|X \geq a)$ on the same figure.

3. *An Exercise on Joint Distributions.... Gray 3.35*

   Suppose that a random vector $\mathbf{X} = \begin{bmatrix} X_0 & X_1 & \dots & X_{k-1} \end{bmatrix}^T$ is i.i.d. with marginal pmf

   $$p_{X_i}(l) = \begin{cases} p & l = 1 \\ 1-p & l = 0 \end{cases}, \tag{2}$$

   for all i.

   (a) Find the pmf of the random variable $Y = \prod_{i=0}^{k-1} X_i$,

   (b) Find the pmf of the random variable $W = X_0 + X_{k-1}$

   (c) Find the pmf of the vector $\begin{bmatrix} Y & W \end{bmatrix}^T$.

4. *Cauchy-Schwartz Inequality for Random Variables*

   Prove the Cauchy-Schwartz Inequality

   $$Cov(X, Y) \leq \sigma_X \sigma_Y$$

   where X, and Y are real random variables with means μ_x and μ_Y and standard deviations σ_X and σ_Y respectively.

   Hint: Define $Z = a(X - \mu_x) + b(Y - \mu_Y)$ for some $a, b \in \Re$. Note that $E(Z^2) \geq 0$ for any $a, b \in \Re$. Use this fact and choose a and b properly to show the desired inequality.

5. *Simple MATLAB Exercise*

   Write a one line MATLAB code to generate 10000 realizations of a Gaussian random variable X_n with mean 1 and variance 4.

(a) Define the running average

$$S_n = \frac{1}{n} \sum_{k=1}^{n} X_k. \tag{3}$$

Obtain S_n from X_n samples generated. Plot $|S_n - 1|$ as a function of n.

(b) Define the time correlation

$$K_n = \frac{1}{n} \sum_{k=1}^{n} X_k X_{k+1} \tag{4}$$

Plot K_n. Based on the observed trend what would you say about the limit of K_n as n goes to infinity?