2

$u_t = k u_{xx}$, $0 < x < 2$, $t > 0$

$u_x(0,t) = 0$, $u_x(2,t) = 0$ $t > 0$

$$u(x,0) = \begin{cases} 0, & 0 < x < \frac{3}{4} \\ \frac{1}{2}, & \frac{3}{4} \le x \le \frac{5}{4} \\ 0, & \frac{5}{4} < x \le 2 \end{cases}$$

$X(x)\dot{T}(t) = k X''(x) T(t)$ Divide by $k X(x) T(t)$

$$\frac{\dot{T}(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$\boxed{\dot{T}(t) + \lambda k T(t) = 0}$ $\boxed{X''(x) + \lambda X(x) = 0}$

ODE S-L problem

$X''(x) + \lambda X(x) = 0$ CE $r = \pm\sqrt{-\lambda}$

Case 1: $\lambda = 0$

$X(x) = A + Bx$
$X(0) = A = 0$ $A = 0$
$X(2) = 2B = 0$ $B = 0$

Come back if time

Case 2: $\lambda < 0$ let $\lambda = -\beta^2$ $r = \pm\beta$

$X(x) = Ae^{\beta x} + Be^{-\beta x} \Rightarrow X(x) = A\cosh(\beta x) + B\sinh(\beta x)$

$X(0) = A + 0 = 0$ $A = 0$
$X(2) = 0 + B\underbrace{\sinh(\beta x)}_{0} = 0$ let $B = 1$

$\boxed{\lambda_0 = 0,\ X_0(x) = 1}$

Case 3: $\lambda > 0$; $\lambda = \beta^2$, then $r = \pm \beta i$

Then $X(x) = A\cos(\beta x) + B\sin(\beta x)$

$X(0) = A + 0 \quad A = 0$

$X(2) = B \underbrace{\sin(2\beta)}_{\text{can be zero}} = 0$

let $\beta = \frac{n\pi}{2}$

let $B = 1$.

Then, $\lambda = \beta^2 = \left(\frac{n\pi}{2}\right)^2$

$$\boxed{\begin{aligned} \lambda_n &= \left(\frac{n\pi}{2}\right)^2 \\ X_n(x) &= \cos\left(\frac{n\pi}{2}x\right) \end{aligned}}$$

$\dot{T}(t) + \lambda k T(t) = 0$

$\dot{T}(t) + \left(\frac{n\pi}{2}\right)^2 k T(t) = 0$

Solution matches longterm behavior as $t \to \infty$, $u(x,t) \to \infty$.
As time goes on, flux goes to zero. This makes sense considering the sides of the rod are insulated, so at very large t values heat flow will be steady/constant, if not necessarily 0. A larger k would mean it takes longer to get to the steady state, but it will get there eventually.