Journal of Economic Dynamics & Control 76 (2017) 109–135

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Monitoring vulnerability and impact diffusion in financial networks[☆]

Thiago Christiano Silva[a,b,c,*], Sergio Rubens Stancato Souza[a], Benjamin Miranda Tabak[c]

[a] Department of Research, Central Bank of Brazil (Banco Central do Brasil), Brasília, Distrito Federal, Brazil
[b] Department of Computing and Mathematics, Faculty of Philosophy, Sciences, and Literatures in Ribeirão Preto, University of São Paulo, Ribeirão Preto, Brazil
[c] Department of Law and Economics, Catholic University of Brasília, Brasília, Brazil

ARTICLE INFO

Article history:
Received 7 April 2016
Revised 29 December 2016
Accepted 2 January 2017
Available online 7 January 2017

Keywords:
Systemic risk
Monitoring
Vulnerability
Impact propagation
Networks

ABSTRACT

In this paper, we propose novel risk-related network measurements to identify the roles that financial institutions play as potential targets or sources of contagion. We derive theoretical properties and provide a clear systemic risk interpretation for the proposed measures. Devised upon the notion of communicability in networks, we introduce the impact susceptibility index, which indicates whether market participants are locally or remotely vulnerable. We show how this index can be used as a financial stability monitoring tool and apply it to analyze the Brazilian financial market. We find that non-banking institutions are potentially remotely vulnerable in certain periods, while banking institutions are not susceptible to indirect impacts. To address the perspective of market participants as sources of contagion, we propose the impact diffusion influence index, which captures the potential influence of financial institutions on propagating impacts in the network. We unveil the presence of a portion of small/medium banking institutions that is consistently more influential than large banks in potentially diffusing impacts throughout the network. Regarding financial system stability, regulators should identify the entities that play these two roles, as they can render the system more risky.

1. Introduction

This paper analyzes in a joint framework two opposite perspectives of the role of a financial institution with respect to financial contagion: (i) the susceptibility to receiving and (ii) the capacity of amplifying random shocks. We propose novel network measurements that are intuitive to economics and finance to capture these two views on systemic risk. They focus on catastrophic events that can trigger financial contagion, more specifically, default cascades, in the network of bank lending and borrowing relationships. These measures do not require calibration of parameters, which is a positive property as

☆ The views expressed in this paper are those of the authors and do not necessarily reflect those of the Banco Central do Brasil. We would like to thank Eduardo José Araújo Lima, João Barata Ribeiro Blanco Barroso, Aquiles Rocha de Farias, Carlos León, Jorge Ponce, Marcos Souto, Carlos Leonardo Kulnig Cinelli, participants at the seminar of the Universidade Católica de Brasília, at the workshop of Financial Stability (RIDGE), and at the Central Bank of Brazil seminars for comments and suggestions. We are also thankful for the valuable comments and suggestions of three anonymous reviewers, editor Herbert Dawid, and the associate editor.

* Corresponding author at: Department of Research, Central Bank of Brazil (Banco Central do Brasil), Depep/Conep, Andar 13, Setor Bancário Sul (SBS) Quadra 3 Bloco B – Ed. Sede, CEP: 70074-900, Brasília, Distrito Federal, Brazil. Tel.: +55 (61) 3414 3870.

E-mail addresses: thiago.silva@bcb.gov.br (T.C. Silva), sergio.souza@bcb.gov.br (S.R.S. Souza), benjaminm.tabak@gmail.com (B.M. Tabak).

http://dx.doi.org/10.1016/j.jedc.2017.01.001
0165-1889/

we can skip the complex task of model selection. We provide a theoretical analysis of the network measurements and derive novel properties with a clear systemic risk interpretation. In this analysis, we propose the concepts of remote vulnerability and contagiousness of financial institutions. These systemic risk measures have nice and strong theoretical properties that make them useful for surveillance, monitoring, and regulation of the financial system.

The financial crisis of 2008–2009 highlighted the high degree of integration between financial systems worldwide. The set of transactions between economic agents gives rise to large financial networks and can represent a variety of financial instruments with differing financial complexity, such as interbank lending, derivatives and other instruments.[1] Though financial interconnectedness potentially imposes amplification and propagation of insolvency, illiquidity and losses in the financial system, which in turn may spillover to the real economy exacerbating or even originating financial crisis, to date, there is no consensus on how we can derive financial connectivity measures that can be useful for financial regulation and surveillance.

Financial networks are complex systems that absorb and respond to external events and thus are liable to evolving over time. Minoiu and Reyes (2013) argue that bank-level connectivity is volatile, which implies that network topology changes as well. In interesting case studies, Martínez-Jaramillo et al. (2014) and Gabrieli and Georg (2014) reveal that interbank exposures have significantly changed in the Mexican and in the euroarea markets, respectively, after the collapse of Lehman Brothers. Therefore, one could expect substantial changes in financial networks over time. As such, monitoring how financial networks unfold over time is crucial for assessing systemic risk and for measuring the financial contagion likelihood in different periods. In this concern, however, the literature is silent in providing guidelines as to which measures would be useful for monitoring these financial networks. This issue is of utmost importance because the failure of financial institutions can have relevant implications to the real economy and hence far-reaching consequences.

A very relevant question is how the financial network topology influences systemic risk and the probability of contagion within the network. Allen and Gale (2000), Allen et al. (2012) and Gai and Kapadia (2010) show that more complete interbank claims networks are more robust to financial contagion. In contrast, Battiston et al. (2012a) show that the relationship between the probability of default, both individually and systemic, and connectivity is U-shaped. Acemoglu et al. (2015) show in a recent paper that dense interconnections can serve either as an amplifier of negative financial shocks or they can enhance financial stability, depending on the size of the shock. In our paper, we contribute to this work, proposing a model that evaluates the degree of susceptibility to and likelihood of shock propagations of individual financial institutions as well as of the financial system. In our model, systemic risk depends not only on the density of interconnections but also on the capital buffers.[2]

Our paper contributes to the literature in three main ways. First, we propose new network measures that are intuitive to economics and finance and focus on capturing potential events of default cascades inside a financial network. Second, we derive theoretical properties for these network measures and propose the novel concepts of local/remote vulnerability and contagiousness with practical implications to financial regulation, surveillance, and monitoring of financial systems. In this sense, we relate these concepts to the effort of monitoring the soundness of financial institutions. Considering the scope of our model, we show that the monitoring of the direct neighborhood is sufficient to identify financial contagion for locally vulnerable entities. In contrast, the monitoring effort must be broader in case entities are remotely vulnerable. Third, we show how these measures evolve over time using a unique data set on complete exposures for the Brazilian financial market. We find that susceptibility and diffusion of banks and non-banks have very different behaviors, which reinforce the complementary role between our measures.

Our model focuses on financial contagion that may arise from interbank exposures and does not take into account liquidity or margin calls processes. Therefore, models that do consider these mechanisms can complement the information given by our model. In this paper, we provide a set of measures that are useful to identifying default cascade paths inside the financial system. In addition, we compare our measures with state-of-the-art indicators in the literature.

We first introduce the impact susceptibility measure, whose underpinnings rely on the concept of network communicability, originally introduced by Estrada and Hatano (2008). In general terms, we define the impact susceptibility as the likelihood of a financial institution (FI) defaulting as a consequence of default events in arbitrary points of the network. Thus, this measurement gauges the potential susceptibility to defaulting from financial contagion of other market participants.

We use the vulnerability matrix of the financial system to identify the potential contagion paths between FIs. We evaluate the impact susceptibility as a weighted combination of shortest paths and walks of several lengths that are mapped out from the vulnerability matrix. In this way, the impact susceptibility highlights how likely potential contagion paths may end up impacting FIs. The motivation of using not only shortest paths but also other longer paths is as follows. The contagion transmission may not occur only through the shortest path between different FIs. For example, it may cascade through longer paths in an isolated or additive manner, using routes that are easier to “breakthrough” because of the presence of FIs with small capital buffers. The impact susceptibility captures both types of contagion transmissions.

[1] A study commissioned by the International Monetary Fund, the Bank for International Settlements, the Financial Stability Board and the G20 has found that size is not the only dimension that prevails when establishing the systemic importance of financial institutions, although it is the most important factor. The study finds that interconnectedness is the second most important factor in the determination of the systemic importance of financial institutions (IMF et al., 2009).

[2] The importance of capital buffers in the context of contagion/losses propagation has also been assessed in the theoretical literature (Amini et al., 2016) and in applied exercises that attempt to measure the extent of financial contagion (Nier et al., 2007).

The impact susceptibility is suitable as a monitoring tool for financial stability, because it identifies potential candidates for a close surveillance. We link the impact susceptibility of each bank to its potential of defaulting due to negative impacts that participants in the network can experience. We demonstrate that, when the impact susceptibility of an FI equals one, the analysis of its local vulnerabilities or equivalently the health of its neighbors is roughly sufficient to determine whether that FI will propagate financial contagion. Opposed to that, when the impact susceptibility of that FI is larger than one, we demonstrate that only assessing the financial health of its neighborhood may not provide sufficient information as to whether that entity will propagate contagion. In this case, the FI is prone to contagion from remote institutions that can communicate through the network interconnections and hence inflict damage on that FI. We relate this phenomenon to the concept of remote vulnerability, which can be understood as the susceptibility of market participants to receiving indirect hits through third counterparties.

One of our main theoretical contributions is in demonstrating that the impact susceptibility of an FI is greater than one if and only if it is remotely vulnerable. Using this result, potential good candidates for a close surveillance are those entities that are remotely vulnerable, because distant defaults are likely to reach them. Note that, however, this is a measure of potential for contagion and not of contagion itself, meaning that the default of the market participant will depend on the occurrence of losses in its direct exposures and on its current capability of absorbing these losses.

The impact susceptibility gives us a sense of how far an FI can be impacted by a random shock. However, it does not convey how important losses of that FI can be to the entire financial system. For this matter, we propose the weighted impact susceptibility that weighs the impact susceptibility of that FI by its economic importance to the financial system. We proxy the economic importance of an FI as its share in the total liabilities of the system. If it defaults, the financial system will lose these liabilities. The weighted impact susceptibility is then related to the potential loss that the financial system would suffer if that FI defaults due to a local or remote vulnerability route.

One natural extension of the impact susceptibility index is to transport its meaning from the bank-level to a network-level measurement. Using this idea, we introduce the notion of impact fluidity of a network. We define the impact fluidity as how easy a potential impact can travel throughout the vulnerability network. It closely relates to the average length of default cascade paths that arrive to each defaulting FI. Thus, it hints supervisors as to what extent they should be concerned about indirect spillovers in the financial system. Domino-like effects due to an onset of a contagion process are more prone to happening in networks with large impact fluidity. This holds true because, in this situation, FIs tend to be very susceptible to receiving impacts from the network due to their large communicability. Therefore, networks with small fluidity of potential impacts are safer than those with large impact fluidity. We show that the network topology and the available capital buffers of FIs are the main contributors for establishing the impact fluidity of the network.

Another contribution of this paper is in defining a new indicator that captures the notion of how influential one institution is in terms of diffusing severe impacts (defaults) over the network. Note that the impact diffusion influence is the opposite perspective of the impact susceptibility. We can understand the diffusion influence of an FI q in terms of the variation it provokes on the communicability indices of all of the participants when we remove q's power of diffusing impacts in the vulnerability matrix. If the communicability decreases to a large extent, then we assume that q plays an important role in diffusing impact throughout the network. Conversely, if its removal slightly modifies the communicability between FIs, then q does not influence the impact diffusion process in the network.

We then decompose the variations of the communicability indices due to that removal mechanism into two complementary terms that are intuitive from a viewpoint of impact propagation. These terms are the influence exerted by q in starting and in intermediating the impact propagation. We also provide guidelines for employing the diffusion influence indices as monitoring tools for financial stability. With this purpose, we introduce the concept of remote contagiousness, which is the possibility of an FI q to lead into default other network members that are not directly exposed to q. In addition, we show that, if q is remote contagious, then its impact diffusion influence must be greater than one. This result enables us to find the most harmful entities in the network in a systematic manner.

An innovation in this work is in the decomposition of the impact diffusion influence into the roles of starting and intermediating financial contagion. The literature often conceives the implied systemic risk of a counterparty simply as the additional loss that it imposes to the financial system in case that counterparty defaults, i.e., when it starts an impact diffusion process. For instance, the DebtRank method (Battiston et al., 2012b), the new differential DebtRank (Bardoscia et al., 2015), the pioneering Eisenberg and Noe's (2001) clearing payment algorithm and its extension (Rogers and Veraart, 2013) only account for the component of starting a shock propagation. Therefore, while the importance of counterparties as shock starters is widely studied, the role that these counterparties play as intermediators of an ongoing shock propagation process is a new perspective of seeing contagion that our measures capture.

In financial networks, institutions with large impact diffusion influence have the potential to propagate and amplify losses to the network. This measure, however, does not quantify how harmful these impacts are to the entire financial system. For instance, one institution can have large influence in propagating impacts, but only to non-important institutions. *A contrario sensu*, one institution may have small influence in propagating impact or losses and still have the potential to cause an important impact on the financial system. Therefore, we also define a weighted version of the impact diffusion influence by modulating each potential impact to a corresponding proxy of economic importance. Note that, while the non-weighted version provides a quantitative measure related to the length of default cascades starting or continuing from a given FI, the weighted version also brings into play the economic importance of the entities taking part of these chains after the FI.

Our empirical exercise provides some guidance on how the methodology works in a financial environment dominated by universal banks (characteristic of the Brazilian banking system). There is some controversy on whether universal bank model leads to higher or lower systemic risk. On the one hand, Demirgüç-Kunt and Huizinga (2010) argue that the universal banking model has emerged as a more desirable structure for financial institutions after the crisis, since it is more resilient to adverse shocks due to risk diversification and increases of return. On the other hand, Battiston et al. (2012a) show that risk diversification can also be problematic.[3] Also, Haldane and May (2011) have been warning the policy community about the danger of the universal banking model. Therefore, our empirical exercise that evaluates the vulnerability of universal banks, from a network perspective, may provide some insights on their inherent risks.

The identification of financial institutions that are potential receivers or diffusers of shock inside a financial network has been studied before in the literature (Greenwood et al., 2015; Drehmann and Tarashev, 2013; Hideaki et al., 2013; Battiston et al., 2016a,b). In the majority of these researches, these two opposite roles are evidenced by allocating a single systemic risk measurement differently across individual institutions. In contrast, we do not fraction and allocate in different manners a single systemic risk measure among counterparties; rather, we compute two genuine distinct systemic risk measures. While the impact susceptibility looks at the pairwise communicability between all counterparties to a reference entity, the impact diffusion influence concerns how the overall communicability behaves when we remove a specific counterparty from the network.

The paper proceeds as follows. Section 2 describes our systemic risk framework. Section 3 compares the systemic risk measures we propose in this paper to existent state-of-the-art indicators in the literature. Section 4 details how we construct our dataset that contains bilateral exposures in the Brazilian financial market. Section 5 discusses results we obtain from applying our framework on Brazilian supervisory data. Section 6 draws conclusions and possible future works.

2. Methodology

In this paper, we move forward in the complex network and finance literature by contributing with a new systemic risk framework that is able to identify potential targets and sources of financial contagion.[4] We show intuitions linking these measures to economics and finance and also provide clear systemic risk interpretations that may be fruitful for the surveillance, monitoring, and regulation of the financial system.

We first set the mathematical notations used in this section. We assume that $\mathbf{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ represents the graph encoding the borrowing and lending relationships between financial counterparties. We consider $\mathcal{V}$ as the set of vertices (FIs) and $\mathcal{E}$ as the set of edges (financial operations). The cardinality of $\mathcal{V}$, $N = |\mathcal{V}|$, represents the number of vertices or FIs in the network. The matrix $\mathbf{L}$ expresses the liabilities matrix (weighted adjacency matrix) in which the (i,j)-th entry corresponds to the liabilities of the FI (vertex) i towards j. We define the set of edges $\mathcal{E}$ by the following filter over $\mathbf{L} : \mathcal{E} = \{\mathbf{L}_{ij} > 0 : (i, j) \in \mathcal{V} \times \mathcal{V}\}$. In our analysis, there is no netting between i and j.[5] As such, if an arbitrary pair of FIs owe to each other, then $\mathbf{L}$ will present two independent edges linking each other but in opposite directions. An interesting property of maintaining the gross exposures in the network is that, if an FI defaults, its debtors remain liable for their debts.

We start by presenting the communicability concept, opening way to understanding the measures that comprise our systemic risk framework. Afterwards, we detail the contributions of this work.

2.1. Relevant background: communicability concept

In this section, we review the concept of communicability in complex networks, which we use to build up our risk-related network measures in this paper. Communicability is defined for every pair of vertices $p \in \mathcal{V}$ and $q \in \mathcal{V}$. In essence, the communicability from p to q quantifies how easily vertex p can communicate with q by means of a combination of shortest paths and random walks with varying lengths. The concept of communicability first appeared in the work of Estrada and Hatano (2008), who define the communicability of vertex p to q as:

$$\mathbf{G}_{pq}(\mathbf{M}) = \frac{1}{s!}\mathbf{P}_{pq} + \sum_{k>s} \frac{1}{k!}\left(\mathbf{M}^k\right)_{pq} = \left(e^{\mathbf{M}}\right)_{pq}, \tag{1}$$

in which $\mathbf{P}_{pq}$ denotes the number of paths with the shortest length starting from p and ending at q; s is the length of such paths; and $\mathbf{M}$ is the binary adjacency matrix that encodes the bilateral relationships between the vertices.

The term $\mathbf{M}_{pq}^{(k)}$ is the (p, q)-th element of the kth power of matrix $\mathbf{M}$, which gives the number of walks of length k from p to q along the binary adjacency matrix $\mathbf{M}$, where $k > s$. The communicability of $\mathbf{G}_{pq}$ and $\mathbf{G}_{qp}$ may be different for directed graphs, such as for the financial market network. A large $\mathbf{G}_{pq}$ reveals that p can reach q by several routes. Conversely, when $\mathbf{G}_{pq}$ is small, there are few possibilities for p reaching q.

---

[3] See e.g. Battiston et al. (2012a) and literature cited therein.

[4] We have implemented all the indicators proposed in this paper using the R language. The code has been published as an R package under the name of "NetworkRiskMeasures" and comes with a handful of documentation files illustrating how to run these measures. We are grateful for Carlos L. K. Cinelli for designing the code. https://CRAN.R-project.org/package=NetworkRiskMeasures.

[5] We do not net out pairwise liabilities so as to maintain consistency with the Brazilian law, because financial compensation is not always legally enforceable.

For a financial system network, the pairwise communicability $\mathbf{G}_{pq}$ measures the potential contagion that counterparty p can transmit to q when we take into account the network interconnections that arise from the lending and borrowing operations. The onset of a default on p may propagate to q by several paths, either by their shortest ones or through other longer walks. For example, there could be a fragile long path of FIs with insufficient capital to absorb the default of p, in a way that the shock propagates through this route towards q. It is worth noting that a large $\mathbf{G}_{pq}$ does not imply a default of q when p defaults; rather, it simply indicates that q's assets are prone to impacts in many different ways. On the other extreme, if $\mathbf{G}_{pq}$ is small, p's default may not impact q at all, as there is a small number of possible paths from p to q through which a potential shock can propagate.

2.2. Risk-related network analysis

In this section, we present the main contributions of this work. Note that all of our proposed measures gauge potential contagion in the network.

2.2.1. Impact susceptibility

The original liabilities matrix $\mathbf{L}$ representing the financial network is not a representative matrix for evaluating possible contagion routes among pairs of FIs, in that it only assumes pairwise exposures. A more depictive network would be one that allowed for computing how well an FI would be able to absorb impacts coming from its exposures. The vulnerability index, which we define as the FI's exposures to its capital buffer ratio, exactly conveys that type of information. With that consideration in mind, we first define a truncated or binary version of the vulnerability matrix $\mathbf{V}$ in the liabilities sense, denoted here as $\overline{\mathbf{V}}$, as follows:

$$\overline{\mathbf{V}}_{ij} = \begin{cases} 1, & \text{if } \dfrac{\mathbf{L}_{ij}}{\mathbf{E}_j} \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Observe that $\overline{\mathbf{V}}_{ij} = 1$ only when bank i's liabilities towards j surpasses j's capital buffer. This configuration illustrates the situation in which the default of i leads to a subsequent default of j, as j is highly exposed to its neighbor i. Conversely, $\overline{\mathbf{V}}_{ij} = 0$ whenever j can absorb the losses in case i defaults.[6] Note now that the graph topology that matrix $\overline{\mathbf{V}}$ encodes quantifies the possible contagion paths delineated by the current conditions and exposures of the market participants.

Eq. (2) models counterparty risk in light of existent unsecured financial operations between counterparties. However, the creditor bank j can also hedge its position against this credit risk by buying protection from a third counterparty in the network using a credit derivative. Appendix A shows a modification on the truncated vulnerability matrix that permits this kind of financial instrument between counterparties.

The graph $\overline{\mathbf{V}}$ provides not only information of direct contagion, but also indirect contagion once we take higher powers of $\overline{\mathbf{V}}$. For instance, the entry $(\overline{\mathbf{V}}^k)_{ij} \in \mathbb{N}$ indicates the quantity of contagion paths of length k that starts from i and reaches j, even if j does not directly connect to i. The contagion transmission, however, may not only occur through the shortest path between different FIs. That is, it may cascade through longer paths in an isolated or additive manner, using, for instance, paths that are easier to "breakthrough" because of the presence of members with small capital buffers. The computation of the communicability index in (1) exactly provides this information, once we take as input the truncated vulnerability matrix in (2) as the binary adjacency matrix $\mathbf{M}$.

We define the impact susceptibility of institution q, here symbolized as $\mathbf{S}_q$, as:

$$\mathbf{S}_q(\overline{\mathbf{V}}) = \begin{cases} \dfrac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \displaystyle\sum_{\substack{p \in \mathcal{V} \\ p \neq q}} \mathbf{G}_{pq}(\overline{\mathbf{V}}), & \text{when } k_q^{(\text{in})}(\overline{\mathbf{V}}) > 0 \\ 0, & \text{otherwise} \end{cases}, \tag{3}$$

where $k_q^{(\text{in})}(\overline{\mathbf{V}})$ is the number of counterparties to which q is directly exposed and can lead it to default, i.e., it is the number of counterparties p such that $\mathbf{G}_{pq}(\overline{\mathbf{V}}) > 0$. $\mathbf{S}_q$ is a linear combination of the communicabilities of all of the other market participants $p \in \mathcal{V}$ to q, except q itself. Note that we use the notation $\mathbf{G}_{pq}(\overline{\mathbf{V}})$ to reinforce the fact we evaluate the communicability over the truncated vulnerability matrix $\overline{\mathbf{V}}$. We remove the elements in the main diagonal of the communicability matrix $\mathbf{G}(\overline{\mathbf{V}})$ because they do not convey a measure of intercommunicability between pairs of different FIs. Instead, Estrada and Rodríguez-Velázquez (2005) show that the i-th entry of the main diagonal represents the subgraph centrality or self-communicability of the FI i. In brief terms, the subgraph centrality measures the importance of a vertex by taking into account the number of closed subgraphs of which that vertex is member. Here, in contrast, we are interested in quantifying how easily pairs of FIs can communicate.

We now provide some useful concepts and mathematical derivations that allow us to better understand the properties of the impact susceptibility index. Note that in all these derivations saying that p and q are neighbors in matrix $\overline{\mathbf{V}}$ means

[6] All of our measures are also well-defined using a continuous stress version of the vulnerability matrix: $\overline{\mathbf{V}}_{ij} = \min(\frac{\mathbf{L}_{ij}}{\mathbf{E}_j}, 1)$. Though the monotonicity still holds, in the sense that larger indices imply higher potential contagion, the theoretical properties and financial concepts that we derive in the next sections do not hold anymore.

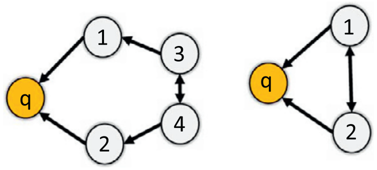

Fig. 1. Possible situations in which q is remotely vulnerable. The connections denote links in the vulnerability network.

that q has equity lower than or equal to its exposure to counterparty p or, equivalently, that q would lose all its equity if p was to default.

Definition 1 (Remote vulnerability)**.** We say that institution q is remotely vulnerable when $\exists p \in \mathcal{V}$ and $k > 1 : (\overline{\mathbf{V}}^k)_{pq} > 0$.

Remark 1. Besides being remotely vulnerable to its indirect neighbors, FI q can be remotely vulnerable to its direct neighbors p whenever there exists an indirect path from p to q. The left panel of Fig. 1 shows an example of remote vulnerability coming from an indirect neighbor. The right panel in Fig. 1 shows a situation in which q is remotely vulnerable to FI 2 through the possible contagion path "$2 \Rightarrow 1 \Rightarrow q$." In this case, all of the members in the path are direct neighbors of q.

Definition 1 states that we can conceive the remote vulnerability of q in terms of the existence of paths linking another arbitrary institution p to q in an indirect manner. We elaborate further on this concept by linking the remote vulnerability to our impact susceptibility index in the next demonstrations.

Lemma 1. *If institution q is a singleton, then it cannot be remotely vulnerable.*

Proof. If q is singleton, then $(\overline{\mathbf{V}}^k)_{pq} = 0$, $\forall p \in \mathcal{V}$ and $k \in \mathbb{N}$. Hence, q cannot be remotely vulnerable in view of Definition 1. □

Lemma 2. *If p is a direct neighbor of q, $p \neq q$, then $\mathbf{G}_{pq}(\overline{\mathbf{V}}) \geq 1$.*

Proof. We can rewrite the definition of the communicability in (1) in terms of a power series as follows:

$$(e^{\overline{\mathbf{V}}})_{pq} = \sum_{k=0}^{\infty} \frac{\left(\overline{\mathbf{V}}^k\right)_{pq}}{k!} = \mathbb{I}_{pq} + (\overline{\mathbf{V}})_{pq} + \frac{\left(\overline{\mathbf{V}}^2\right)_{pq}}{2!} + \cdots, \tag{4}$$

in which $\mathbb{I}$ is the identity matrix. We note that the identity matrix element $\mathbb{I}_{pq}$ has a zero value because $p \neq q$. The second term in the RHS of (4), $(\overline{\mathbf{V}})_{pq}$, results in 1 as q is a neighbor of p by hypothesis. The order-k term yields non-zero entries only if there exist paths of length k from p to q. Putting these facts together in (4), we get:

$$\begin{aligned}(e^{\overline{\mathbf{V}}})_{pq} &= \mathbb{I}_{pq} + \overline{\mathbf{V}}_{pq} + \mathbf{R}_{pq} \\ &= 0 + 1 + \mathbf{R}_{pq} \\ &= 1 + \mathbf{R}_{pq} \geq 1\end{aligned} \tag{5}$$

in which $\mathbf{R}_{pq}$ represents the residual sum of the paths of length $k > 1$ from p to q, whose expression is:

$$\mathbf{R}_{pq} = \sum_{k=2}^{\infty} \frac{\left(\overline{\mathbf{V}}^k\right)_{pq}}{k!}. \tag{6}$$

From (6), we see that $\mathbf{R}_{pq} \geq 0$, because that expression is composed of an infinite weighted linear combination of high-order matrix powers. As the vulnerability matrix only has non-negative entries, high-order powers of such matrix will always yield non-negative numbers. □

Corollary 1. *The equality of Lemma 2, i.e., when $\mathbf{G}_{pq}(\overline{\mathbf{V}}) = 1$, only holds when $\mathbf{R}_{pq} = 0$ in (5). According to (6), this can only happen when no paths of length $k \geq 2$ exist between p and q. Hence, it can only exist paths of unitary length linking p to q. In this case, such path is the direct exposure of q to p (or direct liability of p towards q larger than q's equity).*

Lemma 3. *If $\mathbf{G}_{pq}(\overline{\mathbf{V}}) > 1$, then q is remotely vulnerable.*

Proof. Invoking Lemma 2, the strict inequality only holds in the case $\mathbf{R}_{pq} > 0$ in (5). In accordance to (6), this is true when paths of length $k \geq 2$ exist between p and q. Hence, q must be remotely vulnerable. □

Theorem 1. *$\mathbf{S}_q > 1$ if and only if q is remotely vulnerable.*

Proof. We first note that q cannot be singleton. Otherwise, from Lemma 1, q would not be remotely vulnerable. Therefore, with no loss of generality, we assume that q is not singleton, i.e., $k_q^{(\text{in})} > 0$.

Initially, we prove that the RHS of Theorem 1 implies the LHS: if q is remotely vulnerable, then $\mathbf{S}_q > 1$. We start the proof by manipulating algebraically the expression in (3). These steps are valid for proving both ways of the theorem. The derivation of the referred equation is given as follows:

$$\begin{aligned}
&\mathbf{S}_q(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \mathbf{G}_{pq}(\overline{\mathbf{V}}) \\
&\Leftrightarrow \mathbf{S}_q(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \left(\mathbb{I}_{pq} + \overline{\mathbf{V}}_{pq} + \sum_{k=2}^{\infty} \frac{\left(\overline{\mathbf{V}}^k\right)_{pq}}{k!} \right) \\
&\Leftrightarrow \mathbf{S}_q(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \left(\overline{\mathbf{V}}_{pq} + \sum_{k=2}^{\infty} \frac{\left(\overline{\mathbf{V}}^k\right)_{pq}}{k!} \right).
\end{aligned} \tag{7}$$

Note that:

$$k_q^{(\text{in})}(\overline{\mathbf{V}}) = \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \overline{\mathbf{V}}_{pq} = \sum_{p\in\mathcal{V}} \overline{\mathbf{V}}_{pq} \tag{8}$$

Plugging (8) into (7) results in:

$$\begin{aligned}
&\mathbf{S}_q(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \left[k_q^{(\text{in})}(\overline{\mathbf{V}}) + \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \sum_{k=2}^{\infty} \frac{\left(\overline{\mathbf{V}}^k\right)_{pq}}{k!} \right] \\
&\Leftrightarrow \mathbf{S}_q(\overline{\mathbf{V}}) = 1 + \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \sum_{k=2}^{\infty} \frac{\left(\overline{\mathbf{V}}^k\right)_{pq}}{k!}
\end{aligned} \tag{9}$$

$$\Leftrightarrow \mathbf{S}_q(\overline{\mathbf{V}}) = 1 + \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \mathbf{R}_{pq} \tag{10}$$

We assume, by hypothesis, that q is remotely vulnerable to at least one FI p. Hence, $\exists p \in \mathcal{V}$ and $k > 1 : (\overline{\mathbf{V}}^k)_{pq} > 0$. In other words, $\mathbf{R}_{pq} > 0$, for some p. Incorporating this fact into (10) yields $\mathbf{S}_q(\overline{\mathbf{V}}) > 1$, and the first part of the proof is complete.

Observe that $\mathbf{S}_q(\overline{\mathbf{V}}) \geq 1$ whenever q is not singleton. This is a lower bound for connected institutions in the graph.

Now, we prove that the LHS of Theorem 1 implies the RHS. Using the hypothesis of the LHS of Theorem 1, we get:

$$\begin{aligned}
&\mathbf{S}_q(\overline{\mathbf{V}}) > 1 \\
&\Leftrightarrow \mathbf{S}_q(\overline{\mathbf{V}}) = 1 + \frac{1}{k_q^{(\text{in})}(\overline{\mathbf{V}})} \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \mathbf{R}_{pq} > 1 \\
&\Leftrightarrow \sum_{\substack{p\in\mathcal{V}\\ p\neq q}} \mathbf{R}_{pq} > 0
\end{aligned} \tag{11}$$

Eq. (11) only holds when there exists at least one path of length $k > 1$ between any $p \in \mathcal{V}$, $p \neq q$, and q, i.e., when q is remotely vulnerable to p. Using Lemma 3, we conclude that q is remotely vulnerable and the proof is complete. □

Note that Eq. (3) assumes a non-continuous interval of values. If entity q is invulnerable to any other entities, then $k_q^{(\text{in})}(\overline{\mathbf{V}}) = 0$, i.e., q is singleton in the vulnerability matrix and no other FI can communicate with q. In this situation, q's susceptibility for impacts is $\mathbf{S}_q(\overline{\mathbf{V}}) = 0$. When q is not singleton, the range of values assumed by the impact susceptibility index is $\mathbf{S}_q(\overline{\mathbf{V}}) \in [1, G_N^{\max}]$, where $G_N^{\max}$ is the maximum communicability between any pairs of vertices in a graph with N vertices. Note that the communicability reaches its maximum value in complete graphs. According to Estrada et al. (2008), $G_N^{\max}$ is given by:

$$G_N^{\max} = \frac{1}{Ne}\left(e^N - 1\right), \tag{12}$$

that is, the maximum communicability $G_N^{\max} \to \infty$ as $N \to \infty$.

Eq. (3) provides important indicative information as to whether FI q may be led into default by third counterparties in the network. If $\mathbf{S}_q(\overline{\mathbf{V}})$ is large, the probability that a default of a randomly chosen FI p reaches FI q is higher, whereas if $\mathbf{S}_q(\overline{\mathbf{V}})$ is small, that probability is comparatively lower. The impact susceptibility $\mathbf{S}_q(\overline{\mathbf{V}})$ alone does not determine contagion risk, but rather provides subsidies to systematically identify whether the presence of vulnerable FIs may render the system more or less risky.

2.2.2. Weighted impact susceptibility

In this section, we discuss the weighted impact susceptibility. While its non-weighted version extracts a risk-related indicator that solely depends on the vulnerability network topology, the weighted version incorporates the importance of FIs that can be potentially hit by one or more shocks. Here, we are not concerned whether FIs are remotely or only locally vulnerable. Rather, the weighted impact susceptibility of FI q weighs the possibility that q receives shocks that originate in any part of the network with its importance to the financial system. In this respect, if several vulnerability paths ultimately end in q but q's importance is very low, then the weighted impact susceptibility is small. In turn, if q's importance is very high, the weighted impact susceptibility is large even in the case where it presents very few vulnerability paths that reach it. We can model this behavior by the following expression:

$$\mathbf{S}_q^{(w)}(\overline{\mathbf{V}}, \mathbf{P}_q^{(\text{value})}) = \mathbf{P}_q^{(\text{value})} \sum_{\substack{p \in \mathcal{V} \\ p \neq q}} \mathbf{G}_{pq}(\overline{\mathbf{V}}), \tag{13}$$

in which $\mathbf{S}_q^{(w)}(\overline{\mathbf{V}}, \mathbf{P}_q^{(\text{value})})$ is the weighted impact susceptibility of q and $\mathbf{P}_q^{(\text{value})}$ is a proxy for its importance. Observe that, in the weighted version, we do not divide by the in-degree of q and instead multiply by $\mathbf{P}_q^{(\text{value})}$. The difference is conceptual: the non-weighted impact susceptibility of q is worried with the existence of remote entities that can ultimately impact q. Hence, we divide by q's in-degree to re-scale the index and hence control for the connections that q has to its direct neighbors. In contrast, the weighted impact susceptibility is concerned with the potential loss that the financial system would suffer should q default due to a local or remote vulnerability route. Hence, we weigh by its economic importance rather than re-scaling the coefficient as in the non-weighted version.

2.2.3. Network impact fluidity

In this section, we introduce the impact fluidity, which is an adaptation of the impact susceptibility to a network-level perspective. The potential fluidity of an impact in the network $F(\overline{\mathbf{V}})$ is given by:

$$F(\overline{\mathbf{V}}) = \frac{1}{N} \sum_{q \in \mathcal{V}} \mathbf{S}_q(\overline{\mathbf{V}}), \tag{14}$$

i.e., we quantify the potential fluidity in terms of average total impact susceptibility in the network. It is worth noting that the network impact fluidity, despite being in a similar format, is not equal to the total communicability concept introduced by Benzi and Klymko (2013), in the sense that, in our measure, the impact susceptibilities are weighted adaptively according to the number of investors of the FIs in accordance with (3).

The idea behind such index is as follows. When most FIs have large impact susceptibility, it is easier that defaults from third counterparties to spillover to remote entities in the network. In other terms, the vulnerability network formed by the FIs favors the fluidity of impacts in the network and, hence, $F(\overline{\mathbf{V}})$ is large. In contrast, when most of the FIs have small impact susceptibility, few contagion routes exist that ultimately can impact remote financial counterparties. In this way, the network retains most of the impact at the vicinity of where the default or original impact happened. As such, $F(\overline{\mathbf{V}})$ is small.

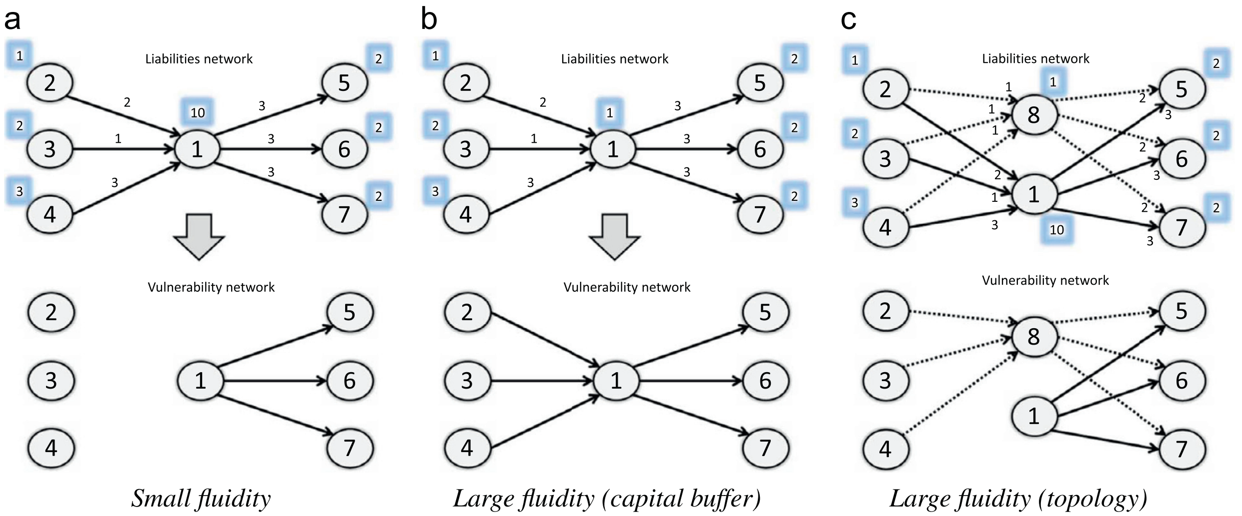

Fig. 2. Illustrative networks with different impact fluidities. The numbers inside the blue glowing rectangles symbolize the FI's capital buffer. The numbers near the edges portray the liability from one counterparty to another in the liabilities network. In the vulnerability network, the existence of an edge between two counterparties implies that a default of the source endpoint will lead the target endpoint into default as well. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

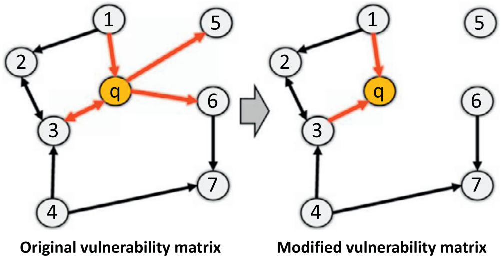

Fig. 3. Illustrative process detailing the vertex removal mechanism used to compute the impact diffusion influence. Vertex q is set to be removed. The bold red edges are removed because q is in one of their endpoints. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

The fluidity of the network can vary mainly due to two reasons: capital buffer and network topology. Fig. 2 a shows a network with small network impact fluidity, as the FIs in the left side are totally incommunicable with FIs located at the right side. This is because FI 1 acts as a shield in preventing the propagation of impacts from one side to another, as it has a large available capital buffer. Note, however, that if FI 1's capital buffer diminishes, the impact fluidity of the network largely increases, as Fig. 2b reveals. Another factor that can lead to the increase of $F(\overline{\mathbf{V}})$ is the network topology. Fig. 2c shows an example in which the introduction of a new FI permits the transmission of shocks from the left to the right side. In practical terms, the shocks always escape from the most vulnerable FI, in case, FI 8, due to its small capital buffer.

Nonetheless, we highlight that the network topology plays a crucial role in both situations. For instance, we can have several FIs with very small capital buffers and still do not have large impact fluidity. This is the case when none of these FIs invests in the market, i.e., they are not exposed to any network members.

2.2.4. Impact diffusion influence

In this section, we detail another contribution of this work, which is the impact diffusion influence index. Until this point, we have explored measures that handle the likelihood of counterparties to being susceptible to default events occurring anywhere in the network. Now, we take the opposite perspective and construct a measure that shows the potential influence that a counterparty exercises on the diffusion or propagation of impacts in the network. Observe that, while the susceptibility conveys the concept of how exposed one member of the network is in relation to the remaining participants, the impact diffusion influence suggests how harmful one member of the network is to the others.

We can understand the impact diffusion influence of counterparty q in terms of the variation it provokes on the communicability indices between all of the participants when q's power of diffusing impacts is removed from the vulnerability network. We can perform this operation by deleting the out-edges that emanate from q. This type of filtering can be thought of as a default whereby q is not any longer able to honor its obligations. Thus, this process transforms q into a sink vertex in the network as every path that reaches q will now end at there. Fig. 3 presents a schematic of the removal mechanism of q and how the vulnerability matrix changes accordingly.

We express the impact diffusion influence of counterparty q, here denoted as $\mathbf{I}_q(\overline{\mathbf{V}})$, as:

$$\mathbf{I}_q(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \sum_{p \in \mathcal{V}} \sum_{\substack{r \in \mathcal{V} \\ r \neq p}} \left[\mathbf{G}_{pr}(\overline{\mathbf{V}}) - \mathbf{G}_{pr}\left(\overline{\mathbf{V}}^{(q-)}\right) \right], \tag{15}$$

where $k_q^{(\text{out})}(\overline{\mathbf{V}})$ is the number of counterparties to which q owes an amount of money at least equal to their capital buffers, and therefore can be led into default whenever q defaults. The term $\overline{\mathbf{V}}^{(q-)}$ denotes the modified truncated vulnerability matrix that we construct by removing the out-edges that originate from q. The factor $[\mathbf{G}_{pr}(\overline{\mathbf{V}}) - \mathbf{G}_{pr}(\overline{\mathbf{V}}^{(q-)})]$ indicates the communicability index of walks from p to r that visit q. This term is evaluated by first computing the communicability index of p to r in the original vulnerability network. From that, we subtract the fraction of that communicability that is not due to a path that has q along the way. Consequently, Eq. (15) effectively quantifies the shortfall occurred in the pairwise network communicability, when q's power of diffusing impacts is disabled. In addition, observe that we do not compute self-communicability indices as they do not convey the notion of forward propagation of impacts.

If q is responsible for diffusing a significant portion of impact throughout the network, then its removal will notably reduce the overall communicability between counterparties in the network. Hence, the impact diffusion influence will be large. In contrast, if q diffuses impact to the network only to a small extent, then the overall communicability between counterparties will remain mostly unaltered. In this scenario, the factor $[\mathbf{G}_{pr}(\overline{\mathbf{V}}) - \mathbf{G}_{pr}(\overline{\mathbf{V}}^{(q-)})]$ in (15) will be small such that the impact diffusion influence q will be small as well.

With a close inspection of (15), the terms inside the summation can be conveniently factored out into two systemic risk roles that are intuitive and important from a viewpoint of impact propagation, which are the influence of q on starting an impact propagation process and on intermediating an ongoing impact propagation process. The next theorem provides the mathematical derivations.

Theorem 2. $\mathbf{I}_q(\overline{\mathbf{V}})$ *can be decomposed into two orthogonal and complementary terms*:

$$\mathbf{I}_q(\overline{\mathbf{V}}) = \mathbf{I}_q^{(\text{start})}(\overline{\mathbf{V}}) + \mathbf{I}_q^{(\text{inter})}(\overline{\mathbf{V}}), \tag{16}$$

in which $\mathbf{I}_q^{(\text{start})}(\overline{\mathbf{V}})$ *quantifies the potential influence of q on starting an impact propagation process and* $\mathbf{I}_q^{(\text{inter})}(\overline{\mathbf{V}})$ *indicates the role of q in intermediating ongoing impacts, i.e., impacts that do not start at q but that necessarily pass through it in the chaining effect.*

The two terms in (16) are given by:

$$\mathbf{I}_q^{(\text{start})}(\overline{\mathbf{V}}) = \begin{cases} \dfrac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \sum\limits_{\substack{p \in \mathcal{V} \\ p \neq q}} \mathbf{G}_{qp}(\overline{\mathbf{V}}), & if\ k_q^{(\text{out})}(\overline{\mathbf{V}}) > 0 \\ 0, & otherwise \end{cases} \tag{17}$$

$$\mathbf{I}_q^{(\text{inter})}(\overline{\mathbf{V}}) = \begin{cases} \dfrac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \sum\limits_{\substack{p \in \mathcal{V} \\ p \neq q}} \sum\limits_{\substack{r \in \mathcal{V} \\ r \notin \{q,p\}}} \left[\mathbf{G}_{pr}(\overline{\mathbf{V}}) - \mathbf{G}_{pr}\left(\overline{\mathbf{V}}^{(q-)}\right)\right], & if\ k_q^{(\text{out})}(\overline{\mathbf{V}}) > 0, \\ 0, & otherwise \end{cases} \tag{18}$$

Proof. The strategy is to start off from (15) and analyze the communicabilities matrices to reach (16). Let $\mathbf{G}(\overline{\mathbf{V}})$ and $\mathbf{G}(\overline{\mathbf{V}}^{(q-)})$ represent the original and modified communicability matrices, respectively. The modified communicability matrix is evaluated by transforming q into a sink vertex. As such, they can be represented by:

$$\mathbf{G}(\overline{\mathbf{V}}) = \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{q1} & g_{q2} & \dots & g_{qN} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \dots & g_{NN} \end{pmatrix}, \quad \mathbf{G}\left(\overline{\mathbf{V}}^{(q-)}\right) = \begin{pmatrix} g'_{11} & g'_{12} & \dots & g'_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g'_{N1} & g'_{N2} & \dots & g'_{NN} \end{pmatrix}, \tag{19}$$

in which g_{pr} and g'_{pr} are the communicability indices from p to r evaluated from the original and modified communicability matrices. Note that the q-th row of $\mathbf{G}(\overline{\mathbf{V}}^{(q-)})$ must be a zero row, because the diffusion power of q is effectively disabled in the modified vulnerability matrix.

The difference of both matrices in (19) is given by:

$$\mathbf{G}(\overline{\mathbf{V}}) - \mathbf{G}\left(\overline{\mathbf{V}}^{(q-)}\right) = \begin{pmatrix} \Delta g_{11} & \Delta g_{12} & \dots & \Delta g_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{q1} & g_{q2} & \dots & g_{qN} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta g_{N1} & \Delta g_{N2} & \dots & \Delta g_{NN} \end{pmatrix}, \tag{20}$$

in which $\Delta g_{pr} = g_{pr} - g'_{pr}$. Note that in the q-th row, we simplify $\Delta g_{qr} = g_{qr} - 0 = g_{qr}, \forall r \in \mathcal{V}$. Attempting to approach the functional form in (15), we divide (20) by $k_q^{(\text{out})}(\overline{\mathbf{V}})$ and remove the self-communicability entries $\Delta g_{pp}, \forall p \in \mathcal{V}$, as follows:

$$\frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})}\left[\mathbf{G}(\overline{\mathbf{V}}) - \mathbf{G}\left(\overline{\mathbf{V}}^{(q-)}\right) - \mathbf{G}^{(\text{self})}(\overline{\mathbf{V}})\right] = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \begin{pmatrix} 0 & \Delta g_{12} & \dots & \Delta g_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{q1} & g_{q2} & \dots & g_{qN} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta g_{N1} & \Delta g_{N2} & \dots & 0 \end{pmatrix}, \tag{21}$$

in which $\mathbf{G}^{(\text{self})}(\overline{\mathbf{V}})$ is a matrix with zero-valued values except for in its main diagonal, whose entries assume the values $\mathbf{G}_{pp}^{(\text{self})}(\overline{\mathbf{V}}) = \Delta g_{pp}, \forall p \in \mathcal{V}$. We introduce this matrix to remove the self-communicabilities entries of the term $\mathbf{G}(\overline{\mathbf{V}}) - \mathbf{G}(\overline{\mathbf{V}}^{(q-)})$.

Note that we can obtain (15) from (21) by simply summing over all entries of the matrix in (21). The summation of these terms can be performed in a formal way using the following matricial expression:

$$\mathbf{I}_q(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \mathbf{1}^T \left[\mathbf{G}(\overline{\mathbf{V}}) - \mathbf{G}\left(\overline{\mathbf{V}}^{(q-)}\right) - \mathbf{G}^{(\text{self})}(\overline{\mathbf{V}})\right]\mathbf{1} \tag{22}$$

where $\mathbf{1}$ is a vector with dimension $N \times 1$, and T is the transpose operator.

With a matricial representation of $\mathbf{I}_q(\overline{\mathbf{V}})$, we are now able to decompose its constituents terms. First, the influence of q on starting a impact diffusion process is related to its communicability to the remainder of the network. This information can be extracted from the summation of the terms in the q-th row in the expression (21):

$$\mathbf{I}_q^{(\text{start})}(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})}\left[g_{q1} + g_{q2} + \dots + g_{qN}\right] \tag{23}$$

$$\mathbf{I}_q^{(\text{start})}(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \sum_{\substack{p \in \mathcal{V} \\ p \neq q}} \mathbf{G}_{qp}(\overline{\mathbf{V}}), \tag{24}$$

which retrieves (17).

The remainder of the rows in the factor $[\mathbf{G}(\overline{\mathbf{V}}) - \mathbf{G}(\overline{\mathbf{V}}^{(q-)}) - \mathbf{G}^{(\text{self})}(\overline{\mathbf{V}})]$ in (21), i.e., rows in the set $\{1, 2, \ldots, q-1, q+1, \ldots, N\}$, account for the influence of q on the role of intermediating impact diffusions that start from other counterparties in the network. That is:

$$\mathbf{I}_q^{(\text{inter})}(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \left[\Delta g_{12} + \cdots + \Delta g_{1N} + \cdots + \Delta g_{(q-1)N} + \Delta g_{(q+1)1} + \cdots + \Delta g_{N(N-1)}\right] \tag{25}$$

$$\mathbf{I}_q^{(\text{inter})}(\overline{\mathbf{V}}) = \frac{1}{k_q^{(\text{out})}(\overline{\mathbf{V}})} \sum_{\substack{p \in \mathcal{V} \\ p \neq q}} \sum_{\substack{r \in \mathcal{V} \\ r \notin \{q,p\}}} \left[\mathbf{G}_{pr}(\overline{\mathbf{V}}) - \mathbf{G}_{pr}\left(\overline{\mathbf{V}}^{(q-)}\right)\right], \tag{26}$$

which retrieves (18) and the proof is complete. □

The potential impact exerted by q on initiating a diffusion process is related to how easily q can reach other counterparties in the network. We can effectively extract this feature from the communicability index, which captures both the shortest and also longer paths in the vulnerability network. In contrast, the intermediation role of q on diffusing impacts is related to how strong it influences the communicability indices of other pairs of counterparties. If the removal of q causes the communicability of the third counterparties p to k to reduce in a significant manner, then q's influence on the contagion paths from p to k is important. However, if the communicability is slightly or not modified at all, then q does not play the role of intermediator of impacts of p to k.

In financial networks, institutions with large impact diffusion influence have the potential to propagate impact or losses to a large number of FIs in the network. This measure, however, does not inform how harmful these impact propagations can be to the financial system. For instance, one institution can have large influence in propagating impacts, but only to non-important institutions. Conversely, one institution may have small influence in propagating impact or losses and still have the potential to cause a devastating impact on the financial system.

Similar to what we have performed for the impact susceptibility, we present the concept of remote contagiousness that have a clear systemic risk interpretation. We then link this concept of remote contagiousness to the impact diffusion influence measure.

Definition 2 (Remote contagiousness)**.** We say that institution q is remotely contagious when $\exists p \in \mathcal{V}$ and $k > 1 : (\overline{\mathbf{V}}^k)_{qp} > 0$.

Lemma 4. $\mathbf{I}_q^{(\text{start})} > 1$ *if and only if q is remotely contagious.*

Proof. Once we transpose the communicability matrix $\mathbf{G}(\overline{\mathbf{V}})$, the same steps followed in proving Theorem 1 can be straightforwardly applied to demonstrate the sufficient and necessary conditions of Lemma 4. □

Theorem 3. *If q is remotely contagious, then* $\mathbf{I}_q > 1$.

Proof. From (16), we know that $\mathbf{I}_q = \mathbf{I}_q^{(\text{start})} + \mathbf{I}_q^{(\text{inter})}$. Employing the hypothesis that q is remotely contagious into Lemma 4, we find that $\mathbf{I}_q^{(\text{start})} > 1$. Thus, the inequality $\mathbf{I}_q > 1$ always holds, because $\mathbf{I}_q^{(\text{inter})} \geq 0$. Hence, q must be remotely contagious and the proof is complete. □

Remark 2. The converse of Lemma 3 is not always true. This is because q can contribute to the diffusion process by both starting and intermediating it. In this process, $\mathbf{I}_q^{(\text{start})} \leq 1$ and $\mathbf{I}_q^{(\text{inter})} \leq 1$ and $\mathbf{I}_q > 1$ can still hold in a way that q is not remotely contagious.

2.2.5. Weighted impact diffusion influence

The impact diffusion influence evaluated in (16) assesses how hazardous one entity can be to another in a quantitative manner: the larger is the number of entities it can lead to default, direct or indirectly, the higher is its influence. The indicator, however, does not take into account the importance or value of each of the institutions that are potentially led into default. It is rather intuitive to give larger weights in the impact propagation process to those exposures that lead more important institutions into default.

In line with that, we define the weighted impact diffusion influence of q as follows:

$$\mathbf{I}_q^{(\text{w})}\left(\overline{\mathbf{V}}, \mathbf{P}^{(\text{value})}\right) = \sum_{p \in \mathcal{V}} \sum_{\substack{r \in \mathcal{V} \\ r \neq p}} \left[\mathbf{G}_{pr}(\overline{\mathbf{V}}) - \mathbf{G}_{pr}\left(\overline{\mathbf{V}}^{(q-)}\right)\right] \cdot \mathbf{P}_r^{(\text{value})}, \tag{27}$$

in which $\mathbf{P}^{(\text{value})}$ is again a proxy for the value or importance of all of the financial institution in the market. Again, we drop the division to q's out-degree because we are no longer looking at how far q can diffuse impacts. Rather, we are estimating how potentially hazardous q can be to the entire financial system.

We reinforce that the importance in (27) is attributed to the final destination reached by the walks, instead of their starting point. This allows one to compute a proxy for the extent of the damage potentially caused by a given institution. This contrasts with the definition of the weighted impact susceptibility, in which we weigh by the starting point rather than the destination points.

3. Comparison of different systemic risk measures

In this section, we compare our measures to state-of-the-art systemic risk and other related network indicators from the literature. We focus on providing advantages and limitations of our indicators in light of the existent literature.

The non-weighted versions of the impact susceptibility and impact diffusion influence entirely rely on the network structure and attempt to extract and count all vulnerability paths that may lead counterparties into default, be it through the shortest or longer paths. We design them in such a way to identify useful concepts, such as the remote/local vulnerability and contagiousness of counterparties, with practical implications to financial regulation, surveillance, and monitoring of financial systems. In contrast, the design of their weighted versions explores another angle of the network by evaluating the amount of economic loss that the system may incur through these default propagation paths. Since we use the truncated vulnerability matrix in the computation, all of these measures focus on catastrophic or default events, i.e., they assume that none of the counterparties will perform debt repayments and thus yield non-zero results only if these defaults lead other counterparties into default.

We first note that the idea of decomposing how counterparties diffuse shocks into the roles of starters and intermediaries in the financial network is novel. The literature often conceives the implied systemic risk of a counterparty simply as the additional loss that it imposes to the financial system in case that counterparty defaults, i.e., when it starts a impact diffusion process. For instance, while the DebtRank method (Battiston et al., 2012b), the new differential DebtRank (Bardoscia et al., 2015),[7] the pioneering Eisenberg and Noe (2001)'s clearing payment algorithm and its extension (Rogers and Veraart, 2013)[8] share close similarities to the weighted impact diffusion influence,[9] they only account for the systemic risk component of starting a shock propagation. Therefore, while the importance of counterparties as shock starters is widely studied, the role that these counterparties play as intermediaries of an ongoing shock propagation process is a new perspective of seeing contagion that our measures capture.

Our work also closely relates to Drehmann and Tarashev (2013)'s approach, who also estimate the systemic importance of interconnected banks in catastrophic scenarios using the perspectives of (i) the impact susceptibility, which they term as vulnerability to propagated shocks, and (ii) the impact diffusion influence, which they refer to propagation of shocks across the system.[10] The authors first quantify a single measurement of systemic risk but allocate differently across individual institutions depending on the viewpoint of receiving or diffusing shocks. This detail is a crucial distinction between our and their approach, because the computation of our impact susceptibility and impact diffusion influence differ by construction. Thus, we do not fraction and allocate in different manners a single systemic risk measure among counterparties; rather, we compute two genuine distinct systemic risk measures. While the impact susceptibility looks at the pairwise communicability between all counterparties to a reference entity, the impact diffusion influence concerns how the overall communicability behaves when we remove a specific counterparty from the network. In addition, it is not evident how to extract concepts such as remote vulnerability or contagiousness in view of Drehmann and Tarashev (2013)'s framework as they deal with tail events over probability distributions.

Hideaki et al. (2013) and Battiston et al. (2016a,b) also study systemic risk by investigating the counterparties' profiles of receiving and diffusing financial stress using an adaptation of Battiston et al. (2012b)'s DebtRank. Along the algorithm, DebtRank computes how an initial shock inflicts financial stress to each individual in the network. In the final step, the method performs a linear combination of these individual stress levels to obtain a scalar and global quantity that Battiston et al. (2012b) define as the DebtRank index. To find the profiles of receiving and diffusing financial stress, the authors skip this final aggregation process and work with that individual level information. Thus, they use information of the same nature—i.e., the set of financial stress levels of each counterparty—but allocate differently between counterparties depending on which view they are looking at: stress receivers or diffusers. In contrast, our approach delivers distinct information when we capture impact susceptibility (receiver) and impact diffusion influence (diffuser).

[7] Battiston et al. (2012b)'s DebtRank may underestimate systemic risk levels, as it blocks second- and high-order rounds of financial stress that may arise from cycles or multiple vulnerability routes in the network. To avoid this problem, Bardoscia et al. (2015) introduce a modified version of DebtRank that we term here as differential DebtRank, in which banks are able to recursively diffuse stress differentials instead of their current stress levels at each iteration. We refer the reader to Silva et al. (2017) for a thorough comparison on the original and differential versions of the DebtRank.

[8] Eisenberg and Noe (2001)'s model shows that there exists a unique payment vector solution that minimizes individual and collective losses, under the assumptions of absence of bankruptcy and liquidation costs. Rogers and Veraart (2013) build upon this model by also considering bankruptcy and liquidation costs. Though these models were defined for clearing a payments network, they can also be transposed to a web of vulnerabilities and thus be comparable to DebtRank and to the weighted impact diffusion influence.

[9] These versions of the DebtRank are better comparable to the weighted version than to the non-weighted impact diffusion influence. While the former estimates the potential loss due to a default event, the last relates to the length of default cascades paths that exist from one FI to all the other counterparties in the system.

[10] Drehmann and Tarashev (2013) propose a generalization of the existing contribution approach that relies on Shapley values. They extend such methodology to also account for interconnected institutions. In general terms, the methodology considers distinct groups of subsystems composed of banks and estimates to what extent each bank in this subgroup contributes to the hypothetical risk of each subsystem.

We can also draw some parallels on the impact susceptibility with regard to Google's PageRank.[11] In general terms, PageRank evaluates the importance of a web page by simulating the behavior of a user browsing the web. Most of the time, the user visits pages just by surfing, i.e., by clicking on links, therefore traversing the network of web pages. Another manner is to directly jump to another page, to which the current page may not necessarily hold a link reference, by typing the URL on the browser or going to a bookmark. In a network, we can model this process by a simple linear convex combination of a random walk with occasional jumps toward randomly selected vertices. If we interpret shocks in a financial network as users browsing the web, the shock transmission rules of PageRank and the impact susceptibility share similarities when we do not allow for random jumps in the network. However, PageRank does not offer clear systemic risk classifications, such as the concepts of remote/local vulnerability. This is because PageRank is a general feedback-based network measure and hence does not account for particularities of financial networks. Moreover, it is not trivial to transform relationships between web pages (links) into financial vulnerabilities that we use to compute impact susceptibility. Therefore, it becomes difficult to interpret and give economic intuition to PageRank estimates, especially when we consider random jumps from the viewpoint of pairwise vulnerabilities between counterparties in a financial network.

The measures we propose in this paper count on the concept of network communicability, which considers not only the shortest paths between counterparties but also longer paths. We can contrast the usage of this technical concept to evaluate dissimilarity between pairs of counterparties to the recently proposed harmonic distance (Acemoglu et al., 2015). The harmonic distance adopts the mean hitting time of a discrete-state Markov chain defined over the network. The mean hitting time from i to j is the expected number of time steps it takes the chain to hit counterparty j conditional on starting from counterparty i. Hence, from the viewpoint of financial networks, the harmonic distance gives a sense of how likely third counterparties, on average, will be occupied or hit by a conditional shock that initiates somewhere in the network. In this way, the harmonic distance relies on computing transition probabilities of paths with arbitrary lengths—i.e., shortest and longer paths—but with no external penalization of longer paths. This point differs from the communicability between counterparties in that it confers more weight to paths that are shorter using a factorial dampening coefficient instead of computing transition probabilities. We believe that the exogenous dampening factor is important because it models the fact that each bank absorbs part of the incoming loss through their positive capital buffers such that the shock dampens as it travels along the network.

Next, we list some strengths and limitations related to our measures. Some of their strengths are:

- These measures provide results that are intuitive to economics and finance.
- These measures do not require calibration of parameters as they extract topological information that the web of vulnerability between counterparties encodes. For instance, we must calibrate the proportion of occasional random jumps in the Google's PageRank algorithm. The output of PageRank is very sensitive to this parameter and hence can lead to very different distributions on the web pages importance. Our measures do not require this parameter fine-tuning step and hence we can skip the complex task of model selection.
- The mathematical underpinnings of the methodology rely on evaluating the truncated vulnerability matrix, rather than the vulnerability matrix that existent systemic risk measures normally employ. This matrix is much sparser than the ordinary vulnerability matrix, and therefore easier to visualize and interpret.
- The easy interpretation of local/remote vulnerability in terms of our theoretical findings. In this matter, whenever the expression $\mathbf{S}_q > 1$ holds, we can directly conclude for remote vulnerability and hence the existence of indirect contagion paths that can lead q into default. Similarly, if $\mathbf{I}_q^{(\text{start})} > 1$ holds, then q can lead into default third counterparties that are not directly connected to q. The simplicity in identifying counterparties that are remotely or locally vulnerable both in the perspectives of receiving and diffusing shocks turns the framework into an interesting monitoring tool for regulators.

We can highlight the following limitations of our model:

- It seems common regulatory practice, and also independently adopted by risk management desks throughout the financial industry, to limit the single largest exposure of an institution to values well below the capital buffer of the institution. This observation implies that the truncated vulnerability matrix yields almost always a completely disconnected network of singletons. From this viewpoint, the proposed measures would be uninformative as the systemic stress is not severe enough to produce default cascades. Therefore, our measures may not be able to measure a possible buildup of stress. Two possible approaches to address this lack of information would be (i) to adapt our framework by reducing the capital buffers in the denominator of (2) proportionally for all counterparties, or (ii) to use another measure from the literature tailored for this specific use, e.g. the DebtRank measure. The original and differential DebtRank are able to single out partially distressed counterparties in a network as a result of different initial shock scenarios of any magnitude, which is useful to identify stress buildup when it is yet in a low level. Our measures identify more precisely destructive stress,

[11] PageRank is a feedback-based network measure that considers a vertex or web page as important whenever (i) there are many neighbors that point to that web page and (ii) those neighbors are also important. See Silva and Zhao (2016) for a compilation and comparison of several network measures in the literature.

therefore our measures and DebtRank-based methodologies offer a complementary view of the evolution of systemic risk in financial networks.

- Acemoglu et al. (2015) and Battiston et al. (2012a) propose models in which different risk sources interact. In both models, losses from failure in debt repayment can combine with liquidity issues and thus cause further financial damage. In contrast, we design a systemic risk framework that focuses on identifying default cascades in which the risk propagation driver is counterparty risk. In this process, we attempt to give economic intuition to the results by presenting how defaults can affect institutions in a local or remote fashion. Under this setting, we also extend this model by taking into account CDS contracts in Appendix A. We show there that CDS contracts can reshape the entire network and hence can significantly attenuate or amplify systemic risk. In this manner, we do not include common or correlated exposures, liquidity issues related to margin calls or runs on debtor FIs, and other mechanisms of risk generation and transmission.
- Acemoglu et al. (2015) show that financial contagion not only depends on the network structure but also on the magnitude of the initial shock. In addition, they find that systemic risk and thus amplification of shocks exhibits a form of phase transition with regard to the shock size. When the shock is small, denser or more complete networks resist better to shocks than sparse networks. In contrast, when the shock is large, then the network serves as a medium of shock propagation and amplification and hence sparser networks lead to lower levels of systemic risk. Thus, we need to conclude that a measure of systemic risk that does not distinguish for shock size is blind to this well-known effect. With this concern in mind, we see that the original definitions of our measures do not allow for the specification of shock size nor breadth. In this respect,
  - For the weighted and non-weighted versions of the impact diffusion influence, this is because they have an implicit initial shock scenario that is the default of a single counterparty. Fig. 3 makes this point clear: by removing edges from vertex q, we are assuming that q does not pay back its debts towards its direct creditors. We can also extend our methodology to account for arbitrary initial scenarios that encompass more than a single counterparty. For the non-weighted impact diffusion influence, we can simply agglomerate all counterparties that will suffer the shock into a single super-vertex in the network and all the theoretical derivations will still hold. In contrast, the algorithm trivially extends for arbitrary initial shock scenarios for the weighted impact diffusion influence, in a way that we do not need any changes on the network structure.
  - The impact susceptibility, in turn, only accounts for the network structure to extract the propensity of counterparties to receiving arbitrary shocks in the network. Thus, it does not look at the shock size nor breadth but at the web of interrelationships that can drive shocks toward specific counterparties. Hence, the impact susceptibility is independent of the initial shock.

The above considerations suggest that presently we do not find the one-size-fits-all model. On the contrary, we find models with different approaches that focus on distinct subsets of mechanisms that influence systemic risk in financial systems. Therefore, the usability of the models strongly depends on the specific purposes for which they are being employed. In this way, it makes sense to speak of a toolbox of models and not a single model for supervisors, who will analyze the current scenarios taking into account the strengths and weaknesses of each tool for the task that is being performed.

4. Data

In this paper, we use a unique Brazilian database with supervisory data. We extract quarterly information on Brazilian domestic financial market exposures, supervisory variables and balance sheet statements. We use accounting information to evaluate the FIs' capital buffers from March 2008 through December 2014. This information is vital to compute some network measurements, such as the impact susceptibility and its derived measures.

Following Souza et al. (2015), we define the capital buffer excess of an FI as the FI's total capital buffer (Tier 1 + Tier 2 capitals) that exceeds 8% of its risk-weighted assets (RWA). In Brazil, the capital requirement is 13% or 15% for specific types of credit unions and 11% for other FIs, including banks. Most FIs hold excess in their capital buffer levels (their regulatory capital exceeds the requirement). FIs that are not compliant with this requirement are warned by the Supervision and must present a plan to recover compliance in a given period. If the plan is not credible or not feasible, the Authority intervenes. We set 8% RWA as a reference when computing capital buffers as we assume that if an FI holds less than what is recommended by the Basel Committee on Banking Supervision (BCBS), i.e., 8% of its RWA, it will take longer to raise its capital to an adequate level and will likely suffer an intervention.

Although exposures among FIs may be related to operations in the credit, capital and foreign exchange markets, here we focus solely on unsecured operations in the money market. The money market comprises operations on private securities. We have information on operations with private securities that is provided by the Cetip:[12] interfinancial deposits, debt

[12] Cetip is a depositary of mainly private fixed income, state and city public securities and other securities representing National Treasury debts. As a central securities depositary, Cetip processes the issue, redemption and custody of securities, as well as, when applicable, the payment of interest and other events related to them. The institutions eligible to participate in Cetip include commercial banks, multiple banks, savings banks, investment banks, development banks, brokerage companies, securities distribution companies, goods and future contracts brokerage companies, leasing companies, institutional investors, non-financial companies (including investment funds and private pension companies) and foreign investors.

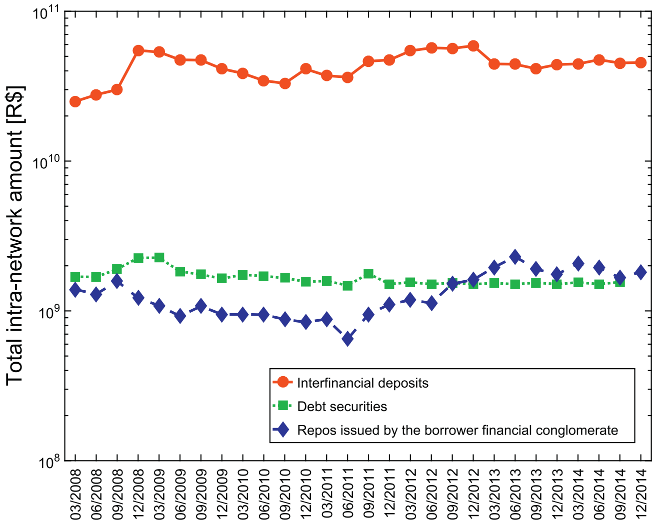

Fig. 4. Total amount of active operations established between members of the financial network in the studied period. The y-axis is in log-scale.

securities and repurchase agreements collateralized by debt securities issued by leasing companies of the same financial conglomerate.[13] In this work, we term the last financial instrument as "repo issued by the borrower financial conglomerate." Fig. 4 portrays the total amount of active operations established between members of the financial network from 2008 to 2014. Noticeably, we see that the amount of interfinancial deposits prevails over operations related to debt securities and repos issued by the borrower financial conglomerate. The total amount invested in this market by its participants varies from R$28.09 billion to R$61.98 billion in the period, corresponding to 1.5% of the FIs' total assets and 14% of their aggregated Tier 1 Capital.

We use exposures among financial conglomerates and individual FIs that do not belong to a conglomerate. Intra-conglomerate exposures are not considered. We take both banking and non-banking financial institutions in our investigation. Banking institutions can be commercial banks, investment banks, savings banks and development banks. Credit unions represent non-banking institutions. Banks and non-banks are classified by size according to the same methodology applied to their groups, i.e., the group of banks and the group of non-banks. We use a simplified version of the size categories defined by the Central Bank of Brazil in the Financial Stability Report published in the second semester of 2012 (see BCB, 2012), as follows:[14] (1) we group together the micro, small, and medium banks into the "small/medium" category, and (2) the official large category is maintained as is in our simplified version. Therefore, instead of four segments representing the banks' sizes, we only employ two.

For each pairwise exposure between financial conglomerates or individual institutions, we remove the share that is guaranteed by the Brazilian Credit Guarantee Fund (FGC).[15] Among the types of financial institutions that we are employing in our analysis, only credit unions are not registered members in the FGC. All of the financial instruments that we are using are covered by the FGC. Until May 2013, the FGC guarantees up to R$70 thousand for each deposit holder against each registered institution. After that date, due to Resolution 4222 published by the National Monetary Council, that amount increased to R$250 thousand. Say that the liability of p to q at time t is $\boldsymbol{L}_{pq}(t)$. If p is not a credit union, then we adjust that liability to $\max[0, \boldsymbol{L}_{pq}(t) - \text{FGC}(t)]$, where:

$$\text{FGC}(t) = \begin{cases} 70,000, & \text{if } t < \text{May}/2013, \\ 250,000, & \text{otherwise}. \end{cases} \tag{28}$$

Fig. 5a displays the evolution of the total number of participants in the financial market. Small/medium banks are the majority in the entire period. Large non-banks and small/medium non-banks are present in similar quantities in the sample.

[13] Recall that repurchase agreements are technically secured operations. However, since the borrower in this type of repo guarantees the operation using collateral of a leasing company of the same financial conglomerate, the collateral bears the same credit risk of the borrower financial conglomerate. Thus, in practical terms, the financial operation turns out to be unsecured.

[14] The Financial Stability Report ranks FIs according to their positions in a descending list ordered by FIs' total assets. The Report builds a cumulative distribution function (CDF) of the FIs' total assets and classifies them depending on the region in which they fall in the CDF. It considers as large FIs that fall in the 0–75% region. Similarly, medium-sized FIs fall in the 75–90% category, small-sized, in the 90-99% mark, and those above 99% are micro-sized.

[15] The Credit Guarantee Fund, whose legal establishment is authorized by the Resolution 2197 issued by the National Monetary Council, is a private institution responsible for the protection of checking/saving account holders and investors against registered financial institutions in case of intervention, liquidation or bankruptcy.

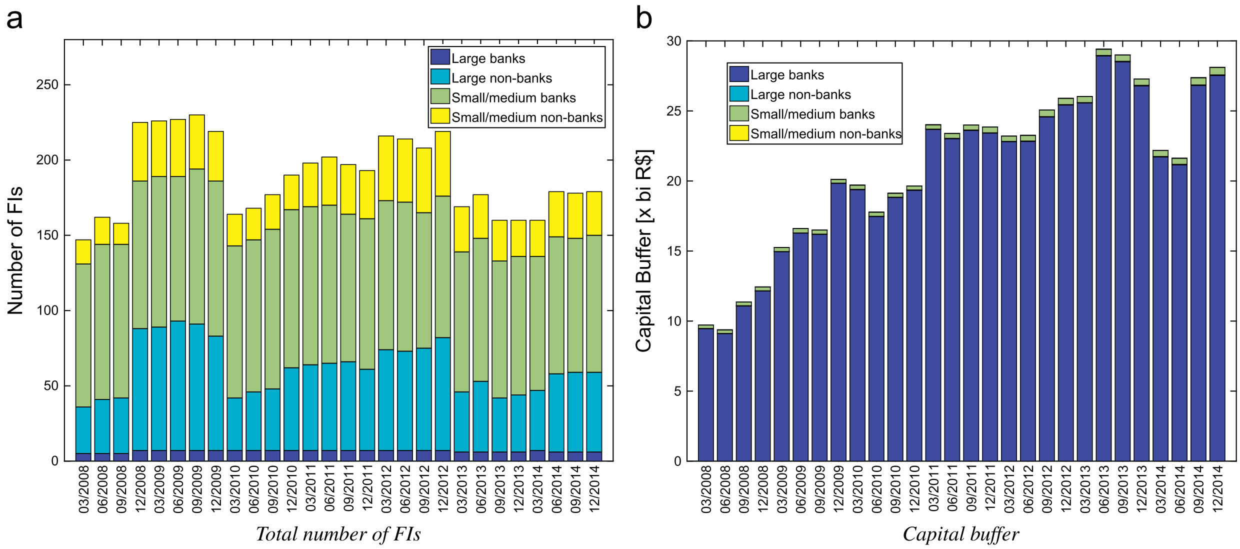

Fig. 5. Evolution of total number and the average capital buffer of FIs in the Brazilian financial network. We discriminate the trajectories by size (large or small/medium) and type of entity (banking or non-banking).

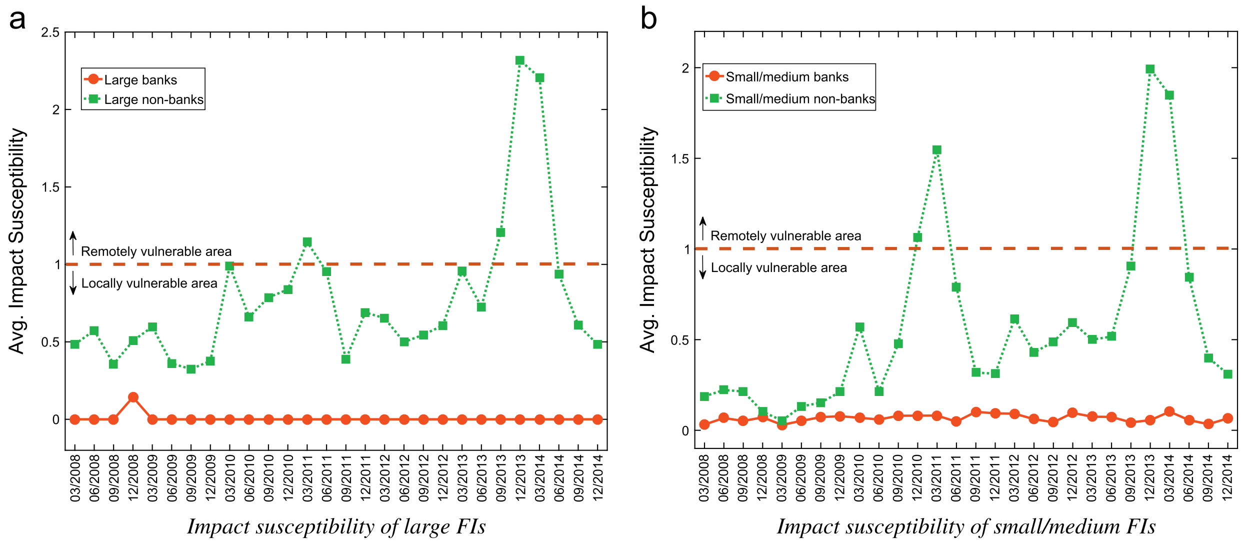

Fig. 6. Trajectory of the average impact susceptibility of large and small/medium FIs as a function of time in the Brazilian interbank network.

The number of large banks is the minority and remains roughly constant throughout the period. Although the proportions of the analyzed segments, which are divided into banks and non-banks and size groups as defined above, remain roughly the same, the total number of FIs does change. In particular, the number of FIs consistently grows until March 2013, date in which it suffers a considerable drop due to a reduction in the number of large non-banks. Afterwards, we can verify again the upward trend on the number of FIs. Fig. 5b presents the mean capital buffer for the same categories of FIs. We can see that large banks have much higher capital buffers than the other entities, which reflects their size differences.

5. Results

In this section, we employ our proposed risk-related measurements on the Brazilian financial market and discuss the main findings. Recall that we build up this financial market using interfinancial deposits, debt securities, and repos issued by the borrower financial conglomerate among banking and non-banking institutions. In addition, we suppose that a default occurs whenever the total capital buffer of a bank, relative to its RWA, falls below the 8% Basel regulatory requirement.

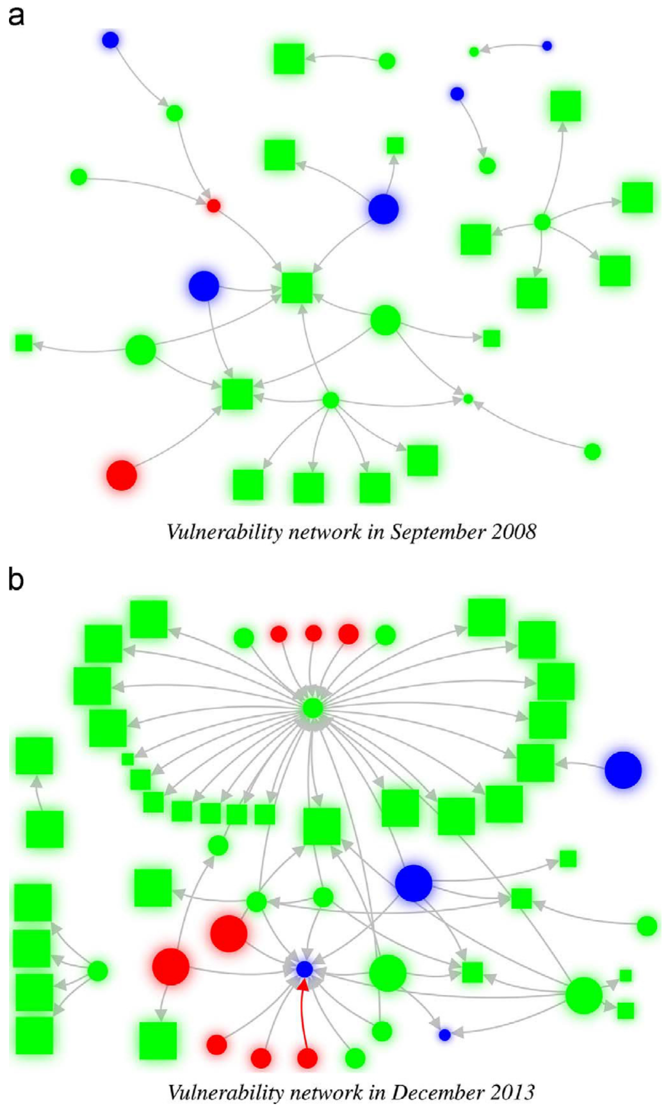

Fig. 7. Vulnerability networks of the Brazilian financial market computed for two dates. An edge from *i* to *j* exists whenever the default of *i* leads *j* into default as well. We define default as the situation in which the total capital buffer of a bank, relative to its RWA, falls below the 8% Basel regulatory requirement. The circles denote banking institutions, while the squares, non-banking institutions. The green color portrays domestic private institutions; the red color, government-owned institutions; the blue color, private with foreign control institutions; and the black color, private with foreign participation institutions. The vertices' sizes are proportional to the corresponding institutions' sizes. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

5.1. Impact susceptibility

We investigate the contribution of the network structure to the fragility of the financial system. To do this, we compute the impact susceptibility of each bank, so that we can identify which banks are more prone to default as a consequence of defaults of other randomly chosen banks. We note that institutions with higher impact susceptibility indices (much higher than 1) should be monitored strictly, especially if they also have large impact diffusion influence.

Fig. 6a and b displays the impact susceptibility coefficient for large and small/medium FIs in the financial market from 2008 to 2014. These figures make clear that non-banking institutions are dominant with respect to the average impact susceptibility throughout the studied period. We distinguish the areas denoting remote and local vulnerabilities. We observe that every remotely vulnerable entity must also be locally vulnerable.

In addition, Fig. 6a shows that large banks have zero impact susceptibility indices for almost the entire interval. As such, they are not susceptible to individual defaults because they have a combination of sufficiently large capital buffers (recall Fig. 5b) and relative small exposures to the financial network. In this way, their connectivity is sparse in the vulnerability matrix domain. Nevertheless, there is a single point in which the impact susceptibility of large banks is non-zero: in December 2008. Out of the six existent large banks in that period, one pair of large banking institutions has a large exposure

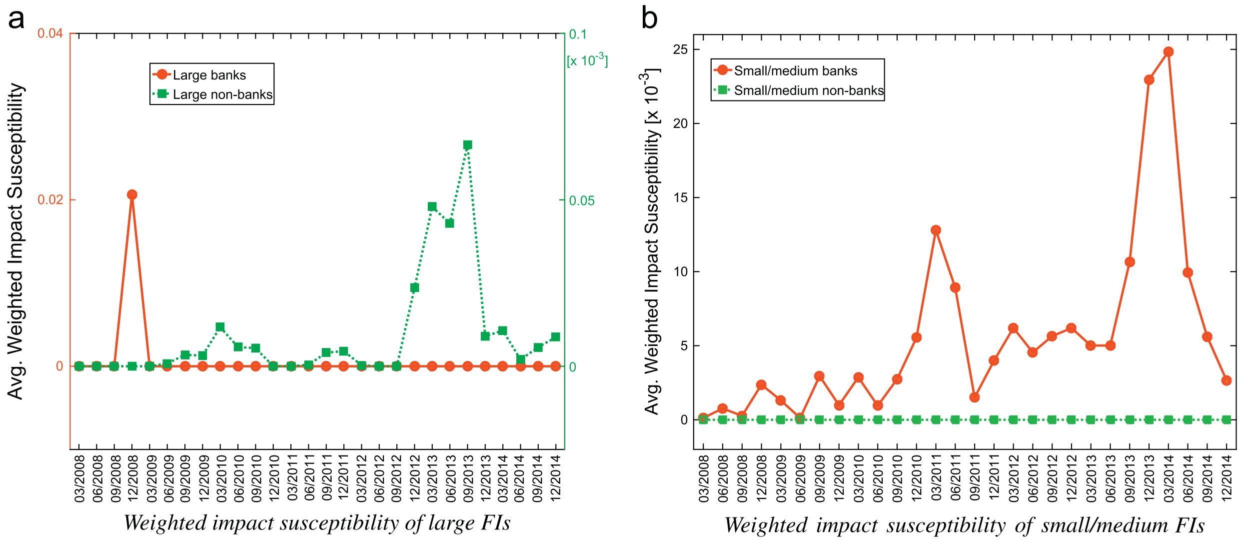

Fig. 8. Trajectory of the average weighted impact susceptibility of large and small/medium FIs as a function of time in the Brazilian interbank network.

between one another that can potentially lead the creditor side into default if the liability is not honored by the debtor. This explains the positive average impact susceptibility of $\frac{1}{6} \approx 0.17$ for large banks in December 2008. Large banks, in all of the other periods, are essentially singletons[16] in the vulnerability network.

Fig. 6b exhibits a similar picture for small/medium banks: very small, but still positive, impact susceptibility values. In addition, for all of the banking institutions we find that the impact susceptibility is less than one, revealing that these entities, in general, are not prone to being remotely impacted from an indirect and communicable neighbor. That is, they are only locally vulnerable. In this way, the analysis of their local vulnerability is generally sufficient to determine as to whether or not they may lead to the amplification of contagion routes.

Still in Fig. 6a and b, we see that non-banking institutions are prevalent in terms of impact susceptibility values when we compare against banking institutions. Having in mind that non-banks are almost purely investors in the Brazilian financial market, they are likely to be more exposed than the others. Note that there are periods in which the impact susceptibility is greater than one. For large non-banks, they are: during the first quarter of 2011 and from the second semester of 2013 to the first quarter of 2014. For small/medium non-banks: from the fourth quarter of 2010 to the first quarter of 2011 and from the fourth quarter of 2013 to the first quarter of 2014. Within these regions, the financial healthiness of these non-banking institutions cannot simply be evaluated by their local vulnerability. They are remotely vulnerable. Even though the health of their neighbors remains important for a first analysis, as they act as shields and can therefore repel the propagation and amplification of contagion paths, it is not sufficient to assess whether or not these entities with large impact susceptibility will be able to absorb losses coming from arbitrary players in the network.

In order to get a better glimpse of the discussed results, we show the networks constructed from the truncated vulnerability matrices for September 2008 and December 2013 in Fig. 7a and b, respectively. Recall that in the vulnerability matrix an edge from p to q exists only if the default of p forces q into default. We choose the network snapshot for September 2008 because it is a date at the heart of the subprime crisis. We also display the network snapshot for December 2013 in that it is an interesting point in which the impact susceptibility of non-banking institutions is larger than one, according to Fig. 6a and b.

These figures pictorially clarify the reason non-banking institutions own the largest impact susceptibilities: the majority of the potential contagion paths end up in them for both periods. They are, thus, leaves or sinks in the graph. This is in line with their investors-only market profiles previously described. Furthermore, we see that it is much more usual that non-banking institutions are potentially vulnerable to small/medium banks than to large banks. In September 2008, one small/medium bank can potentially lead six large non-banks and another small/medium bank institution to default. This influence is accentuated to a large extent in December 2013, when there is a small/medium bank that can potentially impact more than 20 non-banking institutions in a direct hit, and even more indirectly.

In December 2013, we see a higher frequency of pairs of small/medium banks leading one another to default than in September 2008. This is one of the reasons non-banking institutions have an average impact susceptibility greater than one, as these vulnerable pairs of banking institutions almost always communicate with non-banking entities. Hence, the latter can be indirectly impacted in a domino-like effect if the indirect, but communicable, banking institution defaults. As such, non-banking institutions must be aware not only of the health of their direct neighbors, but also of the financial soundness of their indirect ones.

[16] A singleton vertex is defined as a vertex with no out- and in-edges.

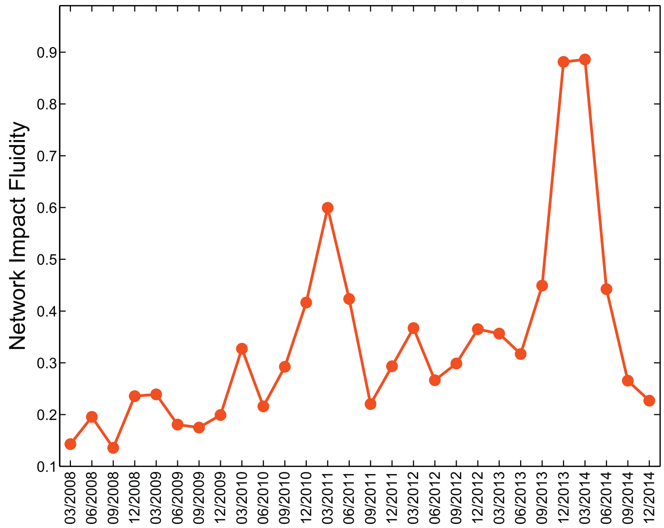

Fig. 9. Trajectory of the network impact fluidity as a function of time in the Brazilian interbank network.

5.2. Weighted impact susceptibility

We can see from the previous investigation that non-banking institutions are the most susceptible entities to random shocks in the financial network. A natural extension of this analysis is to verify whether or not these susceptible entities have significant economic importance inside the interbank market. To capture that risk dimension, we can use the weighted impact susceptibility.

First, we define the proxy for capturing the economic importance of financial institutions as required in (13). For that, we use the share of the financial institution's total liabilities with respect to the entire financial system, i.e.,

$$\mathbf{P}_q^{(\text{value})} = \frac{\sum_{p \in \mathcal{V}} \mathbf{L}_{qp}}{\sum_{q \in \mathcal{V}} \sum_{p \in \mathcal{V}} \mathbf{L}_{qp}} \tag{29}$$

We use liabilities instead of assets because, if q defaults, it may not be able to repay its creditors in full. In fact, the rshort-run recovery rate of the creditors of q is effectively zero.. The normalization in (29) permits us to get rough estimates of the potential monetary loss due to q's susceptibility of being impacted by random shocks inside the network. The estimates are not exact in the sense that the communicability index may possibly contain attenuated cycles in its computation.

Fig. 8a and b portrays the average weighted impact susceptibility values of large and small/medium financial institutions, respectively. Even though non-banking institutions are the most susceptible entities to impacts, their weighted impact susceptibility indices are small. This occurs because they do not cause significant losses inside the interbank market as they mainly invest and do not get funding in that market. Hence, their importance with respect to our proxy in (29) is practically zero. Small/medium banking institutions have amplified fragility due to their considerable funding in the interbank market. We note the peak of the average weighted impact susceptibility for large banks in December 2008. This is consistent with their non-zero impact susceptibility in Fig. 6a. The large amplification in the weighted impact susceptibility (about 2.1% of the total liabilities in the financial system) occurs because large banks often receive large amounts of funding that they use to provide funding to other financial institutions that are with liquidity issues. Hence, they act as liquidity providers in the Brazilian interbank market.[17]

5.3. Network impact fluidity

Given the topology of the vulnerabilities network associated to the financial system, the network impact fluidity measures the network's potential contagion intensity. Put differently, it gauges how far a default of an FI can, on average, propagate, leading other FIs into default. Thus, domino-like effects due to a contagion process are more prone to happening in networks with large impact fluidity. This holds true because, in this situation, FIs tend to be very susceptible to receiving impacts from the network due to their large communicability. Note, however, that the network impact fluidity measures the potential impact and not the realized loss that really occurs. The latter depends on how the agents behave and how large the institutions' capital buffers are when the event occurs.

Fig. 9 exhibits the trajectory of the network impact fluidity evaluated on the Brazilian financial market from 2008 to 2014. Observe that the fluidity remains below the mark of one in the entire period, suggesting that, on average, contagion

[17] See Silva et al. (2016) for a thorough discussion on this topic.

Fig. 10(a). Trajectory of the average impact diffusion influence and its constituent parts (start and intermediate factors) as a function of time in the Brazilian interbank network.

processes only happen to entities that have direct exposures to the defaulted neighbor. That is, contagion processes in an indirect manner are unlikely. As such, the contagion routes are short in general. Note, however, that institution-wise surveillance must be performed, as some entities can be very susceptible to impacts, while the impact fluidity of the network is small. Nonetheless, the network impact fluidity supplies a quick gist as to how the network members are susceptible to random defaults occurring in the market.

Looking back at Fig. 7a and b, which shows the vulnerability networks in September 2008 and December 2013, respectively, it is remarkable that the quantity of edges in December 2013 is significantly larger than in September 2008. In fact, the densities of the vulnerability networks in September 2008 and December 2013 are 0.11% and 0.18%, respectively.[18] In principle, this observation evidences that the network impact fluidity in December 2013 is much higher than in September 2008, as we can see from Fig. 9. In fact, the network in September 2008 marks the smallest network impact fluidity in the studied period. One reason for that stems from a regulatory resolution published by the National Monetary Council[19] and

[18] Contrasting to that, the densities formed by the liabilities networks in September 2008 and December 2013 amount to 1.85% and 1.25%. This means that, even though in September 2008 there are more pairwise relationships between institutions, the lent and borrowed amounts are smaller. This fact further strengthens the argument that impact fluidity in networks is not necessarily correlated to the density of the liabilities network. Instead, it both depends on the network topology and the FIs' capital buffers, for they may provide fragility or robustness to the system.

[19] The National Monetary Council is the major institution of the Brazilian National Financial System. It is in charge of formulating monetary and credit policies, aiming at the preservation of the Brazilian monetary stability, and the promotion of economic and social development.

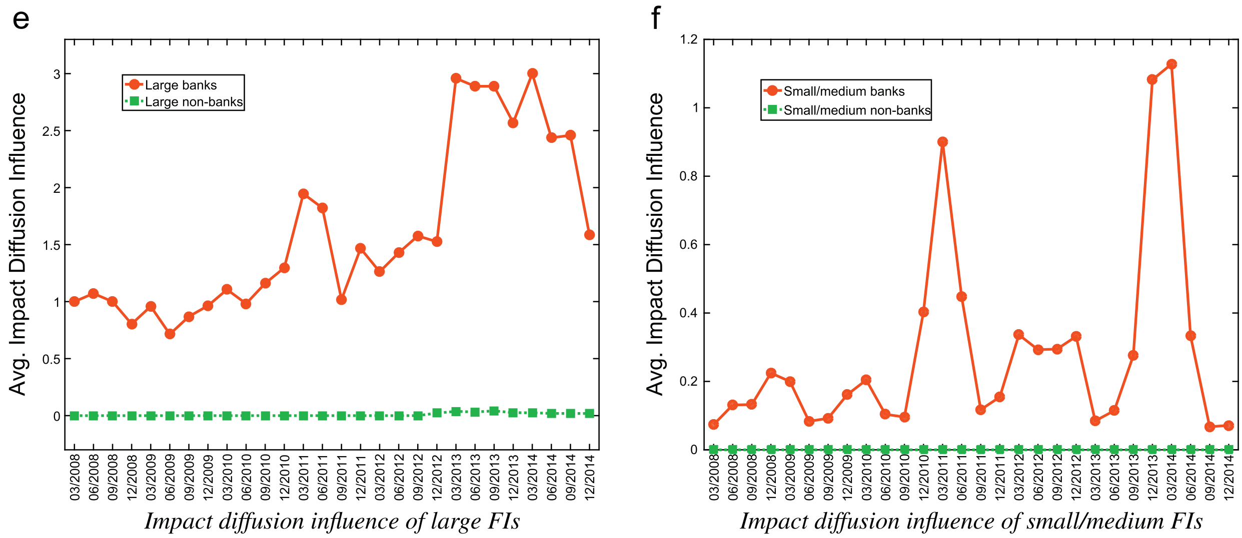

Fig. 10(b). (Continued)

accessory regulations issued by the Central Bank of Brazil that came into effect in July 2008.[20] That resolution modified the computation of the capital requirements of financial institutions. In particular, the resolution largely broadened the risk coverage of supervised institutions by the Central Bank of Brazil. As a result, we observe an increase of the tier 1 capital share of the available capital buffers of FIs. This may explain in part the reason of the robustness of the Brazilian financial market in September 2008, which is captured by its small network impact fluidity in the referred period.

5.4. Impact diffusion influence

Opposed to the impact susceptibility that measures if a bank may default as a consequence of the default of others, the impact diffusion influence measures the potential influence of an institution on the impact propagation over the network. That is, the more institutions it can potentially propagate impacts to, the larger is its exerted influence. In addition, the impact diffusion influence of an institution is composed of two orthogonal terms: the potential influence that it exerts when it starts the diffusion process, i.e., when it is the first institution in the propagation chain, and when it acts as an intermediator by being part of a contagion chain originated by another institution. Fig. 10a and b exhibits the average impact diffusion influence of institutions when they play the role of starters in the diffusion process for large and small/medium institutions, respectively. In turn, Fig. 10c and d depicts the impact diffusion influence for large and small/medium institutions, respectively, when they act as intermediators in the diffusion process. Summing up both contributions, Fig. 10e and f portrays the total impact diffusion influence of large and small/medium institutions, respectively, in the Brazilian financial market from 2008 to 2014.

An interesting finding is that impact diffusion influence exerted by large banks is solely due to its potential influence on starting a diffusion process. In this way, they do not act as potential intermediators in an ongoing contagion process. This is an interesting property that provides robustness for the Brazilian financial market. One reason is that the network presents very strong patterns of disassortative mixing and core–periphery structure,[21] in such a manner that almost all of the small/medium institutions are likely to be directly connected to a large banking institution. In this configuration, the large bank can stop the propagation process at the initial phase of the contagion process. Hence, contagion chains are expected to be small. Another advantage of this setup is that large banks often have very large capital buffers (recall Fig. 5b), in such a way that it is very unlikely for a contagion process to propagate through them.

In contrast to that, the impact diffusion influence imposed by small/medium banking institutions is due to their routes as starters and intermediators in a potential diffusion process in the network. In particular, we see two prominent peaks happening for both diffusion influence terms in the first half of 2011 and from the fourth quarter of 2013 to the first quarter of 2014. We can get a gist as to why there is a peak of the impact diffusion influence for small/medium banks in December 2013 by looking at Fig. 7b, which shows the vulnerability network at that period. In that occasion, there is a small/medium bank that can lead to default more than twenty non-banking institutions. Moreover, such small/medium bank is susceptible to be impacted by several other banking institutions. This setup skyrockets the impact diffusion influence

[20] The reader is referred to BCB (2008) for more information. The resolution number published by the National Monetary Council is 3.490, which was first published in August 29, 2007.

[21] See Silva et al. (2016) for a qualitative discussion on the network topology of the Brazilian financial network.

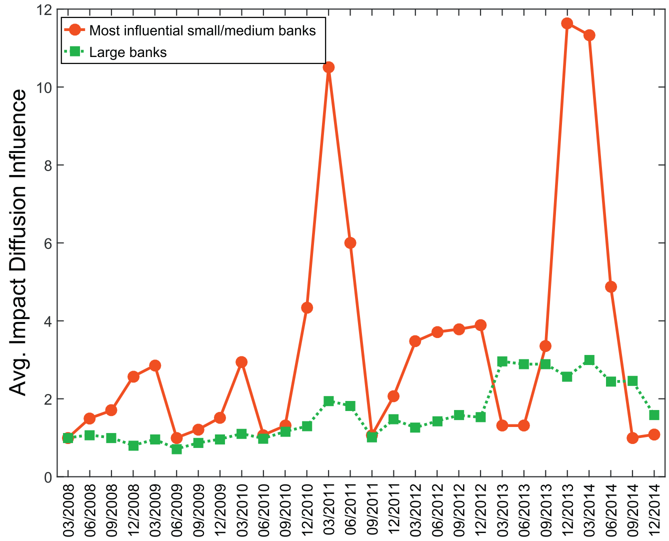

Fig. 11. Comparison of the average impact diffusion influence of large banks with the top $M(t)$ small/medium banks, where $M(t)$ is the number of large banks at instant t.

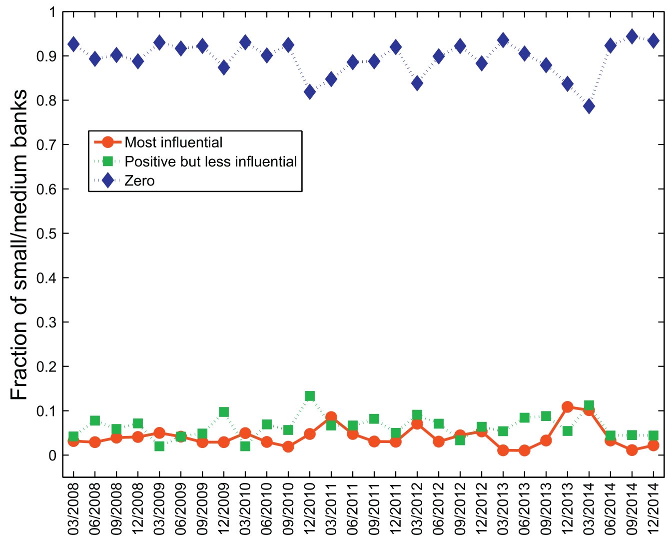

Fig. 12. Fractions of (i) "Most influential:" small/medium banks that are more influential than the average influence exerted by large banks; (ii) "Zero:" small/medium banks that have no influence in diffusing impact in the network; (iii) "Positive but less influential:" small/medium banks that have a smaller, but non-zero, impact diffusion influence than the average influence of large banks.

indices for institutions that are communicable to that specific small/medium bank. Hence, the appearance of that contrasting peak.

We can see that non-banking institutions are not the main actors in diffusing impacts over the network, as they typically cannot communicate to anyone. In fact, inspecting Fig. 7, they correspond to leaf vertices most of the time, because they are "dead-end" vertices. In particular, the impact diffusion influence is zero for small/medium non-banks in the entire period. Large non-banks, notwithstanding, have zero impact diffusion influence from September 2008 to September 2012. After that point, the average influence index is small, but remains positive until December 2014, especially because of the impact diffusion influence that measures how important they are as intermediates in the diffusion process. The region ranging from the fourth quarter of 2012 to the third quarter of 2013 shows that a few large non-banking institutions have non-zero values for intermediating diffusion processes.[22] Examples of these features can be visually noticed in the vulnerability networks depicted in Fig. 7. For instance, inspecting the vulnerability network in September 2008 in Fig. 7a, it is clear that non-banking institutions are incommunicable to the remainder of the network. Diverging from that, looking at the

[22] The average is small because most of those institutions have intermediating diffusion process influences equal to zero.

vulnerability network in December 2013 exhibited in Fig. 7b, we do see few non-banking institutions that can possibly lead other FIs into default, mostly small/medium banks. As such, they can potentially inflict damage to others; hence, the positive impact diffusion influence.

Returning to Fig. 10, we notice that banking institutions have larger impact diffusion influence than non-banking institutions in the entire period. Interestingly, the influence of large banks in potentially propagating impacts is, on average, of the same magnitude order of those of small/medium banks. However, by inspecting the data, on the one hand, we verify that all of the large banks have similar non-zero influence. On the other hand, the distribution of impact diffusion influence exerted by small/medium banks is highly skewed. For instance, looking at Fig. 7, we can see that some small/medium banks play central roles in diffusing impacts, and, in many cases, these roles are much more central than those of large banks. One argument to support that is that the vulnerability network in December 2013 has a small/medium bank that can diffuse potential impact to more than twenty non-banking institutions in a direct hit. The large bank that potentially can inflict more damage, however, only affects seven other institutions.

In order to check for this skewness, we plot in Fig. 11 the average impact diffusion influence of large banks against that of the top $M(t)$ small/medium banks, where $M(t)$ represents the number of large banks existent in the network at time t (recall Fig. 5a). Note that these top small/medium banks influence much more in the process of diffusing impact over the network as the vulnerability networks in Fig. 7 clearly show. We see that non-banking institutions prefer to create relationships with small/medium banks than with large banks. Small/medium banks are often more demanded by them probably because they offer high return rates at the cost of higher risks. In view of that, our proposed index for measuring impact diffusion

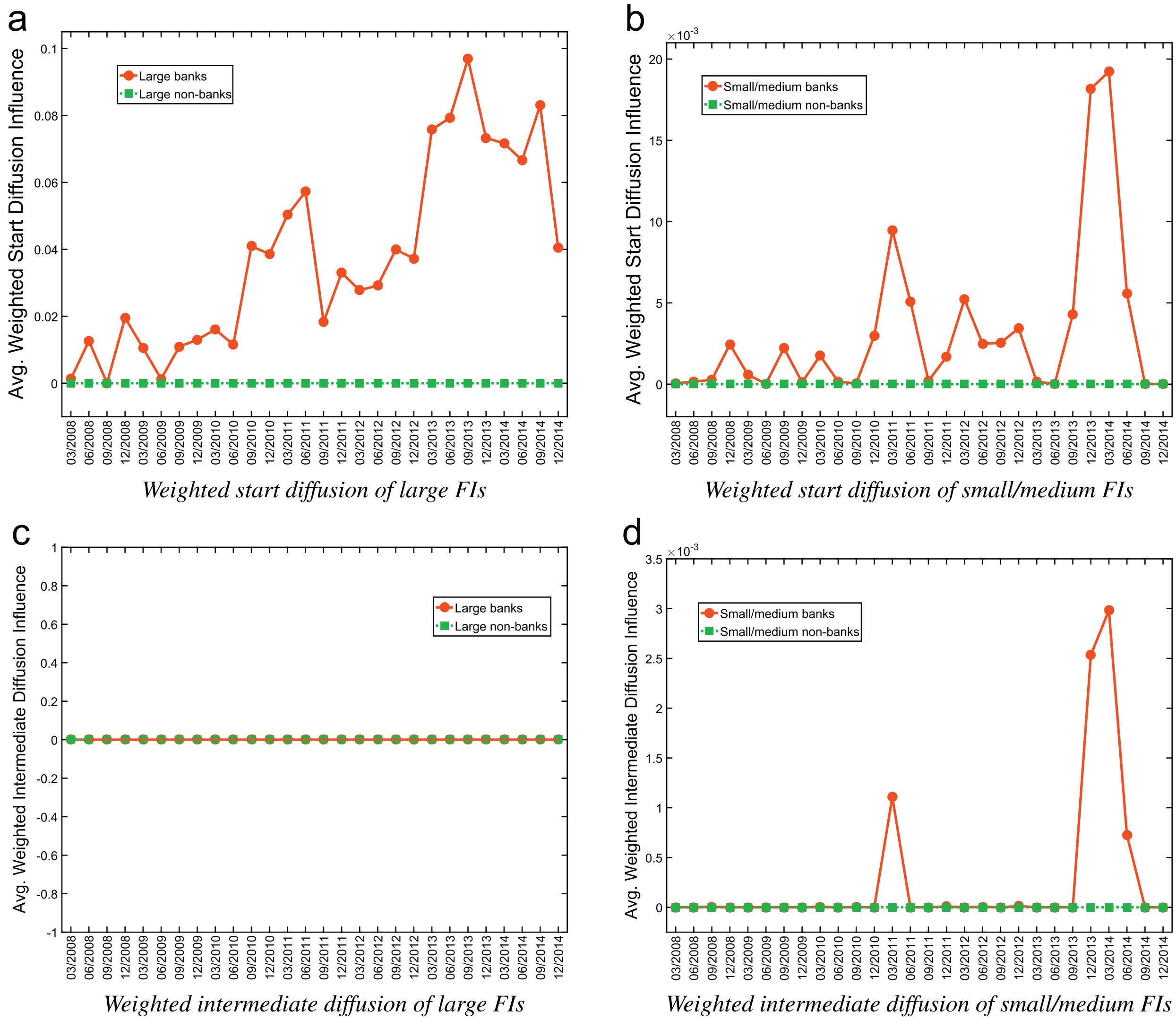

Fig. 13(a). Trajectory of the average weighted impact diffusion influence and its constituent parts (start and intermediate factors) for static snapshots of the financial network in different periods.

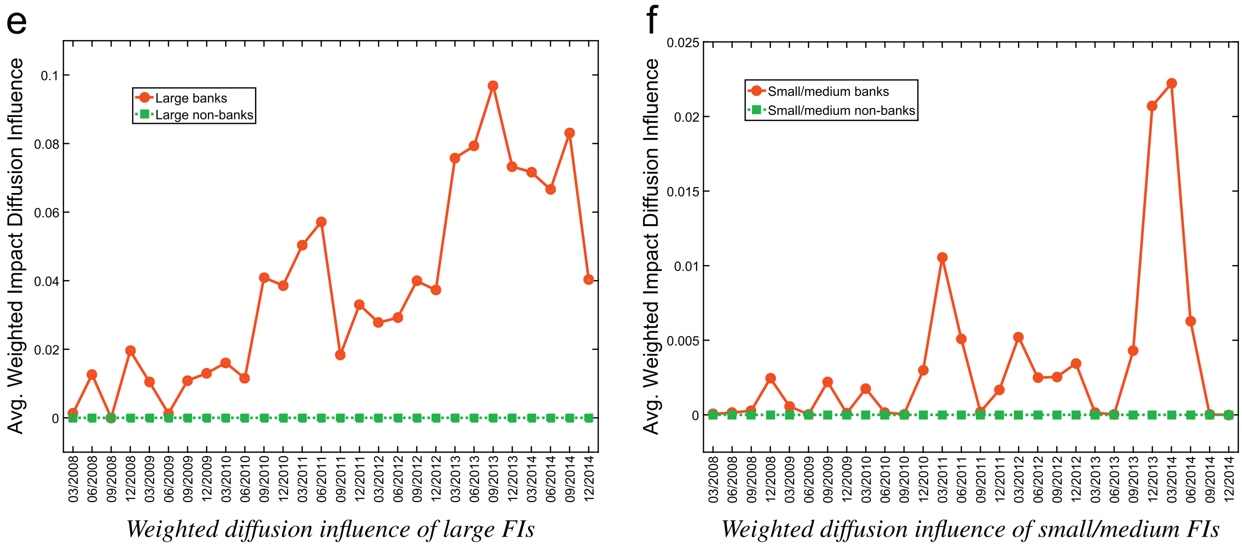

Fig. 13(b). (Continued)

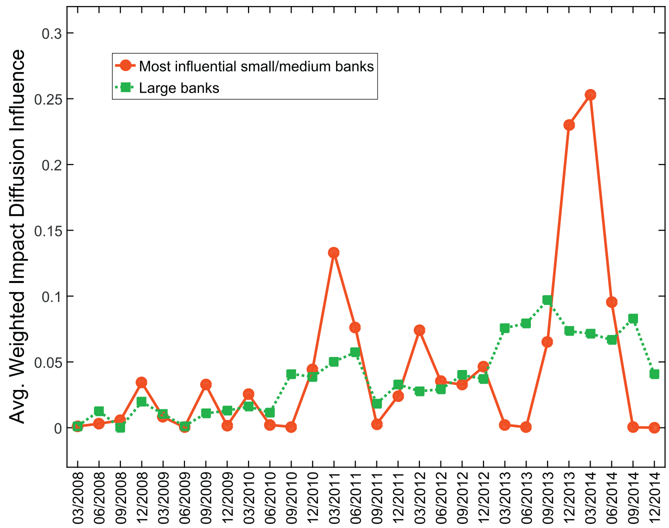

Fig. 14. Comparison of the average weighted impact diffusion influence of large banks with the top $M(t)$ small/medium banks, where $M(t)$ is the number of large banks at instant t.

influence correctly captures this notion that top small/medium banks are candidates of potentially diffusing more impact than large banks.

Fig. 12 provides another evidence of the skewness of the distribution of the small/medium banks' diffusion influences. Numerically, a large portion of the small/medium banks have zero impact diffusion influence in the studied period, which contributes to diminishing their corresponding average influence index. It is clear that, on average, more than 90.20% of small/medium banks do not take part in diffusing potential impacts throughout the network. In contrast, however, there is a small quantity of small/medium banks, which appear in the vulnerability networks depicted in Fig. 7, whose average impact diffusion influence is above the large banks' average. In the period, this portion of most influential small/medium banks is composed, on average, of 3.92% of the small/medium banks. There is a reminiscent of small/medium banks that have a smaller, but non-zero, impact diffusion influence than the average of large banks. They account, on average, for just 5.88% of the small/medium banks. In terms of surveillance on the financial system, regulators must be aware of these most influential entities, as they can render the system more risky. In the Brazilian financial market, they are very few, but may potentially harm the system as an entirety in the sense of the impact diffusion influence index.

5.5. Weighted impact diffusion influence

The impact diffusion influence gives an equal importance for all of the impact diffusing institutions in the network. However, in general, the occurence of a default of a large bank is much more catastrophic than that of a small/medium non-bank in terms of overall assets losses. That considered, providing a measure of value or importance to each of the institutions is essential, as we are able to get quantitative instead of only qualitative information as the non-weighted influence index supplies. To evaluate the weighted impact diffusion influence, we use the same proxy as in (29).

Fig. 13a and b exhibits the average weighted impact diffusion influence of institutions when they play the role of starters in the diffusion process for large and small/medium institutions, respectively. Similarly, Fig. 13c and d depicts the weighted impact diffusion influence for large and small/medium institutions, respectively, when they act as intermediators in the diffusion process. Summing up both contributions, Fig. 13e and f portrays the total weighted impact diffusion influence of large and small/medium institutions, respectively, in the Brazilian financial market from 2008 to 2014. We find that, on average, large banks are about five times more influent than large non-banks, and that non-banking institutions are far less influent than banking ones.

We can verify how the weight in the impact diffusion influence can alter the overall results by comparing Figs. 14 and 11, which portray the weighted and non-weighted results, respectively. We can see that, while the average non-weighted diffusion influence exerted by the most influential small/medium banking institutions mostly remains consistently above the large banks' average in the non-weighted version, this behavior does not extend to the weighted version. We can see that the ratios over time between the impact diffusion influence exerted by large banks and the most influential small/medium banks are larger for the weighted version (Fig. 14) than for the non-weighted version (Fig. 11). This effect stems from small/medium banks influencing a larger quantity of other FIs than large banks. These influenced FIs, however, do not hold large shares of the total interbank market liabilities. Therefore, the weighted version of the impact diffusion influence attenuates the importance of the impact. Large banks, in contrast, influence a smaller quantity of FIs that are, on average, more important, as they hold a relevant share of the total interbank market liabilities.

6. Conclusion

This paper proposes a systemic risk framework that captures two opposite perspectives of the role of a financial institution with respect to financial contagion: (i) the susceptibility to receiving and (ii) the capacity of amplifying random shocks. The measures that constitute the framework are intuitive to economics and finance, do not require the calibration of parameters, take into account the entire set of bilateral exposures between financial institutions, and focus on capturing potential catastrophic events inside a financial network.

We present five new measures of potential contagion in a financial network: weighted and non-weighted impact susceptibility, network impact fluidity, and weighted and non-weighted impact diffusion influence. The non-weighted versions of the impact susceptibility and impact diffusion influence entirely rely on the network structure and attempt to extract and count all vulnerable paths that lead counterparties into default, be it through the shortest or longer paths. In contrast, the design of their weighted versions explore another angle of the network by evaluating the amount of economic loss that the system may incur through these vulnerability paths.

We derive theoretical properties for these network measures and propose the novel concepts of local/remote vulnerability and contagiousness with direct practical implications to financial regulation, surveillance, and monitoring of financial systems. In this sense, we relate these concepts to the effort of monitoring the soundness of financial institutions. For those entities that are remotely vulnerable, supervisors should not only monitor the healthiness of the local neighborhood of these entities but also more distant and communicable financial institutions. Institutions that are remotely contagious should receive special attention as they are potential candidates to harming the financial system in a significant way. Our theoretical derivations provide a quick way to check whether a market participant is remotely vulnerable or contagious by simply inspecting the magnitude of the impact susceptibility and impact diffusion influence. We can complement this strategy by using the weighted versions of the impact susceptibility and impact diffusion influence, which gives us a monetary sense of how important are the fragility of and the damage caused by financial institutions, respectively.

We analyze how these measures evolve over time using a unique dataset on complete exposures for the Brazilian financial market from 2008 to 2014. We find that banks and non-banks share very different susceptibility and diffusion profiles, which reinforce the complementary role that these measures extract inside the financial network.

By employing the impact susceptibility, we find that banking institutions are barely susceptible to indirect impacts, while non-banking institutions show trends of remote vulnerability in some periods. Given that banking institutions are well capitalized and are barely susceptible to indirect impacts, we can say that the Brazilian financial system is rather stable.

We use the impact diffusion influence to gauge the influence of financial institutions, as shocks sources or transmitters, and discover the presence of a portion of small/medium banking institutions that is consistently more influential than large banks in potentially diffusing impacts throughout the network. That is a tool that financial system monitors and regulators can use to identify financial institutions that should receive more attention in times of crisis.

The framework takes into account a single source of vulnerability that may lead to financial contagion, which is counterparty risk. Further research can incorporate other possible sources of vulnerability, such as common or correlated exposures, asset firesales, illiquidity spirals due to marginal calls. Another possible future work can exploit how vulnerability changes with liquidity assistance (Capponi and Chen, 2015). It is worth mentioning that we have to be careful with policy recommendations as they seek to reduce heterogeneity in interconnections within financial networks, which in turn may backfire leading to less robust and resilient financial systems (León and Berndsen, 2014).

Acknowledgments

Thiago C. Silva (Grant no. 302808/2015-9) and Benjamin M. Tabak (Grant no. 305427/2014-8) gratefully acknowledge financial support from the CNPq foundation.

Appendix A. Other types of vulnerability matrices

Acemoglu et al. (2015) highlight that the influence of networks in financial contagion crucially depends on the nature of economic interactions between different counterparties that constitute the network. Our systemic risk framework relies on the truncated vulnerability matrix to capture how default events propagate in the network and provides measures of the default cascade paths that arise in view of these events.

In the main text, we focus on designing the truncated vulnerability matrix in terms of financial instruments that are unconditional on events that may happen on third counterparties. In this sense, the edge (i, j) conveys the notion of how j would be affected should i default by inspecting the amount of the exposure of j toward i against j's current loss absorbing capability, i.e., its capital buffer. The event that unleashes financial loss originates from i, i.e., a counterparty that is in one of the endpoints of the corresponding edge. We can also extend this notion to contingent vulnerabilities that arise due to external events.

Economic agents can hedge unsecured financial assets against credit defaults by buying protection that third counterparties offer through credit derivatives contracts, such as credit default swaps (CDS).[23] By transferring risks to third counterparties, CDS can potentially reshape the entire network topology and can either increase or attenuate systemic risk.

In this appendix, we modify the truncated vulnerability matrix to model credit default swaps, in which the debt instrument issuer that is subject to credit risk also participates in the network. However, our results trivially extend to other types of credit derivatives. We consider both the unsecured financial operations between counterparties as well as CDS that counterparties may use to hedge their positions against the debtor in case of credit default. We can represent this problem as a multilayer financial network, in which one layer denotes the aggregate unsecured financial operations between counterparties—such as the sum of debt securities and interfinancial deposits—and the other layer encompasses CDS operations. Strictly speaking, for N FIs, we would have N sub-layers to model the overall CDS operations layer. The ith layer would represent the hedge operations that two counterparties perform on credit events related to the default of counterparty i.

In our model, we start from the original truncated vulnerability matrix in which the elements (i, j) refer to the local vulnerability of j to i. This local approach allows us to define a new truncated vulnerability matrix in which we aggregate to the elements (i, j) the CDS operations that insure against the default of FI i and have FI j as the buyer or the seller of the swap. If we perform this aggregation for all the CDS layers, consisting of the CDS operations related to all of the possible defaulting issuer FIs, we end up with a single modified truncated vulnerability matrix. Next, we detail the computation of this matrix.

Suppose that FI j agrees to lend to counterparty i in a unsecured financial operation an amount greater than its capital buffer. Consequently, there will be an edge linking i to j in the unsecured financial operations layer to represent the potential impact susceptibility of j in case i defaults. However, suppose j hedges her position against the credit default of i by buying credit protection from a third counterparty, say k. In this case, k would hold the entire counterparty risk of this operation. However, FI j is not exempt from credit losses as it would still incur in losses in case both k and j default. Then, to account for these facts in the truncated vulnerability matrix, we must consider that:

- The purchase of CDS protection contracts attenuates the unsecured exposure and hence the vulnerability by diminishing counterparty risk given that the protector does not default in the entire process.
- The sale of CDS protection contracts increases the unsecured exposure even in the case the protection buyer defaults in the process. This is because third counterparties still remain liable to their liabilities toward defaulted entities.

Hence, we can modify the truncated vulnerability matrix as follows:

$$\overline{\mathbf{V}}_{ij} = \begin{cases} 1, & \text{if } \dfrac{\mathbf{L}_{ij}}{\mathbf{E}_j} - \displaystyle\sum_{k \in \mathcal{S}_i} \dfrac{\mathbf{P}^{(i)}_{jk}}{\mathbf{E}_j} + \displaystyle\sum_{k' \in \mathcal{V}} \dfrac{\mathbf{P}^{(i)}_{k'j}}{\mathbf{E}_j} \geq 1, \\ 0, & \text{otherwise,} \end{cases} \tag{30}$$

in which:

[23] A Credit Default Swap (CDS) is a contract designed to transfer the credit risk of a debt instrument between two or more counterparties. In this contract, the buyer of the swap makes payments to the swap's seller up until the contract's maturity date. In turn, the seller agrees that, in the event that the debt issuer defaults or experiences another credit event, the seller will pay the buyer the security's premium as well as all interest payments that would have been paid between that time and the security's maturity date.

- $\mathbf{P}_{jk}^{(i)}$ represents the protection that j buys from k to hedge against the default of i. Normally, j will protect the whole financial operation, such that $\mathbf{P}_{jk}^{(i)} = \mathbf{L}_{ij}$.
- $\mathcal{S}_i$ is the set of FIs that survive in the event that counterparty i enters default. This is an important step because, even though j can buy credit protection from the default of i from the third counterparty k, this hedge will be useless should the default of i imply in a subsequent default of k. To obtain $\mathcal{S}_i$, we compute $\overline{\mathbf{V}}_{ij}$ using (30), supposing initially that $\mathcal{S}_i$ comprises all FIs except i. After the computation of (30), we may find that additional FIs default as a consequence of the default of FI i. In this case, we take the additional failing FI off of the set $\mathcal{S}_i$, reiterating the process until the set composition remains constant. This algorithm is similar to the Eisenberg and Noe (2001)'s fictitious default algorithm.
- $\mathcal{V}$ is the set of all FIs in the market.

Focusing on the first condition of (30), the term $\frac{\mathbf{L}_{ij}}{\mathbf{E}_j}$ represents the direct exposure that j has toward i; the term $\sum_{k\in\mathcal{S}_i} \frac{\mathbf{P}_{jk}^{(i)}}{\mathbf{E}_j}$ denotes all the protection contracts that j buys from third counterparties to hedge against credit default of i, which we express in terms of j's capital buffer to maintain the notion of pairwise vulnerability; and the term $\sum_{k'\in\mathcal{V}} \frac{\mathbf{P}_{k'j}^{(i)}}{\mathbf{E}_j}$ stands for all the protection contracts that j sells to third counterparties. If the net vulnerability is still greater than or equal to one, then it means that default of i still leads j into default as well, even in light of the CDS protections.

In this setup, note that FI j will only receive back its claims from counterparties that survive from i's default. However, it will still remain liable to all of its sold protections against that same counterparty.

References

Acemoglu, D., Ozdaglar, A., Tahbaz-Salehi, A., 2015. Systemic risk and stability in financial networks. Am. Econ. Rev. 105, 564–608.
Allen, F., Babus, A., Carletti, E., 2012. Asset commonality, debt maturity and systemic risk. J. Financ. Econ. 104, 519–534.
Allen, F., Gale, D., 2000. Financial contagion. J. Polit. Econ. 108, 1–33.
Amini, H., Cont, R., Minca, A., 2016. Resilience to contagion in financial networks. Math. Finance 26, 329–365.
Bardoscia, M., Battiston, S., Caccioli, F., Caldarelli, G., 2015. DebtRank: a microscopic foundation for shock propagation. PLoS One 10, e0130406.
Battiston, S., Caldarelli, G., D'Errico, M., Gurciullo, S., 2016a. Leveraging the Network: A Stress-Test Framework Based on DebtRank. SSRN 2571218.
Battiston, S., D'Errico, M., Gurciullo, S., 2016. DebtRank and the network of leverage. J. Altern. Invest. 18, 471–485.
Battiston, S., Gatti, D.D., Gallegati, M., Greenwald, B., Stiglitz, J.E., 2012. Liaisons dangereuses: increasing connectivity, risk sharing, and systemic risk. J. Econ. Dyn. Control 36, 1121–1141.
Battiston, S., Puliga, M., Kaushik, R., Tasca, P., Caldarelli, G., 2012. DebtRank: too central to fail? Financial networks, the FED and systemic risk. Sci. Rep. 2, 541.
BCB, 2008. Financial Stability Report 7. Central Bank of Brazil, Brasília, Brazil, pp. 1–174.
BCB, 2012. Financial Stability Report 11. Central Bank of Brazil, Brasília, Brazil, pp. 1–63.
Benzi, M., Klymko, C., 2013. Total communicability as a centrality measure. J. Complex Netw. 1, 124–149.
Capponi, A., Chen, P.-C., 2015. Systemic risk mitigation in financial networks. J. Econ. Dyn. Control 58, 152–166.
Demirgüç-Kunt, A., Huizinga, H., 2010. Bank activity and funding strategies: the impact on risk and returns. J. Financ. Econ. 98, 626–650.
Drehmann, M., Tarashev, N., 2013. Measuring the systemic importance of interconnected banks. J. Financ. Intermed. 22, 586–607.
Eisenberg, L., Noe, T.H., 2001. Systemic risk in financial systems. Manag. Sci. 47, 236–249.
Estrada, E., Hatano, N., 2008. Communicability in complex networks. Phys. Rev. E 77, 036111.
Estrada, E., Higham, D.J., Hatano, N., 2008. Communicability and multipartite structures in complex networks at negative absolute temperatures. Phys. Rev. E: Stat. Nonlinear Soft Matter Phys. 78, 026102.
Estrada, E., Rodríguez-Velázquez, J.A., 2005. Subgraph centrality in complex networks. Phys. Rev. E: Stat. Nonlinear Soft Matter Phys. 71, 056103.
Gabrieli, S., Georg, C., 2014. A Network View on Interbank Market Freezes. Deutsche Bundesbank Discussion Paper 44.
Gai, P., Kapadia, S., 2010. Contagion in financial networks. Proc. R. Soc. 4666, 2401–2424.
Greenwood, R., Landier, A., Thesmar, D., 2015. Vulnerable banks. J. Financ. Econ. 115, 68–81.
Haldane, A.G., May, R.M., 2011. Systemic risk in banking ecosystems. Nature 469, 3515.
Hideaki, A., Battiston, S., Yoshi, F., 2013. DebtRank Analysis of the Japanese Credit Network. Discussion Paper Series 13-E-087, Research Institute of Economy, Trade and Industry.
IMF, BIS, FSB, 2009. Guidance to Assess the Systemic Importance of Financial Institutions, Markets and Instruments: Initial Considerations. Report to the G20 Finance Ministers and Governors. International Monetary Fund, Bank for International Settlements, and Financial Stability Board.
León, C., Berndsen, R.J., 2014. Rethinking financial stability: challenges arising from financial networks' modular scale-free architecture. J. Financ. Stab. 15, 241–256.
Martínez-Jaramillo, S., Alexandrova-Kabadjova, B., Bravo-Benítez, B., Solórzano-Margain, J.P., 2014. An empirical study of the Mexican banking system's network and its implications for systemic risk. J. Econ. Dyn. Control 40, 242–265.
Minoiu, C., Reyes, J.A., 2013. A network analysis of global banking: 1978–2010. J. Financ. Stab. 9, 168–184.
Nier, E., Yang, J., Yorulmazer, T., Alentorn, A., 2007. Network models and financial stability. J. Econ. Dyn. Control 31, 2033–2060.
Rogers, L.C.G., Veraart, L.A.M., 2013. Failure and rescue in an interbank network. Manag. Sci. 59, 882–898.
Silva, T.C., de Souza, S.R.S., Tabak, B.M., 2016. Network structure analysis of the Brazilian interbank market. Emerg. Mark. Rev. 26, 130–152.
Silva, T.C., Tabak, B.M., Guerra S.M., 2017. Why do vulnerability cycles Matter in financial networks? Physica A: Stat. Mech. Appl. 471, 592-606.
Silva, T.C., Zhao, L., 2016. Machine Learning in Complex Networks. Springer, Cham, Switzerland.
Souza, S.R.S., Tabak, B.M., Silva, T.C., Guerra, S.M., 2015. Insolvency and contagion in the Brazilian interbank market. Physica A: Stat. Mech. Appl. 431, 140–151.