Math 2940 Worksheet

Markov Chains, Intro to Eigenvalues

Week 10

October 31st, 2019

This worksheet covers material from **Sections 4.9** and **5.1**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. On any given day, the weather in Ithaca is either good or bad. If the weather is good today, then there is a 80% chance the weather will be good tomorrow. However, if the weather is bad today, then there is a 60% chance that the weather will be bad tomorrow.

(a) What is the stochastic matrix for this situation?

(b) Suppose the predicted weather for Monday is a 50% chance of good weather, and a 50% chance of bad weather. What is the probability of good weather on Wednesday?

(c) In the long run, how likely is it for the weather in Ithaca to be good on a given day?

Question 2. The simplest type of inheritance occurs when a trait is governed by a pair of genes, each of which may be of two types, say G and g. An individual may have a GG combination, or Gg (which is the same as gG), or gg.

Often, the GG and Gg genotypes are indistinguishable in appearance, in which case we say that the G gene *dominates* the g gene. An individual is called *dominant* if they have GG genes, a *hybrid* if they have Gg genes, and *recessive* if they have gg.

Consider a process of continued matings. We start with an individual whose genotype is known, and mate it with a hybrid. An offpsring is chosen at random and is mated with a hybrid. This process is repeated through a number of generations.

The genotype of the chosen offpsring in successive generations can be represented by a Markov chain. The states are indicated by GG, Gg, and gg respectively. The stochastic matrix is:

$$P = \begin{array}{c c} & \begin{array}{ccc} GG & Gg & gg \end{array} \\ \begin{array}{c} GG \\ Gg \\ gg \end{array} & \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

(a) Suppose the process is started with a hybrid bred to a hybrid. What is the probability of each outcome (dominant, hybrid, recessive) after one iteration?

(b) What is the probability of each outcome after two iterations?

(c) What is the steady-state vector for this Markov chain?

Question 3.

(a) Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$?
If so, find the eigenvalue.

(b) Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$?
Why or why not?

(c) Construct a 3×3 matrix whose eigenvalues are -1, 5, and 3.

(d) Construct a 2×2 matrix which has an eigenvector of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue -3.

(e) Construct a 2×2 matrix whose only eigenvalue is -3.

Question 4. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

(a) Show that $A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$.

(b) Using your answer from part (a), compute $A^2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(c) Compute $A^{100} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(d) Suppose a new matrix B has an eigenvector $\boldsymbol{x}$ with eigenvalue λ. What, if anything, does this tell you about the eigenvalues and eigenvectors of B^k, where k is a positive integer?

Question 5. Consider a cricket which hops along a one-dimensional grid with N nodes arranged along a circle. At each hop, the cricket can move to its immediate neighbor node clockwise, to its neighbor node counter-clockwise, or stay at its current node. Each movement is equally likely to occur.

(a) Write down the stochastic matrix P associated with this Markov chain.

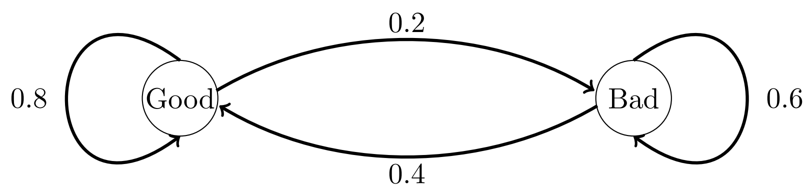
(b) Compute the steady-state vector $\boldsymbol{q}$ for the stochastic matrix P.

(c) Consider the case where the cricket's wing is injured: the cricket now stays at its current node or moves with equal probability and when it does jump, it is twice as likely to move clockwise as it is to move counterclockwise.

Write down the stochastic matrix P and show that the steady-state vector $\boldsymbol{q}$ is the same as in parts (a) and (b). (This is the magic of *periodicity*.)

Answer to Question 1.

(a) For this problem, we can visualize the Markov chain as the following directed graph:



To form the stochastic matrix, the first column will be the transition probabilities starting from good weather, and the second column will be the transition probabilities starting from a day with bad weather. The matrix should be:

$$\boxed{\begin{array}{c c} & \begin{array}{cc}\text{Good} & \text{Bad}\end{array} \\ \begin{array}{c}\text{Good}\\ \text{Bad}\end{array} & \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \end{array}}$$

From here on out, I will refer to this matrix as P.

(b) We start with the initial state $\boldsymbol{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ on Monday.

To figure out the prediction $\boldsymbol{x}_1$ of weather for Tuesday, we then multiply by the stochastic matrix P from part (a):

$$\boldsymbol{x}_1 = P\boldsymbol{x}_0 = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.4 + 0.2 \\ 0.1 + 0.3 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

To figure out the prediction $\boldsymbol{x}_2$ of weather on Wednesday, we multiply by the stochastic matrix P again:

$$\boldsymbol{x}_2 = P\boldsymbol{x}_1 = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.48 + 0.16 \\ 0.12 + 0.24 \end{bmatrix} = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$$

from which we can see that $\boxed{\text{there is a 64\% chance of good weather on Wednesday.}}$

(c) To figure out the behavior in the long run, we want to figure out the steady-state behavior of the Markov chain. The steady state (which I'll call $\boldsymbol{q}$) satisfies the equation $P\boldsymbol{q} = \boldsymbol{q}$. This is equivalent to finding $\boldsymbol{q}$ such that $(P - I)\boldsymbol{q} = 0$, where I is the identity matrix. Therefore we want to find the nullspace of the matrix $P - I$,

$$P - I = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$$

Which we can row reduce to

$$\sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

We can now see that the nullspace of $P - I$ is the set of all vectors such that $-x_1 + 2x_2 = 0$, or in other words $x_1 = 2x_2$.

The entries in the steady state $\boldsymbol{q}$ should add up to one, because they represent probabilities. Therefore, the steady state of this Markov chain is:

$$\boldsymbol{q} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

And so in the long run, $\boxed{\text{the weather has a } \frac{2}{3} \text{ chance of being good on any given day}}$

Answer to Question 2.

(a) Since this process is started with a hybrid, the initial state $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Breeding with another hybrid would correspond to multiplying by the matrix P:

$$\boldsymbol{x}_1 = P\boldsymbol{x}_0 = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$$

Which means:

$$\boxed{\begin{array}{l} \text{The probability of a dominant offspring is 25\%} \\ \text{The probability of a hybrid offspring is 50\%} \\ \text{The probability of a recessive offspring is 25\%} \end{array}}$$

(b) A second iteration would correspond to multiplying by the matrix P again:

$$\boldsymbol{x}_2 = P\boldsymbol{x}_1 = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$$

and therefore:

$$\boxed{\begin{array}{l} \text{The probability of a dominant offspring is 25\%} \\ \text{The probability of a hybrid offspring is 50\%} \\ \text{The probability of a recessive offspring is 25\%} \end{array}}$$

(c) As we've already seen in part (b), the vector $\boldsymbol{q} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$ satisfies $\boldsymbol{q} = P\boldsymbol{q}$, so therefore $\boldsymbol{q}$ is the steady state vector.

In case you didn't realize this, we can still find the steady-state vector by finding the nullspace of $P - I$:

$$P - I = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.25 & 0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0.25 & -0.5 \end{bmatrix}$$

Row reducing,

$$\sim \begin{bmatrix} -0.5 & 0.25 & 0 \\ 0 & -0.25 & 0.5 \\ 0 & 0.25 & -0.5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -0.5 & 0.25 & 0 \\ 0 & -0.25 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So the nullspace is described by the equations:

$$-2x_1 + x_2 = 0$$
$$-x_2 + 2x_3 = 0$$

which means $x_1 = \frac{x_2}{2} = x_3$. To get the steady-state vector, we also want the entries to add up to one since these are supposed to be probabilities.

Therefore the steady-state vector is $\boxed{\boldsymbol{q} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}}$

Answer to Question 3.

(a) To check if it's an eigenvalue, we multiply the matrix by the vector:

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ -1 \end{bmatrix}$$

and since $\begin{bmatrix} 19 \\ -1 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, this means $\boxed{\begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ is not an eigenvector.}}$

(b) If $\lambda = 2$ is an eigenvalue, this means there is a non-zero vector $\boldsymbol{v}$ that satisfies $(A - 2I)\boldsymbol{v} = \mathbf{0}$. So we can check if $\lambda = 2$ is an eigenvalue by computing the nullspace of $A - 2I$, and seeing whether it is non-trivial.

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Row reducing,

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Since $A - 2I$ has a non-trivial nullspace, $\boxed{\lambda = 2 \text{ is an eigenvalue.}}$

(c) One such example is:

$$\boxed{A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

Here, the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) We want to find entries a, b, c, and d such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

One such example is:

$$\boxed{\begin{bmatrix} 1 & -4 \\ 0 & -3 \end{bmatrix}}$$

(e) One such example is:

$$\boxed{A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}}$$

since $A\boldsymbol{v} = -3\boldsymbol{v}$ for every vector $\boldsymbol{v}$.

Answer to Question 4.

(a) This can be verified by matrix multiplication:

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-2+0 \\ 2+2+0 \\ 2-2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

(b) Using our answer from part (a) twice,

$$A^2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = A \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$$

(c) Repeating the process from part (b),

$$A^{100} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2^{100} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2^{100} \\ 2^{101} \\ 0 \end{bmatrix}$$

(d) If B has an eigenvector $\boldsymbol{x}$ with eigenvalue λ, then:

$$B\boldsymbol{x} = \lambda\boldsymbol{x}$$

therefore

$$B^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$$

which means that $\boxed{B^k \text{ has an eigenvector of } \boldsymbol{x} \text{ with eigenvalue } \lambda^k.}$

Answer to Question 5.

(a) Numbering the nodes 1, ..., N and then assigning each node to the respective columns, the stochastic matrix is:

$$\boxed{P = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & \cdots & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & \cdots & \cdots & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \\ 1/3 & 0 & \cdots & \cdots & 0 & 1/3 & 1/3 \end{bmatrix}}$$

(b) Because of the symmetry, we can guess that the steady state vector is $\boldsymbol{q} = \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix}$.

Plugging this in, we can check that

$$P\boldsymbol{q} = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & \cdots & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & \cdots & \cdots & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \\ 1/3 & 0 & \cdots & \cdots & 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix} = \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix} = \boldsymbol{q}$$

Therefore, the steady state vector is $\boxed{\boldsymbol{q} = \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix}}$.

(c) In this case, the stochastic matrix P is:

$$\boxed{P = \begin{bmatrix} 1/2 & 1/4 & 0 & 0 & \cdots & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 & \cdots & \cdots & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \\ 1/4 & 0 & \cdots & \cdots & 0 & 1/4 & 1/2 \end{bmatrix}}$$

We can again check that:

$$P\boldsymbol{q} = \begin{bmatrix} 1/2 & 1/4 & 0 & 0 & \cdots & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 & \cdots & \cdots & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & \\ 1/4 & 0 & \cdots & \cdots & 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix} = \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix} = \boldsymbol{q}$$

and therefore the steady state vector is again $\boxed{\boldsymbol{q} = \begin{bmatrix} 1/N \\ 1/N \\ \\ \vdots \\ \\ 1/N \end{bmatrix}}$.