MATH 2930 DISCUSSION 10/13

Vibrations

(1) A mass of 1 kg stretches a spring 8 cm. The mass is first pushed upward, contracting the spring a distance of 2 cm, and then set in motion with a downward velocity of 60 cm/s. Assuming that there is no damping and no external force is applied,

(a) find the position $u(t)$ of the mass at any time t;

(b) determine the period, the frequency and the amplitude of the motion

(2) Consider a forced oscillator whose motion is modeled by

$$y'' + \gamma y' + y = F_0 \sin(\omega t)$$

where $y = y(t)$ and $\gamma > 0$.

(a) Find the steady state solution (i.e. solution as $t \to \infty$) to this problem.

(b) Does the steady state solution depend on the initial conditions?

(c) For $\gamma << 1$, at what ω is the amplitude of the steady state oscillation maximized?

(d) What is the amplitude of the steady state oscillation as $\omega \to 0$?

Higher-order ODEs

(3) Find the general solution of the 4th order differential equation

$$2y''' - 4y'' - 2y' + 4y = 0$$

(4) Determine the general solution of the given differential equation:

$$y^{(4)} + y^{(3)} = \sin(2t)$$

(5) Determine the general solution of the given differential equation:

$$y^{(4)} + 2y'' + y = 3 + \cos(2t)$$

Answers

(1a) Differential equation is $u'' + \frac{k}{m}u = 0$ with initial conditions $u(0) = u_0$ and $u'(0) = v_0$

Solution to this is

$$u(t) = u_0 \cos(\sqrt{\frac{g}{L}}t) + v_0\sqrt{\frac{L}{g}}\sin(\sqrt{\frac{g}{L}}t)$$

With $g = 10m/s$, and u in meters, we get $u_0 = 0.02m$ and $v_0 = -0.6m/s$, resulting in:

$$u(t) = 0.02\cos(\sqrt{125}t) - 0.6\sqrt{\frac{1}{125}}\sin(\sqrt{125}t)$$

(1b) Frequency is $\omega = \sqrt{\frac{g}{L}} =$, period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$

Amplitude is $R = \sqrt{A^2 + B^2} = \sqrt{u_0^2 + v_0^2\frac{L}{g}} = \frac{2\sqrt{41}}{5}$

Note: don't use a spring constant k in units of N/cm, these won't cancel correctly

(2a) Guess particular solution of the form $Y(t) = A\cos(\omega t) + B\sin(\omega t)$

Solution can be written $y(t) = \frac{F_0}{(1-\omega^2)^2+(\gamma\omega)^2}(-\gamma\omega\cos(\omega t) + (1-\omega^2)\sin(\omega t))$

(2b) No, only the homogenous solution depends on the initial condition

(2c) Amplitude is $R = \frac{F_0}{(1-\omega^2)^2+(\gamma\omega)^2}$ which for $\gamma << 1$ is maximized at $\omega = 1$

(2d) As $\omega \to 0$, $R \to F_0$

(3) Characteristic equation is $2r^3 - 4r^2 - 2r + 4 = 0$

roots are $r = 1, 2, -1$, each with multiplicity $s = 1$

General solution is

$$y(t) = c_1e^t + c_2e^{2t} + c_3e^{-t}$$

(4) Characteristic equation is $r^4 + r^3 = 0$

Roots are $r = 0$ with multiplicity $s = 3$ and $r = -1$ with multiplicity $s = 1$

Homogenous solution is $c_1 + c_2t + c_3t^2 + c_4e^{-t}$

Guess particular solution of the form $Y(t) = A\cos(2t) + B\sin(2t)$

Plug in and solve to get: $A = 1/40$ and $B = 1/20$

Solution is $y(t) = c_1 + c_2t + c_3t^2 + c_4e^{-t} + \frac{1}{20}\sin(2t) + \frac{1}{40}\cos(2t)$

(5) $y = c_1\cos(t) + c_2\sin(t) + c_3t\cos(t) + c_4t\sin(t) + 3 + \frac{1}{9}\cos(2t)$