Math 2930 Worksheet

Final Exam Review

Week 14

May 3rd, 2019

Question 1. Solve the initial value problem

$$y' - y = 2xe^x, \qquad y(0) = 1$$

Question 2. Find the general solution of the differential equation

$$\left(\frac{y}{x} + 6x\right) + \big(\ln(x) - 2\big)\frac{dy}{dx} = 0$$

Question 3. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{x-y}$$

Question 4. Consider the differential equation: $y' = y - y^3$.

(a) Find the equilibrium solutions and determine which of these solutions are asymptotically stable, semistable, and unstable.

(b) Draw the phase line and sketch several solution curves in the ty-plane for $t > 0$.

(c) Assuming that the solution $y(t)$ has the initial value $y(0) = -\frac{1}{2}$, compute the limit of $y(t)$ as $t \to +\infty$.

Question 5. Find the general solution of the differential equation

$$y'' - ty' + y = 0$$

given that one solution is $y_1 = t$.

(It's OK to leave part of your answer in the form of an integral.)

Question 6. Find the general solution to the following ODE:

$$\frac{d^4y}{dt^4} + 4\frac{d^2y}{dt^2} = t^2$$

Question 7. A thin wire coinciding with the interval $[-L, L]$ is bent into the shape of the circle so that the ends $x = -L$ and $x = L$ are joined. Under certain conditions, the temperature $u(x, t)$ in the wire satisfies the boundary-value problem:

$$\begin{aligned}
&u_t = \alpha^2 u_{xx}, && -L < x < L,\ t > 0 \\
&u(-L, t) = u(L, t), && t > 0 \\
&u_x(-L, t) = u_x(L, t), && t > 0 \\
&u(x, 0) = f(x), && -L < x < L
\end{aligned}$$

Find the solution to this problem using the method of separation of variables.

Question 8. The Neumann problem for the Laplace equation in the interior of the circle $r = a$ is given by

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, && 0 \le r < a, \quad 0 \le \theta < 2\pi \\
&u_r(a, \theta) = f(\theta), && 0 \le \theta < 2\pi
\end{aligned}
$$

(a) Using the method of separation of variables, find the solution to this problem.

(b) What condition should one impose on the function $f(\theta)$ for this problem to be solvable?

Answer to Question 1. This is a first-order linear equation, so we can solve it using integrating factors. (You could also use something like method of undetermined coefficients, but this would be harder).
We have the equation

$$y' - y = 2xe^x$$

Multiplying both sides by $e^{\int -1 dx} = e^{-x}$,

$$e^{-x}y' - e^{-x}y = 2x$$

and then using the product rule,

$$\left(e^{-x}y\right)' = 2x$$

Integrating both sides and then solving for y,

$$\begin{aligned} \int \left(e^{-x}y\right)' dx &= \int 2x dx \\ e^{-x}y &= x^2 + C \\ y &= x^2e^x + Ce^x \end{aligned}$$

Now we plug in the initial condition to find C:

$$\begin{aligned} y(0) &= 0^2(1) + C(1) = 1 \\ C &= 1 \end{aligned}$$

so the final answer is:

$$\boxed{y = x^2e^x + e^x}$$

Answer to Question 2. We are given the equation

$$\left(\frac{y}{x} + 6x\right) + \left(\ln(x) - 2\right)\frac{dy}{dx} = 0$$

We can check this equation for exactness. Since

$$M(x,y) = \frac{y}{x} + 6x, \qquad N(x,y) = \ln(x) - 2$$

We compute that

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{y}{x} + 6x\right] = \frac{1}{x} \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left[\ln(x) - 2\right] = \frac{1}{x} \end{aligned}$$

Since $M_y = N_x$, this equation is exact.
Now, for the solution we want to find a function $\Phi(x,y)$ such that

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= M(x,y) = \frac{y}{x} + 6x \\ \frac{\partial \Phi}{\partial y} &= N(x,y) = \ln(x) - 2 \end{aligned}$$

Integrating the first equation with respect to x and the second equation with respect to y, we get

$$\Phi(x,y) = \int \left(\frac{y}{x} + 6x\right) dx = y\ln(x) + 3x^2 + g(y)$$
$$\Phi(x,y) = \int (\ln(x) - 2)\, dy = y\ln(x) - 2y + f(x)$$

(Remember that because we are "undoing" partial derivatives, we get $+f(x)$ or $+g(y)$ instead of $+C$.) Combining those two equations for Φ, we see that

$$\Phi(x,y) = y\ln(x) + 3x^2 - 2y$$

So our final answer is

$$\boxed{y\ln(x) + 3x^2 - 2y = C}$$

Answer to Question 3. We want to find the general solution of the following differential equation:

$$\frac{dy}{dx} = \frac{1}{x-y}$$

Method 1: "Flip" the variables

If we "flip" both sides of the equation by taking the reciprocal, we get the equation:

$$\frac{dx}{dy} = x - y$$

The clever part of this method is realizing that this is actually a first-order linear ODE for finding x as a function of y. Rearranging things into standard form:

$$\frac{dx}{dy} - x = -y$$

Since it's a first order linear ODE, we can solve it using an integrating factor of:

$$\mu(y) = e^{\int -1 dy} = e^{-y}$$

Multiplying the entire equation by this integrating factor, we get

$$e^{-y}\frac{dx}{dy} - e^{-y}x = -ye^{-y}$$

The left hand side is an expression that comes from a product rule, so this can be written

$$\left(e^{-y}x\right)' = -ye^{-y}$$

Integrating both sides (this requires integration by parts for the right hand side),

$$e^{-y}x = \int -ye^{-y}dy = (y+1)e^{-y} + C$$

Then multiplying both sides by e^y to solve for x as a function of y, we get the general solution:

$$\boxed{x = y + 1 + Ce^y}$$

Method 2: Integrating factor to make exact

We can start with the original equation:

$$\frac{dy}{dx} = \frac{1}{x-y}$$

We can rearrange it into something that looks like an exact equation:

$$(x-y)\frac{dy}{dx} = 1$$
$$-1 + (x-y)\frac{dy}{dx} = 0$$

Unfortunately, when we check if this is exact, we get:

$$M = -1 \qquad\qquad N = x - y$$
$$M_y = 0 \qquad\qquad N_x = 1$$

So we see that $M_y \neq N_x$, so this equation is not exact as written. Since

$$\frac{N_x - M_y}{M} = \frac{0-1}{-1} = 1$$

is a function of y (and more importantly, *not* a function of x), we can find an integrating factor of the form $\mu = \mu(y)$.

If we multiply both sides of the equation by $\mu(y)$, we get

$$-\mu + \mu \cdot (x-y)\frac{dy}{dx} = 0$$

So in order for this to be exact, we would need

$$M = -\mu(y) \qquad\qquad N = (x-y)\mu(y)$$
$$M_y = -\frac{d\mu}{dy} \qquad\qquad N_x = \mu(y)$$

$$-\frac{d\mu}{dy} = \mu(y)$$
$$\frac{d\mu}{dy} = -\mu(y)$$

This last line is a separable equation for $\mu(y)$. Which has a solution of:

$$\mu(y) = e^{-y}$$

So this means that if we multiply our original equation by e^{-y}, yielding:

$$-e^{-y} + (x-y)e^{-y}\frac{dy}{dx} = 0$$

which is now an exact equation. That means we want to find a function $\Phi(x, y)$ with the following partial derivatives:

$$\frac{\partial \Phi}{\partial x} = -e^{-y} \qquad\qquad \frac{\partial \Phi}{\partial y} = (x-y)e^{-y}$$
$$\Phi = \int -e^{-y}dx \qquad\qquad \Phi = \int (x-y)e^{-y}dy$$
$$\Phi = -xe^{-y} + f(y) \qquad\qquad \Phi = -xe^{-y} - \int ye^{-y}dy$$
$$\Phi = -xe^{-y} + f(y) \qquad\qquad \Phi = -xe^{-y} + (y+1)e^{-y} + g(x)$$

So in order for $\Phi(x, y)$ to match both of these, we must have $f(y) = (y+1)e^{-y}$ and $g(x) = 0$. This yields a general solution of:

$$\Phi(x, y) = \boxed{-xe^{-y} + (y+1)e^{-y} = C}$$

This can be algebraically rearranged to be in the format of the other answers:

$$\boxed{x = y + 1 + Ce^{y}}$$

Method 3: Substitution
Starting with

$$\frac{dy}{dx} = \frac{1}{x - y}$$

use the substitution:

$$\begin{aligned} v &= x - y \\ \frac{dv}{dx} &= 1 - \frac{dy}{dx} \\ \frac{dy}{dx} &= 1 - \frac{dv}{dx} \end{aligned}$$

Turning the original equation into

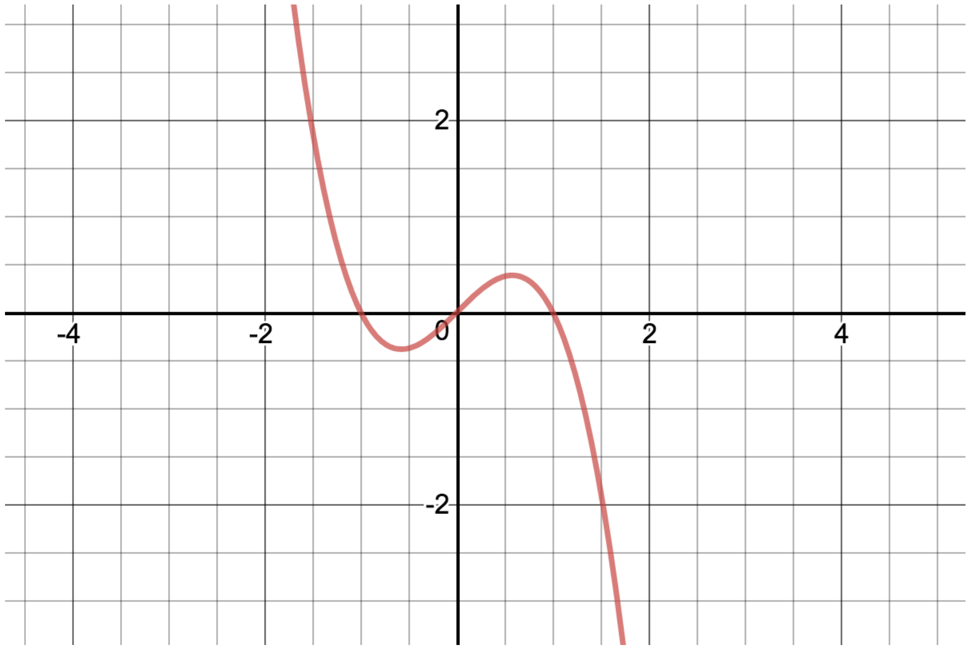
$$1 - \frac{dv}{dx} = \frac{1}{v}$$

This equation is separable:

$$\begin{aligned} \frac{dv}{dx} &= \frac{-1}{v} + 1 = \frac{v-1}{v} \\ \int \frac{v}{v-1} dv &= \int dx \\ \int 1 + \frac{1}{v-1} dv &= x + C \\ v + \ln(v-1) &= x + C \\ x - y + \ln(x-y-1) &= x + C \\ \ln(x-y-1) &= y + C \\ x - y - 1 &= Ce^{y} \end{aligned}$$

$$\boxed{x = y + 1 + Ce^{y}}$$

Answer to Question 4.
(a) Since this is an autonomous equation, we'll first look at the graph of $\frac{dy}{dt}$ (vertical axis) vs y (horizontal axis):



The equilibria occur where this graph crosses the horizontal axis, these are the values of y where $\frac{dy}{dt} = 0$. Solving for them,

$$\frac{dy}{dt} = y - y^3 = 0$$
$$y(1 - y^2) = y(1 - y)(1 + y) = 0$$
$$y = 0, \quad 1, \quad -1$$

To figure out whether these equilibrium solutions are stable, unstable, or semistable, we will look at the sign of $\frac{dy}{dt}$ nearby.
For $y = -1$, we see that for values of y less than -1, $\frac{dy}{dt}$ if positive, so solutions are increasing. For values of y slightly greater than -1, we see $\frac{dy}{dt}$ is negative, so solutions are decreasing. Since solutions below $y = -1$ are increasing and solutions above are decreasing, we get that

$$\boxed{y = -1 \text{ is a } \textit{stable} \text{ equilibrium}}$$

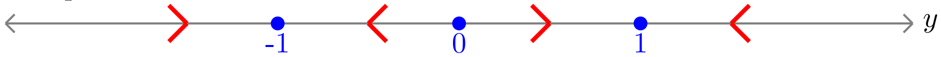
For $y = 0$, solutions slightly below are decreasing and solutions slightly above are increasing, so

$$\boxed{y = 0 \text{ is an } \textit{unstable} \text{ equilibrium}}$$

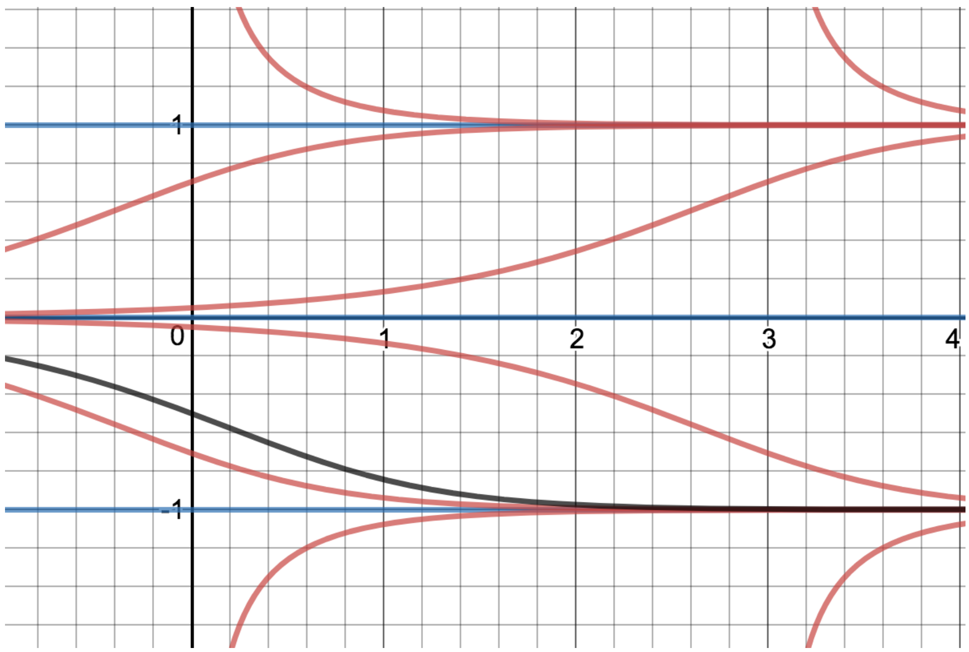
For $y = 1$, solutions slightly below are increasing and solutions slightly above are decreasing, so

$$\boxed{y = 1 \text{ is a } \textit{stable} \text{ equilibrium}}$$

(b)
The phase line looks like:



and the solutions look like:



Here the equilibria are in blue, and solutions are in red and black.

(c) If the initial condition is $y(0) = -\frac{1}{2}$, then the solution $y(t)$ will approach the nearest stable equilibrium at $y = -1$.
This specific solution $y(t)$ is the one graphed in black above.

Answer to Question 5. We want to find the general solution of

$$y'' - ty' + y = 0$$

given that $y_1 = t$ is a solution.

This is a *reduction of order* problem, which means that we will look for a general solution of the form

$$y(t) = y_1(t)v(t) = tv(t)$$

Taking derivatives,

$$\begin{aligned} y &= tv \\ y' &= v + tv' \\ y'' &= 2v' + tv'' \end{aligned}$$

Plugging this back into the original equation,

$$\begin{aligned} y'' - ty' + y &= 0 \\ tv'' + 2v' - t\left(v + tv'\right) + tv &= 0 \\ tv'' + (2 - t^2)v' &= 0 \\ \frac{v''}{v'} &= t - \frac{2}{t} \end{aligned}$$

If we define a new variable $u = v'$, then we have

$$\frac{1}{u}\frac{du}{dt} = t - \frac{2}{t}$$

Multiplying both sides by t and integrating,

$$\begin{aligned} \int \frac{1}{u} du &= \int \left(t - \frac{2}{t}\right) dt \\ \ln(u) &= \frac{t^2}{2} - 2\ln(t) + C_1 \end{aligned}$$

Exponentiating both sides, and replacing u with v',

$$u = v' = \frac{C_1 e^{t^2}}{t^2}$$

Integrating,

$$v = C_1 \int \frac{e^{t^2}}{t^2} dt + C_2$$

And multiplying both sides by t,

$$\boxed{y = tv = C_1 t \int \frac{e^{t^2}}{t^2} dt + C_2 t}$$

Answer to Question 6. First, we want to find the solution y_c of the homogeneous equation:

$$y^{(4)} + 4y'' = 0$$

We do this by solving for the roots of the characteristic equation,

$$\begin{aligned} y^{(4)} + 4y'' &= 0 \\ r^4 + 4r^2 &= 0 \\ r^2(r^2 + 4) &= 0 \\ r = 0,\ 0,\ \pm 2i \end{aligned}$$

So the complementary solution is

$$y_c(t) = c_1 + c_2 t + c_3 \cos(2t) + c_4 \sin(2t)$$

Now for the particular solution Y. Based on the method of undetermined coefficients, our guess for the particular solution will be

$$Y = \left(A + Bt + Ct^2\right) t^2$$

where the $A + Bt + Ct^2$ term is based on the right hand side of the original equation, and the t^2 term shows up because we have a double root of $r = 0$ in the characteristic equation. Multiplying this out and taking derivatives,

$$\begin{aligned} Y &= At^2 + Bt^3 + Ct^4 \\ Y' &= 2At + 3Bt^2 + 4Ct^3 \\ Y'' &= 2A + 6Bt + 12Ct^2 \\ Y''' &= 6B + 24Ct \\ Y^{(4)} &= 24C \end{aligned}$$

Plugging this back into the original equation,

$$\begin{aligned} Y^{(4)} + 4Y'' &= t^2 \\ 24C + 4(2A + 6Bt + 12Ct^2) &= t^2 \\ 48Ct^2 + 24Bt + (8A + 24C) &= t^2 \end{aligned}$$

Comparing like terms, we get a system of three equations for A, B, and C:

$$\begin{aligned} 48C &= 1 \\ 24B &= 0 \\ 8A + 24C &= 0 \end{aligned}$$

The solution to this system of equations is:

$$A = \frac{-1}{16}, \qquad B = 0, \qquad C = \frac{1}{48}$$

so the particular solution is

$$Y(t) = -\frac{t^2}{16} + \frac{t^4}{48}$$

Then the general solution is the combination of the complementary and particular solutions:

$$\boxed{y = y_c + Y = c_1 + c_2 t + c_3 \cos(2t) + c_4 \sin(2t) - \frac{t^2}{16} + \frac{t^4}{48}}$$

Answer to Question 7.
Let $u(x,t) = X(x)T(t)$. Then $u_t = XT'$, $u_{xx} = X''T$.
Plugging them into the PDE, we get

$$XT' = \alpha^2 X''T$$

Dividing both sides by $\alpha^2 XT$,

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X}$$

Since the left hand side depends on t only, and the right hand side depends on x only, both sides must be equal to the same constant, which I will call $-\lambda$:

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda$$

From this, we get two ODEs, which are:

$$T' + \alpha^2 \lambda T = 0$$
$$X'' + \lambda X = 0$$

Also, from the first boundary condition:

$$u(-L, t) = u(L, t)$$
$$X(-L) = X(L)$$

and from the second boundary condition:

$$u_x(-L, t) = u_x(L, t)$$
$$X'(-L) = X'(L)$$

Thus we get the eigenvalue problem:

$$X'' + \lambda X = 0$$
$$X(-L) = X(L), \qquad X'(-L) = X'(L)$$

Case 1: $\lambda < 0$
Let $\lambda = -\mu^2$, where $\mu > 0$. Then our general solution is

$$X(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

and its derivative is

$$X'(x) = \mu c_1 \sinh(\mu x) + \mu c_2 \cosh(\mu x)$$

Plugging in the first boundary condition $X(L) = X(-L)$,

$$X(L) = X(-L)$$
$$c_1 \cosh(\mu L) + c_2 \sinh(\mu L) = c_1 \cosh(-\mu L) + c_2 \sinh(-\mu L)$$

Since sinh() is an odd function and cosh() is an even function, we can rearrange this as follows:

$$c_1 \cosh(\mu L) + c_2 \sinh(\mu L) = c_1 \cosh(\mu L) - c_2 \sinh(\mu L)$$
$$2c_2 \sinh(\mu L) = 0$$

We note that $\sinh(\mu L) = 0$ if and only if $\mu L = 0$, which is not the case since we have defined μ to be positive. This means $c_2 = 0$. Plugging in the second boundary condition $X'(L) = X'(-L)$,

$$X'(L) = X'(-L)$$
$$\mu c_1 \sinh(\mu L) + \mu c_2 \cosh(\mu L) = \mu c_1 \sinh(-\mu L) + \mu c_2 \cosh(-\mu L)$$

which similarly simplifies as:

$$\mu c_1 \sinh(\mu L) + \mu c_2 \cosh(\mu L) = -\mu c_1 \sinh(\mu L) + \mu c_2 \cosh(\mu L)$$
$$2c_1 \sinh(\mu L) = 0$$

which by the same reasoning as before requires that $c_1 = 0$. Therefore we have both $c_1 = c_2 = 0$, so we only get the trivial solution.

Case 2: $\lambda = 0$
For this case we have $X'' = 0$, so the general solution is:

$$X(x) = c_1 + c_2 x$$

Plugging in the first boundary condition,

$$\begin{aligned} X(L) &= X(-L) \\ c_1 + c_2 L &= c_1 - c_2 L \\ 2c_2 L &= 0 \end{aligned}$$

So $c_2 = 0$. For this case, the second boundary condition is actually always satisfied:

$$\begin{aligned} X'(L) &= X'(-L) \\ c_2 &= c_2 \end{aligned}$$

But this places no restrictions on c_1, which can be arbitrary.
Thus we have an eigenvalue of $\lambda = 0$, with eigenfunction $X_0(x) = 1$.
For this value of λ, we also get that $T' = 0$, so $T_0(t)$ is also an arbitrary constant.

Case 3: $\lambda > 0$
For this case, the general solution is:

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Taking its derivative,

$$X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Plugging in the first boundary condition,

$$\begin{aligned} X(L) &= X(-L) \\ c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) &= c_1 \cos(-\sqrt{\lambda}L) + c_2 \sin(-\sqrt{\lambda}L) \\ c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) &= c_1 \cos(\sqrt{\lambda}L) - c_2 \sin(\sqrt{\lambda}L) \\ 2c_2 \sin(\sqrt{\lambda}L) &= 0 \end{aligned}$$

And plugging in the second boundary condition,

$$\begin{aligned} X'(L) &= X'(-L) \\ -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) &= -c_1\sqrt{\lambda}\sin(-\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(-\sqrt{\lambda}L) \\ -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) &= c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) \\ 2c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) &= 0 \end{aligned}$$

Combining both of these conditions, the only we can get a nontrivial solution is if:

$$\begin{aligned} \sin(\sqrt{\lambda}L) &= 0 \\ \sqrt{\lambda}L &= n\pi, & n &= 1, 2, 3, \dots \\ \lambda &= \left(\frac{n\pi}{L}\right)^2, & n &= 1, 2, 3, \dots \end{aligned}$$

So these are our eigenvalues λ. In this case, both c_1 *and* c_2 are arbitrary, so the corresponding eigenfunctions are:

$$X_n(x) = c_1 \cos\left(\frac{n\pi}{L}x\right) + c_2 \sin\left(\frac{n\pi}{L}x\right)$$

For these eigenvalues, the ODE for $T(t)$ becomes:

$$T' + \alpha^2 \left(\frac{n\pi}{L}\right)^2 T = 0$$
$$T' = -\left(\frac{\alpha n\pi}{L}\right)^2 T$$

the solution to which is:

$$T_n(t) = e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}$$

Relabelling the constants for convenience, our fundamental solutions are:

$$u_0(x,t) = X_0(x)T_0(t) = \frac{a_0}{2}$$

and

$$u_n(x,t) = X_n(x)T_n(t) = e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

and the general solution is a linear combination of the fundamental solutions:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

Now for the initial conditions. Plugging in $t = 0$,

$$f(x) = u(x,0)$$
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

from which we see that the coefficients a_n and b_n should be the coefficients of the Fourier series expansion of $f(x)$ on the interval $[-L, L]$. Therefore the complete answer is:

$$\boxed{\begin{aligned} u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) \\ &\text{where} \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}}$$

Answer to Question 8. (a) For the method of separation of variables, we look for solutions of the form:

$$u(r\theta) = R(r)\Theta(\theta)$$

Plugging this into Laplace's equation in polar coordinates,

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

Separating variables,

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta}$$

Since the left hand side depends only on r, and the right hand side only on θ, they both must be equal to the same constant λ:

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

We can use this to get the following ODEs for $R(r)$ and $\Theta(\theta)$:

$$\Theta'' + \lambda\Theta = 0$$
$$r^2R'' + rR' - \lambda R = 0$$

In order for our solution to be well-defined in polar coordinates, we want to make sure that our solution is the same when we increase θ by 2π, as this is just traveling around a circle back to the same point. This means that $u(r, \theta)$ should be periodic in θ with period 2π.
More precisely, we need that for all angles θ:

$$\begin{aligned} u(r, \theta) &= u(r, \theta + 2\pi) \\ R(r)\Theta(\theta) &= R(r)\Theta(\theta + 2\pi) \\ \Theta(\theta) &= \Theta(\theta + 2\pi) \end{aligned}$$

This will serve in the role of boundary conditions for our ODE for $\Theta(\theta)$. In other words, we are looking for non-trivial solutions to:

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(\theta) = \Theta(\theta + 2\pi)$$

<u>For $\lambda < 0$</u>, our solutions for Θ are of the form:

$$\Theta(\theta) = c_1e^{\sqrt{\lambda}\theta} + c_2e^{-\sqrt{\lambda}\theta}$$

In order for this to be periodic,

$$\begin{aligned} \Theta(\theta) &= \Theta(\theta + 2\pi) \\ c_1e^{\sqrt{\lambda}\theta} + c_2e^{-\sqrt{\lambda}\theta} &= c_1e^{\sqrt{\lambda}(\theta+2\pi)} + c_2e^{-\sqrt{\lambda}(\theta+2\pi)} \\ c_1e^{\sqrt{\lambda}\theta} + c_2e^{-\sqrt{\lambda}\theta} &= c_1e^{\sqrt{\lambda}\theta}e^{2\pi\sqrt{\lambda}} + c_2e^{-\sqrt{\lambda}\theta}e^{-2\pi\sqrt{\lambda}} \end{aligned}$$

and matching like terms, we would need that:

$$c_1 = c_1e^{2\pi\sqrt{\lambda}} \quad \text{and} \quad c_2 = c_2e^{-2\pi\sqrt{\lambda}}$$

This only happens when $c_1 = c_2 = 0$, which is the trivial solution.

For $\lambda = 0$, our solutions for Θ are of the form:

$$\Theta(\theta) = c_1 + c_2\theta$$

In order for this to be periodic,

$$\begin{aligned}
\Theta(\theta) &= \Theta(\theta + 2\pi) \\
c_1 + c_2\theta &= c_1 + c_2(\theta + 2\pi) \\
c_1 + c_2\theta &= c_1 + c_2(2\pi) + c_2\theta \\
0 &= c_2(2\pi) \\
c_2 &= 0
\end{aligned}$$

So $c_2 = 0$, but c_1 could be any constant. This means that $\Theta_0(\theta) = c_1$ works for any constant c_1. When $\lambda = 0$, the equation for R is:

$$r^2R'' + rR' = 0$$

This is an Euler equation, so looking for solutions of the form $R = r^m$,

$$\begin{aligned}
m(m-1) + m &= 0 \\
m^2 &= 0
\end{aligned}$$

So we have a repeated root at $m = 0$. This corresponds to a solution of:

$$R_0(r) = c_1 + c_2\ln(r)$$

Since the natural logarithm is not defined at the origin $r = 0$, if we want our solution u to be defined at $r = 0$, we need to set $c_2 = 0$, leaving:

$$R_0(r) = c_1$$

So for $\lambda = 0$, we have that R can also be any constant. Thus we have an eigenvalue-eigenfunction pair of:

$$\lambda = 0, \qquad u_0(r,\theta) = \frac{c_0}{2}$$

where c_0 could be any constant (I'm writing it this way since it will end up being useful later). For $\lambda > 0$, our solutions for Θ are of the form:

$$\Theta(\theta) = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$$

In order for this to be periodic,

$$\begin{aligned}
\Theta(\theta) &= \Theta(\theta + 2\pi) \\
A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right) &= A\cos\left(\sqrt{\lambda}(\theta + 2\pi)\right) + B\sin\left(\sqrt{\lambda}(\theta + 2\pi)\right)
\end{aligned}$$

Using the angle addition trig identities, the right hand side becomes:

$$= A\left[\cos\left(\sqrt{\lambda}\theta\right)\cos\left(2\pi\sqrt{\lambda}\right) - \sin\left(\sqrt{\lambda}\theta\right)\sin\left(2\pi\sqrt{\lambda}\right)\right] + B\left[\sin\left(\sqrt{\lambda}\theta\right)\cos\left(2\pi\sqrt{\lambda}\right) + \sin\left(2\pi\sqrt{\lambda}\right)\cos\left(\sqrt{\lambda}\theta\right)\right]$$

Grouping together the terms by θ,

$$= \left[A\cos\left(2\pi\sqrt{\lambda}\right) + B\sin\left(2\pi\sqrt{\lambda}\right)\right]\cos\left(\sqrt{\lambda}\theta\right) + \left[B\cos\left(2\pi\sqrt{\lambda}\right) - A\sin\left(2\pi\sqrt{\lambda}\right)\right]\sin\left(\sqrt{\lambda}\theta\right)$$

Comparing like terms, A and B should satisfy the equations:

$$\begin{aligned} A &= A\cos\left(2\pi\sqrt{\lambda}\right) + B\sin\left(2\pi\sqrt{\lambda}\right) \\ B &= B\cos\left(2\pi\sqrt{\lambda}\right) - A\sin\left(2\pi\sqrt{\lambda}\right) \end{aligned}$$

This is always true for $A = B = 0$, but this corresponds to the trivial solution. However, for values of λ where the cosine term above is 1 and the sine term is 0, this would be true for any A and B. More precisely, we want the values of λ where:

$$\begin{aligned} &\cos\left(2\pi\sqrt{\lambda}\right) = 1 && \Longrightarrow && 2\pi\sqrt{\lambda} = 2\pi n, \qquad n = 1, 2, 3... \\ &\sin\left(2\pi\sqrt{\lambda}\right) = 0 && \Longrightarrow && 2\pi\sqrt{\lambda} = \pi n, \qquad n = 1, 2, 3... \end{aligned}$$

Since we want *both* to be true, we take the more restrictive condition that:

$$\begin{aligned} 2\pi\sqrt{\lambda} &= 2\pi n, && n = 1, 2, 3... \\ \sqrt{\lambda} &= n, && n = 1, 2, 3... \\ \lambda &= n^2, && n = 1, 2, 3... \end{aligned}$$

So these are our eigenvalues, and they have corresponding eigenfunctions:

$$\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta), \qquad n = 1, 2, 3...$$

Now we want to solve for $R(r)$ at these eigenvalues $\left(\lambda = n^2\right)$. We get an Euler equation:

$$r^2 R'' + rR' - n^2 R = 0$$

Looking for solutions of the form $R = r^m$, we plug this in and solve for m:

$$\begin{aligned} m(m-1) + m - n^2 &= 0 \\ m^2 - n^2 &= 0 \\ m^2 &= n^2 \\ m &= \pm n \end{aligned}$$

So $R(r)$ looks like:

$$R_n(r) = Ar^n + Br^{-n}$$

However, because we want R to be defined at $r = 0$, we then take $B = 0$, leaving:

$$R_n(r) = Ar^n$$

Putting this all together (and renaming some constants), our general solution is of the form:

$$\begin{aligned} u(r,\theta) &= \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) \\ u(r,\theta) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left[A_n\cos(n\theta) + B_n\sin(n\theta)\right] \end{aligned}$$

Now we still have to apply the Neumann boundary conditions. Taking the partial derivative with respect to r,

$$u_r(r,\theta) = \sum_{n=1}^{\infty} nr^{n-1}\big[A_n\cos(n\theta) + B_n\sin(n\theta)\big]$$

Plugging in $r = a$, and setting it equal to $f(\theta)$

$$u_r(a,\theta) = \sum_{n=1}^{\infty} na^{n-1}\big[A_n\cos(n\theta) + B_n\sin(n\theta)\big] = f(\theta), \qquad 0 \le \theta < 2\pi$$

This means that we want $u_r(a,\theta)$ to match the Fourier series for $f(\theta)$ on the interval $[0, 2\pi]$. Using the formula for Fourier series coefficients, we get the following equations for A_n and B_n:

$$\begin{aligned} na^{n-1}A_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \\ na^{n-1}B_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \end{aligned}$$

So our solution to the problem is:

$$\boxed{u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n\big[A_n\cos(n\theta) + B_n\sin(n\theta)\big]}$$

where the coefficients are given by:

$$\boxed{\begin{aligned} A_n &= \frac{1}{na^{n-1}\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \\ B_n &= \frac{1}{na^{n-1}\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \\ c_0 &= \text{ any constant} \end{aligned}}$$

(b) In our solution to part **(a)** , when we were enforcing the boundary conditions on u_r, we got to an equation of the form:

$$u_r(a,\theta) = \sum_{n=1}^{\infty} na^{n-1}\big[A_n\cos(n\theta) + B_n\sin(n\theta)\big] = f(\theta), \qquad 0 \le \theta < 2\pi$$

and then went on to take a Fourier series expansion of $f(\theta)$. However, Fourier series usually have a constant term $\frac{c_0}{2}$. While our formula for $u(r,\theta)$ had this, when we took the derivative, this then makes sure that $u_r(r,\theta)$ does *not* have a constant term.
So for this problem to be solvable, we would need the $\frac{c_0}{2}$ constant term of the Fourier series for $f(\theta)$ to be zero. Rephrasing this in terms of a condition on f,

$$\boxed{\frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta = 0}$$