Math 2940 Worksheet

Bases, Coordinate Systems

Week 7

October 10th, 2019

This worksheet covers material from **Sections 4.3 and 4.4**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$. Then every vector in H is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Is $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ a basis for H?

Question 2. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\boldsymbol{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$.

Question 3. Let V and W be vector spaces, let $T : V \to W$ and $U : V \to W$ be linear transformations, and let $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ be a basis for V. If $T(\boldsymbol{v}_j) = U(\boldsymbol{v}_j)$ for every value of j between 1 and p, show that $T(\boldsymbol{x}) = U(\boldsymbol{x})$ for every vector $\boldsymbol{x}$ in V.

Question 4. Mark each statement True or False, and give a brief explanation why.

(a) If $H = \text{Span}\{\boldsymbol{b}_1, ..., \boldsymbol{b}_p\}$, then $\{\boldsymbol{b}_1, ..., \boldsymbol{b}_p\}$ is a basis for H.

(b) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$

(c) A basis is a spanning set that is as large as possible.

(d) A linearly independent set in a subspace H is a basis for H.

(e) If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

(f) A basis is a linearly independent set that is as large as possible

Question 5. Let $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\boldsymbol{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\boldsymbol{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

(a) Show that the set $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is a basis of $\mathbb{R}^3$.

(b) Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.

(c) Write the equation that relates $\boldsymbol{x} \in \mathbb{R}^3$ to $[\boldsymbol{x}]_{\mathcal{B}}$ (the coordinates of $\boldsymbol{x}$ in terms of the basis $\mathcal{B}$.)

(d) Find $[\boldsymbol{x}]_{\mathcal{B}}$, for the $\boldsymbol{x}$ given above.

Question 6. A matrix A is said to be *symmetric* if $A = A^T$. A certain class of symmetric matrices (known as positive definite matrices) can be factored as $A = LL^T$, where L is a lower triangular matrix. This factorization is called a *Cholesky decomposition*.

(a) In terms of storage, argue why a Cholesky decomposition is better than a LU decomposition. How does this coincide with the symmetry of the matrix A?

(b) If a matrix A has a Cholesky decomposition, what can you say about the diagonal elements of A?

(c) Try to compute a Cholesky decomposition of the matrix $A = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$ directly by writing out the equations for the elements of the lower triangular matrix L. Can you extend this method to larger matrices?

Answer to Question 1. Even though $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly independent and its span contains H, $\boxed{\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \text{ is not a basis for } H}$. This is because $\text{Span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$ is actually too large.

The requirement for a basis would be that $\text{Span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\}) = H$, but these two sets are not equal because the span contains elements that are not in H.

For a specific example, $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \text{Span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$ from the definition of span, but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin H$, because $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is *not* of the form $\begin{bmatrix} s \\ s \\ 0 \end{bmatrix}$.

Answer to Question 2. Since W is a subspace of a three-dimensional space ($\mathbb{R}^3$), and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ has four elements, this is a linearly dependent set. Therefore to find a basis for W, we will want to get rid of any "redundant" vectors. We can do that by creating a matrix with the $\boldsymbol{v}_i$ vectors as its columns, and row reducing.

$$\begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we only have pivots in the first and second columns, this means that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ form a basis for this subspace.

$$\boxed{\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ is a basis of } W}$$

Answer to Question 3. With linear transformations, we only need to know what they do to the basis vectors to determine how they act on every vector. Given a generic vector $\boldsymbol{x}$ in V, we can write it as a linear combination of the basis vectors:

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_p\boldsymbol{v}_p$$

Then we can use the properties of linear transformations to compute

$$\begin{aligned} T(\boldsymbol{x}) &= T\big(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_p\boldsymbol{v}_p\big) \\ &= c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + ... + c_pT(\boldsymbol{v}_p) \end{aligned}$$

Since $T(\boldsymbol{v}_j) = U(\boldsymbol{v}_j)$ for every basis vector $\boldsymbol{v}_j$,

$$\begin{aligned} T(\boldsymbol{x}) &= c_1 U(\boldsymbol{v}_1) + c_2 U(\boldsymbol{v}_2) + ... + c_p U(\boldsymbol{v}_p) \\ &= U\big(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_p\boldsymbol{v}_p\big) \\ &= U(\boldsymbol{x}) \end{aligned}$$

so $\boxed{T(\boldsymbol{x}) = U(\boldsymbol{x})}$ as desired.

Answer to Question 4.

(a) $\boxed{\text{False.}}$ The set $\{\boldsymbol{b}_1, \boldsymbol{b}_2, ...\boldsymbol{b}_p\}$ might not be linearly independent. For an example, consider:

$$H = \mathbb{R}^2, \qquad \boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \boldsymbol{b}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $\boxed{\text{True.}}$ Because an invertible function is one-to-one, the columns are linearly independent, and because an invertible function is onto, the columns must span $\mathbb{R}^n$. This emans the columns satisfy both requirements for being a basis.

(c) $\boxed{\text{False.}}$ A basis must also be linearly independent. (Adding more vectors will only prevent the set from being linearly independent.)

(d) $\boxed{\text{False.}}$ The linearly independent subset also needs to span H.

(e) $\boxed{\text{True.}}$ As in Question 2, we can keep removing linearly dependent vectors from S until the set is linearly independent. The result will be a basis for V.

(f) $\boxed{\text{True.}}$ A basis needs to be a linearly independent set, and if a linearly independent set is as large as possible, it will also span the vector space.

Answer to Question 5.

(a) To show that the set $\mathcal{B}$ is a basis, we need to show that they form the columns of an invertible matrix. This is equivalent to checking that this matrix row reduces to the identity.

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $\boxed{\text{the set } \mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\} \text{ is a basis of } \mathbb{R}^3.}$

(b) To find the change-of-coordinates matrix, we want a linear transformation T that takes each $\boldsymbol{b}_i$ in the $\mathcal{B}$-coordinates to $\boldsymbol{b}_i$ in the standard coordinates. This means:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-3\\4\\0\end{bmatrix},\qquad \text{and}\qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-8\\2\\3\end{bmatrix}$$

The corresponding matrix is:

$$\boxed{\begin{bmatrix}1&-3&3\\0&4&-6\\0&0&3\end{bmatrix}}$$

(c) For this case, we can think of this as finding a vector $[\boldsymbol{x}]_{\mathcal{B}}$ such that when we apply our change of coordinates transformation T, we get out $\boldsymbol{x}$. In mathematical terms,

$$T\left([\boldsymbol{x}]_{\mathcal{B}}\right)=\boldsymbol{x}$$

and using our matrix from part (b) along with $\boldsymbol{x}=\begin{bmatrix}-8\\2\\3\end{bmatrix}$,

$$\boxed{\begin{bmatrix}1&-3&3\\0&4&-6\\0&0&3\end{bmatrix}[\boldsymbol{x}]_{\mathcal{B}}=\begin{bmatrix}-8\\2\\3\end{bmatrix}}$$

(d) Solving our equations from part (c) by row reducing an augmented matrix,

$$\left[\begin{array}{ccc|c}1&-3&3&-8\\0&4&-6&2\\0&0&3&3\end{array}\right]$$
$$\left[\begin{array}{ccc|c}1&-3&3&-8\\0&4&-6&2\\0&0&1&1\end{array}\right]$$
$$\left[\begin{array}{ccc|c}1&-3&0&-11\\0&4&0&8\\0&0&1&1\end{array}\right]$$
$$\left[\begin{array}{ccc|c}1&-3&0&-11\\0&1&0&2\\0&0&1&1\end{array}\right]$$
$$\left[\begin{array}{ccc|c}1&0&0&-5\\0&1&0&2\\0&0&1&1\end{array}\right]$$

so the solution is

$$\boxed{[\boldsymbol{x}]_{\mathcal{B}}=\begin{bmatrix}-5\\2\\1\end{bmatrix}}$$

Answer to Question 6.

(a) A Cholesky decomposition is better from a storage perspective than an LU decomposition, because it requires storing only one triangular matrix L, instead of two triangular matrices U and L.

For a symmetric matrix A, if I know what all the entries on and below the main diagonal are, then I also know what all the entries above the main diagonal are (and vice versa). So it isn't surprising that our factorization only requires half as much storage.

(b) If A has a Cholesky decomposition, then the i-th diagonal element can be computed as the dot product of the i-th column of L (which I will call L_i), and the i-th row of L^T.

Because of how the transpose works, this is just a dot product of the vector L_i with itself:

$$A_{i,i} = L_i \cdot L_i$$

This guarantees that $\boxed{A_{i,i} \geq 0}$, since this dot product will be a sum of squares.

(c) We can write the lower triangular matrix L as:

$$L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$$

Then we can compute A as:

$$\begin{aligned} A = LL^T &= \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \\ \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix} &= \begin{bmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{bmatrix} \end{aligned}$$

This gives us a system of three nonlinear equations:

$$\begin{aligned} a^2 &= 4 \\ ab &= 6 \\ b^2 + c^2 &= 10 \end{aligned}$$

Now, we can find a solution to this system by solving for possible values of a, then b, then c:

$$\begin{aligned} a^2 = 4 &\quad \Longrightarrow \quad a = 2 \\ (2)b = 6 &\quad \Longrightarrow \quad b = 3 \\ 3^2 + c^2 = 10 &\quad \Longrightarrow \quad c = 1 \end{aligned}$$

So the Cholesky decomposition is:

$$\boxed{A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}}$$

In general, we can write down a system of nonlinear equations for the entries in L, and then solve them by starting at the top right, finding the entries in the row from left to right, and then repeating for the next row.