Hopenhayn Model of Firm Heterogeneity

Hopenhayn, H. 1992. "Entry, Exit, and Firm Dynamics in Long Run Equilibrium", *Econometrica*, 60, p. 1127-50. (partial equilibrium, discrete choice problem to derive firm size distributions).

1 Environment

- Population: Industry composed of a continuum of firms (mass not necessarily 1) which produce a homogeneous product.

- Firms behave competitively, taking prices in the output (p_t) and labor input (w_t) markets as given.

- Aggregate Demand given by the *inverse demand* function $P(Q_t)$. <u>Assumption A.1a</u>: P is continuous, strictly decreasing, and $\lim_{Q \to \infty} P(Q) = 0$.

  - For example, let $P = \frac{1}{d+Q}$ where $d = 0.1$.

- <u>Assumption A.1.b</u>: The input price $W(N_t)$, where N_t is total industry labor demand, is assumed to satisfy properties: W is continuous, nondecreasing and strictly bounded above zero.

  - For example, let $w = 1$.

- Technology: output of a given firm is $q_t = f(\varphi_t, n_t)$ where $\varphi_t \in [0,1]$ is a productivity shock which follows a Markov process, independent across firms with conditional distribution $F(\varphi_t | \varphi_{t-1})$.

  - For example, let $f(\varphi, n) = \varphi n^{\alpha}$ where $\alpha = \frac{1}{2}$. It is important that the production function exhibits decreasing returns to scale.

- A fixed cost $w_t c_f$ must be paid every period by an incumbent firm (for exit to take place). Here we will assume that the cost is associated with management.

- Profit function given by $\pi(\varphi_t, p_t, w_t) = max_n p_t q_t - w_t n_t - w_t c_f$.

- Firms discount profits at rate $0 < \beta < 1$.

  - For example, $\beta = .8$ (time period is 5 years at 4%).

- <u>Assumption A.2.</u>(a) q and n are single valued, increasing in φ, and continuous; (b) π is continuous and strictly increasing in φ; (c) $\lim_{Q \to 0} \pi(\varepsilon, P(Q), w) > 0$.

- Assumption A.3(a) F is continuous in φ_{t-1} and φ_t, (b) F is strictly decreasing in φ_{t-1}. (basically just persistence).

- Assumption A.4 (Recurrence) For any $\varepsilon > 0$, there exists an integer k such that $F^k(\cdot|\varphi)$ gives the distribution of φ_{t+k} given $\varphi_{t-1} = \varphi$ (a mixing condition).

  - For example, suppose there are two states. $\varphi \in \{\varphi_L, \varphi_H\}$ with $1 = \varphi_H > \varphi_L > 0$. $F(\varphi' = \varphi_{j\neq i}|\varphi = \varphi_i) = 1 - \theta < \frac{1}{2}$, $i, j \in \{H, L\}$. Thus, the process displays persistence since $\theta > \frac{1}{2}$. We take $\theta = 0.9$.
  - New draws:

- A.2 and A.3 imply expected discounted profits are increasing in φ. Furthermore, if your profits are high (low) today, they are expected to stay high (low) in the near future. This is important for the exit decision.

- Each period, before φ_t is realized, incumbent firms may exit. An exiting firm receives 0 present value.

- In equilibrium, firms exit whenever their state φ_{t-1} falls below a reservation level x_{t-1}. Given this cutoff point, A.4 implies the life span of a firm is almost surely finite (i.e. there is a steady stream of exit).

- An entering firm must pay a fixed cost $w_t c_e$. Then it draws a productivity shock φ_t from initial density function η. For example, let $\Pr\{\varphi = \varphi_H\} = \eta = \frac{1}{2}$.

- Assumption A.5 η has a continuous distribution function G.

- Summary of timing: Incumbent

1. enter period t with last period's shock φ_{t-1}.

2. if $\varphi_{t-1} < x_{t-1}$, exit (to avoid future low productivity)

3. receive this period's shock φ_t from $F(\varphi_t|\varphi_{t-1})$

4. choose labor demand $n_t = g(\varphi_t; p, w)$ and pay fixed cost wc_f

5. enter period $t + 1$ with φ_t

- Summary of timing: Potential Entrant

1. Decide whether to pay fixed cost wc_e and receive this period's shock φ_t from $\eta(\varphi_t)$

2. Same as 4-5 above[1]

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[1] Note that under this timing an entrant may immediately exit (which is actually consistent with (15) below.

- Let the measure $\mu_t(A)$ summarize the number of firms that have $\varphi_t \in A \subset S$. Due to entry and exit, total mass of firms $\mu_t(S)$ is not necessarily 1.

- Notice that under the timing above, even if firms with $\varphi_{t-1} \leq x_{t-1}$ are exiting, the distribution $\mu_t(\varphi_t)$ includes new entrants who may draw low φ_t from the distribution function $G(\varphi_t)$.

- Aggregate output supply and input demand given by

$$Q_t^s(\mu_t; p, w) = \int q_t(\varphi_t; p, w)\mu_t(d\varphi_t) \tag{1}$$

$$N_t^d(\mu_t; p, w) = \int n_t(\varphi_t; p, w)\mu_t(d\varphi_t) \tag{2}$$

2 Equilibrium

- Given that all the uncertainty is idiosyncratic and there is a continuum of firms, we will be considering stationary equilibria where $\{p_t, w_t\} = \{p, w\}$.

- The problem of an incumbent firm at stage 4 in the above timing is really just a decision on exit next period (note that the static labor demand problem has already been solved implicitly in the profit function):

$$v(\varphi; p, w) = \pi(\varphi, p, w) + \beta \max\{0, E[v(\varphi'; p, w)|\varphi]\} \tag{3}$$

where $E[g(\varphi')|\varphi] = \int_{\varphi' \in [0,1]} g(\varphi')F(d\varphi'|\varphi)$.[2]

- The solution to problem (3) is a cutoff rule. Let

$$x = \inf\{\varphi \in S : E[v(\varphi'; p, w)|\varphi] \geq 0\} \tag{4}$$

Thus any firm with $\varphi < x$ will exit.

- Prop. 1. v is a bounded, continuous, and strictly increasing function of φ. Note: present discounted profits are increasing in productivity follows from properties of π and F.

- Since all firms are ex-ante identical, entry occurs provided

$$\int v(\varphi; p, w)\eta(d\varphi) \geq wc_e.$$

- Let M_t denote the mass of entrants in period t. An equilibrium with free entry requires

$$\int v(\varphi; p, w)\eta(d\varphi) \leq wc_e, \tag{5}$$

with equality if $M_t > 0$ (in which case we are free to "choose" any measure M_t we want).

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[2] In Hopenhayn's notation, $F(d\varphi'|\varphi)$ stands for the densitiy function for next period's shock φ' conditional on φ.

- The entry and exit rules imply an evolution for the state of the industry

$$\mu_{t+1}([0,\overline{\varphi}']) = \int_{\varphi' \in [0,\overline{\varphi}']} \int_{\varphi \geq x} dF(\varphi'|\varphi)\mu_t(d\varphi)d\varphi' + M_{t+1}G(\overline{\varphi}') \qquad (6)$$

Note that the first term considers only those firms which do not exit.

- Could write this in terms of Transition function

$$T(\varphi, A) = \left\{ \begin{array}{cc} \int_{s' \in A} dF(s'|\varphi)ds' & \text{if } \varphi \geq x \\ 0 & \text{otherwise} \end{array} \right.$$

so that (6) given by

$$\mu_{t+1} = T\mu_t + M_{t+1}\eta \qquad (7)$$

- *Importantly*, it is simple to see that (7) μ_{t+1} is homogeneous of degree 1 in μ_t and M_{t+1}.

Definition 1 *A **stationary competitive industry equilibrium** is a list* $\{p^*, w^*, Q^*, N^*, M^*, x^*, \mu^*\}$ *such that*

1. $p^* = D(Q^*)$, $w^* = W(N^*)$, $Q^* = Q^s(\mu^*, p^*, w^*)$, $N^* = N^d(\mu^*, p^*, w^*)$ (markets clear)

2. x^* satisfies (4) (optimal exit)

3. (5) satisfied at p^* and w^* with equality if $M^* > 0$ (free entry)

4. $\mu_t = \mu^*$ solves (7)

- To prove existence, find a fixed point of $\mu \to (p(\mu), w(\mu)) \to (v, g) \to \mu$.

- Theorem 2. A SS equilibrium exists (there are many possible equilibria). e.g. no trade is always an equilibrium.

- Theorem 3. A SS equilibrium with entry/exit exists iff entry costs are sufficiently low.

- Theorem 4. Conditions for uniqueness include production functions of the form $f(\varphi, n) = \varphi n^\alpha$, $\alpha \in (0, 1)$. This is condition U.2 on p. 1137.

- An important step in Theorem 2 is existence of a unique invariant measure μ^* which follows from the recurrence or mixing condition of Ass4.

3 Example

- Given only two states for the productivity shock, the example is not rich enough to generate a size distribution over firms. However, it does illustrate entry and exit.
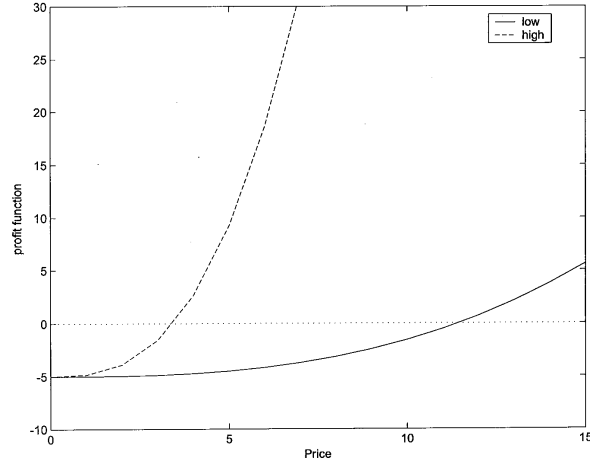
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- Firm's static optimization problem:

$$P\alpha\varphi_i n^{\alpha-1} = 1 \Longleftrightarrow n_i^* = (P\alpha\varphi_i)^{1/(1-\alpha)}, i \in \{H, L\}$$

In this specific case, $n_i^* = \left(\frac{P\varphi_i}{2}\right)^2$. Hence firm size, measured by number of workers, is increasing in productivity.
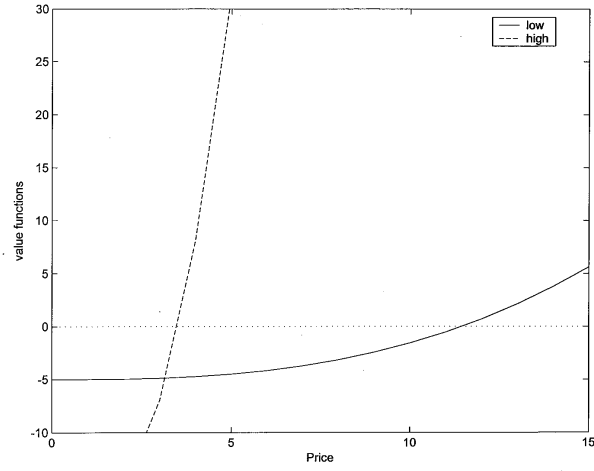
- Thus

$$\begin{aligned} \pi(\varphi_i, P, 1) &= P\varphi_i (n_i^*)^\alpha - n_i^* - c_f \\ &= (P\alpha\varphi_i)^{1/(1-\alpha)} \left[\frac{1}{\alpha} - 1\right] - c_f \end{aligned}$$

Note that with $\alpha = 1/2$, the static profit function is increasing and convex in price (as in Varian) and productivity.

- The two steady state value functions indexed by φ_i are

$$v(\varphi_i; P) = \frac{(P\varphi_i)^2}{4} - c_f + \beta \max\left\{0, \theta v(\varphi_i; P) + (1-\theta)v(\varphi_{j\neq i}; P)\right\}$$

- A firm will exit if $\theta v(\varphi_i; P) + (1 - \theta) v(\varphi_{j \neq i}; P) < 0$.
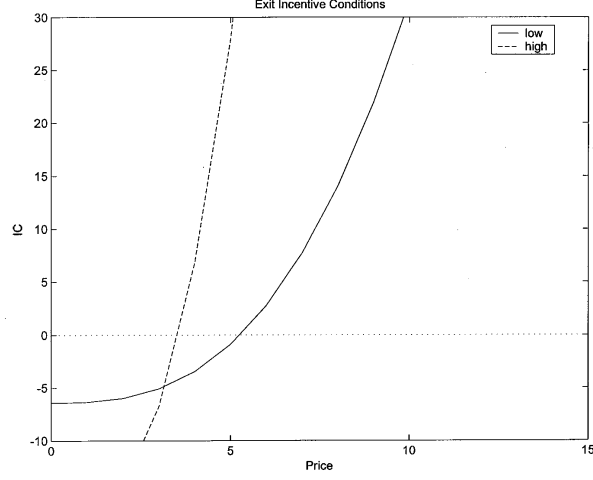
Figure 3

- Here we will "construct" an equilibrium where if a firm receives a low shock, it will exit if a one shot deviation is suboptimal or

$$0 > \theta v(\varphi_L; P) + (1 - \theta)v(1; P) \tag{8}$$

but if it receives a high shock, it will not exit if a one shot deviation is suboptimal or

$$\theta v(1; P) + (1 - \theta)v(\varphi_L; P) > 0. \tag{9}$$

It may be possible to satisfy these one shot deviation conditions ((8) and (9)) since $v(1; P) > v(\varphi_L; P)$ and $\theta > \frac{1}{2}$.

- Conjecture this exit strategy is optimal and then check under what conditions on parameters it is optimal. Under the construction,

$$v(\varphi_L; P) = \frac{(P\varphi_L)^2}{4} - c_f + \beta \cdot 0 \tag{10}$$

$$v(1; P) = \frac{P^2}{4} - c_f + \beta \left\{ \theta v(1; P) + (1 - \theta)v(\varphi_L; P) \right\} \tag{11}$$

That is, 2 linear equations in 2 unknowns.

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- Since we have exit, in order to ensure a stationary equilibrium we need positive entry. If

$$\eta v(1, P) + (1 - \eta)v(\varphi_L, P) = c_e \qquad (12)$$

then since firms are indifferent between entering and staying out of the market, we are free to choose how many firms enter (i.e. M). From (10) and (11), it is clear that (12) is one equation in one unknown P, so that entry pins down prices. Then what pins down the amount of entry?
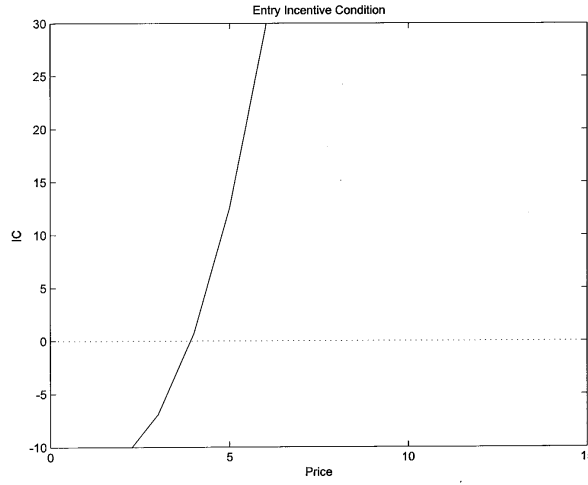

Figure 4

- Under our conjecture, this entry condition can be written

$$\eta \left[\frac{\frac{(P^*)^2}{4} \left[1 + \beta(1 - \theta)\varphi_L^2\right] - c_f \left[1 + \beta(1 - \theta)\right]}{(1 - \beta\theta)} \right] + (1 - \eta) \left[\frac{(P^* \varphi_L)^2}{4} - c_f \right] = c_e \qquad (13)$$

which is one equation in one unknown P. The lhs is continuously increasing in P,is negative if $P = 0$, is ∞ as $P \to \infty$ and is independent of c_e. Thus a solution P^* exists. Note from the previous figure on exit incentive conditions that a necessary condition for the conjectured equilibrium to exist is that P^* lies somewhere between 4 and 5. We can choose c_e to

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make that happen. Another way to think of this is that if we have data on the equilibrium price in the market P^*, this can help us identify c_e which we may not have data on.

- Now we need to pin down the law of motion for the distribution (which will give us Q) and ultimately M. Abusing notation somewhat (i.e. writing the distribution as a pdf rather than a cdf), we have

$$
\begin{aligned}
\mu_{t+1}(1) &= \mu_t(1)\theta + M_{t+1}\eta \\
\mu_{t+1}(\varphi_L) &= \mu_t(1)(1-\theta) + M_{t+1}(1-\eta)
\end{aligned}
\tag{14}
$$

That is, the mass of firms at the beginning of next period at the high productivity level are incumbents who remain high and new entrants who receive a high productivity level. Further, the mass of firms at the beginning of next period (before exit) at the low productivity level are incumbents who were high but received a low productivity draw and new entrants who receive a low productivity draw.

- From (14), we can solve for the invariant distribution

$$
\begin{aligned}
\mu(1) &= \frac{\eta M}{(1-\theta)} \\
\mu(\varphi_L) &= \eta M + M(1-\eta) = M
\end{aligned}
\tag{15}
$$

which makes clear the dependence of the distribution on the mass of new entrants M and θ through x (so that the notation $m(x, M)$ makes sense). Note that under the conjecture that all firms with the low productivity $\mu(\varphi_L)$ exit, then we will need to have entry by M equal that measure if there is to be a stationary distribution (i.e. $\mu(\varphi_L) = M$ is rationalized by the fact that exit has to equal entry).

  - Note further from (15) it is easy to see the homogeneity property. If M doubles, so does μ.

- Output given by

$$
Q^s = \mu(1)\left(n_H^*\right)^{\alpha} + \mu(\varphi_L)\varphi_L\left(n_L^*\right)^{\alpha}
\tag{16}
$$

That is there are some incumbent firms and new entrants who produce in a given period t before they exit in $t+1$ (if they receive a bad shock in t).

- In that case, goods market clearing implies $Q^s = Q$ from (16) and the inverse demand function

$$
P = 1/(d + Q^s)
\tag{17}
$$

implies 2 equations (13), (17) in 2 unknowns P and M. Note that once we know P^* from (13) and substitute it in n^*, then the rhs of (17) only depends on M and the lhs is a number, so we can use it to determine M^*.

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- Finally, we need to verify that at P^*, the one shot deviation conditions (8) and (9) are satisfied (which they are).

- In summary, it may seem counterintuitive, but the free entry condition effectively pins down prices (P) and the market clearing condition pins down the mass of new entrants (M).