

Econ 710 Quarter 1: Problem set 1

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Exercise 1

Suppose $(Y, X')'$ is a random vector with

$$Y = X'\beta_0 \cdot U$$

where $\mathbb{E}[U|X] = 1$, $\mathbb{E}[XX']$ is invertible, and $\mathbb{E}[Y^2 + \|X\|^2] < \infty$. Furthermore, suppose that $\{Y_i, X'_i\}_{i=1}^\infty$ is a random sample from the distribution of $(Y, X')'$, where $\frac{1}{n} \sum_{i=1}^n X_i X'_i$ is invertible and let $\hat{\beta}$ be the OLS estimator:

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^n X_i X'_i \right)^{-1} \frac{1}{n} \sum_{i=1}^n X_i Y_i$$

- (i) **Interpret the entries of β_0 in this model.** Here, when we take the derivative with respect to X , we will get a value/slope that is affected by the unobservables as well. Therefore β_0 is the value of the slope if it were unaffected by unobservable factors. **there's probably a better way to say this**
- (ii) **Show that $Y = X'\beta_0 + \tilde{U}$ where $\mathbb{E}[\tilde{U}|X] = 0$.**

$$\begin{aligned} Y &= X'\beta_0 + \tilde{U} \\ X'\beta_0 \cdot U &= X'\beta_0 + \tilde{U} \\ \iff \mathbb{E}[\mathbb{E}[X'\beta_0 \cdot U|X]] &= \mathbb{E}[\mathbb{E}[X'\beta_0 + \tilde{U}|X]] \\ \mathbb{E}[X'\beta_0] &= \mathbb{E}[X'\beta_0 + \mathbb{E}[\tilde{U}|X]] \\ \mathbb{E}[X'\beta_0] &= \mathbb{E}[X'\beta_0] \end{aligned}$$

incorrect use of expectations i think

- (iii) **Show that $\mathbb{E}[X(Y - X'\beta)] = 0$ if and only if $\beta = \beta_0$ and use this to derive OLS as a method of moments estimator.**

I worked on this Problem set with Sarah Bass, Michael Nattinger, Alex von Hafften, and Danny Edgel.

First, let $\beta = \beta_0$. Then

$$\begin{aligned}
\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[X(Y - X'\beta_0)] \\
&= \mathbb{E}[XY - XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0 \cdot U] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[\mathbb{E}[XX'\beta_0 \cdot U|X]] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0 \mathbb{E}[U|X]] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0] - \mathbb{E}[XX'\beta_0] \\
&= 0
\end{aligned}$$

Now suppose that $\mathbb{E}[X(Y - X'\beta)] = 0$. Then

$$\begin{aligned}
0 &= \mathbb{E}[XY] - \mathbb{E}[XX'\beta] \\
&= \mathbb{E}[X(X'\beta_0 + \tilde{U})] - \mathbb{E}[XX'\beta] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] + \mathbb{E}[X\tilde{U}] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] + \mathbb{E}[\mathbb{E}[X\tilde{U}|X]] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] \\
&\iff \beta = \beta_0
\end{aligned}$$

unsure about if the tilde-U assumption from ii carries over to this. unsure about moments estimator.

(iv) Show that the OLS estimator is conditionally unbiased.

(v) Show that the OLS estimator is consistent. First note that

$$\begin{aligned}
\hat{\beta} &= \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^n X_i Y_i \\
&= \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \beta_0 + X_i \tilde{U}_i \right) \\
&= \beta_0 + \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^n X_i \tilde{U}_i
\end{aligned}$$

Which converges **in distribution** to $\beta_0 + \mathbb{E}[X_i X_i']^{-1} \mathbb{E}[X_i \tilde{U}_i]$. Now, because $\mathbb{E}[\tilde{U}|X] = 0$,

$$\begin{aligned}
\hat{\beta} &\xrightarrow{p} \beta_0 + \mathbb{E}[X_i X_i']^{-1} \mathbb{E}[X_i \tilde{U}_i] \\
&= \beta_0
\end{aligned}$$

definitely missing things on these last two parts

Exercise 2

Let X be a random variable with $\mathbb{E}[X^4] < \infty$ and $\mathbb{E}[X^2] > 0$. Furthermore, let $\{X_i\}_{i=1}^n$ be a random sample from the distribution of X .

- (i) Which of the following four statistics can you use the law of large numbers and continuous mapping theorem to show convergence in probability as $n \rightarrow \infty$?

(a)

$$\frac{1}{n} \sum_{i=1}^n X_i^3 \rightarrow^p \mathbb{E}[X_i^3]$$

(b)

$$\max_{1 \leq i \leq n} X_i$$

cannot use CMT?

(c)

$$\frac{\sum_{i=1}^n X_i^3}{\sum_{i=1}^n X_i^2}$$

(d)

$$1 \left\{ \frac{1}{n} \sum_{i=1}^n X_i > 0 \right\}$$