Econ 710 Quarter 1: Problem set 1

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Exercise 1

Suppose $(Y, X')'$ is a random vector with

$$Y = X'\beta_0 \cdot U$$

where $\mathbb{E}[U|X] = 1$, $\mathbb{E}[XX']$ is invertible, and $\mathbb{E}[Y^2 + ||X||^2] < \infty$. Furthermore, suppose that $\{Y_i, X_i'\}_{i=1}^{\infty}$ is a random sample from the distribution of $(Y, X')'$, where $\frac{1}{n}\sum\limits_{i=1}^{n} X_i X_i'$ is invertible and let $\hat{\beta}$ be the OLS estimator:

$$\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$$

(i) **Interpret the entries of β_0 in this model.** Here, when we take the derivative with respect to X, we will get a value/slope that is affected by the unobservables as well. Therefore β_0 is the value of the slope if it were unaffected by unobservable factors. **there's probably a better way to say this**

(ii) **Show that $Y = X'\beta_0 + \tilde{U}$ where $\mathbb{E}[\tilde{U}|X] = 0$.**
Notice that $\tilde{U} = Y - X'\beta_0 = X'\beta_0 \cdot U - X'\beta_0 = X'\beta_0(U-1)$. So, when $\tilde{U} = X'\beta_0(U-1)$, the statement is true. Also,

$$\begin{aligned}
\mathbb{E}[\tilde{U}|X] &= \mathbb{E}[X'\beta_0(U-1)|X] \\
&= X'\beta_0\mathbb{E}[U|X] - X'\beta_0 \\
&= X'\beta_0 - X'\beta_0 = 0
\end{aligned}$$

(iii) **Show that $\mathbb{E}[X(Y - X'\beta)] = 0$ if and only if $\beta = \beta_0$ and use this to derive OLS as a method of moments estimator.**

First, let $\beta = \beta_0$. Then

$$\begin{aligned}
\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[X(Y - X'\beta_0)] \\
&= \mathbb{E}[XY - XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0 \cdot U] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[\mathbb{E}[XX'\beta_0 \cdot U|X]] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0\mathbb{E}[U|X]] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0] - \mathbb{E}[XX'\beta_0] \\
&= 0
\end{aligned}$$

I worked on this Problem set with Sarah Bass, Michael Nattinger, Alex von Hafften, and Danny Edgel.

Now suppose that $\mathbb{E}[X(Y - X'\beta)] = 0$. Then

$$\begin{aligned}
0 &= \mathbb{E}[XY] - \mathbb{E}[XX'\beta] \\
&= \mathbb{E}[X(X'\beta_0 + \tilde{U})] - \mathbb{E}[XX'\beta] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] + \mathbb{E}[X\tilde{U}] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] + \mathbb{E}[\mathbb{E}[X\tilde{U}|X]] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] \\
\iff \beta &= \beta_0
\end{aligned}$$

unsure about moments estimator?

(iv) **Show that the OLS estimator is conditionally unbiased.**

We need to show that $\mathbb{E}[\hat{\beta}|X_1, ..., X_n] = \beta_0$:

$$\begin{aligned}
\mathbb{E}[\hat{\beta}|X_1, ..., X_n] &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_iY_i \middle| X_1, ..., X_n\right] \\
&= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_i\mathbb{E}[Y_i|X_i] \\
&= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_i\mathbb{E}[X_i'\beta_0 + \tilde{U}|X_i] \\
&= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_i(X_i'\beta_0 + \mathbb{E}[\tilde{U}|X_i]) \\
&= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_iX_i'\beta_0 \\
&= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)\beta_0 \\
&= \beta_0
\end{aligned}$$

(v) **Show that the OLS estimator is consistent.** First note that

$$\begin{aligned}
\hat{\beta} &= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_iY_i \\
&= \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\beta_0 + X_i\tilde{U}_i\right) \\
&= \beta_0 + \left(\frac{1}{n}\sum_{i=1}^n X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n X_i\tilde{U}_i
\end{aligned}$$

Which converges **in distribution** to $\beta_0 + \mathbb{E}[X_iX_i']^{-1}E[X_i\tilde{U}_i]$. Now, because $\mathbb{E}[\tilde{U}|X] = 0$,

$$\begin{aligned}
\hat{\beta} &\to^p \beta_0 + \mathbb{E}[X_iX_i']^{-1}E[X_i\tilde{U}_i] \\
&= \beta_0
\end{aligned}$$

need to polish this

Exercise 2

Let X be a random variable with $\mathbb{E}[X^4] < \infty$ and $\mathbb{E}[X^2] > 0$. Furthermore, let $\{X_i\}_{i=1}^n$ be a random sample from the distribution of X.

(i) **Which of the following four statistics can you use the law of large numbers and continuous mapping theorem to show convergence in probability as $n \to \infty$?**

(a)

$$\frac{1}{n}\sum_{i=1}^{n} X_i^3 \to^p \mathbb{E}[X_i^3]$$

(b)

$$\max_{1\leq i\leq n} X_i$$

cannot use CMT?

(c)

$$\frac{\sum_{i=1}^{n} X_i^3}{\sum_{i=1}^{n} X_i^2}$$

(d)

$$1\left\{\frac{1}{n}\sum_{i=1}^{n} X_i > 0\right\}$$

(ii) **For which of the following three statistics can you use the central limit theorem and continuous mapping to show convergence in distribution as $n \to \infty$?**

(a)

(b)

(c)

Exercise 3