Econ 710 Quarter 1: Problem set 1

Emily Case

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Exercise 1

Suppose $(Y, X')'$ is a random vector with

$$Y = X'\beta_0 \cdot U$$

where $\mathbb{E}[U|X] = 1$, $\mathbb{E}[XX']$ is invertible, and $\mathbb{E}[Y^2 + ||X||^2] < \infty$. Furthermore, suppose that $\{Y_i, X_i'\}_{i=1}^{\infty}$ is a random sample from the distribution of $(Y, X')'$, where $\frac{1}{n}\sum_{i=1}^{n} X_i X_i'$ is invertible and let $\hat{\beta}$ be the OLS estimator:

$$\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$$

(i) **Interpret the entries of β_0 in this model.** Here, when we take the derivative with respect to X, we will get a value/slope that is affected by the unobservables as well. Therefore β_0 is the value of the slope if it were unaffected by unobservable factors.

(ii) **Show that $Y = X'\beta_0 + \tilde{U}$ where $\mathbb{E}[\tilde{U}|X] = 0$.**
Notice that $\tilde{U} = Y - X'\beta_0 = X'\beta_0 \cdot U - X'\beta_0 = X'\beta_0(U-1)$. So, when $\tilde{U} = X'\beta_0(U-1)$, the statement is true. Also,

$$\begin{aligned}
\mathbb{E}[\tilde{U}|X] &= \mathbb{E}[X'\beta_0(U-1)|X] \\
&= X'\beta_0\mathbb{E}[U|X] - X'\beta_0 \\
&= X'\beta_0 - X'\beta_0 = 0
\end{aligned}$$

(iii) **Show that $\mathbb{E}[X(Y - X'\beta)] = 0$ if and only if $\beta = \beta_0$ and use this to derive OLS as a method of moments estimator.**

I worked on this Problem set with Sarah Bass, Michael Nattinger, Alex von Hafften, and Danny Edgel.

First, let $\beta = \beta_0$. Then[1]

$$\begin{aligned}
\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[X(Y - X'\beta_0)] \\
&= \mathbb{E}[XY - XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0 \cdot U] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[\mathbb{E}[XX'\beta_0 \cdot U|X]] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0 \mathbb{E}[U|X]] - \mathbb{E}[XX'\beta_0] \\
&= \mathbb{E}[XX'\beta_0] - \mathbb{E}[XX'\beta_0] \\
&= 0
\end{aligned}$$

Now suppose that $\mathbb{E}[X(Y - X'\beta)] = 0$. Then

$$\begin{aligned}
0 &= \mathbb{E}[XY] - \mathbb{E}[XX'\beta] \\
&= \mathbb{E}[X(X'\beta_0 + \tilde{U})] - \mathbb{E}[XX'\beta] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] + \mathbb{E}[X\tilde{U}] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] + \mathbb{E}[\mathbb{E}[X\tilde{U}|X]] \\
\mathbb{E}[XX'\beta] &= \mathbb{E}[XX'\beta_0] \\
\iff \beta &= \beta_0
\end{aligned}$$

To construct the estimator:

$$\begin{aligned}
0 &= \frac{1}{n}\sum_{i=1}^{n} X_i(Y_i - X_i'\hat{\beta}) \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \frac{1}{n}\sum_{i=1}^{n} X_i X_i'\hat{\beta} \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \hat{\beta} \\
\hat{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i
\end{aligned}$$

(iv) **Show that the OLS estimator is conditionally unbiased.**

[1] Note that $\mathbb{E}[XX']$ is invertible

We need to show that $\mathbb{E}[\hat{\beta}|X_1, ..., X_n] = \beta_0$:

$$\begin{aligned}
\mathbb{E}[\hat{\beta}|X_1, ..., X_n] &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \middle| X_1, ..., X_n\right] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i \mathbb{E}[Y_i|X_i] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i \mathbb{E}[X_i'\beta_0 + \tilde{U}|X_i] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i (X_i'\beta_0 + \mathbb{E}[\tilde{U}|X_i]) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i X_i'\beta_0 \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)\beta_0 \\
&= \beta_0
\end{aligned}$$

(v) **Show that the OLS estimator is consistent.** By the law of large numbers, $\frac{1}{n}\sum_{i=1}^{n} X_i X_i' \to^p \mathbb{E}[XX']$ and $\frac{1}{n}\sum_{i=1}^{n} X_i Y_i \to^p \mathbb{E}[XY]$. We know that $\mathbb{E}[XX']$ is invertible. Now, by the continuous mapping theorem, we get that

$$\begin{aligned}
\hat{\beta} &\to^p \mathbb{E}[XX']^{-1}\mathbb{E}[XY] \\
&= \mathbb{E}[XX']^{-1}\mathbb{E}[X(X'\beta_0 + \tilde{U})] \\
&= \mathbb{E}[XX']^{-1}\mathbb{E}[XX'\beta_0 + X\tilde{U})] \\
&= \beta_0 + \mathbb{E}[X\tilde{U})] \\
&= \beta_0
\end{aligned}$$

Exercise 2

Let X be a random variable with $\mathbb{E}[X^4] < \infty$ and $\mathbb{E}[X^2] > 0$. Furthermore, let $\{X_i\}_{i=1}^n$ be a random sample from the distribution of X.

(i) **Which of the following four statistics can you use the law of large numbers and continuous mapping theorem to show convergence in probability as $n \to \infty$?**

(a) $\frac{1}{n}\sum_{i=1}^{n} X_i^3$
Since the fourth moment is finite, the third moment is also finite, so $\mathbb{E}[|X^3|] < \infty$. Now we can apply the law of large numbers:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^3 \to^p \mathbb{E}[X_i^3]$$

(b) $\max_{1 \le i \le n} X_i$

We cannot use CMT here, because some distributions don't even have a maximum (i.e. Normal distributions). It also doesn't have any kind of sample average in it.

(c) $\frac{\sum_{i=1}^{n} X_i^3}{\sum_{i=1}^{n} X_i^2}$

LLN[2] gives us that $\sum_{i=1}^{n} X_i^3 \to^p \mathbb{E}[X^3]$ and $\sum_{i=1}^{n} X_i^2 \to^p$. Then using CMT,

$$\frac{\sum_{i=1}^{n} X_i^3}{\sum_{i=1}^{n} X_i^2} \to^p \frac{\mathbb{E}[X^3]}{\mathbb{E}[X^2]}$$

(d) $1\left\{\frac{1}{n}\sum_{i=1}^{n} X_i > 0\right\}$

Notice that if $\mathbb{E}[X] = 0$, we have a discontinuity in the function. We don't know for sure what $\mathbb{E}[X]$ is, so we can't use CMT.

(ii) **For which of the following three statistics can you use the central limit theorem and continuous mapping to show convergence in distribution as $n \to \infty$?**

(a) $W_n := \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i^2 - \mathbb{E}[X_1^2])$

Since we have $\mathbb{E}[X_i^4] < \infty$, $Var(X_1^2) < \infty$ and we can use Lindeberg-Levy, it will converge in distribution to $N(0, Var(X_1^2))$.

(b) W_n^2

Since $f(x) = x^2$ is continuous everywhere,

$$W_n^2 \to^d N(0, Var(X_i^2)^2)$$

(c) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i^2 - \overline{X_n^2})$ simplifies to

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2 - \frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j^2 = 0$$

so we cannot use CLT.

(iii) **Show that** $\max_{1 \le i \le n} X_i \to^p 1$ **if** $X \sim uniform(0,1)$

$$\begin{aligned}
Pr(\max_{1\le i\le n} |X_i - 1| \le \epsilon) &= Pr(\max_{1\le i\le n} 1 - X_i \le \epsilon)\\
&= Pr(\max_{1\le i\le n} X_i \ge 1-\epsilon)\\
&= 1 - Pr(\max_{1\le i\le n} X_i < 1-\epsilon)\\
&= 1 - Pr(X_i < 1 - \epsilon \forall X_i)\\
&= 1 - \Pi_{i=1}^{n} Pr(X_i < 1-\epsilon)\\
&= 1 - \Pi_{i=1}^{n}(1-\epsilon)\\
&= 1 - (1-\epsilon)^n \qquad \to 1
\end{aligned}$$

[2]Using similar logic to part a.

(iv) **Show that Pr**$(\max_{1\leq i\leq n} X_i > M \to 1$ **for any** $M \geq 0$ **if** $X \sim exponential(1)$

$$\begin{aligned}
Pr(\max_{1\leq i\leq n} X_i > M) &= 1 - Pr(\max_{1\leq i\leq n} X_i \leq M) \\
&= 1 - Pr(X_i \leq M \forall X_i) \\
&= 1 - (1-e^{-M})^n && \to 1
\end{aligned}$$

Exercise 3

Suppose that $\{X_i\}_{i=1}^n$ is an i.i.d. sequence of N(0,1) random variables. Let W be independent of $\{X_i\}_{i=1}^n$ with $\Pr(W=1) = Pr(W=-1) = 1/2$. Let $Y_i = X_i W$.

(i) Show that $\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \to^d N(0,1)$ as $n \to \infty$. First, note that $\mathbb{E}[X_i^2] = Var(X_i) = 1 < \infty$. This meets the conditions for Lindeberg-Levy:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mathbb{E}[X_1]) \to^d N(0, Var(X_1)) = N(0,1)$$

(ii) Show that $\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \to^d N(0,1)$ as $n \to \infty$. There are two cases, either $Y_i = X_i$ (W=1) in which the answer is the same as (i), or $Y_i = -X_i$ (W=-1). Normal distributions are symmetric, and so $Y_i = -X_i$ will have the same distribution. In either case,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \to^d N(0,1)$$

(iii) Show that $Cov(X_i, Y_i) = 0$. Note that $\mathbb{E}[X_i W] = 0$.

$$\begin{aligned}
Cov(X_i, Y_i) &= \mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])] \\
&= \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_i W - \mathbb{E}[X_i W])] \\
&= \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_i W)] \\
&= \mathbb{E}[X_i^2 W] \\
&= \mathbb{E}[X_i^2]\mathbb{E}[W] \\
&= 1 * 0 = 0
\end{aligned}$$

(iv) Does $V := \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i, Y_i)' \to^d N(0, I_2)$ as $n \to \infty$? W makes V not continuous, which is necessary, so it does not converge to $N(0, I_2)$.

(v) How does this exercise relate to the Cramer-Wold device introduced in lecture 2? We showed individual convergence in parts (i) and (ii), but the joint limit distribution was not possible.