metrics PS3

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① I have <u>no</u> idea how to do this one :)

② $Y = X\beta_1 + u \qquad \mathbb{E}[u|Z] = 0 \qquad \mathbb{E}[h(Z)^4] < \infty$

(i) $\mathbb{E}[h(z)(Y - X\beta)]$

$= \mathbb{E}[\mathbb{E}[h(z)(Y - X\beta) | z]]$

$= \mathbb{E}[h(z)\mathbb{E}[Y - X\beta | z]]$

$= \mathbb{E}[h(z)\mathbb{E}[u | z]]$ iff $\beta = \beta_1$ and $h(z) \neq 0$

$= 0.$

(ii) $\mathbb{E}[h(z)(Y - X\hat{\beta}_1^n)] = 0$

$\mathbb{E}[h(z)Y - h(z)X\hat{\beta}_1^n] = 0$

$\mathbb{E}[h(z) \cdot Y] = \mathbb{E}[h(z)X\hat{\beta}_1^n]$

$\mathbb{E}[h(z) \cdot Y] = \mathbb{E}[h(z)X]\hat{\beta}_1^n$

$\Longrightarrow \hat{\beta}_1^n = \mathbb{E}[h(z)X]^{-1}\mathbb{E}[h(z) \cdot Y]$

(iii) show $\sqrt{n}(\hat{\beta}_1^n - \beta_1) \xrightarrow{d} N(0, \Omega^n)$

$$\text{CLT} \Longrightarrow \hat{\beta}_1^n = \frac{\frac{1}{n}\sum_i^n h(z_i)Y_i}{\frac{1}{n}\sum_i^n h(z_i)X_i} = \beta_1 + \frac{\sum_{i=1}^n h(z_i)u}{\sum_{i=1}^n h(z_i)X_i}$$

$$\frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \frac{1/\sqrt{n}}{1/n}$$

$$\Longrightarrow \sqrt{n}(\hat{\beta}_1^n - \beta_1) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n h(z_i)u}{\frac{1}{n}\sum_{i=1}^n h(z_i)X_i}$$

$$\xrightarrow{d} \frac{\mathbb{E}[h(z)u]}{\mathbb{E}[h(z)X]} N(0,1)$$

$$= N(0, \Omega^h)$$

where $\Omega^h = \dfrac{\mathbb{E}[h(Z)u]^2}{\mathbb{E}[h(Z)X]^2} = \dfrac{\mathbb{E}[h(Z)^2 u^2]}{\mathbb{E}[h(Z)X]^2}$.

(iv) $$\Omega^h = \frac{\mathbb{E}[h(Z)^2 u^2]}{\mathbb{E}[h(Z)X]^2}.$$

$$= \frac{\mathbb{E}[h(Z)^2 \mathbb{E}[u^2 | Z]]}{\mathbb{E}[h(Z)\mathbb{E}[X|Z]]^2} \quad \text{by LIE}$$

$$\geq \mathbb{E}\left[\frac{\cancel{h(Z)^2}\, \mathbb{E}[u^2|Z]}{\cancel{h(Z)^2}\, \mathbb{E}[X|Z]^2}\right] \quad \text{by cauchy-schwarz.}$$

$$= \mathbb{E}\left[\frac{\mathbb{E}[X|Z]^2}{\mathbb{E}[u^2|Z]}\right]^{-1}$$

now let $h(Z) = \dfrac{\mathbb{E}[X|Z]}{\mathbb{E}[u^2|Z]}$, then:

$$\Omega^h = \frac{\mathbb{E}\left[\frac{\mathbb{E}[X|Z]^2}{\mathbb{E}[u^2|Z]^{\cancel{2}}}\, \mathbb{E}[\cancel{u^2}|Z]\right]}{\mathbb{E}\left[\frac{\mathbb{E}[X|Z]}{\mathbb{E}[u^2|Z]}\, \mathbb{E}[X|Z]\right]^2} = \frac{1}{\mathbb{E}\left[\frac{\mathbb{E}[X|Z]^2}{\mathbb{E}[u^2|Z]}\right]^2}$$

then $\tilde{h}(Z)$ clearly attains the lower bound.

③ $\log(wage) = \beta_0 + educ \cdot \beta_1 + \sum_{t=31}^{39} \mathbb{1}\{yob = t\}\beta_t + \sum_{s=1}^{50} \mathbb{1}\{sob = s\}\gamma_s + u$

See attached matlab code

$\Longrightarrow \hat{\beta}_1 = 0.1084$

$\sqrt{Vb} = 0.0195$