Econ 714 Quarter 1: Problem set 2

Emily Case

February 6, 2021

Problem 1

Consider a growth model with preferences $\sum_{t=0}^{\infty} \beta^t \log C_t$, production function $Y_t = AK_t^{\alpha}$,the capital law of motion $K_{t+1} = K_t^{1-\delta} I_t^{\delta}$, and the resource constraint $Y_t = C_t + I_t$.

(1) Write down the social planner's problem and derive the Euler equation. Provide the intuition to this optimality condition using the perturbation argument.

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta}$$

Set up the lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log C_t - \lambda_t[K_t^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta} - K_{t+1}]$$

FOC[C_t]:

$$\frac{\beta^t}{C_t} = -\lambda_t K_t^{1-\delta}(-1)\delta(AK_t^{\alpha} - C_t)^{\delta-1}$$

which also gives us

$$\lambda_t = \frac{-\beta^t}{\delta C_t K_t^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta-1}}$$

FOC[K_{t+1}]:

$$\lambda_t = \lambda_{t+1}[k_{t+1}^{-\delta}(1-\delta)(AK_{t+1}^{\alpha} - C_{t+1})^{\delta} + \delta K_{t+1}^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta-1}\alpha AK_{t+1}^{\alpha-1}]$$

I worked on this Problem set with Sarah Bass, Michael Nattinger, Alex von Hafften, and Danny Edgel.

Combining, and replacing the I_t's, we get the Euler Equation:

$$\frac{-\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{-\beta^{t+1}}{\delta C_{t+1} K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}} \left[k_{t+1}^{-\delta}(1-\delta)I_{t+1}^{\delta} + \delta K_{t+1}^{1-\delta} I_t^{\delta-1} \alpha A K_{t+1}^{\alpha-1}\right]$$
$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1} K_{t+1}} \left[(1-\delta)I_{t+1} + \delta\alpha A K_{t+1}^{\alpha}\right] \tag{1}$$

need to do perturbation argument

(2) Derive the system of equations that pins down the steady state of the model.

Using the Euler Equation (1) and the capital law of motion,

$$K_{t+1} = K_t^{1-\delta} I_t^{\delta} \tag{2}$$

now with $C_t = C_{t+1} = \bar{C}$, $K_t = K_{t+1} = \bar{K}$, and $I_t = I_{t+1} = \bar{I}$[1], we get:

$$(1) \Rightarrow 1 = \beta \bar{K}^{-\alpha} \bar{I}^{\delta-1} [\delta\alpha A \bar{K}^{\alpha} + (1-\delta)\bar{I}] \tag{3}$$
$$(2) \Rightarrow 1 = \bar{K}^{-\delta} \bar{I}^{\delta} \tag{4}$$

which define the steady state. Notice also that (4) implies $\bar{K} = \bar{I}$, which will be useful.

(3) Log-linearize the equilibrium conditions around the steady state.

First log-linearize $I_t = AK_t^{\alpha} - C_t$:

$$i_t = \frac{\alpha A \bar{K}^{\alpha}}{\bar{I}} k_t - \frac{\bar{C}}{\bar{I}} c_t = \alpha A \bar{K}^{\alpha-1} k_t - \frac{\bar{C}}{\bar{I}} c_t$$

Now I log-linearize equation (2):

$$\begin{aligned} \bar{K}(1 + k_{t+1}) &= \bar{K}^{1-\delta}[1 + (1-\delta)k_t]\bar{I}^{\delta}(1 + \delta i_t) \\ k_{t+1} &= (1-\delta)k_t + \delta i_t \\ &= (1-\delta)k_t + \delta\left(\alpha A \bar{K}^{\alpha-1} k_t - \frac{\bar{C}}{\bar{I}} c_t\right) \\ &= k_t\left(1 - \delta + \delta\alpha A \bar{K}^{\alpha-1}\right) - \delta\frac{\bar{C}}{\bar{I}} c_t \end{aligned}$$

check this against someone's
And finally, equation (1):

[1] Where $\bar{I} = A\bar{K}^{\alpha} - \bar{C}$

(4) Write down a dynamic system with one state variable and one control variable. Use the Blanchard-Kahn method to solve this system for a saddle path.

(5) Show that the obtained solution is not just locally accurate, but is in fact the exact solution to the planner's problem.

(6) Generalize the (global) solution to the case of stochastic productivity shocks A_t.

(7) The analytical tractability of the model is due to special functional form assumptions,which however, have strong economic implications. What is special about consumption behavior in this model? Provide economic intuition.

(8) *Bonus task:* can you introduce labor into preferences and production function without compromising the analytical tractability of the model?

I will probably not do this