② firm i takes other prices P_{jk} $j \neq i$ as given (Bertrand)

firm i wants to max profits: $\pi_{ik} = P_{ik} \underbrace{A_{ik} L_{ik}}_{Y_{ik} = C_{ik}} - w L_{ik}$

also s.t. $C_{ik} = \left[\sum P_{ik}^{1-\theta}\right]^{\frac{\theta-\rho}{\theta-1}} \frac{P^{\rho}}{P_{ik}^{\theta}} C$

notice we can do $C_{ik} = A_{ik} L_{ik}$

$\Rightarrow L_{ik} = C_{ik} A_{ik}^{-1}$ $\qquad C_{ik}\left[P_{ik} - W A_{ik}^{-1}\right]$

then $$\pi_{ik} = P_{ik}^{1-\theta}\left[\sum P_{ik}^{1-\theta}\right]^{\frac{\theta-\rho}{\theta-1}} \frac{P^{\rho}}{\cancel{P_{ik}^{\theta}}} C$$
$$- W A_{ik}^{-1}\left[\sum P_{ik}^{1-\theta}\right]^{\frac{\theta-\rho}{\theta-1}} \frac{P^{\rho}}{P_{ik}^{\theta}} C$$

$$\pi'_{ik}(P_{ik}) = P^{\rho} C \Bigg[(1-\theta) P_{ik}^{-\theta} \left[\cancel{\sum} P_{ik}^{\cancel{1-\theta}}\right]^{\frac{-\theta+\rho}{\cancel{\theta-1}}}$$
$$\cancel{+}\ P_{ik}^{1-2\theta}\left(\frac{\theta-\rho}{\cancel{\theta-1}}\right)\cancel{(1-\theta)}\left[\cancel{\sum} P_{ik}^{1-\theta}\right]^{\frac{\cancel{1-\rho}}{\cancel{\theta-1}}\ \rho-1} \cancel{P_{ik}^{-\theta}}$$
$$- \frac{W}{A_{ik}}\left(-\theta P_{ik}^{-\theta-1}\left[\sum P_{ik}^{1-\theta}\right]^{\frac{\theta-\rho}{\theta-1}} \cancel{+}\ P_{ik}^{-2\theta}\left(\frac{\theta-\rho}{\cancel{\theta-1}}\right)\cancel{(1-\theta)}\left(\sum P_{ik}^{1-\theta}\right)^{\frac{1-\rho}{\theta-1}} \cdot \cancel{P_{ik}^{-\theta}}\right)\Bigg] = 0$$

$$\Rightarrow (1-\theta) P_{ik}^{-\theta} P_K^{\theta-\cancel{\rho}} - P_{ik}^{1-\theta}(\theta-\rho) P_K^{2\theta-\cancel{\rho}-1} P_{ik}^{-\theta}$$
$$= \frac{W}{A_{ik}}\left[(-\theta) P_{ik}^{-\theta-1} P_K^{\theta-\cancel{\rho}} - P_{ik}^{-\theta}(\theta-\rho) P_K^{\theta-1+\rho}\right]$$

$$(1-\theta)\left(\frac{P_K}{P_{ik}}\right)^{\theta} - \left(\frac{P_{ik}}{P_K}\right)^{1-\theta}\left(\frac{P_K}{P_{ik}}\right)^{\theta} = \frac{W}{A_{ik}}\left[-\theta\left(\frac{P_K}{P_{ik}}\right)^{\theta} P_{ik}^{-1} - \left(\frac{P_{ik}}{P_K}\right)^{1-\theta} P_{ik}(\theta-\rho)\right]$$

denote $S_{ik} := \left(\frac{P_{ik}}{P_K}\right)^{1-\theta}$; $\qquad P_{ik}^{-1} \cdot X = P_{ik}^{1}$

$$\rightarrow P_{ik} = \frac{W}{A_{ik}}\left[\frac{-\theta}{(1-\theta) - S_{ik}} - \frac{P_{ik}^{2}(\theta-\rho) S_{ik}}{(1-\theta) S_{ik}^{1-2\theta} - S_{ik}^{\theta}}\right]$$

I messed up some algebra and can't find where.