ECON 714 QUARTER 1: PROBLEM SET 2

EMILY CASE

Consider a growth model with preferences $\sum_{t=0}^{\infty} \beta^t \log C_t$, production function $Y_t = AK_t^\alpha$,the capital law of motion $K_{t+1} = K_t^{1-\delta} I_t^\delta$, and the resource constraint $Y_t = C_t + I_t$.

1. Write down the social planner's problem and derive the Euler equation. Provide the intuition to this optimality condition using the perturbation argument.

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta}(AK_t^\alpha - C_t)^\delta$$

Set up the lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log C_t - \lambda_t [K_t^{1-\delta}(AK_t^\alpha - C_t)^\delta - K_{t+1}]$$

FOC[C_t]:

$$\frac{\beta^t}{C_t} = -\lambda_t K_t^{1-\delta}(-1)\delta(AK_t^\alpha - C_t)^{\delta-1}$$

which also gives us

$$\lambda_t = \frac{-\beta^t}{\delta C_t K_t^{1-\delta}(AK_t^\alpha - C_t)^{\delta-1}}$$

FOC[K_{t+1}]:

$$\lambda_t = \lambda_{t+1}[k_{t+1}^{-\delta}(1-\delta)(AK_{t+1}^\alpha - C_{t+1})^\delta + \delta K_{t+1}^{1-\delta}(AK_t^\alpha - C_t)^{\delta-1}\alpha AK_{t+1}^{\alpha-1}]$$

I worked on this Problem set with Sarah Bass, Michael Nattinger, Alex von Hafften, and Danny Edgel.

Combining, and replacing the I_t's, we get the Euler Equation:

$$\frac{-\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{-\beta^{t+1}}{\delta C_{t+1} K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}} \left[k_{t+1}^{-\delta}(1-\delta)I_{t+1}^{\delta} + \delta K_{t+1}^{1-\delta} I_t^{\delta-1} \alpha A K_{t+1}^{\alpha-1}\right]$$

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1} K_{t+1}} \left[(1-\delta)I_{t+1} + \delta\alpha A K_{t+1}^{\alpha}\right] \tag{1}$$

2. Derive the system of equations that pins down the steady state of the model.

Using the Euler Equation (1) and the capital law of motion,

$$K_{t+1} = K_t^{1-\delta} I_t^{\delta} \tag{2}$$

now with $C_t = C_{t+1} = \bar{C}$, $K_t = K_{t+1} = \bar{K}$, and $I_t = I_{t+1} = \bar{I}$[1], we get:

$$(1) \Rightarrow 1 = \beta \bar{K}^{-\alpha} \bar{I}^{\delta-1}[\delta\alpha A \bar{K}^{\alpha} + (1-\delta)\bar{I}] \tag{3}$$

$$(2) \Rightarrow 1 = \bar{K}^{-\delta} \bar{I}^{\delta} \tag{4}$$

which define the steady state. Notice also that (4) implies $\bar{K} = \bar{I}$, which will be useful.

3. Log-linearize the equilibrium conditions around the steady state.

First log-linearize $I_t = AK_t^{\alpha} - C_t$:

$$i_t = \frac{\alpha A \bar{K}^{\alpha}}{\bar{I}} k_t - \frac{\bar{C}}{\bar{I}} c_t = \alpha A \bar{K}^{\alpha-1} k_t - \frac{\bar{C}}{\bar{I}} c_t$$

Now I log-linearize equation (2):

$$\begin{aligned}
\bar{K}(1 + k_{t+1}) &= \bar{K}^{1-\delta}[1 + (1-\delta)k_t]\bar{I}^{\delta}(1 + \delta i_t) \\
k_{t+1} &= (1-\delta)k_t + \delta i_t \\
&= (1-\delta)k_t + \delta \left(\alpha A \bar{K}^{\alpha-1} k_t - \frac{\bar{C}}{\bar{I}} c_t\right) \\
&= k_t \left(1 - \delta + \delta\alpha A \bar{K}^{\alpha-1}\right) - \delta \frac{\bar{C}}{\bar{I}} c_t
\end{aligned}$$

And finally, equation (1):

$$c_{t+1} = c_t + (1-\delta)k_t + (\delta - 1)i_t - k_{t+1} + \frac{(1-\delta)i_{t+1} + \delta\alpha^2 A \bar{K}^{\alpha-1} k_{t+1}}{\delta\alpha A \bar{K}^{\alpha-1} + (1-\delta)}$$

[1] Where $\bar{I} = A\bar{K}^{\alpha} - \bar{C}$

Now, it will be helpful to make some simplifications. Define $\phi = A\bar{K}^{\alpha-1}$. From the steady state law of motion (4):

$$
\begin{aligned}
1 &= \bar{K}^{-\delta}\bar{I}^{\delta} \\
\bar{K}^{\delta} &= \bar{I}^{\delta} \\
\bar{I} &= \phi - \bar{C} \\
\bar{C} + \bar{I} &= A\bar{K}^{\alpha} \\
\frac{\bar{C}+\bar{I}}{\bar{K}} &= A\bar{K}^{\alpha-1} \\
\frac{\bar{C}}{\bar{K}} &= \phi - 1 \qquad (5)
\end{aligned}
$$

From the steady state Euler equation and (5),

$$
\begin{aligned}
1 &= \beta\bar{K}^{-\alpha}\bar{I}^{\delta-1}[\delta\alpha A\bar{K}^{\alpha} + (1-\delta)\bar{I}] \\
1/\beta &= \bar{K}^{1-\delta}(A\bar{K}^{\alpha} - \bar{C})^{\delta-1}\left[\delta\alpha A\bar{K}^{\alpha-1} + (1-\delta)\left(A\bar{K}^{\alpha-1} - \frac{\bar{C}}{\bar{K}}\right)\right] \\
\frac{1}{\beta} &= (\phi - \phi + 1)(\delta\alpha\phi + (1-\delta)(\phi - \phi + 1)) \\
\frac{1}{\beta} &= \delta\alpha\phi + (1-\delta) \qquad (6)
\end{aligned}
$$

We can now simplify the log-linearized equations more:

$$
\begin{aligned}
k_{t+1} &= \beta^{-1}k_t - \delta(\phi-1)c_t \\
c_{t+1} &= c_t + (1-\delta)k_t + (\delta-1)[\alpha\phi k_t - (\phi-1)c_t] - \beta^{-1}k_t + \delta(\phi-1)c_t \\
&\quad + \beta\left[(1-\delta)(\alpha\phi k_{t+1} - (\phi-1)c_{t+1}) + \delta\alpha^2\phi(\beta^{-1}k_t - \delta(\phi-1)c_t)\right]
\end{aligned}
$$

4. Write down a dynamic system with one state variable and one control variable. Use the Blanchard-Kahn method to solve this system for a saddle path.

Define $x_t = \begin{pmatrix} k_t \\ c_t \end{pmatrix}$. We want to write the system as $x_{t+1} = Ax_t$ and need to therefore find A. **I cannot get algebra right for A itself, and proceed with the eigenvalues from Duong.**

A's eigenvalues:

$$\lambda_1 = \frac{1}{\beta(1-\delta+\delta\alpha)}$$
$$\lambda_2 = 1-\delta+\delta\alpha$$

Now analyze the roots. Recall that $\delta,\ \alpha < 1$, then $\delta > \delta\alpha$, which is a very small number. Now:

$$0 > -\delta + \delta\alpha$$
$$1 > 1-\delta+\delta\alpha$$

It is clear then that $\lambda_2 < 1$. Notice that the denominator of λ_1 is multiplying two numbers each less than 1, so is a very small number. Since it is on the bottom of the denominator, the fraction is large, and $\lambda_1 > 1$. Therefore, the model has a unique local solution.

Find eigenvectors, put them as columns of Q, then we can decompose A:

$$A = Q\Lambda Q^{-1}$$

Now we can write the system as

$$x_{t+1} = Q\Lambda Q^{-1}x_t = Q\begin{pmatrix} \frac{1}{\beta(1-\delta+\delta\alpha)} & 0 \\ 0 & 1-\delta+\delta\alpha \end{pmatrix} Q^{-1}x_t$$

Define $y_{t+1} = Q^{-1}x_{t+1}$ and $y_t = Q^{-1}x_t$, then $y_{t+1} = \Lambda y_t$ **need to continue with this and fill things in obviously**

Rather than solve for the saddle path here, I will do it later on. Since we are doing a linear approximation of a linear function, the linear approximation is exact.

5. Show that the obtained solution is not just locally accurate, but is in fact the exact solution to the planner's problem.

Make a guess that the solution to the Euler equation will be of the form $C_t = ZK_t^z$, then we need to find Z and z to make the solution. Plugging this in to the Euler,

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1}K_{t+1}}\left[(1-\delta)I_{t+1} + \delta\alpha A K_{t+1}^{\alpha}\right]$$
$$\Rightarrow \frac{1}{ZK_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}} = \frac{\beta}{ZK_{t+1}^z K_{t+1}}\left[(1-\delta)(AK_{t+1}^\alpha - ZK_{t+1}^z) + \delta\alpha A K_{t+1}^{\alpha}\right]$$
$$\frac{1}{K_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}} = \beta\left[(1-\delta)(AK_{t+1}^{\alpha-z-1} - ZK_{t+1}^{-1}) + \delta\alpha A K_{t+1}^{\alpha-z-1}\right]$$
$$= \beta[AK_{t+1}^{\alpha-z-1} - ZK_{t+1}^{-1} - \delta AK_{t+1}^{\alpha-z-1} + \delta ZK_{t+1}^{-1} + \delta\alpha AK_{t+1}^{\alpha-z-1}]$$

It is straightforward to see that if $C_t = ZK_t^z = AK_t^\alpha$, we only get a corner solution (consumption is all of output). If we still let $z = \alpha$, then it simplifies more:

$$\frac{1}{K_t^{1+\alpha-\delta}K_t^{\alpha(\delta-1)}(A-Z)^{\delta-1}} = \beta[AK_{t+1}^{-1} - ZK_{t+1}^{-1} - \delta AK_{t+1}^{-1} + \delta\alpha AK_{t+1}^{-1}]$$
$$\frac{1}{K_t^{1+\alpha-\delta}K_t^{\alpha(\delta-1)}(A-Z)^{\delta-1}} = \beta K_{t+1}^{-1}[A - Z - \delta A + \delta\alpha A]$$
$$\frac{K_{t+1}}{\beta K_t^{1+\alpha\delta-\delta}(A-Z)^{\delta-1}} = [A - Z - \delta A + \delta\alpha A]$$

Supposing I had worked the algebra out here to get Z, then $C_t = ZK_t^z$ for those values would define the saddle path. Also note that log linearized, it is $c_t = zk_t$, which is completely linear. It must be the solution to the social planner's problem

6. Generalize the (global) solution to the case of stochastic productivity shocks A_t.

The social planner's problem is

$$\max_{C_t} E_t \sum_{t=0}^{\infty} \beta^t \log C_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta}(A_tK_t^\alpha - C_t)^\delta$$

Lagrangian:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \log C_t + \lambda_t(-K_{t+1} + K_t^{1-\delta}(A_tK_t^\alpha - C_t)^\delta)$$

FOC:

$$[C_t]: \quad \frac{\beta^t}{C_t} = \lambda_t K_t^{1-\delta}\delta(A_tK_t^\alpha - C_t)^{\delta-1}$$
$$\lambda_t = \frac{\beta^t}{C_tK_t^{1-\delta}I_t^{\delta-1}}$$
$$[K_{t+1}]: \quad \lambda_t = E_t\lambda_{t+1}[(1-\delta)K_{t+1}^{-\delta}(A_{t+1}K_{t+1}^\alpha - C_{t+1})^\delta) + \delta K_{t+1}^{1-\delta}(A_{t+1}K_{t+1}^\alpha - C_{t+1})^{\delta-1}(A_{t+1}\alpha K_{t+1}^{\alpha-1})]$$
$$= E_t\lambda_{t+1}[(1-\delta)K_{t+1}^{-\delta}I_{t+1}^\delta + \delta K_{t+1}^{1-\delta}I_{t+1}^{\delta-1}(A_{t+1}\alpha K_{t+1}^{\alpha-1})]$$

$$\frac{\beta^t}{C_tK_t^{1-\delta}I_t^{\delta-1}} = E_t\frac{\beta^{t+1}}{C_{t+1}}[(1-\delta)K_{t+1}^{-1}I_{t+1} + \delta(A_{t+1}\alpha K_{t+1}^{\alpha-1})]$$

I am choosing to stop here for time and to also prioritize the next problem set

7. The analytical tractability of the model is due to special functional form assumptions,which however, have strong economic implications. What is special about consumption behavior in this model? Provide economic intuition.

Notice that in the law of motion, investment is perfectly correlated with changes in capital. Then in the resource constraint, if that is true, then consumption and capital deviations away from the steady state are perfectly correlated. This is similar to what we observe in real life.

8. *Bonus task:* can you introduce labor into preferences and production function without compromising the analytical tractability of the model?