Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in AIML.

Course Number	AIMLC ZC416
Course Name	Mathematical Foundations for Machine Learning
Nature of Exam	Closed Book (Mid Sem - Regular)
Weightage for grading	30%
Duration	90 minutes
Date of Exam	20/07/2023 (AN)

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# Questions	4

Instructions

1. All questions are compulsory.
2. All parts of a question should be answered consecutively. Each answer should start from a fresh page.

Q Answer the following questions with justifications.

(1) Given the characteristic equation of a matrix A, can we compute the characteristic equation of cA where c is a non-zero scalar without knowing the entries of A? If so, show how to do it using detailed calculations. Otherwise explain why it is not possible. Clearly state all your assumptions

(2 Marks)

(2) Consider an inner product space with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y}$ defined with the help of matrix $\mathbf{A}$ defined below.

$$\mathbf{A} = \begin{bmatrix} 5.5 & -1.5 \\ -1.5 & 5.5 \end{bmatrix}$$

Consider two vectors $\mathbf{a} = [1 \quad 5]^T$ and $\mathbf{b} = [2 \quad 7]^T$ in the inner product space.

(a) Find the distance $d(\mathbf{a}, \mathbf{b}) = \|a - b\|$ between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space where $\| \cdot \|$ is the norm induced by the inner product.

(2 marks)

(b) Find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space.

(2 marks)

Q Answer the following for the given matrix:

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(A) Using elementary row operations, write the matrix in its row echelon form

(2 marks)

(B) Let $\mathbf{V}$ be a vector subspace spanned by the columns of matrix $\mathbf{A}$. Find the basis and dimension of $\mathbf{V}$.

(2 marks)

(C) Let $\mathbf{V}$ be a vector subspace spanned by vectors $\mathbf{x}$, such that $\mathbf{Ax} = 0$. Find the basis and dimension of $\mathbf{V}$.

(2 marks)

(D) Give the set of linearly independent rows of $\mathbf{A}$. What is the number of vectors in this set?

(2 marks)

Q Answer the following for the given matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(A) Obtain the left-singular vectors of $\mathbf{A}$.

(3 marks)

(B) Obtain the right-singular vectors of $\mathbf{A}$.

(3 marks)

(C) Obtain the singular value matrix $\mathbf{\Sigma}$. What is the spectral norm of $\mathbf{A}$?

(2 marks)

Q Answer the following

(1) Compute the following for the function $f(x_1, x_2) = e^{x_1} + x_1 x_2 - \log(1 + x_2)$.

(A) The expression for gradient, its dimension and its value at $(1, 2)$.
(2 marks)

(B) The expression for the Hessian matrix, its dimension and its value at $(1, 2)$.
(2 marks)

(C) The derivative $\frac{df}{dt}$ using chain rule of differentiation when $x_1 = t^2 + 2at,\ x_2 = \sin(t)$.
(2 marks)

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(\boldsymbol{x}) = (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Find Taylor's polynomial of degree 1 of f at $[1, 1]$.
(2 Marks)