Answer Key and Marking Scheme

Q1 Answer

(1) We need to prove 3 properties of distance metric

Property 1: $d(x, y) \geq 0$ as $d(x, y)$ involves taking sum of non-negative numbers due to the use of $|\cdot|$, $\sum_{i=1}^{n} |x_i - y_i|$ can be zero only when each component of form $x_i - y_i = 0$. This only happens when $x = y$. Hence $d(x, y) = 0$ only for $x = y$. (0.5 marks)

Property 2: Since $|x_i - y_i| = |y_i - x_i|$, it can be concluded that $d(x, y) = d(y, x)$. (0.5 marks)

Property 3: Note $d(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$. Now using triangle inequality of ℓ_1 norm.

$$\sum_{i=1}^{n} |x_i - z_i + z_i - y_i| \leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| = d(x, z) + d(z, y)$$

Hence $d(x, y) \leq d(x, z) + d(z, y)$. (1 marks)

In summary d(x,y) is a distance metric.

(2) $\cos^2(\theta) = \frac{(x_1^T A x_2)^2}{(x_1^T A x_1).(x_2^T A x_2)}$. Substitute values given in question to get $\cos^2(\theta) = \frac{(37+26\alpha)^2}{(5\alpha^2+12\alpha+9).(154)}$ (1 marks)

(i) $cos(45°) = \frac{1}{\sqrt{2}}$. The previous equation after substitution of θ can be rearranged to get $291\alpha^2 + 1000\alpha + 676 = 0$ (0.5 marks)

Finally solution by quadratic formula is $\alpha = \frac{-1000 \pm \sqrt{1000^2 - 4*676*291}}{2*291}$

(ii) $cos(0°) = 1$. The previous equation after substitution of θ can be rearranged to get $94\alpha^2 - 76\alpha + 17 = 0$ (0.5 marks)
Finally solution by quadratic formula is $\alpha = \frac{76 \pm \sqrt{76^2 - 4*17*94}}{2*94}$. Observe that here α do not have real solution.

NOTE: Give 0.5 marks in the previous 2 steps if the student has written correctly upto the quadratic equation in each case. The final step in two subparts using quadratic formula is not compulsory

(3) a) Recall that $det(C_1)$ = Product of its eigenvalues. Since 0 is one of its eigenvalues, hence $det(C_1) = 0$ (0.5 marks).
Since $det(C_1) = det(AB) = det(A).det(B) = det(B).det(A) = det(BA) = det(C_2)$. Hence $det(C_2) = 0$. This means 0 is one of eigenvalues of C_2 as determinant is product of eigenvalues. (0.5 marks)

b) $G = PA$. Recall that $A = U\Sigma V^T$. So $G = PU\Sigma V^T$.
Now since P, U are orthogonal matrices $P^{-1} = P^T$ and $U^{-1} = U^T$. This means $(PU)^T = U^T P^T = U^{-1} P^{-1} = (PU)^{-1}$. Hence PU is orthogonal matrix. (0.25 marks)

Hence SVD of G is $(PU)\Sigma V^T$ (0.25 marks)

$E = AP$. Recall that $A = U\Sigma V^T$. So $E = U\Sigma V^T P$.
Now since P, V are orthogonal matrices $P^{-1} = P^T$ and $V^{-1} =$

V^T. This means $(V^TP)^T = P^TV = P^{-1}(V^{-1})^T = P^{-1}(V^T)^{-1} = (V^TP)^{-1}$. Hence V^TP is orthogonal matrix. (0.25 marks)

Hence SVD of E is $U\Sigma(V^TP)$ (0.25 marks)

(4) Taking partial derivatives we get $\frac{\partial f}{\partial x} = 2x + 3y$, $\frac{\partial f}{\partial y} = 3x + 3y^2$.
Now taking derivatives of fuctions $x(r)$ and $y(r)$ with respect to r we get $\frac{dx}{dr} = -\sin(r)$ and $\frac{dy}{dr} = \cos(r)$. (0.5 marks)
Now using chain rule

$$\frac{df}{dr} = \frac{\partial f}{\partial x}\frac{dx}{dr} + \frac{\partial f}{\partial y}\frac{dy}{dr} = (2x+3y)(-sin(r)) + (3x+3y^2)(cos(r))$$

(0.25 marks)

Now substituting definition of $x(r)$ and $y(r)$, we get

$$\frac{df}{dr} = (2cos(r) + 3sin(r))(-sin(r)) + (3cos(r) + 3(sin(r))^2)(cos(r))$$

Finally, simplifying we get

$$\frac{df}{dr} = -2sin(r)cos(r) - 3sin^2(r) + 3cos^2(r) + 3sin^2(r)cos(r)$$

(0.25 marks)

Q2

1) $f(\boldsymbol{X}) = \boldsymbol{a}^T\boldsymbol{X}\boldsymbol{a}$

i. Clearly, using identity, $\dfrac{\partial f}{\partial \boldsymbol{X}} = \boldsymbol{a}\boldsymbol{a}^T$. (1 mark)

ii. If $\boldsymbol{a}$ is a zero vector, then $\dfrac{\partial f}{\partial \boldsymbol{X}}$ is zero matrix and hence of rank 0. Therefore rank 1 approximation doesnot exist. (1 mark)
If $\boldsymbol{a}$ is a nonzero vector, then there exist $a_i \neq 0$.

Then $\dfrac{\partial f}{\partial \boldsymbol{X}} = \boldsymbol{a}\boldsymbol{a}^T = \begin{bmatrix} a_1\boldsymbol{a}^T \\ \vdots \\ a_i\boldsymbol{a}^T \\ \vdots \\ a_{1000}\boldsymbol{a}^T \end{bmatrix}$.

Now $R_j - \frac{a_j}{a_i}R_i$, $\forall j = 1, \cdots, 1000$ and $j \neq i$, will give $\begin{bmatrix} \mathbf{0}^T \\ \vdots \\ a_i\boldsymbol{a}^T \\ \vdots \\ \mathbf{0}^T \end{bmatrix}$.

By interchanging i^{th} and 1^{st} row we get $\begin{bmatrix} a_i\boldsymbol{a}^T \\ \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \end{bmatrix}$, which is REF and hence $\dfrac{\partial f}{\partial \boldsymbol{X}} = \boldsymbol{a}\boldsymbol{a}^T$ is rank 1 matrix. (2 marks)

Therefore, the rank 1 approximation exists and is the same as the actual matrix. (1 mark)

2) Now $f : \mathbb{R}^4 \to \mathbb{R}$ is defined as

$$f(\boldsymbol{x}) = [x_1, x_2, x_3, x_4] \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{x}^T \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \boldsymbol{x} \text{ when } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

i. $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \dfrac{\partial f}{\partial \boldsymbol{x}} = \boldsymbol{x}^T \left(\begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 6 & 3 & 1 \\ 2 & 4 & 0 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \right)$

$\Rightarrow \nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \boldsymbol{x}^T \left(\begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 12 & 7 & 3 \\ 5 & 7 & 0 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix} \right)$ (1 mark)

ii. $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \mathbf{0}^T$

$$\Rightarrow \quad \boldsymbol{x}^T \left(\begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 12 & 7 & 3 \\ 5 & 7 & 0 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix} \right) = \mathbf{0}^T$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 12 & 7 & 3 \\ 5 & 7 & 0 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix} \boldsymbol{x} = \mathbf{0}$$

To solve this, we will convert coefficient matrix into REF which is equal to

$$\begin{bmatrix} 2 & 5 & 5 & 2 \\ 0 & -0.5 & -5.5 & -2 \\ 0 & 0 & 48 & 20 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}$$

The rank of the matrix is 4 and the number of variables is also 4. Therefore, the solution is unique and the unique solution is $[0, 0, 0, 0]^T$. (2 marks)

Q3

a We see that when the transform $\mathbf{A}$ is applied four times, it results in the input vector itself. So, this can be rotation by 90° in the counter clockwise direction. (2 marks)

So, $\mathbf{A} = \begin{bmatrix} cos(90^\circ) & -sin(90^\circ) \\ sin(90^\circ) & cos(90^\circ) \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (2 marks) b For the matrix **A**, eigen values are:

$\lambda_1 = i$, $\lambda_2 = -i$ (1 mark)

eigen vector corresponding to eigen value $\lambda_1 = i$ is $[i, 1]^T$

eigen vector corresponding to eigen value $\lambda_2 = -i$ is $[-i, 1]^T$

(1 mark)

c **A** is rotation in counter clockwise. Hence, $\mathbf{A}^{-1}$ will be rotation in clockwise direction. (1 mark)

Yes, $(\mathbf{A}^{-1})^4$ will also be I, as applying 90° clockwise rotation successively to a vector 4 times gives back the same vector. (1 mark)

Q4

(1) (a) $dim(V) = \frac{n(n+1)}{2}$ (1.5 marks)

(b) $dim(W) = \frac{n(n+1)}{2}$ (1.5 marks)

(2) $\boldsymbol{k} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ (1.25 marks)

$R = \begin{bmatrix} 1 & -0.5 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (2.75 marks)