Final Exam AIML 2023

Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in Artificial Intelligence and Machine Learning

I Semester 2022-23

Course Number	AIMLC ZC416
Course Name	Mathematical Foundations for Machine Learning
Nature of Exam	Open Book
Weightage for grading	40%
Duration	150 minutes
Date of Exam	02/04/2023 (14:00 - 16:00)

# Pages	3
# Questions	8

Instructions

1. All questions are compulsory.
2. All parts of a question should be answered consecutively. Each answer should start from a fresh page.

(1) Let $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \ldots (\boldsymbol{x}_N, y_N)$ be N points on which we find a SVM classifier using the dual SVM formulation which is given below:

$$\text{maximize} \sum_{i=1}^{i=N} \alpha_i - \sum_{i=1}^{i=N} \sum_{j=1}^{j=N} \alpha_i \alpha_j y_i y_j \boldsymbol{x}_i . \boldsymbol{x}_j \text{ subject to}$$

$$\sum_{i=1}^{N} \alpha_i y_i = 0$$

$$\alpha_i \geq 0 \forall i$$

Let O_A be the value of the objective function at the optimal solution returned by the dual SVM formulation for this problem. Now we add a new point $(\boldsymbol{x}_{N+1}, y_{N+1})$ and find a SVM classifier by solving the dual formulation again. Let O_B be the value of the objective function at the optimal solution for this problem. Considering the following three relationships (a) $O_A > O_B$, (b) $O_A = O_B$, (c) $O_A < O_B$ determine which of these relationships are possible and give a mathematical argument for your answer in each case.

[5 Marks]

(2) Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_N$ be N points on which we perform Principle Components Analysis leading to the discovery of the principle component directions $\boldsymbol{b}_1, \boldsymbol{b}_2 \ldots \boldsymbol{b}_D$. The given data is now transformed to the points $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots \boldsymbol{y}_N$ where $\boldsymbol{y}_i = \boldsymbol{Q}\boldsymbol{x}_i + \boldsymbol{\mu}$ where $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ and $\boldsymbol{\mu}$ is a constant vector. Determine the principle components for the transformed set of points in terms of the old principle components. How much variance is accounted for by the first principal component for the transformed

set of points in terms of the variance accounted for by the first principle component for the original set of points? Justify your answer with mathematical arguments.

[5 Marks]

(3) Consider three linearly independent vectors in $\mathbb{R}^n$ named a_1, a_2 and a_3. Now construct three vectors $b_1 = a_2 - a_3$, $b_2 = a_1 - a_3$ and $b_3 = a_1 - a_2$. Now consider the set $\mathcal{Q} = \{b_1, b_2, b_3\}$. Prove or disprove that the set $\mathcal{Q}$ is linearly independent.

[5 Marks]

(4) Consider two sets named $\mathcal{H}_1$ and $\mathcal{H}_2$. It is known that these two sets are convex sets.
  (a) Prove or disprove that $\mathcal{H}_1 \cap \mathcal{H}_2$ is a convex set. Here $\cap$ represents the set intersection operation.
  (b) Prove or disprove that $\mathcal{H}_1 \cup \mathcal{H}_2$ is a convex set. Here $\cup$ represents the set union operation.

[5 Marks]

(5) A linear Algebra student arrived at an $n \times n$ real matrix given below

$$A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

Help the student to find singular value decomposition of full rank matrix A if the columns of A are orthogonal.

[5 Marks]

(6) A data analyst modeled the objective function $f : \mathbb{R}^n \to \mathbb{R}$ as product of squares of n feature x_i s and $n \geq 2$. He has to maximize the objective function such that sum of squares of n features is less than or equal to c^2 where $c \in \mathbb{R}$. Write the mathematical formulation of the problem and solve it. Using the above result prove the inequality

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n} \text{ for any } a_i > 0, i = 1, \cdots, n$$

[5 Marks]

(7) You are given the quadratic polynomial $f(x, y, z) = 2x^2 - 2xy - 4xz + y^2 + 2yz + 3z^2 - 2x + 2z$:
  (a) Write $f(x, y, z)$ in the form $f(x, y, z) = x^T Ax - b^T x$ where $x = (x, y, z)$, A is a real symmetric matrix, and b is constant vector.
  (b) Find the point (x, y, z) where f(x, y, z) is at an extremum.
  (c) Is this point a minimum, maximum, or a saddle point of some kind?

[5 Marks]

(8) Solve the System of equations by Gaussian elimination method

$2x_1 + 5x_2 + 2x_3 - 3x_4 = 3$

$3x_1 + 6x_2 + 5x_3 + 2x_4 = 2$

$4x_1 + 5x_2 + 14x_3 + 14x_4 = 11$

$5x_1 + 10x_2 + 8x_3 + 4x_4 = 4$

[5 Marks]