Answer Key and Marking Scheme

Q1

1) Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. Now $\{\boldsymbol{v} \in \mathbb{R}^m | \boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{A}_i,\ \alpha_i \in \mathbb{R},\ \forall i = 1 \cdots n\} = \{\boldsymbol{v} \in \mathbb{R}^m | \boldsymbol{v} = \sum_{i=1}^{n} \beta_i \boldsymbol{C}_i,\ \beta_i \in \mathbb{R},\ \forall i = 1 \cdots n\}$ as $\{\boldsymbol{C}_1, \cdots \boldsymbol{C}_n\} = \{\boldsymbol{A}_1, \cdots \boldsymbol{A}_n\}$.
Therefore, rank $(\boldsymbol{A})$ = rank $(\boldsymbol{C})$ and rank $(\boldsymbol{A}|\boldsymbol{b})$ = rank $(\boldsymbol{C}|\boldsymbol{b})$. (1 mark)
Hence the consistency of the new system of equations $\boldsymbol{CX} = \boldsymbol{b}$ is same as that of $\boldsymbol{AX} = \boldsymbol{b}$. (0.5 marks)

2) Let $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{r}_1^T \\ \boldsymbol{r}_2^T \\ \boldsymbol{r}_3^T \end{bmatrix}$ and so we have $\boldsymbol{B}^T = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 \end{bmatrix}$.
Therefore $\boldsymbol{BB}^T(p,q) = \boldsymbol{r}_p^T \boldsymbol{r}_q = \boldsymbol{B}(p,1)\boldsymbol{B}(q,1) + \boldsymbol{B}(p,2)\boldsymbol{B}(q,2)$ (0.5 marks)

$$
\begin{aligned}
\Rightarrow \frac{\partial \boldsymbol{BB}^T(p,q)}{\partial \boldsymbol{B}(i,j)} &= \delta_{pqij} \text{ where} \\
\delta_{pqij} &= \boldsymbol{B}(q,j) \text{ if } i = p, p \neq q \\
\delta_{pqij} &= \boldsymbol{B}(p,j) \text{ if } i = q, p \neq q \\
\delta_{pqij} &= 2\boldsymbol{B}(q,j) \text{ if } i = p = q \\
\delta_{pqij} &= 0 \text{ otherwise.}
\end{aligned}
$$

where $p, q, i = 1, 2, 3$ and $j = 1, 2$ (2 marks)

3) Let $\boldsymbol{A}, \boldsymbol{B}$ be two square matrices of order n and $f(x_1, x_2) = x_2 \cos(x_1)$
   i. Now $(\boldsymbol{A}^T\boldsymbol{BA})^T = \boldsymbol{A}^T\boldsymbol{B}^T\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{BA}$ since $\boldsymbol{B}$ is symmetric.
   Let $\boldsymbol{x} \neq \boldsymbol{0}$

$$
\begin{aligned}
\Rightarrow \quad & \boldsymbol{Ax} \neq \boldsymbol{0} \text{ as } \boldsymbol{A} \text{ is a full rank square matrix.} \\
\Rightarrow \quad & (\boldsymbol{Ax})^T\boldsymbol{BAx} > 0 \text{ as } \boldsymbol{B} \text{ is a positive definite matrix.} \\
\Rightarrow \quad & \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{BA})\boldsymbol{x} > 0 \\
\Rightarrow \quad & \boldsymbol{A}^T\boldsymbol{BA} \text{ is a positive definite matrix.}
\end{aligned}
$$

(1 mark)

   ii. $\nabla f(x_1, x_2) = \begin{bmatrix} -x_2 \sin(x_1), & \cos(x_1) \end{bmatrix}$, (0.5 marks)
   $\boldsymbol{A} = \nabla^2 f(x_1, x_2) = \begin{bmatrix} -x_2 \cos(x_1) & -\sin(x_1) \\ -\sin(x_1) & 0 \end{bmatrix}$. (0.5 marks)
   Now, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_C = \boldsymbol{x}^T\boldsymbol{Cy}$ is an inner product when $\boldsymbol{C}$ is symmetric, positive definite. But $\boldsymbol{C} = \boldsymbol{A}^T\boldsymbol{BA}$ and $\boldsymbol{B}$ is symmetric positive definite matrix of order 2. Clearly $\boldsymbol{C}$ is symmetric. From i. it is clear that $\boldsymbol{C}$ is positive definite if $\boldsymbol{A}$ is of rank 2.
   Now if $\sin(x_1) = 0$ then rank $(\boldsymbol{A}) < 2$ and if $\sin(x_1) \neq 0$ then rank$(\boldsymbol{A}) = 2$. Therefore, the condition for $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_C = \boldsymbol{x}^T\boldsymbol{Cy}$ to be

an inner product is $\sin(x_1) \neq 0$ which means $x_1 \neq n\pi, \forall n \in \mathbb{Z}$.
(1 mark)

iii. Now Taylor's 2nd degree polynomial approximation of f at $[\frac{\pi}{2}, 1]^T$ is given by

$$
\begin{aligned}
& f(\frac{\pi}{2}, 1) + \nabla f(\frac{\pi}{2}, 1) \begin{bmatrix} x_1 - \frac{\pi}{2} \\ x_2 - 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - \frac{\pi}{2} & x_2 - 1 \end{bmatrix} \nabla^2 f(\frac{\pi}{2}, 1) \begin{bmatrix} x_1 - \frac{\pi}{2} \\ x_2 - 1 \end{bmatrix} \\
= \quad & x_2(\frac{\pi}{2} - x_1)
\end{aligned}
$$

(1 mark)

Q2 Answer

1) Considering the positive definite property first. Let $\mathbf{x} \in \mathbb{R}^n$, then $f(\mathbf{x})$ will always be a positive number as long as $\mathbf{x} \neq 0$ as it simply counts the number of non-zero entries. (0.5 marks).

When $\mathbf{x} = 0$, then $f(\mathbf{x}) = 0$ and in no other case can $f(\mathbf{x})$ take the value zero. Hence $f(\mathbf{x})$ **satisfy both the components of positive definite property**. (0.5 marks).

Now consider, absolutely homegeneous property. It says $g(\alpha.\mathbf{x}) = |\alpha|.g(\mathbf{x})$. This cannot be true in the case of given $f(\mathbf{x})$ as scaling by $\alpha \neq 0$, will not change the count of non-zero entries. Hence $f(\mathbf{x})$ **do not satisfy absolutely homegeneous property**. (0.5 marks)
As an example consider, $x = [1 \quad 0 \quad 3]^T$ and $\alpha = 10$. $f(\alpha.\mathbf{x}) = 2$ but $\alpha.f(\mathbf{x}) = 20$. (0.5 marks)

2) i) Yes, It is possible to find the exact value of λ by deriving the constant β. This is because $\mathbf{A^T} = \mathbf{B^T B} = \mathbf{A}$. (0.5 marks)
Hence $\mathbf{A}$ is symmetric matrix and hence has only real eigenvalues. Hence $\beta = 0$ is the only possible value of β. Hence $\lambda = 7$. (0.5 marks)

ii) Observe that $\mathbf{A} = \mathbf{B^T B}$. Hence

$$\mathbf{x^T A x} = \mathbf{x^T B^T B x} = \mathbf{(Bx)^T (Bx)} \quad \mathbf{(0.5 marks)}$$

Now recall that $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|_2^2$. Hence $\mathbf{(Bx)^T(Bx)} = \langle \mathbf{Bx}, \mathbf{Bx} \rangle = \|\mathbf{Bx}\|_2^2$. But by positive definite property of norms $\|\mathbf{Bx}\|_2 \geq \mathbf{0}$. This implies that $\mathbf{x^T A x} \geq \mathbf{0}$. Hence the matrix $\mathbf{A}$ a positive semi-definite matrix. (0.5 marks)

3) i) To derive σ_1, we construct $\mathbf{A^T A}$ as $\mathbf{A^T A} = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$
Now, we derive the eigenvalues of $A^T A$ by constructing the characteristic equation $|\mathbf{A^T A} - \lambda \mathbf{I}| = \mathbf{0}$. By solving the characteristic equation we get $\lambda_1 = 25$ and $\lambda_2 = 0$. Hence, the largest singular value $\sigma_1 = 5$ (1 marks)

ii) Now we derive the eigenvalue corresponding to λ_1 by solving $\mathbf{A^T A x} = \mathbf{25}.\mathbf{x}$ to obtain a representative eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Hence the right singular vector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (1 marks)

iii) Now, recall that $u_1 = \frac{1}{\sigma_1} A v_1$.

$$\mathbf{u_1} = \frac{\mathbf{1}}{\mathbf{5}} \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

(1 marks)

iv)

$$\mathbf{B_1} = \sigma_1(\mathbf{u_1 v_1^T}) = \mathbf{5}. \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} . \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$$

(1 marks)

Q3

(1) $\mathbf{P^{-1}QP} = \mathbf{D}$ (diagonal matrix) (0.5)

Trace($\mathbf{Q}$)=sum of eigen values of $\mathbf{Q}$ = sum of elements of $\mathbf{D}$=6 (0.5)

(2) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$.

$det(A - \lambda * I)=0 \Rightarrow \lambda^2 - 10=0$

Every matrix satisfies characteristic equation

$\mathbf{A^2} = 10\mathbf{I} \Rightarrow \mathbf{A^8} = 10^4\mathbf{I}$ (1mark)

Multiplying A on both the sides $\mathbf{A^9} = \begin{bmatrix} 30000 & 10000 \\ 10000 & -30000 \end{bmatrix}$. (1 mark)

Alternate If they use eigendecomposition and obtain the result, give them the full marks.

(3) (a) $W = \{(a, b, c)|b = a + c\}$

$(0, 0, 0) \in W$ as $0 = 0 + 0$ (0.25 Mark)

Let $X = (x, y, z) \in W$ and $A = (a, b, c) \in W$, then let $\alpha \in R$ and $\beta \in R$

then consider $\alpha X + \beta A = \alpha(x, y, z) + \beta(a, b, c) = (\alpha x + \beta a, \alpha y + \beta b, \alpha z + \beta c)$

Consider $\alpha y + \beta b = \alpha(x + z) + \beta(a + c) = \alpha x + \alpha z + \beta a + \beta c = \alpha x + \beta a + \alpha z + \beta c$ (0.5 Mark)

Hence W s the subspace.

Basis(W) = $\{(1, 1, 0), (0, 1, 1)\}$ dim(W)=2 (0.5 Mark)

(b) Its not subspace since $(0, 0, 0)$ is not in subspace (0.5 Mark)

(4) $\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix}$ Compute Rank (A) as 4 using REF Alternate students can take the transpose of above matrix and find the rank. So dimension = 4. (2 Marks)

(5) $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}$ (1.25 Marks)

$\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 0 & 3.5 \end{bmatrix}$ (0.75 Mark)

Q4

(1) (a) alternative 1:

$D'A'$ and $D'C'$ in $P1$ are the standard basis unit vectors. So, we can find the matrix $\mathbf{A}^{-1}$ that can be applied to these to get DA and DC in $P0$ as follows:

$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}.$

So, the required matrix $\mathbf{A} = (\mathbf{A}^{-1})^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}.$ (2 Marks)

(b) alternative 2:

Let $A = \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$. be the matrix which transforms the corners in $P0$ to $P1$.

$\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix} * \begin{bmatrix} 0.5 & 0 & 1 & 1.5 \\ 1 & 0 & 0 & 1 \end{bmatrix}. = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow a11 * 0.5 + a12 * 1 = 0$

$\Rightarrow a21 * 0.5 + a22 * 1 = 1$

$\Rightarrow a11 * 1 = 1$

$\Rightarrow a21 * 1 = 0$

$\Rightarrow a11 * 1.5 + a12 * 1 = 1$

$\Rightarrow a21 * 1.5 + a22 * 1 = 1$

solving: $a11$=1, $a12$=−0.5, $a21$=0, $a22$=1 (2 Marks)

(2) $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}.$

$det(A - \lambda * I){=}0 \Rightarrow (1-\lambda)^2{=}0 \Rightarrow \lambda{=}1, 1$

$nullspace(A{-}\lambda_i{*}I) \Rightarrow$ (RREF of augmented matrix) $\Rightarrow \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$

eigenvector $= \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$, where t is the free variable. Thus, eigenvector is in the direction of x-axis. (2 Marks)

(3) algebraic multiplicity of $\lambda = 1$ is 2

geometric multiplicity of the eigenspace corresponding to eivenvalue of $\lambda = 1$ is dim of eigenspace (= 1) (2 Marks)