Emanuel Casiano-Diaz
PHYS 256 (Computational Physics) Final Project Report
Prof. Adrian Del Maestro

I. Introduction

In this project, string profiles of ideal, perfectly flexible strings are modeled as functions of time. It turns out that these profiles can be obtained by solving the wave equation, thus making it, in principle, a suitable problem to solve with numerical methods. Knowing this profiles lets us obtain the signal that the string motion produces. With this signal, some properties of the sound can be studied by computing the power spectrum. Even though in this project we shall be analyzing ideal strings, the goal is to effectively simulate the profiles of real instrument strings taking into account all of the imperfections that the real world imposes, such as string stiffness. Let's begin our discussion by introducing the model from which our results arise.

II. Model

Consider an ideal, perfectly flexible string. The profile of such a string in a range of time can be ultimately model by first solving the wave equation:

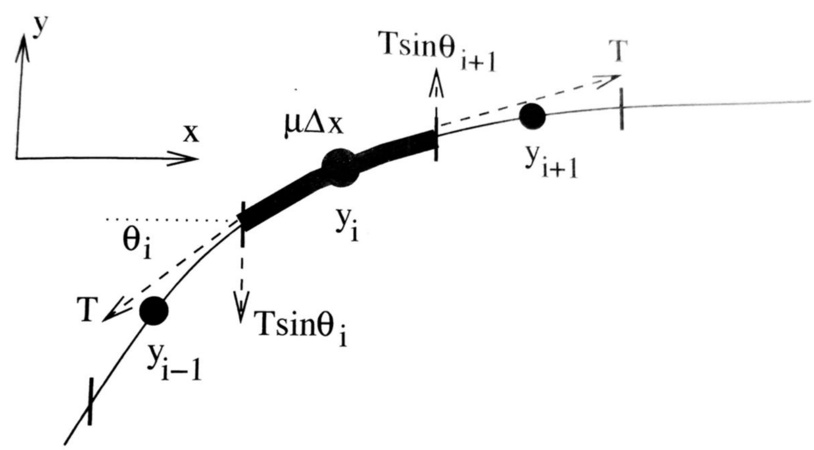
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{(Wave Equation)}$$

where c is the propagation speed of the wave, y is the transverse displacement from the rest position and x is the positions across the length of the string.

We can use Newton's Second Law to demonstrate that the motion of the string is governed by the wave equation. The following proof will follow from reference [1].

Figure1 - Force diagram on a string segment

Computational Physics, N. Giordano and H. Nakanishi



The above figure represents the forces acting on a segment of some string with linear mass density μ. Here, y_{i-1}, y_i *and* y_{i+1} represent the vertical displacements of the string at the spatial indices i across the length of the string. The is a force of tension T that is pulling the string segment from both sides. We can obtain the vertical components of such tensions by multiplying by the angles they form with respect to the horizontal. Now, let's setup the summation of forces on the **bold** string segment.

Newton's Second Law:

The total force acting on an segment Δx is:

$$ma_{y_i} = T sin\theta_{i+1} - T sin\theta_i$$

or

$$(\mu\Delta x)\ \frac{\partial^2 y_i}{\partial t^2} = T sin\theta_{i+1} - T sin\theta_i \qquad \textbf{(Eq. 1)}$$

Assume small angles and small vertical displacements such that we can approximate the sines as:

$$sin\theta_i \approx \frac{y_i - y_{i-1}}{\Delta x} \text{ , } sin\theta_{i+1} \approx \frac{y_{i+1} - y_i}{\Delta x}$$

Now substituting in Equation 1 we get:

$$(\mu\Delta x)\frac{\partial^2 y_i}{\partial t^2} = T(\frac{y_i - y_{i-1}}{\Delta x}) - T(\frac{y_{i+1} - y_i}{\Delta x})$$

$$= T[\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x}]$$

Solving for $\frac{\partial^2 y_i}{\partial t^2}$**:**

$$\frac{\partial^2 y_i}{\partial t^2} \approx \frac{T}{\mu}[\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}] \qquad \textbf{(Eq. 2)}$$

which is the equation of motion of the infinitesimal string segment i**.** Recall that the speed of propagation of a wave on a string is given by $c = \sqrt{\frac{T}{\mu}}$. Also, the fraction after this factor in **Eq 2** is none other than the centered 2nd derivative expression that we saw in class in our discussion of the solution to Laplace's Equation.Taking these into account, we can rewrite **Eq. 2** and notice that it is none other than the wave equation:

$$\frac{\partial^2 y_i}{\partial t^2} \approx c^2 \frac{\partial^2 y_i}{\partial x^2} \qquad Q.E.D$$

III. Numerical Method

It is noted in reference[1] that even though our problem seems to be almost suitable to be solved using methods like *Simultaneous Over Relaxation (SOR),* just like it is common to do in finding a numerical solution to Laplace's Equation , these are actually not suitable in our case. The reason for this is that our solutions are time-dependent, whereas this relaxation methods were used to get time-independent or stationary solutions. We will therefore employ a slightly different numerical approach.

Let's define a time-step Δt and spatial-step Δi such $t = n\Delta t$ and $x = i\Delta x$, where i,n = 0,1,2,3… We can rewrite **Eq 2** in finite difference form using centered derivatives:

$$\frac{y(n+1,i) - 2y(n,i) + y(n-1,i)}{(\Delta t)^2} = c^2\left[\frac{y(n,i+1) - 2y(n,i) + y(n,i-1)}{(\Delta x)^2}\right]$$

Multiply both sides by $(\Delta t)^2$:

$$y(n+1,i) - 2y(n,i) + y(n-1,i) = c^2\frac{(\Delta t)^2}{(\Delta x)^2}[y(n,i+1) - 2y(n,i) + y(n,i-1)]$$

Let $c^2\frac{(\Delta t)^2}{(\Delta x)^2} = r^2$ and solve for $y(n+1,i)$:

$$y(n+1,i) = r^2y(n,i+1) - 2r^2y(n,i) + r^2y(n,i-1) - y(n-1,i) + 2y(n,i)$$

Now with some factoring out of r^2 and regrouping, we get:

$$y(n+1,i) = 2y(n,i)[1-r^2] - y(n-1,i) + r^2[y(n,i+1)+y(n,i-1)] \qquad \textbf{(Eq. 3)}$$

Eq. 3 is of utmost importance, as it represents the function that will be updating the vertical displacement of each point across the string at the multiple time-steps.

The r^2 should be chosen to be 1 in our simulations. The higher order terms that are ignored when obtaining **Eq.3** are mostly cancelled out when this value is chosen to be 1.

Modifications of **Eq. 3** can be done such that we can correct for factors that would otherwise make our string more realistic. For example, in reference[3], a stiffness parameter is added in order to model piano strings more accurate. Nevertheless, we will focus mostly on perfectly flexible string for purposes of this project. There is plenty of interesting physics that we can analyze even with such an ideal model.

IV. Implementation

We now have an approximate solution of the wave equation that we can use to simulate string profiles over some time. I will go through the algorithm below, like a recipe. Also, I am just rephrasing what reference[1] has suggested in my own terms. In no way am I reinventing the wheel here with this algorithm.

Algorithm:

- Set the parameters. Make sure that $dt = \frac{dx}{c}$ such that $r = c\frac{\Delta t}{\Delta x} = 1$.

- Set an initial wave form $y0$ (i.e the initial 'pluck') of the string

-Update the position of the *ith* string element using **Eq.3** by looping from $i = 1 \ to \ i = M - 1$

*Note: Ends should be fixed across the simulation. In other words, don't update.

-Repeat for desired number of time steps.

As explained in section III, the choosing $r = 1$, takes care that higher order terms that were ignored in deriving **Eq.3** are canceled out. If, for example, $r > 1$, we would notice that the y values diverge and our simulation is ruined. On the other hand, if $r < 1$, our simulation doesn't diverge but will yield less accurate results.

In regards to our initial waveform, we can basically choose it to be any well behaved function that we want. In order to have a closer model of what a real string profile just before it is released from the 'pluck' position, we will use a gaussian function as the initial pluck in one case and a triangular pluck in another simulation. The two functions that we used for each initial string profile were:

$$f(x) = \exp[-k(x - x_o)^2] \qquad \text{(Gaussian Pluck)}$$

Where k tells us how 'wide' the gaussian bell is and x_0 is the value of x at which the bell is centered at.

And:

$$f(x) = \begin{cases} \dfrac{hx}{d}, & 0 \leq x \leq d \\ \dfrac{h(L-x)}{L-d}, & d < x \leq L \end{cases}$$

(Triangular Pluck)

Where h is the vertical displacement at the plucking center from the rest position, d is the x position of the plucking center and L is the length of the string.

If you're wondering why we are running updating spatial values only from $i = 1$ to $i = M - 1$, (where M is the number of spatial subdivisions we want), it is because we want to simulate strings that have fixed ends. Be careful not to choose the endpoints to be always zero. Instead, just make sure they are the same as whatever their initial values where. If they happen to be zero or not will depend of your selection of initial string profile.

Now let's make some comparisons with our implementation of the code to results from reference[1] to make sure everything is working correctly.

V. Benchmark Calculations

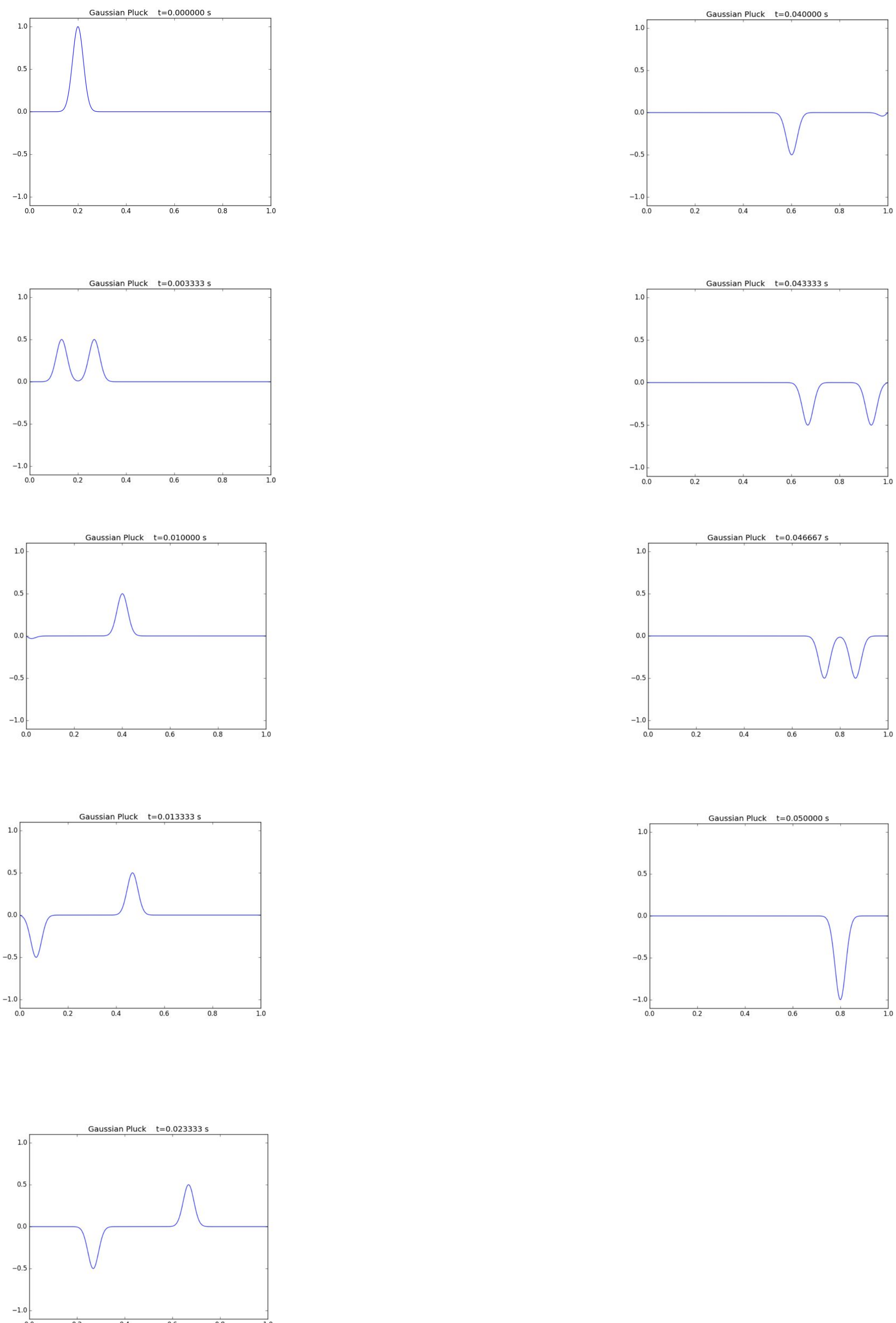
The following tables summarizes the parameters used to do our benchmarking. We are testing our string profiles behave the same way across time as those in reference[1]. The tests were done using a gaussian string profile. The rest of the parameters are summarized in this table.

In the following page, **Figure 2** shows the results for our simulation of the string at subsequent times. Initially, the wave is split into two wavefronts travelling in opposite directions. Reflection and inversion occurs when these fronts touch the fixed ends. Notice also how they combine when they meet. Our results are in agreement with reference[1]. **NOTE: .mp4 simulation in in folder OUTPUT**

Table 1: Parameters for benchmarking

Parameters	Value
String Length (L)	1 m
Propagation Speed (c)	300 $\frac{m}{s}$
Spatial Step Size (Δx)	0.01 m
Gaussian Bell Width (k)	1000 m^{-2}
Plucking Center (x_o)	0.3 m
Spatial Subdivisions (M)	1500

Figure 2: String profiles at subsequent times using a Gaussian Pluck (Benchmarking)

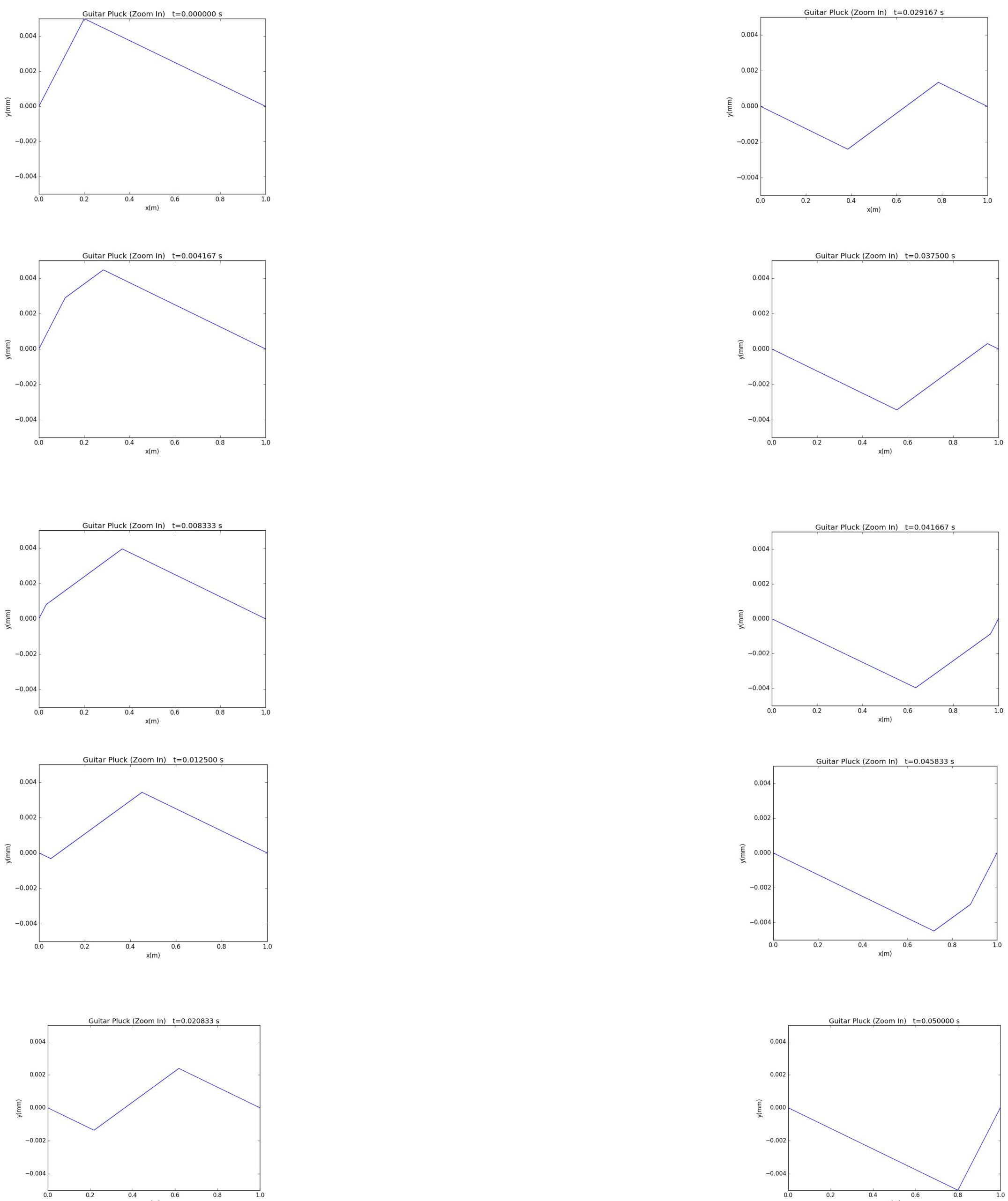


VI. Results and analysis

The triangular pluck (Guitar String)

Now that we have seen the behavior of the string profile when the initial profile is modeled as a gaussian, let's observe the behavior when the initial string profile is a triangular function instead. This model, will mostly resemble the string of a guitar, or any other plucked instrument for that matter.

Figure 3: String Profiles of a Guitar Pluck over subsequent times



The parameters used for the simulation above where the same as in **Figure 2** but using the triangular function that we talked about in section IV. The plucking center was at $d = 0.2\ m\ \ or\ (L/5)$ and the initial displacement was $h = 0.005\ mm$.

Now let’s observe the motion of the string. Just after letting go of the string, the initial kink is split into two kinks that travel in opposite directions. In the same manner as in **Figure 2** of the Gaussian Pluck, the waves are inverted and reflected at the endpoints. In **Figure 3** we only see half a period. After a whole period, the string just goes back to it’s initial profile.

Spectral Analysis

With our string profiles ready, we can study the sound properties of the string.

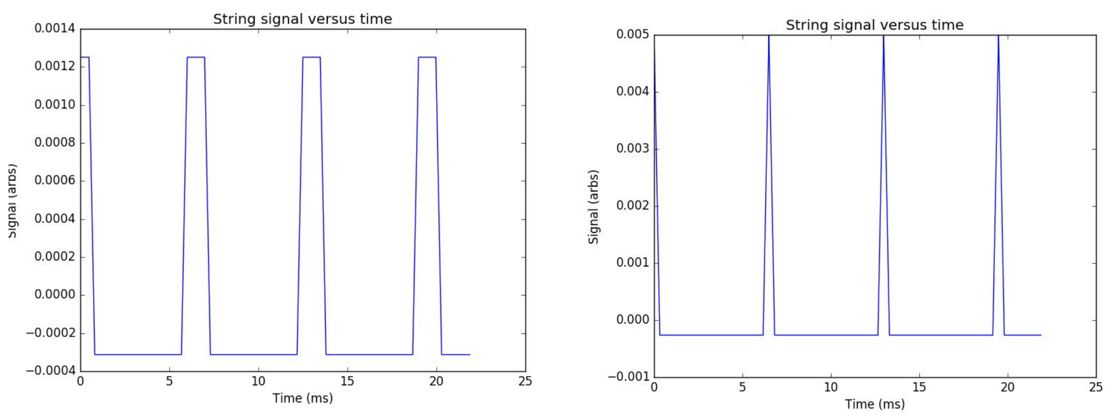
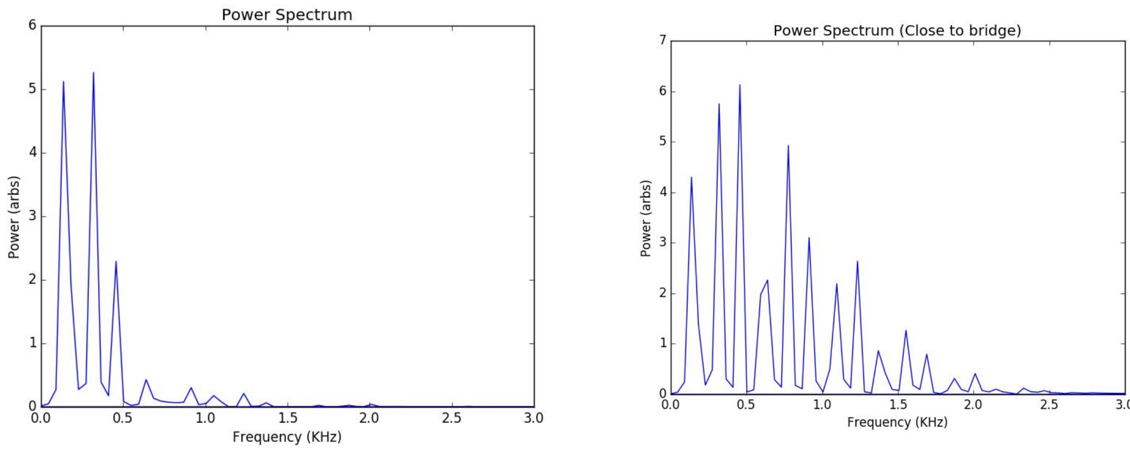


Figure 4 Motion of single point across time

This is the motion of a point on the string we modeled in **Figure 3.** Such point has been chosen to be is usually taken to be around 5% away from one endpoint. In the plot to the right, all the parameters have been kept the same except the plucking point which is now only at a position $\frac{L}{20}$.We can now do a Fourier Transform on the previous two plots and observe their power spectra.

Figure 5 Power Spectra

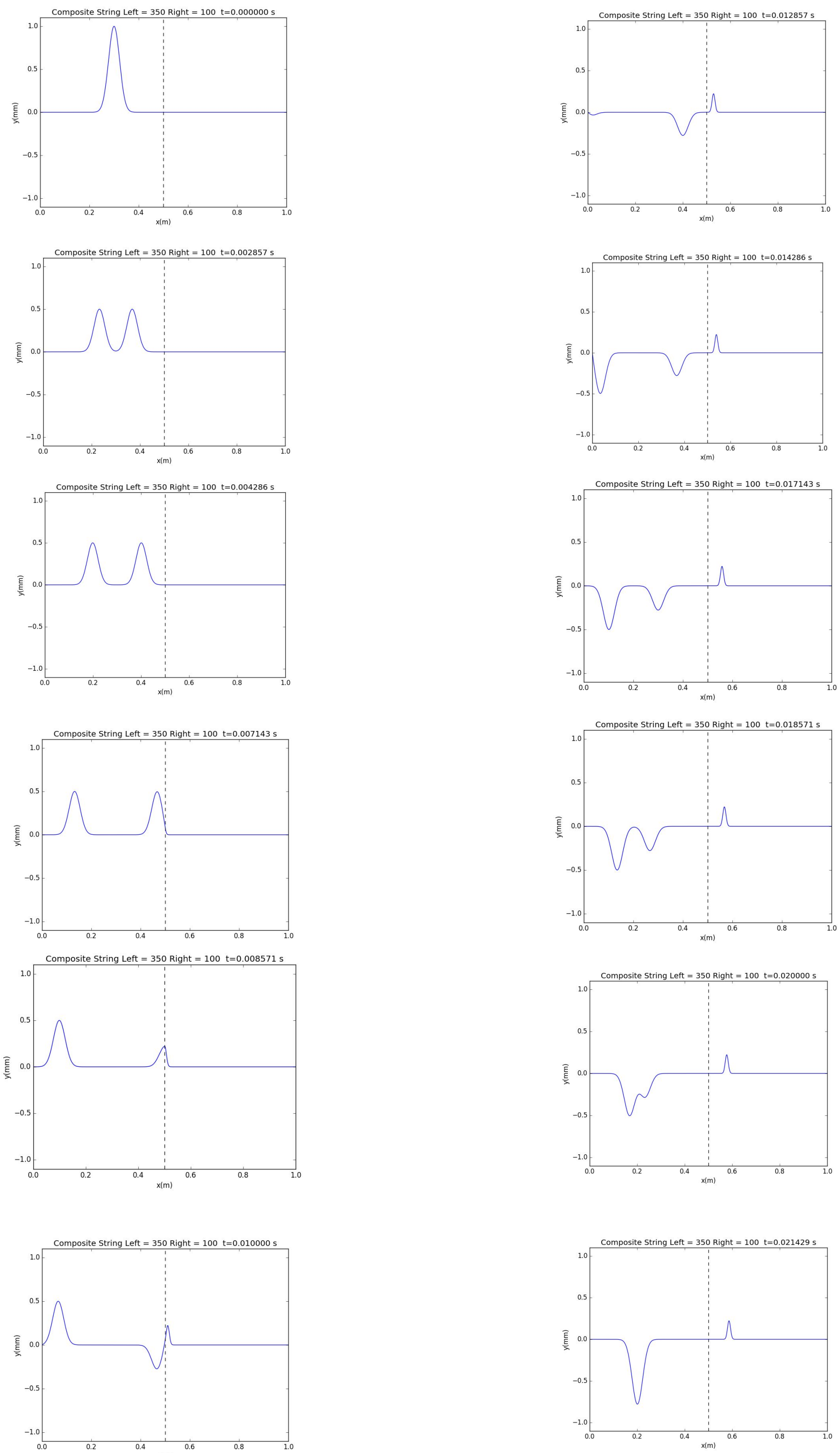


Again, the plot on the left corresponds to plucking at a distance $\frac{L}{5}$ from one end of the string, while the plot on the right is for a plucking point of $\frac{L}{20}$. The peaks on these plots correspond to the harmonic frequencies. Note that the closer we pluck to the bridge, the more peaks we have, which is what gives guitars that timbre when plucking near the bridge. What's happening is that the more sound power is being carried by the harmonics because of how close these frequencies are to the fundamental frequency. Let's finish our discussion of waves on strings by considering the very special case of composite strings.

Composite Strings

A composite string is a string in which a wave will travel with some speed at some segments, while at another speed in other segments. In reference[1], composite strings are modelled such that a wave packet goes from a region in which the propagation speed is high to another where it's lower. We shall take a look at the interesting string profiles that we end up with by considering such systems.

Figure 6 Composite String Profiles



In **Figure 6** we can see a few profiles at subsequent time steps. We can observe that when the wave crosses to the right side, some of the wave packet is transmitted while some of it is reflected. In the figure above, the propagation speed in the left side is $300\frac{m}{s}$ while $100\frac{m}{s}$ on the right side.

VII. Discussion

By employing our numerical scheme, we have seen that solutions to the wave equation can be obtained such that we can model de behavior of ideal strings across a range of time. We saw that parameters such as the approximate shape of the string initially and the plucking center, affect the overall string profile and, therefore, sound. Also, by tweaking a few lines of code, we were able to simulate a composite string, which may prove useful in studying, for example, how electromagnetic waves propagate through different media. We can base ourselves on the conclusions we got from this project if we want to simulate more realistic string in the future.

VIII. References

[1]Vibrations, Waves and the Physics of Musical Instruments. (2006). In *Computational Physics*. Upper Saddle River, NJ: Pearson Prentice Hall.

[2]Gulla, J. (2011). *Modeling the Wave Motion of a Guitar String*. Trondheim, Norway: Trondheim Katedralskole

[3]Saitis, C. (2008). *Physical modelling of the piano: An investigation into the effect of string stiffness on the hammer-string interaction*. Belfast, Northern Ireland: Sonic Arts Research Centre.

[4]Rossing, T. D. (2010). *The science of string instruments*. New York: Springer.