Assignment 07: Numerical Integration and Random Numbers

Due Date: 12/02/2016

1. Electric Potential Above a Conducting Plane [10 points]

An infinite, thin plane sheet of conducting material has a circular hole of radius a cut in it and lies in the xy-plane. A thin flat disk of the same material, but a slightly smaller radius lies in the plane, filling the hole, but separated by a thin insulating ring. The disk is maintained at fixed potential V_d, while the infinite sheet is kept grounded at $V = 0$.

(a) Solve for the potential everywhere in three dimensions (using a suitable grid) for $V_d = 1$ and $a = 1$. Compare your result for the potential a distance z above the center of the disk ($x = y = 0$) with the analytical result:

$$V(z) = V_d \left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right). \tag{1}$$

(b) The potential V a perpendicular distance z above the *edge* of the disk (i.e. when $\sqrt{x^2 + y^2} = a$) can be written in terms of an elliptic integral:

$$V(z) = \frac{V_d}{2} \left(1 - \frac{kz}{\pi a} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}\right) \tag{2}$$

where $k = 2a/\sqrt{z^2 + 4a^2}$. Using numerical integration, evaluate $V(z)$ as a function of z for $V_d = 1$ and $a = 1$ and compare with your results for the potential on a grid.

BONUS: Using appropriate cylindrical coordinates, derive the expressions in Eqs. (1) and (2).

2. Power Law Singularities [10 points]

Consider the integral:

$$I(\alpha) = \int_0^1 \frac{dx}{(1 - x)^\alpha}$$

where $0 \leq \alpha < 1$.

(a) In class, we saw how to use a mapping to transform an infinite integral onto a finite region. Using a similar approach, derive a mapping to deal with the integrable power law singularity in I.

(b) Numerically compute the integral using Simpson's method as a function of α and plot the result, comparing with the analytical value.

(c) Determine how many intervals N are needed to obtain an accuracy of $\epsilon \leq 10^{-4}$ for each α, *i.e.* make a plot of N vs. α. Discuss your result.

3. Testing for Pseudorandomness [5 points]

One of the easiest ways to determine if your random number generator is working is to visually inspect a plot of successive random numbers, i.e. $(x_i, y_i) = (r_i, r_{i+1})$ for N random numbers r_i. Compare $N = 10^3$ uniformly distributed random numbers on $[0, 1]$ generated with the Linear Congruential Generator with $m = 2^{31} - 1, a = 6, c = 7$ and $r_0 = 3$ with those generated by `np.random.random()`. Can you construct a set of parameters that work better?

4. White Noise **[5 points]**

Use the Box-Müller method we discussed in class to generate an array consisting of 2^{10} random numbers distributed according to the Gaussian distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$:

$$p(y)dy = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2} dy.$$

Treat these values as a time series and use the Fast Fourier Transform to compute the power spectrum. Plot your results; you have just generated some white noise.

5. Project Update **[5 points]**

Produce a single slide using Keynote, Beamer, Powerpoint, etc. describing your project, save it as a `pdf` file named `Lastname_FirstInitial.pdf` and upload it to BlackBoard. During class on Friday December 2nd, you will be asked to give a two minute summary of what you have accomplished thus far, and what you plan to achieve by December 16th. The two minute time limit will be strictly enforced.