Code used throughout the assignment can be found in the following git repository: https://github.com/ecasiano/PrinciplesOfComplexSystems

1 Exercise 1

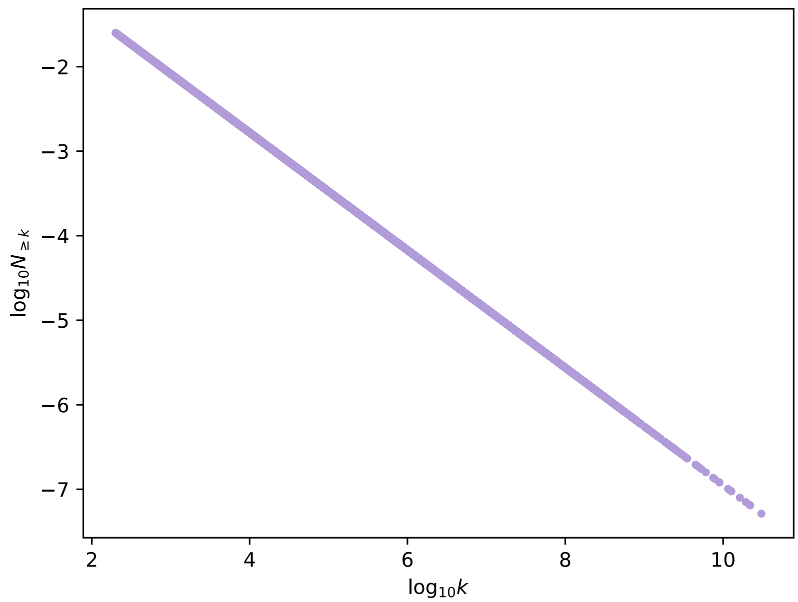


Figure 1: Logarithm of the complementary cumulative distribution of N_k with respect to the frequencies, k.

2 Exercise 2

The following values correspond to the N_k values in the previous figure:

CCDF exponent: $\gamma - 1 = 0.6951$ with $r^2 = 0.99455$
Mean*: $\mu_{N_k} = 3.36372e+06$
Standard Deviation*: $\sigma_{N_k} = 1.26384e+08$
95% Confidence Interval** : $[2.84624e+06, 3.88120e+06]$

$*$ Mean and Standard Deviation correspond to the full distribution, not only on the range shown.

$**$ Obtained via $\mu_{N_k} \pm \frac{2\sigma_{N_k}}{\sqrt{size(k)}}$

3 Exercise 3

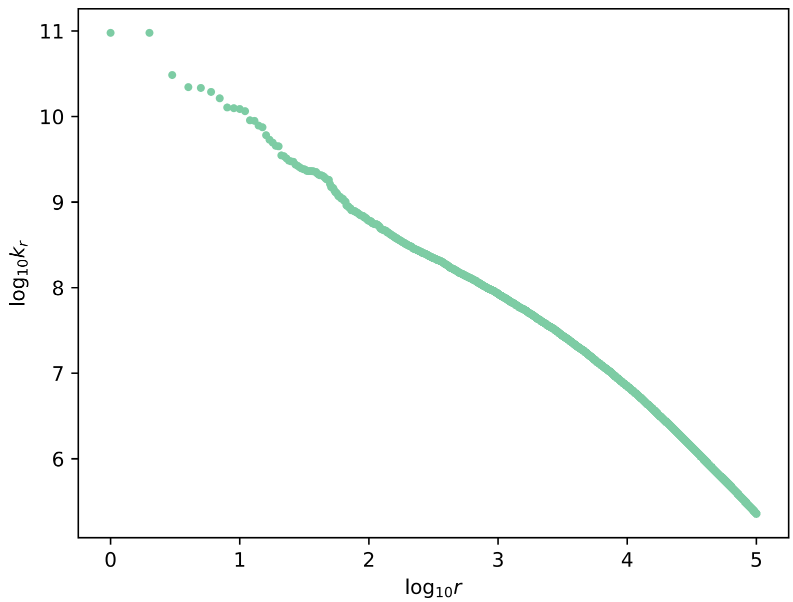


Figure 2: Logarithm of the word frequencies, k_r with respect to the logarithm of their rank, r.

4 Exercise 4

The following values correspond to the k_r values in the previous figure:

Zipf's exponent: $\alpha = 1.41427563$ with $r^2 = 0.99961232$
Mean*: $\mu_{k_r} = 7.54253e + 04$
Standard Deviation*: $\sigma_{k_r} = 4.01540e + 07$
95% Confidence Interval** : $[5.36394e + 04, 9.72111e + 04]$

$*$ Mean and Standard Deviation correspond to the full distribution, not only on the range shown.

$**$ Obtained via $\mu_{k_r} \pm \frac{2\sigma_{k_r}}{\sqrt{size(k)}}$

5 Exercise 5

The rank distribution exponent and the size distribution exponents are related via:

$$\alpha = \frac{1}{\gamma - 1}$$

In the previous questions, it was seen that the Zipf exponent was $\alpha \approx 1.41427563$ and $\gamma - 1 \approx 0.6951$. Thus the right hand side becomes:

$$\frac{1}{\gamma - 1} \approx 1.43864192$$

and

$$\alpha \approx 1.41427563$$

The relative error between the two expressions is then:

$$|\frac{1.43864192 - 1.41427563}{1.43864192}| X100 \approx 1.69\%$$

6) a)

g_{+1} = # of (+1) moves

g_{-1} = # of (-1) moves

$$\frac{(g_{+1}+g_{-1})!}{g_{+1}!\,g_{-1}!}$$

gets rid of permutations of (+1) & (-1) moves

i) $g_{+1}+g_{-1}=2n$ $\Rightarrow 2g_{+1}=2(n+K)$

ii) $(+1)g_{+1}+(-1)g_{-1}= \quad g_{+1}-g_{-1}=2K$ $\Rightarrow 2g_{-1}=2(n-K)$

OR $g_{+1}=n+K$

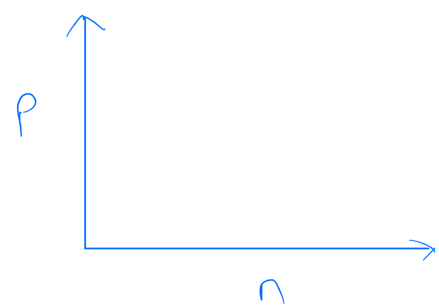
$g_{-1}=n-K$

$$\boxed{N(0,2K,2n)=\frac{(2n)!}{(n+K)!\,(n-K)!}} \equiv N_0$$

b) $$\frac{\text{\# of paths ending at } 2K}{\text{\# of total paths}} = \text{Probability of landing at } 2K$$

$$\Rightarrow P(X_{2n}=2K)=\frac{N(0,2K,2n)}{2^{t=2n}}$$

$$P(2K)=\frac{(2n)!}{2^{2n}(n+K)!\,(n-K)!}=P$$



$$\Rightarrow \ln P=\ln[(2n)!]-\ln[(n+K)!]-\ln[(n-K)!]-2n\ln 2$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$ (Stirling's Approximation)

$$\ln n! \approx \ln\sqrt{2\pi n} + n\ln n - n\ln e$$

$$\approx \ln\sqrt{2\pi} + \ln(n)^{1/2} + n\ln n - n$$

$$\approx \frac{1}{2}\ln n + n\ln n - n + \frac{1}{2}\ln 2\pi$$

$$\ln n! \approx \ln n\left[\frac{1}{2} + n\right] - n + \frac{1}{2}\ln 2\pi$$ (Stirling's accounting for error)

$$\ln P = \ln[(2n)!] - \ln[(n+k)!] - \ln[(n-k)!] - 2n\ln 2$$

$$\ln[(2n)!] = \left[\frac{1}{2} + 2n\right]\ln(2n) - 2n + \frac{1}{2}\ln 2\pi$$

$$\ln[(n+k)!] = \left[\frac{1}{2} + n + k\right]\ln(n+k) - (n+k) + \frac{1}{2}\ln 2\pi$$

$$\ln[(n-k)!] = \left[\frac{1}{2} + n - k\right]\ln(n-k) - (n-k) + \frac{1}{2}\ln 2\pi$$

$$\ln P = \left[\frac{1}{2} + 2n\right]\ln 2n - 2n + \cancel{\frac{1}{2}\ln}\, 2\pi$$

$$- \left[\frac{1}{2} + n + k\right]\ln(n+k) + (n+k) - \cancel{\frac{1}{2}\ln}\, 2\pi$$

$$- \left[\frac{1}{2} + n - k\right]\ln(n-k) + (n-k) - \frac{1}{2}\ln 2\pi$$

$$- 2n\ln 2$$

let $\ln(n+k) = \ln[n(1+\frac{k}{n})] = \ln n + \ln(1+\frac{k}{n}) \approx \ln n + \frac{k}{n} - \frac{k^2}{2n^2}$

& $\ln(n-k) = \ln[n(1-\frac{k}{n})] = \ln n + \ln(1-\frac{k}{n}) \approx \ln n - \frac{k}{n} - \frac{k^2}{2n^2}$

$$\ln P \approx [\tfrac{1}{2} + 2n]\ln 2n - \cancel{2n} + \cancel{n} + \cancel{k} + \cancel{n} - \cancel{k} - \tfrac{1}{2}\ln 2\pi - 2n\ln 2$$
$$+ [-\tfrac{1}{2} - n - k][\ln n + \tfrac{k}{n} - \tfrac{k^2}{2n^2}]$$
$$+ [-\tfrac{1}{2} - n + k][\ln n - \tfrac{k}{n} - \tfrac{k^2}{2n^2}]$$

$$\ln P \approx \tfrac{1}{2}\ln 2n + 2n\ln 2n - \tfrac{1}{2}\ln 2\pi - 2n\ln 2$$
$$-\tfrac{1}{2}\ln n - n\ln n - \cancel{k\ln n} - \cancel{\tfrac{1}{2}\tfrac{k}{n}} - \cancel{k} - \frac{k^2}{n} + \frac{k^2}{4n^2} + \frac{k^2}{2n} + \cancel{\frac{k^3}{2n^2}}$$
$$-\tfrac{1}{2}\ln n - n\ln n + \cancel{k\ln n} + \cancel{\tfrac{1}{2}\tfrac{k}{n}} + \cancel{k} - \frac{k^2}{n} + \frac{k^2}{4n^2} + \frac{k^2}{2n} - \cancel{\frac{k^3}{2n^2}}$$

$$\approx \tfrac{1}{2}\ln 2n + 2n\ln 2n - \tfrac{1}{2}\ln 2\pi - 2n\ln 2 - \ln n - 2n\ln n$$
$$-\frac{2k^2}{n} + \frac{k^2}{2n^2} + \frac{k^2}{n}$$

$$\approx \tfrac{1}{2}\ln 2n + 2n\ln 2n - \tfrac{1}{2}\ln 2\pi - 2n\ln 2 - \ln n - 2n\ln n - \frac{k^2}{n} + \overset{\approx 0}{\tfrac{1}{2}\left(\frac{k}{n}\right)^2}$$

$$\approx \ln[(2n)^{1/2}(2n)^{2n}(2\pi)^{-1/2}(2)^{-2n}(n)^{-1}(n)^{-2n}] - \frac{k^2}{n}$$

$$\approx \ln[2^{1/2}2^{-1/2}2^{2n}2^{-2n}n^{1/2}n^{-1}n^{2n}n^{-2n}\pi^{-1/2}] - \frac{k^2}{n}$$

$$\ln P \approx \ln[\tfrac{1}{\sqrt{n\pi}}] - k^2/n$$

$$\Rightarrow P = \frac{1}{\sqrt{n\pi}} e^{-k^2/n} \quad ; \quad \begin{array}{l} t = 2n \Rightarrow n = \frac{t}{2} \\ x = 2k \Rightarrow k = \frac{x}{2} \end{array}$$

$$= \sqrt{\frac{2}{\pi t}}\, e^{-\frac{\frac{x^2}{4}}{\frac{t}{2}}}$$

$$\boxed{P(x) = \frac{2}{\sqrt{\pi t}}\, e^{-\frac{x^2}{2t}}}$$

7)

$$N = \frac{(g_{+1} + g_{-1})!}{g_{+1}!\, g_{-1}!} \longrightarrow$$ Same distribution as previous problems but with different conditions on displacement & time.

$$\left.\begin{aligned} &\text{i)}\ g_{+1} + g_{-1} = t \\ &\text{ii)}\ g_{+1} - g_{-1} = j - i \end{aligned}\right\} \Rightarrow \begin{aligned} 2g_{+1} &= t + j - i \\ 2g_{-1} &= t - j + i \end{aligned}$$

$$\Rightarrow g_{+1} = \frac{1}{2}(t + j - i)$$

$$g_{-1} = \frac{1}{2}(t - j + i)$$

$$N = \frac{\left[\frac{1}{2}(2t)\right]!}{\left[\frac{1}{2}(t+j-i)\right]!\left[\frac{1}{2}(t-j+i)\right]!}$$

$$\boxed{N = \frac{[t]!}{\left[\frac{1}{2}(t+j-i)\right]!\left[\frac{1}{2}(t-j+i)\right]!} = \binom{t}{\frac{1}{2}(t+j-i)}}$$

8) $$N(1,1,2n-2) = \binom{2n-2}{(2n-2+1-1)/2}$$

$$N(1,1,2n-2) = \binom{2n-2}{n-1}$$

$$N(-1,1,2n-2) = \binom{2n-2}{n}$$

$$N_{fr}(2n) = \binom{2n-2}{n-1} - \binom{2n-2}{n}$$

$$= \frac{(2n-2)!}{(n-1)!(n-1)!} - \frac{(2n-2)!}{n!(n-2)!}$$

$$= \frac{n!(n-2)!(2n-2)! - (2n-2)!(n-1)!(n-1)!}{n!(n-2)!(n-1)!(n-1)!}$$

$$= \frac{n(\cancel{n-1})!(n-2)!(2n-2)! - (2n-2)!(\cancel{n-1})!(n-1)!}{n!(n-2)!(\cancel{n-1})!(n-1)!}$$

$$= (2n-2)!\left[\frac{(n-2)! - (n-1)!}{n!(n-2)!(n-1)!}\right]$$

$$= (2n-2)! \left[\frac{\cancel{(n-2)!} - (n-1)\cancel{(n-2)!}}{n!\,\cancel{(n-2)!}\,(n-1)!} \right]$$

$$= (2n-2)! \left[\frac{1 - n + 1}{n!\,(n-1)!} \right]$$

$$N_{fr}(2n) = \frac{(2n-2)!}{n!\,(n-1)!}\,(2-n)$$

$$\Rightarrow \ln N_{fr} = \ln (2n-2)! - \ln n! - \ln (n-1)! + \ln (2-n)$$

Recall: $\ln n! \approx \left[\frac{1}{2} + n\right] \ln n - n + \frac{1}{2} \ln 2\pi$ (Stirling's Approximation)

$$\ln N_{fr} \approx \left[\tfrac{1}{2} + 2n-2\right] \ln (2n-2) - (2n-2) + \cancel{\tfrac{1}{2} \ln 2\pi}$$

$$- \left[\tfrac{1}{2} + n\right] \ln n + n - \cancel{\tfrac{1}{2} \ln 2\pi}$$

$$- \left[\tfrac{1}{2} + n - 1\right] \ln (n-1) + (n-1) - \tfrac{1}{2} \ln 2\pi + \ln (2-n)$$

$$\approx \left[2n - \tfrac{3}{2}\right] \ln (2n-2) - \cancel{2n} + 2$$

$$- \left[n + \tfrac{1}{2}\right] \ln n + \cancel{n}$$

$$- \left[n - \tfrac{1}{2}\right] \ln (n-1) + \cancel{n} - 1 - \tfrac{1}{2} \ln 2\pi + \ln (2-n)$$

$$\approx \left[2n-\tfrac{3}{2}\right] \ln(2n-2) + 1$$

$$-\left[n+\tfrac{1}{2}\right] \ln n$$

$$-\left[n-\tfrac{1}{2}\right] \ln(n-1) - \tfrac{1}{2}\ln 2\pi + \ln(2-n)$$

$$\approx 2n\ln(2n-2) - \tfrac{3}{2}\ln(2n-2) - n\ln n - \tfrac{1}{2}\ln n$$

$$- n\ln(n-1) + \tfrac{1}{2}\ln(n-1) - \tfrac{1}{2}\ln 2\pi + \ln(2-n) + 1$$

$$\approx \ln\left[(2n-2)^{2n}(2n-2)^{-3/2} n^{-n} n^{-1/2} (n-1)^{-n}(n-1)^{1/2}(2\pi)^{-1/2}(2-n)\right] + 1$$

$$\approx \ln\left[(2n-2)^{2n-3/2} n^{-(n+\frac{1}{2})} (n-1)^{-n+\frac{1}{2}} (2\pi)^{-1/2}\right] + 1$$

$$\Rightarrow N_{fr}(2n) \approx (2n-2)^{2n-3/2} n^{-(n+\frac{1}{2})} (n-1)^{-n+\frac{1}{2}} (2\pi)^{-1/2} e$$

$$\approx 2^{2n-3/2} (n-1)^{2n-3/2} (n-1)^{-n+\frac{1}{2}} n^{-(n+\frac{1}{2})} (2\pi)^{-1/2} e$$

$$\approx 2^{2n-3/2} (n-1)^{n-1} n^{-n-\frac{1}{2}} (2\pi)^{-1/2} e$$

$$\approx 2^{2n-3/2} n^{n-1} \left(1-\tfrac{1}{n}\right)^{n-1} n^{-n-\frac{1}{2}} (2\pi)^{-1/2} e$$

$$\approx 2^{2n-3/2} n^{-3/2} (2\pi)^{-1/2} \left(1-\tfrac{1}{n}\right)^{n-1} e$$

$$N_{fr}(2n) \approx \frac{2^{2n-3/2}}{\sqrt{2\pi}\, n^{3/2}} \left(1-\tfrac{1}{n}\right)^{n-1} e$$

$$\lim_{n\to\infty} N_{fr}(2n) \approx \lim_{n\to\infty} \frac{2^{2n-3/2}}{\sqrt{2\pi}\, n^{3/2}} \left(1-\frac{1}{n}\right)^{n-1} e$$

$$\approx \lim_{n\to\infty} \frac{2^{2n-3/2}}{\sqrt{2\pi}\, n^{3/2}} \underbrace{\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n-1}}_{\left(\frac{1}{e}\right) \text{ exponential definition!}} e$$

$$\approx \lim_{n\to\infty} \frac{2^{2n-3/2}}{\sqrt{2\pi}\, n^{3/2}} \left(\frac{1}{e}\right) e$$

$$\lim_{n\to\infty} N_{fr}(2n) \approx \lim_{n\to\infty} \frac{2^{2n-3/2}}{\sqrt{2\pi}\, n^{3/2}}$$

$$\boxed{\therefore\ N_{fr}(2n) \approx \frac{2^{2n-3/2}}{\sqrt{2\pi}\, n^{3/2}}}$$