1) Find $P_{\geq}(x)$ & $P_{\geq}(A)$

Firs, the CCDF's of $p(x)$ will be determined

a) $p(x) = c x^{-(q+1)}$

$$\Rightarrow P_{\geq}(x) = c \int_{x}^{\infty} t^{-(q+1)} dt$$

$$= c \left. \frac{t^{-q}}{-q} \right|_{t=x}^{t=\infty}$$

$$= \frac{c}{q} x^{-q} \quad ; \quad \frac{c}{q} \equiv A$$

$$\boxed{P_{\geq}(x) = A x^{-q}}$$

b) $p(x) = c e^{-x}$

$$\Rightarrow P_{\geq}(x) = c \int_{x}^{\infty} e^{-t} dt$$

$$= -c e^{-t} \Big|_{t=x}^{t=\infty}$$

$$= -c e^{-x} \quad ; \quad -c \equiv A$$

$$\boxed{P_{\geq}(x) = A e^{-x}}$$

c) $p(x) = Ce^{-x^2}$

$$\Rightarrow P_{\geq}(x) = C\int_x^\infty e^{-t^2}\,dt$$

Let $w = t^2 \Rightarrow t = \pm\sqrt{w} \equiv \sqrt{w}$ (Remember there's a +/-)

$dw = 2t\,dt$ OR $dw = 2\sqrt{w}\,dt \Rightarrow dt = \dfrac{dw}{2\sqrt{w}}$

$w_{low} = x^2$, $w_{high} = \infty$

$$P_{\geq}(x) = A\int_{x^2}^\infty w^{-1/2}e^{-w}\,dw \quad ; \text{ where } \frac{C}{2} \equiv A \qquad (1)$$

Use I.P.P: Let $u = w^{-1/2}$ $\quad dv = e^{-w}dw$

$du = -\frac{1}{2}w^{-3/2}$ $\quad v = -e^{-w}dw$

$$\int_{x^2}^\infty w^{-1/2}e^{-w}\,dw = -w^{-1/2}e^{-w}\Big|_{w=x^2}^{w=\infty} - \frac{1}{2}\int_{x^2}^\infty w^{-3/2}e^{-w}\,dw$$

$$= +x^{-1}e^{-x^2} - \frac{1}{2}\int_{x^2}^\infty w^{-3/2}e^{-w}\,dw$$

$$\Rightarrow P_{\geq}(x) = Ax^{-1}e^{-x^2} - \frac{A}{2}\int_{x^2}^\infty w^{-3/2}e^{-w}\,dw$$

For $x \gg 1$, the left term dominates (Thanks Mathematica)

$$\boxed{P_{\geq}(x) \approx Ax^{-1}e^{-x^2}}$$

(for $x \gg 1$, which is probably the case since we're @ the tail of the distribution)

Now, $P_{\geq}(A)$ will be determined for each $p(x)$ & $p_{\geq}(x)$.

Recall: $P_{\geq}(A) = \int_{p^{-1}(A^{-\gamma})}^{\infty} p(x)\,dx = p_{\geq}(p^{-1}(A^{-\gamma}))$

For now, I'll ignore the proportionality constants

b) $p(x) = ce^{-x}$

$\Rightarrow p^{-1}(x) = -\log\left(\frac{x}{c}\right)$

$p_{\geq}(x) = ce^{-x}$

$P_{\geq}(A) = ce^{-p^{-1}(A^{-\gamma})}$

$= ce^{+\log\left(\frac{A^{-\gamma}}{c}\right)}$

$= \cancel{c}\,\frac{A^{-\gamma}}{\cancel{c}}$

$$\boxed{P_{\geq}(A) = A^{-\gamma}}$$

c) $p(x) = e^{-x^2}$

$p^{-1}(x) = \sqrt{-\log x}$, $p_{\geq}(x) = x^{-1}e^{-x^2}$

$e^{-y^2} = x$

$\Rightarrow -y^2 = \log x$

$\Rightarrow y^2 = -\log x$

$\Rightarrow y = \sqrt{-\log x} = p^{-1}$

$P_{\geq}(A) = (p^{-1})^{-1} e^{-(p^{-1})^2}$

$= (-\log x)^{-1/2} e^{+\log x}$

$= (-\log x)^{-1/2}\, x$; $x = A^{-\gamma}$

$= (-\log A^{-\gamma})^{-1/2} A^{-\gamma}$

$= (\gamma \log A)^{-1/2} A^{-\gamma}$; $C \equiv \gamma^{-1/2}$

$$\boxed{P_{\geq}(A) = C A^{-\gamma} [\log A]^{-1/2}}$$

a) $p(x) = x^{-(\varepsilon+1)}$

$$p^{-1}(x) = x^{-\left(\frac{1}{\varepsilon+1}\right)}$$

$$P_{\geq}(x) = x^{-\varepsilon}$$

$$P_{\geq}(A) = (p^{-1})^{-\varepsilon}$$

$$= \left(x^{-\left(\frac{1}{\varepsilon+1}\right)}\right)^{-\varepsilon}$$

$$= x^{\left(\frac{\varepsilon}{\varepsilon+1}\right)} \quad ; \quad x = A^{-\gamma}$$

$$= \left(A^{-\gamma}\right)^{\left(\frac{\varepsilon}{\varepsilon+1}\right)}$$

$$\boxed{P_{\geq}(A) = A^{-\gamma\left(\frac{\varepsilon}{\varepsilon+1}\right)}}$$

2) Discrete HOT

Cost: Expected fire size in a d-dimensional lattice

$$C_{fire} \propto \sum_{i=1}^{N_{sites}} p_i a_i \equiv f(a_i)$$

a_i → size of cluster where i belongs

p_i → probability of failure in site i over a given time period

Constraint:

$$C_{firewalls} \propto \sum_{i=1}^{N_{sites}} a_i^{\frac{(d-1)}{d}} a_i^{-1} \equiv g(a_i)$$

Show that: $p_i \propto a_i^{-\gamma}$, where $\gamma = 1 + \frac{1}{d}$

$$\frac{\partial f}{\partial a_i} = \lambda \frac{\partial g}{\partial a_i}$$

$$\sum_{i=1}^{N_{sites}} p_i = \lambda \sum_{i=1}^{N_{sites}} \frac{d}{da_i}\left[a_i^{\left(\frac{d-1}{d}\right)} a_i^{-1}\right]$$

$$\Rightarrow p_i = \lambda \frac{d}{da_i}\left[a_i^{(d-1)/d} a_i^{-1}\right]$$

$$= \lambda \frac{d}{da_i}\left[a_i^{\left(\frac{d-1}{d}\right)-1}\right]$$

$$= \lambda \frac{d}{da_i}\left[a_i^{\frac{-1}{d}}\right]$$

$$= \lambda \left(\frac{-1}{d} a_i^{\frac{-1}{d}-1}\right)$$

$$p_i = \frac{-\lambda}{d} a^{-\left(1+\frac{1}{d}\right)} \Longrightarrow \boxed{p_i \propto a^{-\left(1+\frac{1}{d}\right)}}$$

3) $P(i,j) = c\, e^{-i/\ell}\, e^{-j/\ell}$

let the characteristic scale be $\ell = \frac{L}{10}$

$$1 = \sum_{i,j} c\, e^{-i/\ell} e^{-j/\ell}$$

$$= c \sum_{i=1}^{L} e^{-i/\ell} \sum_{j=1}^{L} e^{-j/\ell}$$

$$1 = c\left[- \frac{1 - e^{-\frac{L}{\ell}}}{1 - \sqrt[\ell]{e}} \right]^2 \quad ; \ell = \frac{L}{10}$$

$$\Rightarrow \quad c = \left[\frac{1 - e^{1/\ell}}{1 - e^{-L/\ell}} \right]^2$$

General Normalization Constant of the Spark Probability

For $\ell = \frac{L}{10}$:

$$1 = c\left[- \frac{1 - e^{-10}}{1 - e^{10/L}} \right]^2$$

$$\Rightarrow \boxed{c = \left[\frac{1 - e^{10/L}}{1 - e^{-10}} \right]^2}$$

ρ = tree rate

$1-\rho$ = rate of no-tree

Design parameters to test: $D = [1, 2, L, L^2]$

The design parameter represents how many coords of the next planted tree will be randomly tested.

The yield is the average density of trees left unburned in a configuration after a single spark hits. It is expressed as:

$$Y = \rho - \langle f \rangle \quad \text{cost}$$

$$Y = \rho - \sum_{i,j} P(i,j)\, S(i,j)$$

Cluster size of the cluster where i,j lives.

Recipe to create a H.O.T forest:

1. Create $L \times L$ grid with no trees
2. Set the design parameter D ($D = [1, 2, L, L^2]$)
3. Repeat the following process $L \times L$ times: → Initialize yield: $Y = 0$

   a) Repeat the following process D times:

      i) Select a test-tree coordinate randomly

      If no-tree:

      ii) For the test tree coordinate, calculate the average cost:

      $$\langle C \rangle = \sum_{i,j} P(i,j)\, S(i,j)$$

      iii) Calculate the yield: $Y_{new} = \rho - \langle C \rangle$

      If tree: select another coord.

iv) If $Y_{new} > Y$, then $Y = Y_{new}$ & Let imax, jmax = i, j

b) Plant a tree @ imax, jmax