1 Exercise 1

a) $p(x) = cx^{-(q+1)}$

$$p_{\geq}(x) = c\int_{x}^{\infty} t^{-(q+1)}dt \tag{1}$$

$$= c\frac{t^-q}{-q}|_{t=x}^{t=\infty} \tag{2}$$

$$p_{\geq}(x) = \frac{c}{q}x^{-q} \tag{3}$$

For simplicity, let $\frac{c}{q} \equiv 1$. Recall also that:

$$P_{\geq}(A) = \int_{p^{-1}(A^{-\gamma})}^{\infty} p(x)dx = p_{\geq}(p^{-1}(A^{-\gamma})) \tag{4}$$

The inverse function of $p(x) = cx^{-(q+1)}$ is $p^{-1}(x) = x^{-(\frac{1}{q+1})}$. Thus,

$$P_{\geq}(A) = (p^{-1})^{-q} \tag{5}$$

$$= (x^{-(\frac{1}{q+1})})^{-q}|_{x=A^{-\gamma}} \tag{6}$$

$$P_{\geq}(A) = A^{-\gamma(\frac{q}{q+1})} \tag{7}$$

b) $p(x) = ce^{-x}$

$$p_{\geq}(x) = c\int_{x}^{\infty} e^{-t}dt \tag{8}$$

$$= -ce^{-t}|_{t=x}^{t=\infty} \tag{9}$$

$$p_{\geq}(x) = ce^{-x} \tag{10}$$

For simplicity, let $c \equiv 1$. The inverse function of $p(x) = e^{-x}$ is $p^{-1}(x) = -\log x$. Thus,

$$P_{\geq}(A) = e^{-p^{-1}(A^{-\gamma})} \tag{11}$$

$$= e^{\log A^{\gamma}} \tag{12}$$

$$P_{\geq}(A) = A^{-\gamma} \tag{13}$$

c) $p(x) = e^{-x^2}$

The Complementary Cumulative Distribution Function (CCDF) is:

$$p_{\geq}(x) = x^{-1}e^{-x^2} \tag{14}$$

(For full derivation, click here: INSERT LINK).

The inverse function of $p(x) = e^{-x^2}$ is $p^{-1}(x) = \sqrt{-\log x}$. Thus:

$$\begin{aligned} P_{\geq}(A) &= (p^{-1})^{-1}e^{-(p^{-1})^2} && (15) \\ &= (-\log x)^{-1/2}e^{\log x}|_{x=A^{-\gamma}} && (16) \\ P_{\geq}(A) &= A^{-\gamma}[\log A]^{-1/2} && (17) \end{aligned}$$

2 Exercise 2

The cost function for our discrete H.O.T is:

$$C_{fire} = K \sum_{i=1}^{N_{sites}} p_i a_i \tag{18}$$

And the constraint function is:

$$C_{firewalls} = D \sum_{i=1}^{N_{sites}} a_i^{\frac{(d-1)}{d}} a_i^{-1} \tag{19}$$

where K and D are proportionality constants.

Via the Lagrange Multiplier method, we get the equation:

$$\frac{\partial C_{fire}}{\partial a_i} = \lambda \frac{\partial C_{firewalls}}{\partial a_i} \tag{20}$$

Substituting the expressions for C_{fire} and $C_{firewalls}$ into the equation above:

$$K \sum_{i=1}^{N_{sites}} p_i = \lambda D \sum_{i=1}^{N_site s} \frac{d}{da_i}[a_i^{(\frac{d-1}{d})} a^{-i}] \tag{21}$$

Applying the product rule of derivatives to the right hand side and dropping the summations:

$$K p_i = \lambda D (\frac{-1}{d} a_i^{\frac{-1}{d}-1}) \tag{22}$$

which implies that:

$$p_i = \lambda \frac{D}{K} (\frac{-1}{d} a_i^{\frac{-1}{d}-1}) \tag{23}$$

$$\therefore p_i \propto a_i^{-(1+\frac{1}{d})} \tag{24}$$

3 Exercise 3

a) The probability of a starting a forest fire at site (i, j) is:

$$P(i,j) = ce^{-i/\ell}e^{-j/\ell} \tag{25}$$

where c is a normalization constant. Let the characteristic scale be $\ell = \frac{L}{10}$, where L denotes the linear size of the square lattice. Summing $P(i, j)$ over all possible lattice sites, the normalization constant can be determined:

$$1 = \sum_{i=1,j=1}^{L} ce^{-i/\ell}e^{-j/\ell} \tag{26}$$

$$= c\sum_{i=1}^{L} e^{-i/\ell} \sum_{j=1}^{L} e^{-j/\ell} \tag{27}$$

$$= c(-\frac{1-e^{-\frac{L}{\ell}}}{1-e^{\frac{1}{\ell}}})(-\frac{1-e^{-\frac{L}{\ell}}}{1-e^{\frac{1}{\ell}}})1 \qquad = c(-\frac{1-e^{-\frac{L}{\ell}}}{1-e^{\frac{1}{\ell}}})^2 \tag{28}$$

Solving for c and then substituting $\ell = \frac{L}{10}$, the normalization constant becomes:

$$c = (\frac{1-e^{\frac{10}{L}}}{1-e^{-10}})^2 \tag{29}$$