Cotargiu Georgiana - Cătălina
info engleză, 932/1
2491

Assignment bonus

Cod matricol: $24\underline{91} \to x^5+2x^4+x^3+x^2+2x+2 \in \mathbb{Z}_3[x]$

$g = x^5+2x^4+x^3+x^2+2x+2$

$g' = 5x^4+8x^3+3x^2+2x+2 = 2x^4+2x^3+2x+2 = x^4+x^3+x+1$

$\hookrightarrow (g, g') = 1 \Rightarrow g, g'$ - square free

$\alpha = (g_{ik}) \in M_5(\mathbb{Z}_3)$, where g_{ik}'s are given by:

$x^{3k} = \sum_{i=0}^{4} g_{ik} x^i \pmod{f}, \ k = 0, \dots, 4$

$V = \mathbb{Z}_3[x]/(f) = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 / a_0, \dots a_4 \in \mathbb{Z}_3\}$

$B = (1, x, \dots x^4)$

$1, x^3 \in B \to \begin{cases} 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 + 0\cdot x^4 \\ x^3 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + 1\cdot x^3 + 0\cdot x^4 \end{cases}$

$x^5+2x^4+x^3+x^2+2x+2 = 0 \Rightarrow x^5 = x^4 - x^3 - x^2 + x + 1$

$\Rightarrow x^6 = x^5 - x^4 - x^3 + x^2 + x =$
$(x \cdot x^5) = x^4 - x^3 - x^2 + x + 1 - x^4 - x^3 + x^2 + x = x^3 - x + 1$

$x^9 = x^6 - x^4 + x^3 =$
$(x^3 \cdot x^6) = x^3 - x + 1 - x^4 + x^3 = -x^4 - x^3 - x + 1$

$x^{12} = -x^7 - x^6 - x^4 + x^3$
$(x^3 x^9) = -x(x^3 - x + 1) - x^3 + x - 1 - x^4 + x^3$
$= -x^4 + x^2 - x - x^3 + x - 1 - x^4 + x^3$
$= x^4 + x - 2$

Hence, we get the matrix $Q = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

$$\Rightarrow Q - I_5 = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Let $\varphi : V \to V$, $\varphi(h) = h^3 - h \pmod{f}$

$\Rightarrow \varphi$ is a linear map and $[\varphi]_B = Q - I_5$

$\Rightarrow r = \dim \operatorname{Ker} \varphi = n - \operatorname{rank}(Q - I_5)$

$\operatorname{rank}(Q - I_5) = 3 \Rightarrow$ there exist 2 irreductible factors.

↳ the number of non-zero rows from an echalon form of the matrix $Q - I_5$

Since $\dim V = \deg(f) = 5 \Rightarrow V \cong \mathbb{Z}_3^5$

$\varphi : \mathbb{Z}_3^5 \to \mathbb{Z}_3^5$, $\operatorname{Ker} \varphi = \{ a \in \mathbb{Z}_3^5 \mid \varphi(a) = 0 \}$

$\Rightarrow \operatorname{Ker} \varphi = \{ a = (a_0, \dots, a_4) \in \mathbb{Z}_3^8 \mid (Q - I_5)[a] = [0] \}$

We get the following system:

$$\begin{cases} a_2 + a_3 - a_4 = 0 \Rightarrow a_2 = a_4 = -a_1 \\ -a_1 - a_2 - a_3 = 0 \\ -a_2 + a_4 = 0 \\ a_1 + a_2 + a_3 = 0 \Rightarrow a_1 = -a_2 \\ -a_3 = 0 \Rightarrow a_3 = 0 \end{cases}$$

that has the solution: $a_3 = 0$, $a_2 = a_4 = -a_1$, $a_0 \in \mathbb{Z}_3$

$\operatorname{Ker} \varphi = \{ (a_0, a_1, -a_1, 0, -a_1) / a_0, a_1 \in \mathbb{Z}_3 \}$

$= \langle (1,0,0,0,0), (0,-1,1,0,1) \rangle = \langle v_1, v_2 \rangle$

A basis of $\operatorname{Ker} \varphi$ is (v_1, v_2).

The associated polynomials are : $\begin{cases} h_1 = 1 \\ h_2 = -x + x^2 + x^4 \end{cases}$

$\gcd(g, h_2 + 1) = x^4 + x^2 + 2x + 1$

Hence $g = (x+2)(x^4 + x^2 + 2x + 1)$

2 : $g = (x+2)(x^4 + x^2 + 2x + 1)$