Lecture 2: More non-homotopical semantics; higher categories

Homotopical semantics of type theory

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1 Finite limit categories to models of type theory

First, we'll finish our discussion of the Clairambault and Dybjer's biequivalence between democratic models of type theory with Σ types and extensional Id types and finite limit categories. In the first lecture, we went in one direction: we showed that if $(\mathcal{C}, \pi)$ is a democratic natural model with these type constructors, then its category of contexts $\mathcal{C}$ has finite limits. This construction is due to Hofmann [Hof95], who describes it as an extension of a fibration-splitting construction due to Bénabou (see also Jacobs [Jac91, Proposition 1.3.6], Streicher [Str18, Theorem 3.1]).

Before giving the actual definition, we can get some guidance by looking back at last week's adventures. Starting from a model $(\mathcal{C}, \pi)$, we observed that the set of types in context $\Gamma \in \mathcal{C}$ forms a category, $\mathbf{Ty}_\Gamma$, that context extension defines a functor $\mathbf{Ty}_\Gamma \to \mathcal{C}/\Gamma$ from types over Γ to contexts over Γ, and that when $(\mathcal{C}, \pi)$ is democratic these functors are actually equivalences of categories. If we want to *construct* a democratic model $(\mathcal{C}, \pi^{\mathcal{C}} \colon \mathrm{Tm}^{\mathcal{C}} \to \mathrm{Ty}^{\mathcal{C}})$ from a category of contexts $\mathcal{C}$, then, we essentially need to take $\mathrm{Ty}^{\mathcal{C}}_\Gamma$ to be the collection of substitutions $\mathrm{Ob}(\mathcal{C}/\Gamma)$. Indeed, Seely [See84] does exactly this—but it does not quite work.

Remember that $\mathrm{Ty}^{\mathcal{C}}$ is supposed to be a presheaf over $\mathcal{C}$. Having chosen $\mathrm{Ty}^{\mathcal{C}}_\Gamma \in \mathbf{Set}$ for $\Gamma \in \mathcal{C}$, we must now choose $\mathrm{Ty}^{\mathcal{C}}_\gamma \colon \mathrm{Ty}^{\mathcal{C}}_\Gamma \to \mathrm{Ty}^{\mathcal{C}}_{\Gamma'}$ for $\gamma \colon \Gamma' \to \Gamma$. If we are defining $\mathrm{Ty}^{\mathcal{C}}_\Gamma \coloneqq \mathrm{Ob}(\mathcal{C}/\Gamma)$, then the only natural candidate for $\mathrm{Ty}^{\mathcal{C}}_\gamma$ is pullback along γ:[1]

$$
\begin{tikzcd}
\Delta \arrow[d, "\sigma"', "\mathrm{Ob}(\mathcal{C}/\Gamma) \ni"' near start] \\
\Gamma
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\gamma^*\Delta \arrow[r, dashed] \arrow[d, dashed, "\mathrm{Ty}^{\mathcal{C}}_\gamma(\sigma) \coloneqq \gamma^*\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \arrow[d, "\sigma"] \\
\Gamma' \arrow[r, "\gamma"'] & \Gamma.
\end{tikzcd}
$$

The definition of presheaf also requires that for $\Gamma'' \xrightarrow{\gamma'} \Gamma' \xrightarrow{\gamma} \Gamma$, we have $\mathrm{Ty}^{\mathcal{C}}_{\gamma\gamma'} = \mathrm{Ty}^{\mathcal{C}}_{\gamma'} \circ \mathrm{Ty}^{\mathcal{C}}_\gamma$: for our candidate definition, this means we have to choose our pullbacks such that for $(\sigma \colon \Delta \to \Gamma) \in \mathcal{C}/\Gamma$, the two leftmost vertical maps in the diagrams

$$
\begin{tikzcd}
(\gamma\gamma')^*\Delta \arrow[r] \arrow[d, "(\gamma\gamma')^*\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \arrow[d, "\sigma"] \\
\Gamma'' \arrow[r, "\gamma\gamma'"'] & \Gamma.
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\gamma'^*\gamma^*\Delta \arrow[r] \arrow[d, "\gamma'^*\gamma^*\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma^*\Delta \arrow[r] \arrow[d, "\gamma^*\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \arrow[d, "\sigma"] \\
\Gamma'' \arrow[r, "\gamma'"'] & \Gamma' \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
$$

coincide exactly. Thinking syntactically, the point here is that models of type theory we have a strict equation $A[\gamma][\gamma'] = A[\gamma\gamma']$ for combining substitutions applied to types. While the universal

[1] If we are working constructively, this assumes that by "finite limit category" we mean a category with a *choice* of pullback for each span. Note, however, that the problem with this definition will have nothing to do with constructivity.

property of pullbacks guarantees that $(\gamma\gamma')^*\sigma$ and ${\gamma'}^*\gamma^*\sigma$ are *isomorphic* as objects of $\mathcal{C}/\Gamma''$, there is no reason to expect they are *equal*. Besides it being unnatural in the first place to talk about equality of objects in a category, there are finite limit categories for which it is simply impossible to choose pullbacks in a way that satisfies this property [Lum17].

Hofmann solves this problem by modifying the definition of $\mathrm{Ty}^{\mathcal{C}}_{\Gamma}$: a type is equipped by definition with a coherent choice of representatives for its pullbacks.

Definition 1.1. Given a finite limit category $\mathcal{C}$, we define a natural model $(\mathcal{C}, \pi^{\mathcal{C}} \colon \mathrm{Tm}^{\mathcal{C}} \to \mathrm{Ty}^{\mathcal{C}})$ as follows:

- for $\Gamma \in \mathcal{C}$, $\mathrm{Ty}^{\mathcal{C}}_{\Gamma}$ is the set of functors $A \colon \mathcal{C}/\Gamma \to \mathcal{C}^{\to}_{\mathrm{cart}}$ making the diagram

$$\begin{tikzcd} \mathcal{C}/\Gamma \arrow[rr, "A"] \arrow[dr, "\mathrm{dom}"'] & & \mathcal{C}^{\to}_{\mathrm{cart}} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

  commute, where $\mathcal{C}^{\to}_{\mathrm{cart}}$ is the category of morphisms in $\mathcal{C}$ and cartesian squares between them. In other words, A sends each $\gamma \colon \Gamma' \to \Gamma$ to a morphism $A(\gamma) \colon \Gamma_A(\gamma) \to \Gamma$, and each morphism

$$\begin{tikzcd} \Gamma'' \arrow[rr, "\gamma'"] \arrow[dr, "\gamma\gamma'"'] & & \Gamma' \arrow[dl, "\gamma"] \\ & \Gamma & \end{tikzcd}$$

  in $\mathcal{C}/\Gamma$ to a pullback square

$$\begin{tikzcd} \Gamma_A(\gamma\gamma') \arrow[r] \arrow[d, "A(\gamma\gamma')"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma_A(\gamma) \arrow[d, "A(\gamma)"] \\ \Gamma'' \arrow[r, "\gamma'"'] & \Gamma'. \end{tikzcd}$$

  For every $\gamma \colon \Gamma' \to \Gamma$, we define $\mathrm{Ty}^{\mathcal{C}}_{\gamma} \colon \mathrm{Ty}^{\mathcal{C}}_{\Gamma} \to \mathrm{Ty}^{\mathcal{C}}_{\Gamma'}$ to be precomposition with the postcomposition functor $\gamma_! \colon \mathcal{C}/\Gamma' \to \mathcal{C}/\Gamma$: we send $A \colon \mathcal{C}/\Gamma \to \mathcal{C}^{\to}_{\mathrm{cart}}$ to $\sigma \mapsto A(\gamma\sigma) \colon \mathcal{C}/\Gamma' \to \mathcal{C}^{\to}_{\mathrm{cart}}$.

- For $\Gamma \in \mathcal{C}$, $\mathrm{Tm}^{\mathcal{C}}(\Gamma)$ is the set of pairs (A, a) where $A \in \mathrm{Ty}^{\mathcal{C}}(\Gamma)$ and a is a section of $A(\mathrm{id}_\Gamma) \colon \Gamma_A(\mathrm{id}_\Gamma) \to \Gamma$, that is, a morphism making the diagram

$$\begin{tikzcd} \Gamma \arrow[r, dashed, "a"] \arrow[dr, equal] & \Gamma_A(\mathrm{id}_\Gamma) \arrow[d, "A(\mathrm{id}_\Gamma)"] \\ & \Gamma \end{tikzcd}$$

  commute. For $\gamma \colon \Gamma' \to \Gamma$, $\mathrm{Tm}^{\mathcal{C}}_{\gamma} \colon \mathrm{Tm}^{\mathcal{C}}_{\Gamma} \to \mathrm{Tm}^{\mathcal{C}}_{\Gamma'}$ sends a section a to the section

$$\begin{tikzcd} \Gamma' \arrow[r, "\gamma"] \arrow[dr, dashed] \arrow[ddr, bend right, equal] & \Gamma \arrow[dr, "a"] & \\ & \Gamma_A(\gamma) \arrow[r] \arrow[d, "A(\gamma)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma_A(\mathrm{id}_\Gamma) \arrow[d, "A(\mathrm{id}_\Gamma)"] \\ & \Gamma' \arrow[r, "\gamma"'] & \Gamma \end{tikzcd}$$

  induced by the universal property of the pullback. (Note that $A(\gamma) = \mathrm{Ty}^{\mathcal{C}}_{\gamma}(A)(\mathrm{id}_{\Gamma'})$ by definition.) The natural transformation $\pi^{\mathcal{C}} \colon \mathrm{Tm}^{\mathcal{C}} \to \mathrm{Ty}^{\mathcal{C}}$ sends (A, a) to A.

- For a type $A\colon \maltese\Gamma \to \mathrm{Ty}^{\mathcal{C}}$, the context extension and its projection $\mathsf{p}_A\colon \Gamma.A \to \Gamma$ is $A(\mathrm{id}_\Gamma)\colon \Gamma_A(\mathrm{id}_\Gamma) \to \Gamma$. I leave it to you to fill in the remaining details.

We can now check that this natural model has the type formers we are looking for.

Proposition 1.2. For a finite limit category $\mathcal{C}$, the natural model $(\mathcal{C}, \pi^{\mathcal{C}}\colon \mathrm{Tm}^{\mathcal{C}} \to \mathrm{Ty}^{\mathcal{C}})$ has extensional Id types.

Proof. According to the description in the previous lecture, we need to fill in the dashed maps in a pullback square

$$\begin{tikzcd} \mathrm{Tm} \arrow[r, dashed, "\mathsf{refl}"] \arrow[d, "\langle \mathrm{id}_{\mathrm{Tm}}, \mathrm{id}_{\mathrm{Tm}}\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tm} \arrow[d, "\pi"] \\ \mathrm{Tm} \times_{\mathrm{Ty}} \mathrm{Tm} \arrow[r, dashed, "\mathsf{Id}"'] & \mathrm{Ty}. \end{tikzcd}$$

We define the natural transformation $\mathsf{Id}\colon \mathrm{Tm} \times_{\mathrm{Ty}} \mathrm{Tm} \to \mathrm{Ty}$ as follows: for $\Gamma \in \mathcal{C}$, $A \in \mathrm{Ty}^{\mathcal{C}}_{\Gamma}$ and sections $a_0, a_1\colon \Gamma \to \Gamma_A(\mathrm{id}_\Gamma)$ of $A(\mathrm{id}_\Gamma)$, we define $\mathsf{Id}_\Gamma(a_0, a_1) \in \mathrm{Ty}^{\mathcal{C}}_{\Gamma}$ as sending $\gamma\colon \Gamma' \to \Gamma$ to the equalizer

$$\begin{tikzcd} E \arrow[r, dashed, "\mathsf{Id}_\Gamma(a_0,a_1)(\gamma)"] & \Gamma' \arrow[r, shift left, "\mathrm{Tm}^{\mathcal{C}}_{\gamma}(a_0)"] \arrow[r, shift right, "\mathrm{Tm}^{\mathcal{C}}_{\gamma}(a_1)"'] & \Gamma_A(\gamma). \end{tikzcd}$$

I leave it to you to check that this is indeed natural in Γ provided that we choose our equalizers consistently (this is instructive!), to define $\mathsf{refl}\colon \mathrm{Tm} \to \mathrm{Tm}$, and to check the pullback condition. □

Exercise 1.3. For a finite limit category $\mathcal{C}$, show that the natural model $(\mathcal{C}, \pi^{\mathcal{C}}\colon \mathrm{Tm}^{\mathcal{C}} \to \mathrm{Ty}^{\mathcal{C}})$ has Σ types.

Now we have constructions in both directions, building categories from models and models from categories. Clairambault and Dybjer prove something stronger: a biequivalence of 2-categories

$$\begin{tikzcd} \mathbf{ETT}_{\Sigma,\mathsf{Id}} \arrow[r, shift left, "U"] & \mathbf{FL} \arrow[l, shift left, "H"] \end{tikzcd}$$

between a 2-category of models $\mathbf{ETT}_{\Sigma,\mathsf{Id}}$ and the 2-category $\mathbf{FL}$ of categories with finite limits.[2] We only described the *objects* of these two 2-categories and how to translate between them.

A 2-category consists of a collection of objects, sets of morphisms between objects, and sets of 2-morphisms between morphisms. The collection of finite limit categories comes with a natural 2-category structure: morphisms are finite-limit-preserving functors and 2-morphisms between functors are natural transformations. It is less obvious what the 1- and 2-morphisms of $\mathbf{ETT}_{\Sigma,\mathsf{Id}}$ should be. We *could* say that a morphism $(\mathcal{C}, \pi) \to (\mathcal{D}, \varpi)$ is simply a finite-limit-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ between the underlying categories of contexts, and a 2-morphism a natural transformation between these. Then we would get a biequivalence straight from the object-level maps we've already built, but in a somewhat trivial way.

We would rather describe the morphisms of $\mathbf{ETT}_{\Sigma,\mathsf{Id}}$ in terms of how they preserve the type-theoretic structures. Clairambault and Dybjer do exactly this, defining *pseudo cwf-morphisms* between models and *pseudo cwf-transformations*[3] between them. It is important that the structure-preservation conditions on these 1- and 2-morphisms be sufficiently weak that we have equivalences $HU(\mathcal{C}, \pi) \simeq (\mathcal{C}, \pi)$. In the homotopical version of this story, we will likewise need to find an appropriate notion of homotopy equivalence between models of intensional type theory.

[2] "Biequivalence" is a notion of equivalence between weak 2-categories ("bicategories"); in this case the two 2-categories happen to be strict, but the biequivalence is not.

[3] Corrected in an erratum [CD].

2 Quasicategories

Now we leave the world of extensional type theory behind. We already discussed type theory with intensional identity types in the first lecture, so now we will introduce the other end of the correspondence we hope to establish: $(\infty, 1)$-*categories with finite limits*. An $(\infty, 1)$-category is, intuitively, a weak kind of category where instead of *equality* of morphisms, we speak of "2-cells" which relate morphisms to each other and which are themselves related by 3-cells and so on. The "∞" refers to the fact that this structure goes on forever, while the "1" refers to the fact that the n-cells will be "invertible" for $n \geq 1$.

There are many different "models" for $(\infty, 1)$-categories, that is, definitions of what an $(\infty, 1)$-category is, which are all in some sense equivalent. We will start with the most commonly-used notion, that of *quasicategory*. Quasicategories were introduced by Boardman and Vogt (as simplicial sets satisfying the "restricted Kan condition") [BV73, Definition 4.8]; their theory was developed further by Joyal [Joy02; Joy08], and they are used extensively by Lurie [Lur09] in his work on higher topos theory.

I will follow closely Szumiło's presentation [Szu17, §2] (of the dual case of quasicategories with colimits), which nicely singles out the aspects of the theory most relevant to our journey.

Definition 2.1 (Simplex category)**.** Write Δ for the category of finite nonempty linear orders, $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for $n \geq 0$, and monotone functions between them.

The simplex category can be seen as a full subcategory of **Cat**: any partially ordered set can be seen as a category with a unique morphism $a \to b$ whenever $a \leq b$, and monotone maps between posets correspond exactly to functors between their corresponding categories. We think of the object $[n] \in \Delta$ as a "n-dimensional triangle", drawing the simplices [0], [1], [2], and [3] as a point, line, triangle, and tetrahedron like so:

$$0 \qquad 0 \longrightarrow 1 \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[d] \arrow[ddr] & \\ & 2 \arrow[dr] & \\ 0 \arrow[uur] \arrow[ur] \arrow[rr] & & 3. \end{tikzcd}$$

Definition 2.2 (Simplicial set)**.** A *simplicial set* is a presheaf $X \in \mathrm{PSh}(\Delta)$ over the simplex category. We write $\mathbf{sSet} \coloneqq \mathrm{PSh}(\Delta)$ for the category of simplicial sets, and we write $\Delta^n \coloneqq \text{よ}[n]$ for the representables.

A simplicial set X thus consists of sets $X_n \coloneqq X_{[n]} \in \mathbf{sSet}$ for each n, together with maps between them. We will define quasicategories to be certain simplicial sets, with each X_n defining the set of n-dimensional cells.

Definition 2.3 (Face maps)**.** For every $n \geq 0$ and $0 \leq i \leq n+1$, we have a monomorphism $d^i \colon [n] \rightarrowtail [n+1]$ defined by

$$d^i(j) \coloneqq \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i. \end{cases}$$

This is the unique monomorphism $[n] \rightarrowtail [n+1]$ whose image does not contain $i \in [n+1]$.

Definition 2.4 (Boundaries, horns)**.** For $n \geq 0$, we define the *boundary of the n-simplex* $\partial\Delta^n \in \mathrm{PSh}(\Delta)$ and the (n, k)-horn $\Lambda^n_k \in \mathrm{PSh}(\Delta)$ for $0 \leq k \leq n$ as follows:

$$\partial\Delta^n \coloneqq \operatorname*{colim}_{\substack{f \colon [m] \rightarrowtail [n] \\ m < n}} \Delta^m \qquad \Lambda^n_k \coloneqq \operatorname*{colim}_{\substack{f \colon [m] \rightarrowtail [n] \\ m < n \\ f \neq d^k}} \Delta^m$$

We have $\Lambda^n_k \rightarrowtail \partial\Delta^n \rightarrowtail \Delta^n$. We think of $\partial\Delta^n$ as an n-dimensional triangle missing its interior, and Λ^n_k as the n-dimensional triangle missing its interior and the $(n-1)$-dimensional face opposite the vertex labeled k. For example, Λ^2_0, Λ^2_1, and Λ^2_2 are drawn like so:

$$
\begin{tikzcd}[column sep=small]
 & 1 & \\
0 \arrow[ur] \arrow[rr] & & 2
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & 1 \arrow[dr] & \\
0 \arrow[ur] & & 2
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & 1 \arrow[dr] & \\
0 \arrow[rr] & & 2.
\end{tikzcd}
\tag{2.1}
$$

Definition 2.5. An *inner horn* is a horn Λ^n_k where $n \geq 2$ and $0 < k < n$. The horns Λ^n_0 and Λ^n_n for $n \geq 2$ are called *outer horns*.

Among the 2-horns (2.1), for example, the sole inner horn is the middle one (Λ^2_1). We observe that in a category, any diagram of morphisms that looks like Λ^2_1 can be completed or "filled" to a commutative triangle:

$$
\begin{tikzcd}[column sep=small]
 & b \arrow[dr, "g"] & \\
a \arrow[ur, "f"] & & c
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}[column sep=small]
 & b \arrow[dr, "g"] \arrow[d, phantom, "="] & \\
a \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & {} & c.
\end{tikzcd}
$$

This is not the case for the outer horns. It turns out that this filling property, generalized from Λ^2_1 to all inner horns Λ^n_k, is the right notion of composition for $(\infty, 1)$-categories.

Definition 2.6 (Quasicategory)**.** A *quasicategory* X is a simplicial set with the property that for every inner horn in X, *i.e.*, every $x \colon \Lambda^n_k \to X$ with $n \geq 2$ and $0 < k < n$, there exists an extension

$$
\begin{tikzcd}
\Lambda^n_k \arrow[r, "x"] \arrow[d, tail] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}
$$

to an n-simplex in X.

For a quasicategory X, an *object* $x \in X$ is a 0-simplex $x \colon \Delta^0 \to X$. A *morphism* $f \colon x \to y$ is a 1-simplex $f \colon \Delta^1 \to X$ with $f(0) = x$ and $f(1) = y$. Every 1-dimensional category can be seen as a particularly strict quasicategory by the following construction.

Definition 2.7. The *nerve* of a category $\mathcal{C} \in \mathbf{Cat}$ is the simplicial set $N\mathcal{C} \coloneqq \mathbf{Cat}(-, \mathcal{C})$.

The nerve of a category $\mathcal{C}$ is always a quasicategory. Its 0-simplices correspond to the objects of $\mathcal{C}$, while its 1-simplices correspond to the morphisms.

Definition 2.8. A *functor* between quasicategories X, Y is a morphism of simplicial sets $F \colon X \to Y$.

We expect that the collection of quasicategories itself has the structure of an $(\infty, 1)$-category (or $(\infty, 2)$-category), with quasicategories as objects and functors as 1-morphisms. For now, we just introduce a little bit of the 2-dimensional structure.

Definition 2.9. Write $E[1] \in \mathbf{Cat}$ for the *walking isomorphism*, the category

$$0 \xrightarrow{\cong} 1$$

consisting of two objects $0, 1$ and an isomorphism between them.

In 1-category theory, natural isomorphisms $\eta \colon F \cong G$ between functors $F, G \colon \mathcal{C} \to \mathcal{D}$ correspond exactly to functors $H \colon \mathcal{C} \times E[1] \to \mathcal{D}$ such that $H(-, 0) = F$ and $H(-, 1) = G$. We can translate this definition to quasicategories as follows.

Definition 2.10. Given quasicategories X, Y and functors $F, G\colon X \to Y$, an $E[1]$-*homotopy* $H\colon F \sim G$ is a morphism $H\colon X \times NE[1] \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[r, "X \times 0"] \arrow[dr, "F"'] & X \times NE[1] \arrow[d, "H"] & X \arrow[l, "X \times 1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commute, *i.e.*, with $H(-,0) = F$ and $H(-,1) = G$.

Definition 2.11. A functor $F\colon X \to Y$ between quasicategories is an $E[1]$-*homotopy equivalence* when there exists a functor $G\colon Y \to X$ with homotopies $GF \sim \mathrm{Id}_X$ and $FG \sim \mathrm{Id}_Y$.

Among 1-categories, we have the *groupoids*: categories where every morphism is invertible. The higher analogue for quasicategories is the *Kan complex.*

Definition 2.12 (Kan complex)**.** A *Kan complex* X is a simplicial set with the property that for every horn $x\colon \Lambda^n_k \to X$ with $n \geq 2$ and $0 \leq k \leq n$, there exists an extension

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "x"] \arrow[d, tail] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

to an n-simplex in X.

Proposition 2.13. A quasicategory X is a Kan complex if and only if every morphism in X is invertible, *i.e.*, every $f\colon \Delta^1 \to X$ factors through the inclusion $\Delta^1 \rightarrowtail NE[1]$:

$$\begin{tikzcd} \Delta^1 \arrow[r, "x"] \arrow[d, tail] & X. \\ NE[1] \arrow[ur, dashed] & \end{tikzcd}$$

We also think of Kan complexes as *spaces*: a 0-cell is a point, a 1-cell is a path, and 2-cell is some kind of surface, *etc.* The topological intuition inspires the following terminology.

Definition 2.14. A Kan complex X is *contractible* if the unique map $X \to 1$ to the terminal simplicial set (which is also a Kan complex) is an $E[1]$-homotopy equivalence.

Remark 2.15. For Kan complexes, we could equivalently speak of Δ^1-homotopy and Δ^1-homotopy equivalence, using Δ^1 instead of $NE[1]$, but for general quasicategories $NE[1]$-homotopy is stronger than Δ^1-homotopy (in particular, Δ^1-homotopy is not a symmetric notion).

Whereas 1-categories of hom-sets of morphisms, quasicategories have spaces of morphisms.

Definition 2.16. Given a quasicategory X and $x, y\colon 1 \to X$, the *mapping space* $X(x, y)$ is the simplicial set

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta^1} \arrow[d] \\ 1 \arrow[r, "{\langle x, y \rangle}"'] & X \times X \end{tikzcd}$$

Proposition 2.17. For any quasicategory X and $x, y\colon 1 \to X$, the mapping space $X(x, y)$ is a Kan complex.

We are interested in $(\infty, 1)$-categories with finite limits. Using what we have so far, we can define what it means for an $(\infty, 1)$-category (in the form of a quasicategory) to have a terminal object. Recall that a terminal object in an ordinary category $\mathcal{C}$ is an object $a \in \mathcal{C}$ such that for every other object $x \in \mathcal{C}$, there is exactly one morphism $x \to a$. Another, somewhat contrived way of putting this is that a is terminal if and only if for every $x \in \mathcal{C}$, the unique function $\mathcal{C}(x, a) \to 1$ from the set of morphisms $x \to a$ to the one-element set is a bijection. The latter definition generalizes gracefully to quasicategories.

Definition 2.18. In a quasicategory X, an object $a \in X$ is *terminal* when for every other object $x \in X$, the mapping space $X(x, a)$ is contractible.

In the next lecture, we will define what it means for a quasicategory to have other kinds of limits.

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