Problem 1. *Show that there is a rearrangement of* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ *which diverges to* ∞. $\Diamond$

SOLUTION:

Our proof will be by induction.

Base case $(n = 1)$: *Choose* $O_1 \in \mathbb{N}$ *such that* $\exists\ k_1$ *such that*

$$\sum_{i=1}^{k_1} \frac{1}{2i-1} > 2.$$

Then $\left(\sum_{i=1}^{k_1} \frac{1}{2i-1}\right) - \frac{1}{2} > 1$.

Induction Assumption: *Assume that* $k_1, k_2, \cdots, k_n$ *has been constructed such that*

$$\left[\left(\sum_{i=1}^{k_1} \frac{1}{2i-1}\right) - \frac{1}{2}\right] + \left[\left(\sum_{i=k_1+1}^{k_2} \frac{1}{2i-1}\right) - \frac{1}{4}\right] + \cdots + \left[\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1} - \frac{1}{2n}\right] > n. \tag{1}$$

Observe that the summation in inequality (1) is a rearrangement of all of the terms of the Alternating Harmonic Series up through $-\frac{1}{2n}$.

Since $\sum_{i=1}^{\infty} \frac{1}{2i-1} = \infty$ *we can find an integer* $k+1$ *such that*

$$\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1} > 2.$$

And since $\frac{1}{2i-1} < 1\ \forall\ n \in \mathbb{N}$ *we see that* $\left(\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1}\right) - \frac{1}{2(k+1)} > 1$. *Therefore*

$$\left[\left(\sum_{i=1}^{k_1} \frac{1}{2i-1}\right) - \frac{1}{2}\right] + \left[\left(\sum_{i=k_1+1}^{k_2} \frac{1}{2i-1}\right) - \frac{1}{4}\right] + \cdots + \left[\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1} - \frac{1}{2n}\right] + \left[\sum_{i=k_n+1}^{k_{n+1}} \frac{1}{2i-1} - \frac{1}{2(n+1)}\right] > n+1.$$

Observe that the summation in inequality (1) is a rearrangement of all of the terms of the Alternating Harmonic Series up through $-\frac{1}{2(n+1)}$. *Therefore,* *taking* $k_0 = 0$ *we see that the rearrangement*

$$\sum_{n=k_0}^{\infty} \left(\sum_{i=k_n+1}^{k_{n+1}} \frac{1}{2i-1} - \frac{1}{2n} \right)$$

diverges to infinity.

END OF SOLUTION