How We Got From There To Here:
A Story of Real Analysis
Solution Manual

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Contents

Part I

In Which We Raise a Number of Questions

Prologue: Three Lessons Before We Begin

Problem 1. *As you saw when you filled in the details of our development of the Quadratic Formula*[1] *the substitution* $x = y - \frac{b}{2a}$ *was crucial because it turned*

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

into

$$y^2 = k$$

where k *depends only on* a, b, *and* c. *In the sixteenth century a similar technique was used by Ludovico Ferrari (1522-1565) to reduce the general cubic equation*

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

into the so-called "depressed cubic"

$$y^3 + py + q = 0$$

where p, *and* q *depend only on* a, b, c, *and* d.

The general depressed cubic[2] *had previously been solved by Tartaglia (the Stutterer, 1500-1557) so converting the general cubic into a depressed cubic provided a path for Ferrari to compute the "Cubic Formula" – like the Quadratic Formula but better.*

Ferrari also knew how to compute the general solution of the "depressed quartic" so when he and his teacher Girolomo Cardano (1501-1576) figured out how to depress a general quartic they had a complete solution of the general quartic as well. Alas, their methods broke down entirely when they tried to solve the general quintic equation. Unfortunately the rest of this story belongs in a course on Abstract Algebra, not Real Analysis. But the lesson in this story applies to all of mathematics: Every problem solved is a new theorem which then

[1] If you didn't fill in those details you're being stupid (or at least unduly stubborn). There is a good reason for putting these three lessons first. Stop wasting your time and intellect! Go do it now.

[2] It is not entirely clear why eliminating the quadratic term should be depressing, but there it is.

becomes a tool for later use. Depressing a cubic would have been utterly useless had not Tartaglia had a solution of the depressed cubic in hand. The technique they used, with slight modifications, then allowed for a solution of the general quartic as well.

Keep this in mind as you proceed through this course and your mathematical education. Every problem you solve is really a theorem, a potential tool that you can use later. We have chosen the problems in this text deliberately with this in mind. Don't just solve the problems and move on. Just because you have solved a problem does not mean you should stop thinking about it. Keep thinking about the problems you've solved. Internalize them. Make the ideas your own so that when you need them later you will have them at hand to use.

(a) *Find M so that the substitution $x = y - M$ depresses equation 1, the general cubic equation. Then find p and q in terms of a, b, c, and d.*

SOLUTION:

Let $x = y - M$. Then

$$\begin{aligned} &a(y^3 - 3y^2M + 3yM^2 - M^3) + b(y^2 - 2yM + M^2) + cy - cM \\ &\qquad = y^3 + \left(\frac{-3M}{a} + b\right)y^2 + \left(\frac{3M^2}{a} - \frac{2M}{b} + c\right)y - \frac{M^3}{a} + \frac{M^2}{b} - cM. \end{aligned}$$

Setting the coefficient of y^2 equal to zero and solving gives $M = \frac{ab}{3}$.

Since p and q are the coefficients of y and the constant term respectively we have

$$\begin{aligned} p &= \frac{3M^2}{a} - \frac{2M}{b} + c \\ &= \frac{3\left(\frac{ab}{3}\right)^2}{a} - \frac{2\left(\frac{ab}{3}\right)}{b} + c \\ &= \frac{ab^2}{3} - \frac{2a}{3} + c \\ &= \frac{1}{3}(ab^2 - 2a + 3c), \end{aligned}$$

and

$$\begin{aligned} q &= -\frac{M^3}{a} + \frac{M^2}{b} - cM \\ &= -\frac{\left(\frac{ab}{3}\right)^3}{a} + \frac{\left(\frac{ab}{3}\right)^2}{b} - c\left(\frac{ab}{3}\right) \\ &= \frac{1}{9}(-ab^3 - a^2b - 3abc) \end{aligned}$$

END OF SOLUTION

(b) *Find K so that the substitution $x = y - K$ depresses the general quartic equation. Make sure you demonstrate how you obtained that value or why it works (if you guessed it).*

SOLUTION:

In this case set $x = y - \frac{b}{4a}$.

END OF SOLUTION

(c) *Find N so that the substitution $x = y - N$ depresses a polynomial of degree n. Ditto on showing that this value works or showing how you obtained it.*

SOLUTION:

In general set $x = y - \frac{b}{na.}$

END OF SOLUTION

$\Diamond$

Problem 2. *Here is yet another way to solve a quadratic equation. Read the development below with pencil and paper handy. Confirm all of the computations that are not completely transparent to you. Then use your notes to present the solution with* ***all*** *steps filled in.*[3]

Suppose that r_1 and r_2 are solutions of $ax^2 + bx + c = 0$. Suppose further that $r_1 \geq r_2$. Then

$$\begin{aligned} ax^2 + bx + c &= a(x - r_1)(x - r_2) \\ &= a\left[x^2 - (r_1 + r_2)x + (r_1 + r_2)^2 - (r_1 - r_2)^2 - 3r_1 r_2\right]. \end{aligned}$$

Therefore

$$r_1 + r_2 = -\frac{b}{a} \tag{2}$$

and

$$r_1 - r_2 = \sqrt{\left(\frac{b}{a}\right)^2 - \frac{4c}{a}}. \tag{3}$$

Equations 2 and 3 can be solved simultaneously to yield

$$\begin{aligned} r_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

◊

SOLUTION:

Suppose that r_1 and r_2 are solutions of $ax^2 + bx + c = 0$. Suppose further that $r_1 \geq r_2$. Then since r_1 and r_2 are solutions,

$$\begin{aligned} ax^2 + bx + c &= a(x - r_1)(x - r_2) \\ a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a(x^2 - (r_1 + r_2)x + r_1 r_2) \end{aligned}$$

Therefore

$$r_1 + r_2 = -\frac{b}{a} \tag{4}$$

[3] Be sure you are clear on the purpose of this problem before you begin. This is not about solving the Quadratic Equation. You already know how to do that. Our purpose here is to give you practice filling in the skipped details of mathematical exposition. We've chosen this particular problem because it should be a comfortable setting for you, but this particular solution is probably outside of your previous experience.

and $r_1 r_2 = \frac{c}{a}$. *Proceeding we have*

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[x^2 - (r_1 + r_2)x + \underbrace{(r_1 + r_2)^2 - (r_1 - r_2)^2 - 3r_1 r_2}_{=r_1 r_2}\right]$$

Therefore

$$\begin{aligned} \frac{c}{a} &= (r_1 + r_2)^2 - (r_1 - r_2)^2 - 3r_1 r_2 \\ &= \frac{b^2}{a^2} - (r_1 - r_2)^2 - 3r_1 r_2. \end{aligned}$$

Solving for $(r_1 - r_2)^2$, *gives*

$$\begin{aligned} (r_1 - r_2)^2 &= \frac{b^2}{a^2} - \frac{c}{a} - 3\frac{c}{a} \\ &= \frac{b^2}{a^2} - 4\frac{c}{a} \\ (r_1 - r_2)^2 &= \frac{b^2 - 4ac}{a^2}. \end{aligned}$$

Since $r_1 \geq r_2$, *we get*

$$r_1 - r_2 = \sqrt{\left(\frac{b}{a}\right)^2 - \frac{4c}{a}}. \tag{5}$$

Equations 4 and 5 can be solved simultaneously to yield

$$\begin{aligned} r_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

◊

END OF SOLUTION

Problem 3. *Let p be a prime number and a, b positive integers such that $p\,|(a{\cdot}b)$. Show that $p\,|a$ or $p\,|b$. [Hint: If $p\,|a$ then we are done. If not then notice that p is a prime factor of $a \cdot b$. What does the Fundamental Theorem of Arithmetic say about the prime factors of $a \cdot b$ compared to the prime factors of a and b?]*

SOLUTION:
Suppose that p is prime, that $a, b \in \mathbb{N}$ and that $p\,|(a \cdot b)$.
To Show: *Either $p\,|a$ or $p\,|b$.*

Case 1, $p\,|a$ **:** *If $p\,|a$ then we are done.*

Case 2, $p \not| a$**:** *Suppose that $p \not| a$.*
To show: $p \not| a \;\Rightarrow\; p\,|b$.
Proof:
Let

$$\begin{aligned} a &= p_1 p_2 \ldots p_n \text{ and} \\ b &= q_1 q_2 \ldots q_m \text{ so that} \\ ab &= (p_1 p_2 \ldots p_n)(q_1 q_2 \ldots q_m). \end{aligned}$$

Since $p\,|ab$ either $p = p_i$ for some i, $1 \le i \le n$ or $p = q_j$ for some j, $1 \le j \le m$.
Since $p \not| a$ $p \ne p_i$ $\forall\, i = 1, \ldots, n$.
Therefore $p = q_j$ for some $j, 1 \le j \le m$.
Therefore $a\,|b$.
END OF SOLUTION

Problem 4. *Let p be a prime number and let $a_1, a_2, \ldots, a_n$ be positive integers such that $p \,|\, (a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n)$. Show that $p \,|a_k$ for some $k \in \{1, 2, 3, \ldots, n\}$. [Hint: Use induction on n and the result of the previous problem.]*

SOLUTION:

Proof by Induction:

Base Case: *Supose $a = a_1$. Then $p \,|a \quad \Rightarrow \quad p \,|a_1$.*

Induction Hypothesis: *If $p \,|(a_1 a_2 \ldots a_{n-1})$ then $p \,|a_k$ for some $k \in \{1, 2, \ldots, n-1\}$*

Supose $p \,|[(a_1 a_2 \ldots a_{n-1}) a_n]$. Then by Problem 3 $p \,|(a_1 a_2 \ldots a_{n-1})$ or $p \,|a_n$.

Therefore $p \,|a_k$, $k \in \{1, 2, \ldots, n-1\}$ by our Induction Hypothesis, or $p \,|a_n$.

Therefore $p \,|a_k$, $k \in \{1, 2, \ldots, n\}$

END OF SOLUTION

Problem 5. *Let p be a prime number and let k be an integer with $1 \leq k \leq p-1$. Prove that $p \,\Big|\, \binom{p}{k}$, where $\binom{p}{k}$ is the binomial coefficient $\frac{p!}{k!(p-k)!}$. [Hint: We know $p \,|\, p!$, so $p \,|\, \binom{p}{k} k!(p-k)!$. How does the previous result apply?]*

SOLUTION:
Since $\binom{p}{k} = \frac{p!}{k!(p-k)!}$ we have $p! = \binom{p}{k} k!(p-k)!$. Clearly $p \,|p!$. Therefore $p \,\Big|\binom{p}{k} k!(p-k)!$.

Thus by problem 4, either

$$p \,\left|\, \binom{p}{k}\right.$$

or

$$p \,|k!$$

or

$$p \,|(p-k)! \,.$$

Observe that every factor of $k!$ is less than p. Thus since p is prime $p \nmid k!$.

Similarly $p \nmid (p-k)!$.

Therefore $p \,\Big|\binom{p}{k}$.

END OF SOLUTION

Problem 6. *Prove Fermat's Little Theorem. [Hint: Use induction on n. To get from n to $n+1$, use the binomial theorem on $(n+1)^p$.]*

SOLUTION:

Proof by Induction:

Base Case: *If $n = 1$ then $n^p - n = 1 - 1 = 0$ so $p \mid n^p - n$ when $n = 1$.*

Induction Hypothesis: *Suppose $p \mid n^p - n$.*

Observe that

$$(n+1)^p = n^p + \underbrace{\binom{p}{p-1} n^{p-1} + \ldots + \binom{p}{2} n^2 + \binom{p}{1} n}_{=\sum_{k=1}^{p-1} \binom{p}{k} n^k} + 1.$$

Therefore

$$(n+1)^p - (n+1) = n^p + \sum_{k=1}^{p-1} \binom{p}{k} n^k + 1 - (n+1).$$

$$(n+1)^p - (n+1) = (n^p - n) + \sum_{k=1}^{p-1} \binom{p}{k} n^k.$$

By our Induction Hypothesis,

$$p \mid n^p - n$$

and by problem 5 p divides each term of

$$\sum_{k=1}^{p-1} \binom{p}{k} n^k.$$

Therefore $p \mid (n+1)^p - (n+1)$.

END OF SOLUTION

Chapter 1

Numbers, Real ($\mathbb{R}$) and Rational ($\mathbb{Q}$)

Problem 7. *Let $a, b, c, d \in \mathbb{N}$ and find a rational number between a/b and c/d.*

<u>SOLUTION:</u>

There are two cases:

Case 1 $\left(\frac{a}{b} < \frac{c}{d}\right)$: *In this case $\frac{c}{d} - \frac{a}{b} > 0$ so*

$$\frac{a}{b} < \frac{a}{b} + \frac{1}{2}\left(\frac{c}{d} - \frac{a}{b}\right) \text{ and } \frac{c}{d} > \frac{c}{d} - \frac{1}{2}\left(\frac{c}{d} - \frac{a}{b}\right).$$

Moreover

$$\frac{a}{b} + \frac{1}{2}\left(\frac{c}{d} - \frac{a}{b}\right) = \frac{1}{2}\left(\frac{c}{d} + \frac{a}{b}\right) = \frac{c}{d} - \frac{1}{2}\left(\frac{c}{d} - \frac{a}{b}\right).$$

Therefore

$$\frac{a}{b} < \frac{1}{2}\left(\frac{c}{d} + \frac{a}{b}\right) < \frac{c}{d}.$$

Finally, $\frac{1}{2}\left(\frac{c}{d} + \frac{a}{b}\right)$ is rational since

$$\frac{1}{2}\left(\frac{c}{d} + \frac{a}{b}\right) = \frac{bc - ad}{2bd}$$

and $bc - ad$ and $2bd$ are integers.

Case 2 $\left(\frac{c}{d} < \frac{a}{b}\right)$: *The proof of this case is similar. Just reverse the roles of $\frac{a}{b}$ and $\frac{c}{d}$.*

<u>END OF SOLUTION</u>

$\Diamond$

Theorem 1. *Let $a, b, c,$ and d be integers. There is a number $\alpha \in \mathbb{Q}$ such that $M\alpha = a/b$ and $N\alpha = c/d$ where M and N are also integers.* ▲

Proof:

To prove this theorem we will display α, M and N. It is your responsibility to confirm that these actually work. Here they are: $\alpha = 1/bd$, $M = ad$, and $N = cb$. ■

Problem 8. *Confirm that $\alpha, M,$ and N as given in the proof of theorem 1 satisfy the requirements of the theorem.*

SOLUTION:

$$M\alpha = ad\left(\frac{1}{bd}\right) = \frac{a}{b}$$
$$N\alpha = cb\left(\frac{1}{bd}\right) = \frac{c}{d}.$$

END OF SOLUTION

◊

Problem 9. *Show that each of the following numbers is irrational:*
SOLUTION:

(a) $\sqrt{3}$
Proof:

[by Contradiction]

Suppose that $\sqrt{3} = \frac{a}{b}, a, b \in \mathbb{N}$, *where* $\frac{a}{b}$ *is in lowest terms. Then*

$$3b^2 = a^2,$$

which implies that $3 \,|a^2$, *which implies that* $3 \,|a$ *(why?).*

Therefore $a = 3k$ *for some integer,* k. *Therefore*

$$3b^2 = 9k^2$$
$$b^2 = 3k^2,$$

which implies that $3 \,|b^2$, *which implies that* $3 \,|b$ *(why?).*

Therefore, since $3 \,|a$ *and* $3 \,|b$ $\frac{a}{b}$ *is not in lowest terms, which is a contradiction.*

Therefore $\sqrt{3}$ *is irrational.* ■

(b) $\sqrt{5}$
Proof:

[by Contradiction]

Suppose that $\sqrt{5} = \frac{a}{b}, a, b \in \mathbb{N}$, *where* $\frac{a}{b}$ *is in lowest terms. Then*

$$5b^2 = a^2,$$

which implies that $5 \,|a^2$, *which implies that* $5 \,|a$ *(why?).*

Therefore $a = 5k$ *for some integer,* k. *Therefore*

$$5b^2 = 25k^2$$
$$b^2 = 5k^2,$$

which implies that $5 \,|b^2$, *which implies that* $5 \,|b$ *(why?).*

Therefore, since $5 \,|a$ *and* $5 \,|b$ $\frac{a}{b}$ *is not in lowest terms, which is a contradiction.*

Therefore $\sqrt{5}$ *is irrational.* ■

(c) $\sqrt[3]{2}$
Proof:

[by Contradiction]

Suppose that $\sqrt[3]{2} = \frac{a}{b}, a, b \in \mathbb{N}$, *where* $\frac{a}{b}$ *is in lowest terms. Then*

$$2b^3 = a^3,$$

which implies that $2 \left| a^3 \right.$, *which implies that* $2 \left| a \right.$ *(why?).*

Therefore $a = 2k$ *for some integer,* k. *Therefore*

$$\begin{aligned} 2b^3 &= 8k^3 \\ b^3 &= 4k^3, \end{aligned}$$

which implies that $2 \left| b^3 \right.$, *which implies that* $2 \left| b \right.$ *(why?).*

Therefore, since $2 \left| a \right.$ *and* $2 \left| b \right.$ $\frac{a}{b}$ *is not in lowest terms, which is a contradiction.*

Therefore $\sqrt[3]{2}$ *is irrational.* ■

(d) $i(= \sqrt{-1})$
Proof:

[by Contradiction]

Suppose that $i = \frac{a}{b}$, $a, b \in \mathbb{N}$. *Then* $i^2 = \frac{a^2}{b^2}$ *which is impossible since* $\frac{a^2}{b^2}$ *is positive, but* i^2 *is negative.*

Therefore i *is irrational.* ■

(e) *The square root of every positive integer which is not the square of an integer.*
Proof:

This problem can be restated as: If $n \in \mathbb{N}$ *then* $\sqrt{n}$ *is either an integer or it is irrational.*

We will prove the contrapositive: If $\sqrt{n} = \frac{a}{b}$, *where* $a, b \in \mathbb{N}$, $b \neq 1$, *then* $n \notin \mathbb{N}$.

We assume that $\frac{a}{b}$ *is in lowest terms so that* a *and* b *have no prime factors in common. Therefore* a^2 *and* b^2 *also have no prime factors in common, so that* $n = \frac{a^2}{b^2}$ *is also in lowest terms. Since* $b^2 \neq 1$, $n \notin \mathbb{N}$. ■

END OF SOLUTION ◊

Theorem 2.

(a) *Between any two real numbers there is a rational number.*

(b) *Between any two real numbers there is an irrational number.* ▲

Problem 10.

(a) *Prove that the product of a nonzero rational number and an irrational number is irrational.*

SOLUTION:

Proof:

The proof is by contradiction, so suppose α is rational, β is irrational and $\alpha\beta$ is rational. Since α is rational we have $\alpha = a/b$ for some $a, b \in \mathbb{N}$. Similarly, since $\alpha\beta$ is rational we have $\alpha\beta = c/d$ for some $c, d \in \mathbb{N}$.

Therefore

$$\begin{aligned} \alpha\beta &= c/d \\ (a/b)\beta &= c/d \\ \beta &= (bc)/(ad). \end{aligned}$$

Therefore β is rational. This is a contradiction since we assumed that β was irrational.

■

END OF SOLUTION

(b) *Turn the above ideas into a proof of Theorem 2.* ◊

SOLUTION:

(a) *Between any two real numbers there is a rational number.*

Proof:

Let α and β be real numbers with $\alpha < \beta$.

Case 1: $\beta - \alpha > 1$.

In this case there is an integer k, such that $\alpha < k < \beta$. Since all integers are rational we are done.

Case 2: $\beta - \alpha \leq 1$.

In this case, by the Archimedean Property there is an integer, n such that $n(\beta - \alpha) > 1$. By Case 1 there is an integer, k, such that

$$n\alpha < k < n\beta$$

so that

$$\alpha < k/n < \beta.$$

Since k/n is rational, we are done. ■

(b) *Between any two real numbers there is an irrational number.*

Proof:

Let α and β be as before and suppose that $p > 0$ is irrational. By part (a) of Theorem 3 there is a rational number, say q, such that

$$p\alpha < q < p\beta$$

and so

$$\alpha < q/p < \beta.$$

By part (a) of this problem q/p is irrational so we have found an irrational number between α and β.

■

End of Solution

Problem 11. *Determine if each of the following is always rational or always irrational. Justify your answers.*

(a) *The sum of two rational numbers.*

SOLUTION:
The sum of two rational numbers is also rational.
Proof:
Let $a, b, c, d \in \mathbb{N}$ and form the rational numbers, a/b and c/d. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Q}.$$

■

END OF SOLUTION

(b) *The sum of two irrational numbers.*

SOLUTION:
The sum of two irrational numbers, may be rational or irrational since $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is irrational, whereas $\sqrt{2} + (-\sqrt{2}) = 0$ is rational.
END OF SOLUTION

(c) *The sum of a rational and an irrational number.*
SOLUTION:
The sum of a rational and an irrational number will always be irrational.
Proof:
Suppose $q \in \mathbb{Q}$ and α is irrational, and suppose that $q + \alpha = p \in \mathbb{Q}$. Then $\alpha = p - q \in \mathbb{Q}$ which contradicts our assumption that α is irrational.

Therefore the sum of a rational and an irrational number will always be irrational. ■

END OF SOLUTION

♢

Problem 12. *Is it possible to have two rational numbers, a and b, such that a^b is irrational. If so, display an example of such a and b. If not, prove that it is not possible.*

SOLUTION:
Yes. Let $a = 2$ and $b = 1/2$.
END OF SOLUTION

$\Diamond$

Problem 13. *Decide if it is possible to have two irrational numbers, a and b, such that a^b is rational. Prove it in either case.*

Solution:
Let $a = b = \sqrt{2}$ and observe that $\sqrt{2}$ is irrational. There are two cases:

Case 1: $\left(\sqrt{2}\right)^{\sqrt{2}}$ *is rational.*
Since $\sqrt{2}$ is known to be irrational we are done.

Case 2: $\left(\sqrt{2}\right)^{\sqrt{2}}$ *is irrational.*
In this case we have

$$\left(\left(\sqrt{2}\right)^{\sqrt{2}}\right)^{\sqrt{2}} = 2.$$

Since both $\left(\sqrt{2}\right)^{\sqrt{2}}$ and $\sqrt{2}$ are known to be irrational we are done.

End of Solution

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Chapter 2

Calculus in the 17th and 18th Centuries

Problem 14.

(a) *Use Leibniz's product rule* $\mathrm{d}\,(xv) = x\,\mathrm{d}v + v\,\mathrm{d}x$ *to show that if* n *is a positive integer then* $\mathrm{d}\,(x^n) = nx^{n-1}\,\mathrm{d}x$

SOLUTION:

Proof:

Our proof is by induction.

Base Case: *Supose*

$$y = x^2.$$

Then

$$y = x \cdot x$$

so that by the Product Rule,

$$\begin{aligned}\mathrm{d}y &= x\,\mathrm{d}x + x\,\mathrm{d}x \\ &= 2x\,\mathrm{d}x.\end{aligned}$$

Induction Hypothesis: *Next assume that if* $y = x^{n-1}$ *then* $\mathrm{d}y = (n-1)x^{n-2}$. *So if*

$$y = x^n = x^{n-1} \cdot x$$

we have by the Product Rule,

$$\begin{aligned}\mathrm{d}y &= x^{n-1}\,\mathrm{d}x + x\,\mathrm{d}\left(x^{n-1}\right) \\ &= x^{n-1}\,\mathrm{d}x + x(n-1)x^{n-2}\,\mathrm{d}x \\ &= nx^{n-1}.\end{aligned}$$

Therefore if n is a positive integer then $\mathrm{d}\,(x^n) = nx^{n-1}\,\mathrm{d}x$

■

END OF SOLUTION

(b) *Use Leibniz's product rule to derive the quotient rule*

$$\mathrm{d}\left(\frac{v}{y}\right) = \frac{y\ \mathrm{d}v - v\ \mathrm{d}y}{yy}.$$

SOLUTION:

First we compute $\mathrm{d}\left(\frac{1}{y}\right)$. *Observe that*

$$\begin{aligned} 0 &= \mathrm{d}\left(\frac{y}{y}\right) \\ &= \mathrm{d}\left(y \cdot \frac{1}{y}\right). \end{aligned}$$

By the Product Rule we have

$$= y\,\mathrm{d}\left(\frac{1}{y}\right) + \frac{1}{y}\,\mathrm{d}y$$

so that

$$-y\,\mathrm{d}\left(\frac{1}{y}\right) = \frac{1}{y}\,\mathrm{d}y$$

and

$$\mathrm{d}\left(\frac{1}{y}\right) = \frac{-1}{y^2}\,\mathrm{d}y.$$

Next observe that

$$\mathrm{d}\left(\frac{v}{y}\right) = \mathrm{d}\left(v \cdot \frac{1}{y}\right)$$

so that, by the Product Rule,

$$\begin{aligned} \mathrm{d}\left(\frac{v}{y}\right) &= v\,\mathrm{d}\left(\frac{1}{y}\right) + \frac{1}{y}\,\mathrm{d}v \\ &= v\left(\frac{-1}{y^2}\,\mathrm{d}y\right) + \frac{1}{y}\,\mathrm{d}v \\ &= \mathrm{d}\left(\frac{v}{y}\right) = \frac{y\ \mathrm{d}v - v\ \mathrm{d}y}{yy}. \end{aligned}$$

END OF SOLUTION

(c) *Use the quotient rule to show that if n is a positive integer, then*

$$\mathrm{d}\left(x^{-n}\right) = -nx^{-n-1}\,\mathrm{d}x. \qquad \Diamond$$

Solution:

$$\begin{aligned}
\mathrm{d}\left(x^{-n}\right) &= \mathrm{d}\left(\frac{1}{x^n}\right) \\
&= \frac{x^n\,\mathrm{d}(1) - 1\,\mathrm{d}(x^n)}{x^{2n}} \\
&= \frac{-nx^{n-1}\,\mathrm{d}x}{x^{2n}} \\
&= -nx^{n-1-2n}\,\mathrm{d}x \\
&= -nx^{-n-1}\,\mathrm{d}x.
\end{aligned}$$

End of Solution

Problem 15. *Let p and q be integers with $q \neq 0$. Show* $\mathrm{d}\left(x^{\frac{p}{q}}\right) = \frac{p}{q}x^{\frac{p}{q}-1}\,\mathrm{d}x \quad \Diamond$

SOLUTION:
Let

$$y = x^{p/q}$$

and suppose we wish to find $\mathrm{d}y$*. In that case we have*

$$y^q = x^p.$$

Taking the differential of both sides gives, since p *and* q *are integers:*

$$qy^{q-1}\,\mathrm{d}y = px^{p-1}\,\mathrm{d}x.$$

As before the expression $\mathrm{d}y$ *appears in our formula and is easy to solve for. So we solve:*

$$\begin{aligned}\mathrm{d}y &= \frac{px^{p-1}\,\mathrm{d}x}{qy^{q-1}} \\ &= \frac{p}{q}\left(\frac{x^{p-1}}{y^{q-1}}\right)\mathrm{d}x.\end{aligned}$$

Since $y = x^{\frac{p}{q}}$ *we see that* $y^{q-1} = x^{\frac{p(q-1)}{q}}$ *so that* $\frac{x^{p-1}}{y^{q-1}} = \frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}} = x^{\left[(p-1)-\left(\frac{p(q-1)}{q}\right)\right]} = x^{\left[\frac{q(p-1)}{q}-\frac{p(q-1)}{q}\right]} = x^{\frac{-q+p}{q}} = x^{\frac{p}{q}-1}$*. Substituting this into the equation above gives:*

$$\mathrm{d}y = \frac{p}{q}x^{\frac{p}{q}-1}\,\mathrm{d}x$$

END OF SOLUTION

(2.1)

Problem 16. *Show that the equations* $x = \frac{t-\sin t}{4gc^2}$, $y = \frac{1-\cos t}{4gc^2}$ *satisfy equation 2.1. Bernoulli recognized this solution to be an inverted cycloid, the curve traced by a fixed point on a circle as the circle rolls along a horizontal surface.*

SOLUTION:

$$\begin{aligned}
\frac{\mathrm{d}x}{\sqrt{2gy\left[\mathrm{d}x^2 + \mathrm{d}y^2\right]}} &= \frac{\mathrm{d}t - \cos t\,\mathrm{d}t}{4gc^2\sqrt{2g\frac{1-\cos t+1}{4gc^2}\left[\frac{\mathrm{d}t^2(1-2\cos t+1)}{(4gc^2)^2}\right]}} \\
&= \frac{\mathrm{d}t(1-\cos t}{\frac{\mathrm{d}t}{c}\sqrt{\frac{1-\cos t}{2}(2-2\cos t)}} \\
&= \frac{\mathrm{d}t(1-\cos t)}{\frac{\mathrm{d}t}{c}\sqrt{(1-\cos t)^2}} \\
&= c.
\end{aligned}$$

END OF SOLUTION ◊

Problem 17. *Prove Property 3: If* $m \in \mathbb{N}$ *and* $x \in \mathbb{R}$ *then* $E(mx) = (E(x))^m$.

SOLUTION:
Proof:
Proof by Induction:
Base Case: *By Property 2* $E(x + x) = E(x)E(x) = (E(x))^2$.
Induction Hypothesis: *If* $m \in \mathbb{N}$ *and* $x \in \mathbb{R}$ *then* $E((m-1)x) = (E(x))^{m-1}$.

$$\begin{aligned} E(mx) &= E((m-1)x + x) \\ &= E((m-1)x)E(x) \ \textit{by Property 2} \\ &= E(x)^{m-1}E(x) \ \textit{by our Induction Hypothesis.} \end{aligned}$$

Therefore

$$E(mx) = (E(x))^m.$$

■

END OF SOLUTION ◊

Problem 18. *Prove Property 4:* $E(-x) = \frac{1}{E(x)}$. $\Diamond$

SOLUTION:

Proof:

Observe that

$$\begin{aligned} 1 &= E(0) \\ &= E(x + (-x)) \\ &= E(x)E(-x) \text{ by Property 2.} \end{aligned}$$

Therefore

$$E(-x) = \frac{1}{E(x)}.$$

$\blacksquare$

END OF SOLUTION

Problem 19. *Prove Property 5: If n is an integer with $n \neq 0$ then*

$$E\left(\frac{1}{n}\right) = (E(1))^{\frac{1}{n}}$$

SOLUTION:

Proof:

Observe that

$$\begin{aligned} E(1) &= E\left(\frac{n}{n}\right) \\ &= E\left(n \cdot \frac{1}{n}\right) \\ &= E\left(\frac{1}{n}\right)^{n} \quad \textit{by Property 3.} \end{aligned}$$

Therefore

$$E\left(\frac{1}{n}\right) = (E(1))^{\frac{1}{n}}.$$

■

◊

END OF SOLUTION

Problem 20. *Prove Property 6: If m and n are integers with $n \neq 0$, then*

$$E\left(\frac{m}{n}\right) = (E(1))^{\frac{m}{n}}.$$

SOLUTION:

Proof:

Observe that

$$\begin{aligned} E\left(\frac{m}{n}\right) &= E\left(m \cdot \frac{1}{n}\right) \\ &= E\left(\frac{1}{n}\right)^m \quad \textit{by Property 3} \\ &= \left(E(1)^{\frac{1}{n}}\right)^m \quad \textit{by Property 5} \\ &= (E(1))^{\frac{m}{n}}. \end{aligned}$$

$\blacksquare$

$\Diamond$

END OF SOLUTION

Problem 21. **(a)** *Show that if* $y = \sum_{n=0}^{\infty} a_n x^n$ *satisfies the differential equation* $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y$, *then*

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n$$

and conclude that

$$y = a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 - \frac{1}{6!} a_0 x^6 - \frac{1}{7!} a_1 x^7 + \cdots .$$

SOLUTION:

If $y = \sum_{n=0}^{\infty} a_n x^n$ *then* $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ *so that*

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= -y \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{n=0}^{\infty} a_n x^n \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n &= 0. \end{aligned}$$

Therefore

$$(n+2)(n+1) a_{n+2} - a_n = 2$$

or

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n$$

as required.

END OF SOLUTION

(b) *Since* $y = \sin x$ *satisfies* $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y$, *we see that*

$$\sin x = a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 - \frac{1}{6!} a_0 x^6 - \frac{1}{7!} a_1 x^7 + \cdots$$

for some constants a_0 *and* a_1. *Show that in this case* $a_0 = 0$ *and* $a_1 = 1$ *and obtain*

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \qquad \Diamond$$

Solution:
Evaluating

$$\sin x = a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!}\, a_1 x^3 + \frac{1}{4!}\, a_0 x^4 + \frac{1}{5!}\, a_1 x^5 - \frac{1}{6!}\, a_0 x^6 - \frac{1}{7!}\, a_1 x^7 + \cdots,$$

at $x = 0$ *gives*

$$\sin(0) = a_0 + a_1(0) - \frac{1}{2!} a_0(0)^2 - \frac{1}{3!} a_1(0)^3 + \frac{1}{4!} a_0(0)^4 + \frac{1}{5!} a_1(0)^5 - \frac{1}{6!} a_0(0)^6 - \frac{1}{7!} a_1(0)^7 + \cdots$$

or

$$0 = a_0$$

Differentiating both sides and evaluating at $x = 0$ *will give* $a_1 = 1$. *Thus*

$$\sin x = x - \frac{1}{3!}\, x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

End of Solution

Problem 22.

(a) *Use the series*

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

to obtain the series

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.$$

SOLUTION:

Differentiating both sides of

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

gives

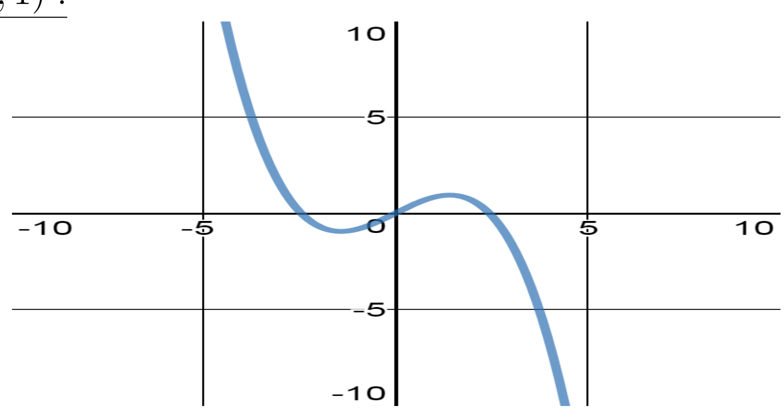
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.$$

END OF SOLUTION

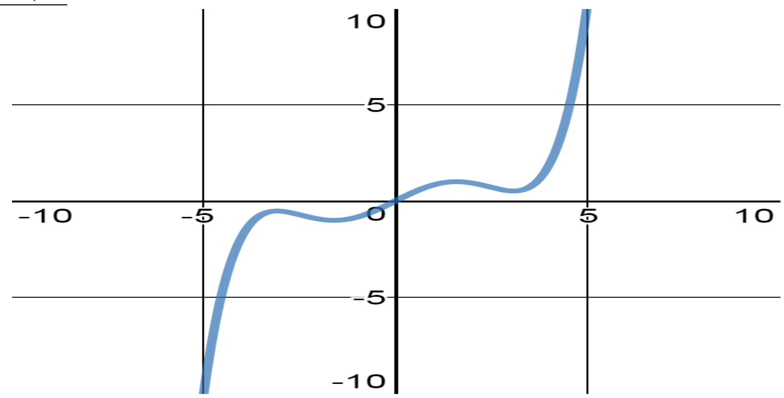
(b) *Let* $s(x,N) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ *and* $c(x,N) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n)!}x^{2n}$ *and use a computer algebra system to plot these for* $-4\pi \le x \le 4\pi, N = 1, 2, 5, 10, 15$. *Describe what is happening to the series as* N *becomes larger.* ◊

SOLUTION:

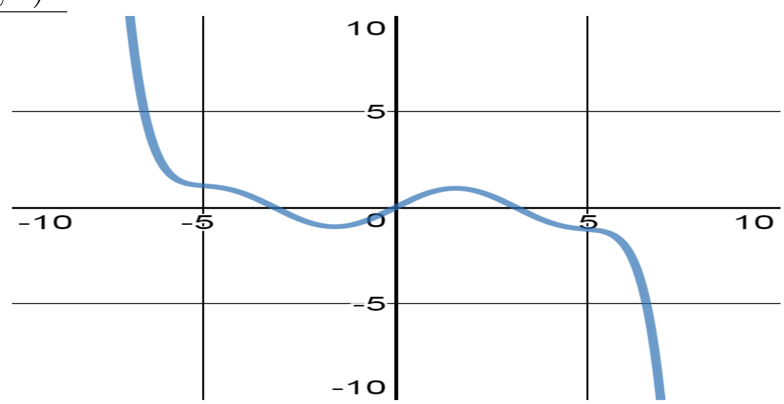
$s(x,1)$:



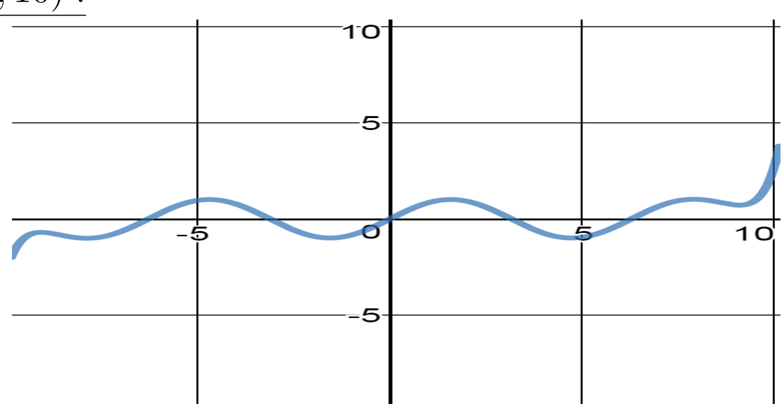
$\underline{s(x, 2):}$



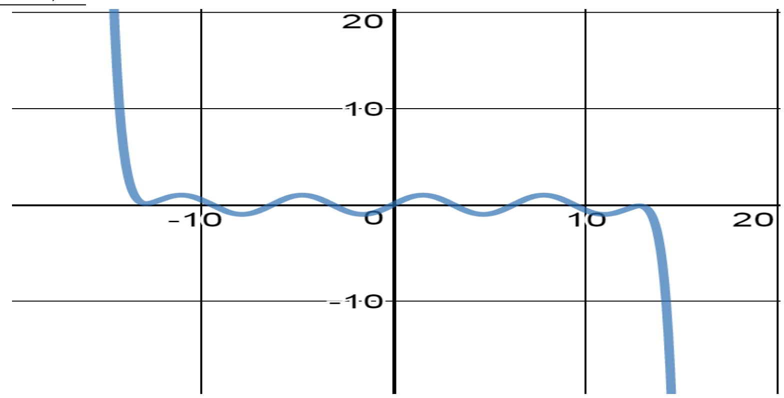
$\underline{s(x, 5):}$



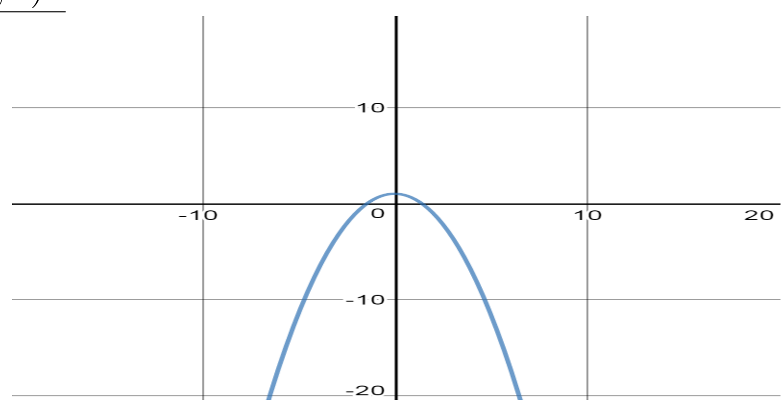
$\underline{s(x, 10):}$



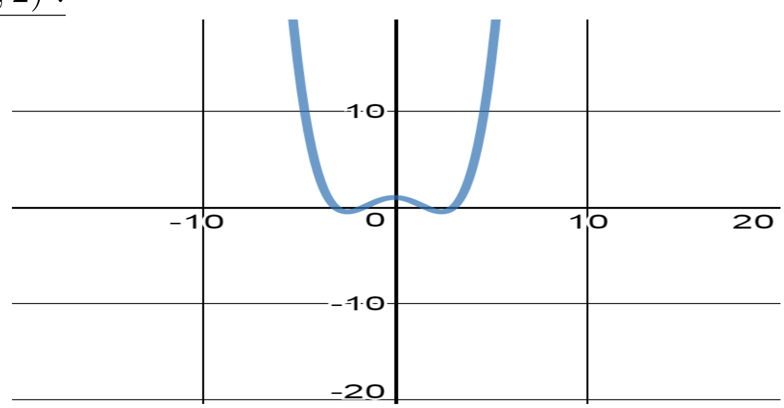
$s(x, 15)$:



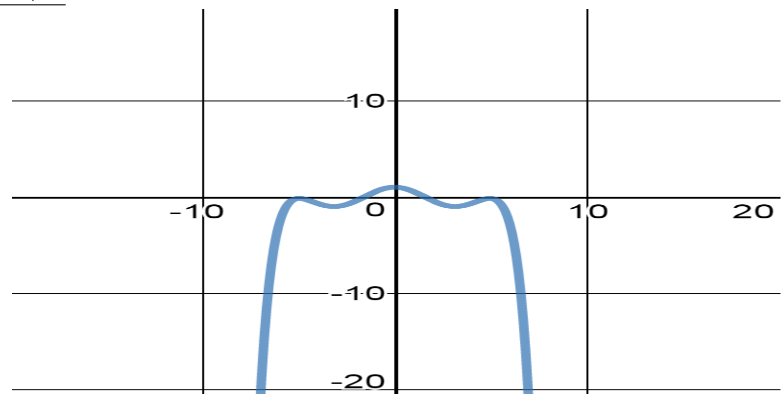
$c(x, 1)$:



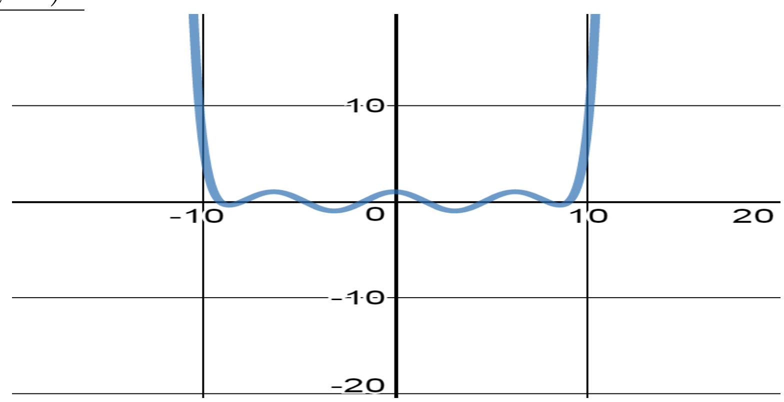
$c(x, 2)$:



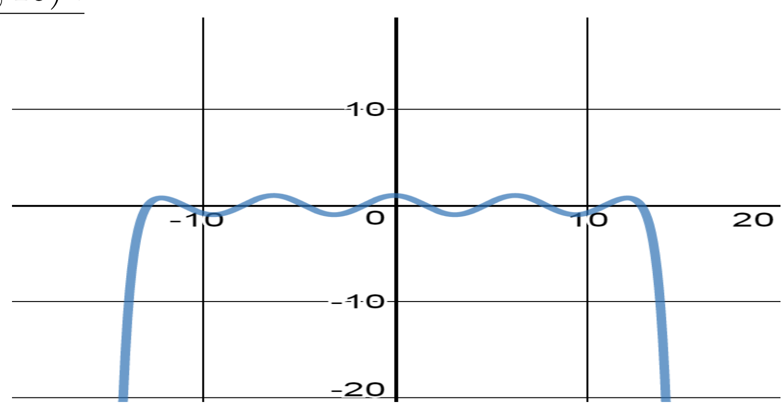
$c(x,5):$



$c(x,10):$



$c(x,15):$



End of Solution

Problem 23. *Use the geometric series,* $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, *to obtain a series for* $\frac{1}{1+x^2}$ *and use this to obtain the series*

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$$

Use the series above to obtain the series $\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$.

SOLUTION:
Since

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= \sum_{n=0}^{\infty} x^n \end{aligned}$$

we see that

$$\begin{aligned} \frac{1}{(1+x)} &= \sum_{n=0}^{\infty} (-x)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{(1+x^2)} &= \sum_{n=0}^{\infty} (-1)^n (x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n}. \end{aligned}$$

Integrating gives:

$$\begin{aligned} \int \frac{1}{(1+x^2)}\, \mathrm{d}x &= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, \mathrm{d}x \\ \arctan x &= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \end{aligned}$$

Taking $x = 1$ *gives*

$$\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1}$$
$$\pi/4 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

or

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \dots.$$

End of Solution

$\Diamond$

Problem 24. *Consider the series representation*

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1+\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n\\ &= \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n.\end{aligned}$$

Multiply this series by itself and compute the coefficients for x^0, x^1, x^2, x^3, x^4 in the resulting series.

SOLUTION:
The first four terms of

$$\begin{aligned}(1+x)^{\frac{1}{2}}]^2 = \Bigg[1+\frac{1}{2}x+\frac{1/2(1/2-1)}{2!}x^2\\ +\frac{1/2(1/2-1)(1/2-2)}{3!}x^3\\ +\frac{1/2(1/2-1)(1/2-2)(1/2-3}{4!}x^4+\dots\Bigg]^2\end{aligned}$$

Simplifying gives

$$\begin{array}{llllll}(1+x)^{\frac{1}{2}}]^2 = 1+\frac{1}{2}x & -\frac{1}{2^2\cdot 2!}x^2 & +\frac{3}{2^3\cdot 3!}x^3 & -\frac{5\cdot 3}{2^4\cdot 4!}x^4 & \dots\\ \qquad\qquad +\frac{1}{2}x & +\frac{1}{2^2}x^2 & -\frac{1}{2^3\cdot 2!}x^3 & +\frac{3}{2^4\cdot 3!}x^4 & \dots\\ & -\frac{1}{2^2\cdot 2!}x^2 & -\frac{1}{2^3\cdot 2!}x^3 & +\frac{1}{2^4\cdot 2!\cdot 2!}x^4 & \dots\\ & & +\frac{3}{2^3\cdot 3!}x^3 & +\frac{3}{2^4\cdot 3!}x^4 & \dots\\ & & & -\frac{5\cdot 3}{2^4\cdot 4!}x^4 & +\dots\\ & & \vdots & & \end{array}$$

and so

$$
\begin{aligned}
(1+x)^{\frac{1}{2}}]^2 = 1 + \underbrace{\left(\frac{1}{2}+\frac{1}{2}\right)}_{=1} x \\
+ \underbrace{\left(\frac{-1}{2^2\cdot 2}+\frac{1}{2^2}-\frac{1}{2^2\cdot 2}\right)}_{=0} x^2 \\
+ \underbrace{\left(\frac{3}{2^3\cdot 3!}-\frac{1}{2^3\cdot 2!}-\frac{1}{2^3\cdot 2!}+\frac{3}{2^3\cdot 3!}\right)}_{=0} x^3 \\
+ \underbrace{\left(\frac{-5\cdot 3}{2^4\cdot 4!}+\frac{3}{2^4\cdot 3!}+\frac{1}{2^4\cdot 2!\cdot 2!}+\frac{3}{2^4\cdot 3!}-\frac{5\cdot 3}{2^4\cdot 4!}\right)}_{=0} x^4 + \ldots .
\end{aligned}
$$

So finally

$$[(1+x)^{\frac{1}{2}}]^2 = 1 + x$$

as expected.

End of Solution

◊

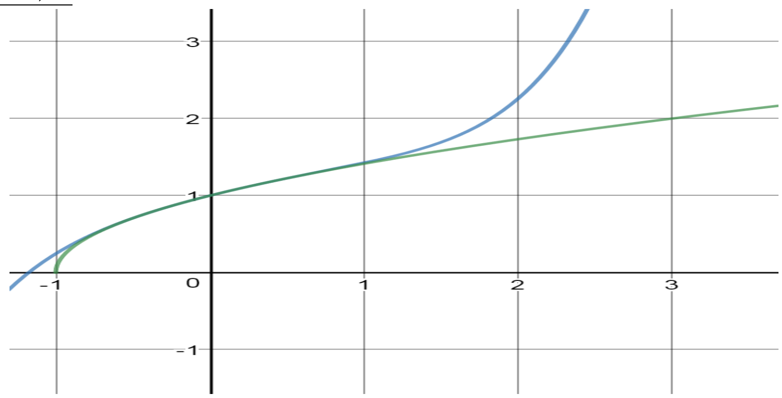
Problem 25. *Let*

$$S(x, M) = \sum_{n=0}^{M} \frac{\prod_{j=0}^{n-1}\left(\frac{1}{2} - j\right)}{n!} x^n.$$

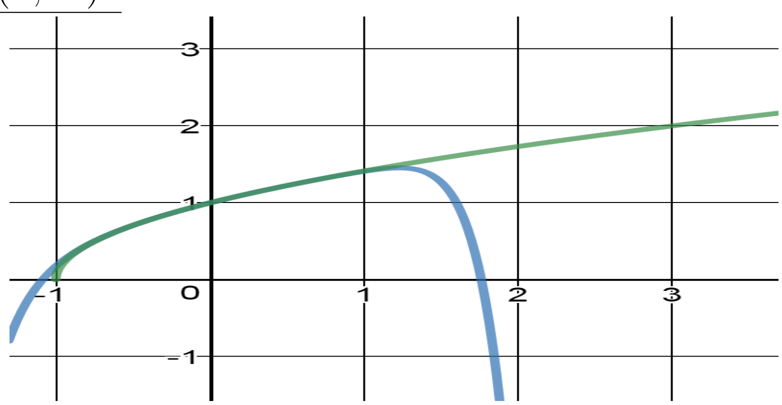
Use a computer algebra system to plot $S(x, M)$ *for* $M = 5, 10, 15, 95, 100$ *and compare these to the graph for* $\sqrt{1+x}$*. What seems to be happening? For what values of* x *does the series appear to converge to* $\sqrt{1+x}$*?* ◊

SOLUTION:

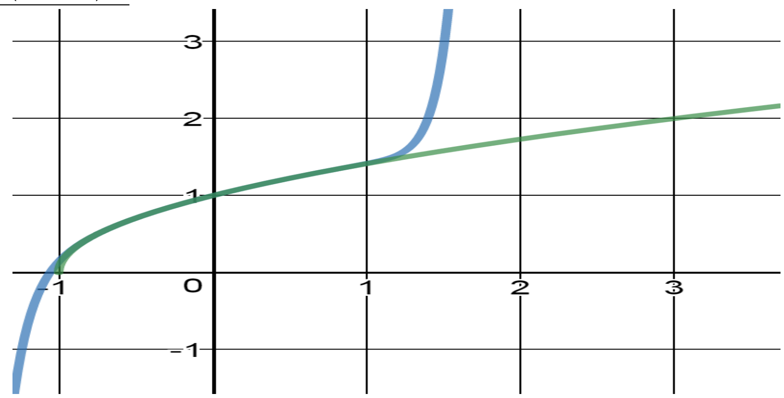
$S(x, 5)$:



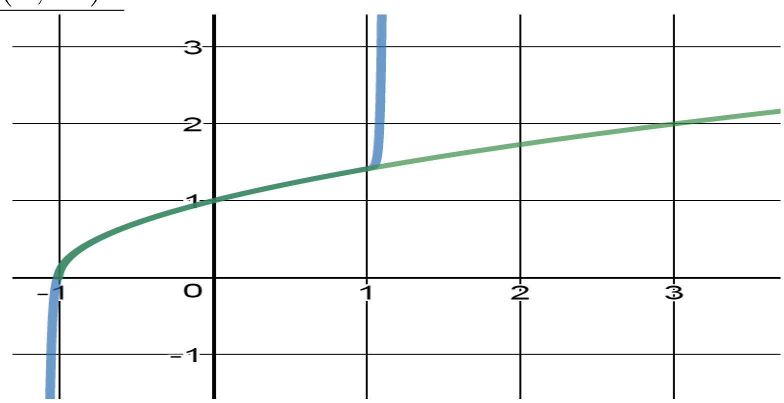
$S(x, 10)$:



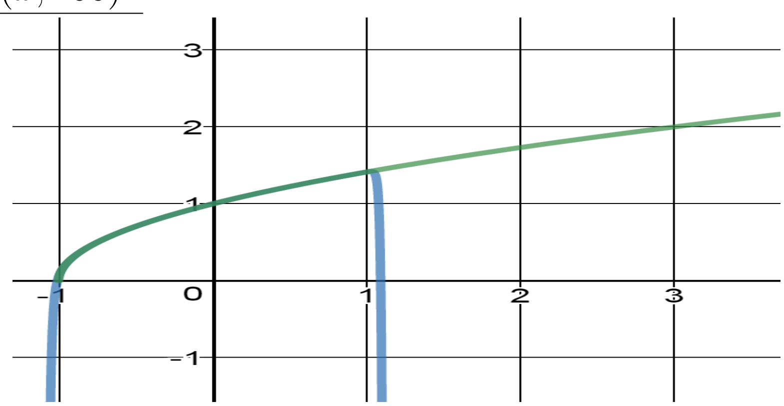
$\underline{S(x,15):}$



$\underline{S(x,95):}$



$\underline{S(x,100):}$



End of Solution

Problem 26. *Use the series* $(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n$ *to obtain the series*

$$\begin{aligned}\frac{\pi}{4} &= \int_{x=0}^{1} \sqrt{1-x^2}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty}\left(\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}\right)\left(\frac{(-1)^n}{2n+1}\right) \\ &= 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \cdots .\end{aligned}$$

Use a computer algebra system to sum the first 100 terms of this series and compare the answer to $\frac{\pi}{4}$. ◊

Solution:

Observe that $(1-x^2)^{1/2} = \left(1-(-x^2)\right)^{1/2}$. *Thus*

$$\begin{aligned}\frac{\pi}{4} &= \int_{x=0}^{1} \sqrt{1-x^2}\,\mathrm{d}x \\ &= \int_{n=0}^{1} \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(1/2-j)}{n!}(-x^2)^n\,\mathrm{d}x \\ &= \int_{n=0}^{1} \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(1/2-j)}{n!}(-1)^n x^{2n}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(1/2-j)}{n!}(-1)^n \int_{x=0}^{1} x^{2n}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(1/2-j)}{n!}(-1)^n \left.\frac{x^{2n+1}}{2n+1}\right|_{x=0}^{1} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(1/2-j)}{n!}\frac{(-1)^n}{2n+1} \\ &= 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \cdots .\end{aligned}$$

Summing the first 100 *terms gives*

$$\frac{\pi}{4} \approx 0.785491,$$

whereas the first 6 *digits of* $\frac{\pi}{4}$ *is given by*

$$\frac{\pi}{4} \approx 0.785398.$$

End of Solution

Problem 27. **(a)** *Show that*

$$\int_{x=0}^{1/2} \sqrt{x - x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{\sqrt{2}\, n!\,(2n+3)\,2^n}$$

and use this to show that

$$\pi = 16 \left(\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{\sqrt{2}\, n!\,(2n+3)\,2^n} \right).$$

(b) *We now have two series for calculating* π : *the one from part (a) and the one derived earlier, namely*

$$\pi = 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \right).$$

We will explore which one converges to π *faster. With this in mind, define* $S1(N) = 16 \left(\sum_{n=0}^{N} \frac{(-1)^n \prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{\sqrt{2}\, n!(2n+3)2^n} \right)$ *and* $S2(N) = 4 \left(\sum_{n=0}^{N} \frac{(-1)^n}{2n+1} \right)$. *Use a computer algebra system to compute* $S1(N)$ *and* $S2(N)$ *for* $N = 5, 10, 15, 20$. *Which one appears to converge to* π *faster?* ♢

Solution:

M	*S1(N)*	*S2(N)*
5	3.14196750207	2.97604617605
10	3.14159565022	3.23231580941
15	3.14159269292	3.0791533942
20	3.14159265424	3.18918478228

End of Solution

Problem 28. *Let k be a positive integer. Find the power series, centered at zero, for $f(x) = (1-x)^{-k}$ by*

(a) *Differentiating the geometric series $(k-1)$ times.*

Solution:

Let $f(x) = \frac{1}{1-x} = (1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ so that

$$
\begin{aligned}
f^{(1)}(x) &= (1-x)^{-2} = \sum_{n=0}^{\infty}(n+1)x^n \\
f^{(2)}(x) &= 2!(1-x)^{-3} = \sum_{n=0}^{\infty}(n+2)(n+1)x^n \\
f^{(3)}(x) &= 3!(1-x)^{-4} = \sum_{n=0}^{\infty}\left(\prod_{i=1}^{3}(n+i)\right)x^n
\end{aligned}
$$

and in general,

$$f^{(k)}(x) = k!(1-x)^{-k} = \sum_{n=0}^{\infty}\left(\prod_{i=1}^{k}(n+i)\right)x^n$$

so that

$$(1-x)^{-k} = \sum_{n=0}^{\infty}\left(\frac{\prod_{i=1}^{k-1}(n+i)}{k!}\right)x^n$$

End of Solution

(b) *Applying the binomial series.*

Solution:

The Binomial Series is given by:

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}(\alpha-j)}{n!}x^n.$$

For this problem we have

$$
\begin{aligned}
\frac{1}{(1-x)^k} = (1+(-x))^{-k} &= \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}(-k-j)}{n!}(-x)^n \\
&= \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}(-k-j)}{n!}(-1)^n x^n \\
&= \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}(k+j)}{n!}x^n
\end{aligned}
$$

END OF SOLUTION

(c) *Compare these two results.* ◊

SOLUTION:

Since $\frac{\prod_{i=1}^{k-1}(n+i)}{k!} = \frac{\prod_{j=0}^{n-1}(k+j)}{n!}$ *(They are both* $\binom{n+k-1}{k}$*) we see that the two series in (a) and (b) are the same.*

END OF SOLUTION

Problem 29. **(a)** *Show that the power series for* $\frac{\sin x}{x}$ *is given by* $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$.

SOLUTION:

$$\begin{aligned}\frac{\sin x}{x} &= \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}{x} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \\ &= 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots .\end{aligned}$$

END OF SOLUTION

(b) *Use (a) to infer that the roots of* $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$ *are given by*

$$x = \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \ldots$$

SOLUTION:

The roots of $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$ *will be the same as the roots of* $\sin x$, *which are as given.*

END OF SOLUTION

(c) *Suppose* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ *is a polynomial with roots* $r_1, r_2, \ldots, r_n$. *Show that if* $a_0 \neq 0$, *then all the roots are non-zero and*

$$p(x) = a_0 \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right) \cdots \left(1 - \frac{x}{r_n}\right).$$

SOLUTION:

Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

and

$$q(x) = a_0 \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right) \cdots \left(1 - \frac{x}{r_n}\right).$$

Clearly $p(x)$ *and* $q(x)$ *have the same roots, so one must be a constant multiple of the other. Since* $p(0) = a_0 = q(0)$ *they are equal.*

END OF SOLUTION

(d) *Assuming that the result in c holds for an infinite polynomial power series, deduce that*

$$1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots = \left(1 - \left(\frac{x}{\pi}\right)^2\right)\left(1 - \left(\frac{x}{2\pi}\right)^2\right)\left(1 - \left(\frac{x}{3\pi}\right)^2\right)\cdots$$

SOLUTION:
This conclusion is immediate with the given assumption and part (c).
END OF SOLUTION

(e) *Expand this product to deduce*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \Diamond$$

SOLUTION:
Expanding the right side of the formula in part (d) and comparing the coefficients of the x^2 term we have

$$\begin{aligned} -\frac{1}{3!}x^2 &= -\left(\frac{x}{\pi}\right)^2 - \left(\frac{x}{2\pi}\right)^2 - \left(\frac{x}{3\pi}\right)^2 - \left(\frac{x}{4\pi}\right)^2 - \cdots \\ &= \left[-\left(\frac{1}{\pi}\right)^2 - \left(\frac{1}{2\pi}\right)^2 - \left(\frac{1}{3\pi}\right)^2 - \left(\frac{1}{4\pi}\right)^2 - \cdots\right] x^2 \end{aligned}$$

Therefore

$$\frac{1}{3!} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

END OF SOLUTION

Problem 30. *Use the geometric series to obtain the series*

$$\begin{aligned}\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}.\end{aligned}$$ ◊

SOLUTION:

The geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

so that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

so, finally

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

END OF SOLUTION

Problem 31. *<u>Without</u> using Taylor's Theorem, represent the following functions as power series expanded about 0 (i.e., in the form $\sum_{n=0}^{\infty} a_n x^n$).*

(a) $\ln\left(1-x^2\right)$

SOLUTION:

From the geometric series we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

so that

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \ldots$$

and

$$\begin{aligned}\frac{-2x}{1-x^2} &= -2x(1 + x^2 + x^4 + x^6 + \ldots).\\ &= -2(x + x^3 + x^5 + x^7 + \ldots).\end{aligned}$$

Integrating both sides gives:

$$\begin{aligned}\ln(1-x^2) &= -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \ldots\right)\\ &= -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \ldots - \frac{x^{2n}}{n} - \ldots.\end{aligned}$$

So

$$\ln(1-x^2) = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}.$$

END OF SOLUTION

(b) $\frac{x}{1+x^2}$

SOLUTION:

From part (a) we see that

$$\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \ldots$$

so that

$$\begin{aligned}\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} &= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \ldots\\ \frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + \ldots\end{aligned}$$

and, finally

$$\frac{x}{1+x^2} = x - x^3 + x^5 - x^7 + \dots$$

END OF SOLUTION

(c) $\arctan\left(x^3\right)$

SOLUTION:

Integrating both sides of

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

gives

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

and so

$$\begin{aligned}\arctan(x^3) &= (x)^3 - \frac{(x^3)^3}{3} + \frac{(x^3)^5}{5} - \frac{(x^3)^7}{7} + \dots \\ &= x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \dots .\end{aligned}$$

END OF SOLUTION

(d) $\ln(2+x)$ *[Hint:* $2 + x = 2\left(1 + \frac{x}{2}\right)$*]* $\Diamond$

SOLUTION:

$$\begin{aligned}\ln(2+x) &= \ln(2(1+x/2)) \\ &= \ln(2) + \ln(1+x/2)\end{aligned}$$

and from problem 30 we have

$$\begin{aligned}&= \ln(2) + \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{3}\left(\frac{x}{2}\right)^3 - \frac{1}{4}\left(\frac{x}{2}\right)^4 + \dots \\ &= \ln(2) + \frac{x}{2} - \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3\cdot 2^3} - \frac{x^4}{4\cdot 2^4} + \dots\end{aligned}$$

END OF SOLUTION

Problem 32. *Let a be a positive real number. Find a power series for a^x expanded about 0. [Hint: $a^x = e^{\ln(a^x)}$]* ◊

SOLUTION:

$$\begin{aligned} a^x &= e^{\ln(a^x)} \\ &= e^{x\ln(a)} \\ &= \sum_{n=0}^{\infty} \frac{(x\ln(a))^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\ln(a))^n}{n!} x^n \end{aligned}$$

END OF SOLUTION

Problem 33. *Represent the function* $\sin x$ *as a power series expanded about* a *(i.e., in the form* $\sum_{n=0}^{\infty} a_n (x-a)^n$*). [Hint:* $\sin x = \sin(a + x - a)$*.]* $\Diamond$

Solution:

$$\sin a + \cos a(x-a) - \sin a(x-a)^2 - \cos a(x-a)^3 + \sin a(x-a)^4 + \cos a(x-a)^5 - \dots$$

End of Solution

Problem 34. *__Without__ using Taylor's Theorem, represent the following functions as a power series expanded about a for the given value of a (i.e., in the form $\sum_{n=0}^{\infty} a_n (x-a)^n$).*

(a) $\ln x$, $a = 1$
SOLUTION:

$$\begin{aligned}\ln x &= \ln(1-(1-x)) \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.\end{aligned}$$

END OF SOLUTION

(b) e^x, $a = 3$
SOLUTION:

$$\begin{aligned}e^x &= e^{3+(x-3)} \\ &= e^3 e^{x-3} \\ &= e^3\left(1 + (x-3) + \frac{(x-3)^2}{2!} + \frac{(x-3)^3}{3!} + \cdots\right) \\ &= \sum_{n=0}^{\infty} \frac{e^3(x-3)^n}{n!}.\end{aligned}$$

END OF SOLUTION

(c) $x^3 + 2x^2 + 3$, $a = 1$

$$x^3 + 2x^2 + 3 = 6 + 7(x-1) + 5(x-1)^2 + (x-1)^3$$

(d) $\frac{1}{x}$, $a = 5$
SOLUTION:

$$\begin{aligned}\frac{1}{x} &= \frac{1}{5+(x-5)} \\ &= \frac{1}{5}\left(\frac{1}{1+\left(\frac{x-5}{5}\right)}\right) \qquad = \frac{1}{5}\left(1 + \frac{-(x-5)}{5} + \frac{(x-5)^2}{5^2} + \frac{-(x-5)^3}{5^3}\right) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n(x-5)^n}{5^{n+1}}\end{aligned}$$

END OF SOLUTION $\Diamond$

Problem 35. *Evaluate the following integrals as series.*

(a) $\displaystyle\int_{x=0}^{1} e^{x^2}\,\mathrm{d}x$

Solution:

$$\begin{aligned}\int_{x=0}^{1} e^{x^2}\,\mathrm{d}x &= \int_{x=0}^{1}\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!}\,\mathrm{d}x\\ &= \int_{x=0}^{1}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}\,\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}\Bigg|_{x=0}^{1}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!(2n+1)}\end{aligned}$$

End of Solution

(b) $\displaystyle\int_{x=0}^{1} \frac{1}{1+x^4}\,\mathrm{d}x$

Solution:

$$\begin{aligned}\int_{x=0}^{1} \frac{1}{1+x^4}\,\mathrm{d}x &= \int_{x=0}^{1}\sum_{n=0}^{\infty}(-1)^n(x^4)^n\,\mathrm{d}x\\ &= \int_{x=0}^{1}\sum_{n=0}^{\infty}(-1)^n x^{4n}\,\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{4n+1}\Bigg|_{x=0}^{1}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1}\end{aligned}$$

End of Solution

(c) $\displaystyle\int_{x=0}^{1} \sqrt[3]{1-x^3}\,\mathrm{d}x$ ◊

Chapter 3

Questions Concerning Power Series

Theorem 3. *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, *then* $a_n = \frac{f^{(n)}(a)}{n!}$, *where* $f^{(n)}(a)$ *represents the* n^{th} *derivative of* f *evaluated at* a. ▲

Problem 36. *Prove Theorem 3.*
Hint: $f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + \cdots = a_0$, *differentiate to obtain the other terms.* ◊

SOLUTION:

Observe that

$$f^{(n)}(x) = \sum_{n=0}^{\infty} \frac{a_k k!}{(n-k)!}(x-a)^{n-k}$$

so that

$$f^{(n)}(a) = a_n n!$$

or

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

END OF SOLUTION

Problem 37.

(a) *Adopting the notation $y = x^{-1}$ and $f^{(n)}(x) = p_n(y)e^{-y^2}$, find $p_{n+1}(y)$ in terms of $p_n(y)$. [Note: Don't forget that you are differentiating with respect to x, not y.]*

SOLUTION:

First observe that if $y = x^{-1}$ then $\frac{dy}{dx} = -x^{-2} = -y^2$.

If

$$f^{(n)}(x) = p_n(y)e^{-y^2}$$

then

$$\begin{aligned} f^{(n+1)}(x) &= p_n(y)\left(-2y\frac{dy}{dx}\right)e^{-y^2} + p_n'(y)\frac{dy}{dx}e^{-y^2} \\ &= e^{-y^2}\underbrace{\left(p_n(y)(-2y)(-y^2) + p_n'(y)(-y^2)\right)}_{=p_{n+1}(y)} \end{aligned}$$

Therefore $p_{n+1}(y) = p_n(y)(-2y)(-y^2)+p_n'(y)(-y^2)$ and the expression on the right is clearly a polynomial since both a polynomial and its derivative are polynomials.

END OF SOLUTION

(b) *Use induction on n to show that $p_n(y)$ is a polynomial for all $n \in \mathbb{N}$.* $\Diamond$

SOLUTION:

Clearly $p_0(y) = 1$ is a polynomial and part (a) gives the inductive step.

END OF SOLUTION

Problem 38.

(a) *Let m be a nonnegative integer. Show that $\lim_{y\to\pm\infty}\frac{y^m}{e^{y^2}}=0$. [Hint: Induction and a dash of L'Hôpital's rule should do the trick.]*

SOLUTION:

Applying L'Hôpital's rule m times gives us a constant in the numerator and, by the previous problem $p_{n+1}(y)e^{-y^2}$ in the denominator. Thus $\lim_{y\to\pm\infty}\frac{y^m}{e^{y^2}}=0$.

END OF SOLUTION

(b) *Prove that $\lim_{y\to\pm\infty}\frac{q(y)}{e^{y^2}}=0$ for any polynomial q.*

SOLUTION:

This follows immediately from part (a).

END OF SOLUTION

(c) *Show that for every nonnegative integer n, $f^{(n)}(0)=0$.* $\Diamond$

SOLUTION:

Proof by Induction:

Base Case: $f(0)=0$.

Inductive Step: *Assume that $f^{(n)}(0)=0$. We show that $f^{(n+1)}(0)=0$.*

$$f^{(n+1)}(0)=\lim_{h\to 0}\frac{f^{(n)}(h)-f^{(n)}(0)}{h}.$$

From the previous problem, and substituting $y=h^{-1}$ this becomes:

$$f^{(n+1)}(0)=\lim_{h\to\infty}p_n(y)e^{-y^2}$$

and from part (a) of this problem we see that

$$f^{(n+1)}(0)=0.$$

END OF SOLUTION

Problem 39. *Use Taylor's formula to obtain the general binomial series*

$$(1+x)^{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(\alpha - j)}{n!} x^n. \qquad \Diamond$$

Solution:

End of Solution

Problem 40. *Use Taylor's formula to obtain the Taylor series for the functions* e^x*, sin* x*, and cos* x *expanded about* a*.* $\Diamond$

SOLUTION:

END OF SOLUTION

Theorem 4. *If f', f'', ..., $f^{(n+1)}$ are all continuous on an interval containing a and x, then*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t. \quad \blacktriangle$$

Problem 41. *Provide a formal induction proof for Theorem 4.* ◊

Solution:

End of Solution

Problem 42. *Use the fact that*

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{2k+1}}{2k} \leq \ln 2 \leq 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{2k+2}}{2k+1}$$

to determine how many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *should be added together to approximate* $\ln 2$ *to within* $.0001$ *without actually computing what* $\ln 2$ *is.* $\Diamond$

SOLUTION:

In general, if $a \leq x \leq b$ *then* $|x-a| \leq b-a$ *and* $b - x \leq b - a$*, so* $\left|\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \ln(2)\right| < .0001$ *when*

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{2k+1}}{2k} + \frac{(-1)^{2k+2}}{2k+1}\right) - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{2k+1}}{2k}\right) \leq .0001.$$

Since all but the one term of the left side of the above equation cancels out we have:

$$\frac{(-1)^{2k+2}}{2k+1} \leq .0001$$

Solving this for k *gives* $k > 5000$*, so the number of term required is* $2k = 10,000$*.*

END OF SOLUTION

Problem 43. *Show that there is a rearrangement of* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ *which diverges to* ∞. ◊

SOLUTION:

Our proof will be by induction.

<u>**Base case** $(n = 1)$:</u> *Choose* $O_1 \in \mathbb{N}$ *such that* $\exists\ k_1$ *such that*

$$\sum_{i=1}^{k_1} \frac{1}{2i-1} > 2.$$

Then $\left(\sum_{i=1}^{k_1} \frac{1}{2i-1}\right) - \frac{1}{2} > 1.$

<u>**Induction Assumption:**</u> *Assume that* $k_1, k_2, \cdots, k_n$ *has been constructed such that*

$$\begin{aligned}\left[\left(\sum_{i=1}^{k_1} \frac{1}{2i-1}\right) - \frac{1}{2}\right] + \left[\left(\sum_{i=k_1+1}^{k_2} \frac{1}{2i-1}\right) - \frac{1}{4}\right] + \cdots \\ + \left[\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1} - \frac{1}{2n}\right] > n.\end{aligned} \tag{3.1}$$

Observe that the summation in inequality (3.1) is a rearrangement of all of the terms of the Alternating Harmonic Series up through $-\frac{1}{2n}$.

Since $\sum_{i=1}^{\infty} \frac{1}{2i-1} = \infty$ *we can find an integer* $k+1$ *such that*

$$\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1} > 2.$$

And since $\frac{1}{2i-1} < 1\ \forall\ n \in \mathbb{N}$ *we see that* $\left(\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1}\right) - \frac{1}{2(k+1)} > 1$. *Therefore*

$$\begin{aligned}\left[\left(\sum_{i=1}^{k_1} \frac{1}{2i-1}\right) - \frac{1}{2}\right] + \left[\left(\sum_{i=k_1+1}^{k_2} \frac{1}{2i-1}\right) - \frac{1}{4}\right] \\ + \cdots \\ + \left[\sum_{i=k_{n-1}+1}^{k_n} \frac{1}{2i-1} - \frac{1}{2n}\right] \\ + \left[\sum_{i=k_n+1}^{k_{n+1}} \frac{1}{2i-1} - \frac{1}{2(n+1)}\right] > n+1.\end{aligned}$$

Observe that the summation in inequality (3.1) is a rearrangement of all of the terms of the Alternating Harmonic Series up through $-\frac{1}{2(n+1)}$. *Therefore,* *taking* $k_0 = 0$ *we see that the rearrangement*

$$\sum_{n=k_0}^{\infty} \left(\sum_{i=k_n+1}^{k_{n+1}} \frac{1}{2i-1} - \frac{1}{2n} \right)$$

diverges to infinity.
End of Solution

Problem 44. *Show that there is a rearrangement of* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ *which diverges to* $-\infty$. $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 45. *Use Taylor's formula to find the Taylor series of the given function expanded about the given point a.*

(a) $f(x) = \ln(1+x)$, $a = 0$

SOLUTION:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

END OF SOLUTION

(b) $f(x) = e^x$, $a = -1$

SOLUTION:

$$e^x = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$$

END OF SOLUTION

(c) $f(x) = x^3 + x^2 + x + 1$, $a = 0$

SOLUTION:

This function is its own Taylor series.

END OF SOLUTION

(d) $f(x) = x^3 + x^2 + x + 1$, $a = 1$ ◊

SOLUTION:

$$f(x) = 4 + 6(x-1) + 4(x-1)^2 + (x-1)^3$$

END OF SOLUTION

Part II

Interregnum

Problem 46. *Show that* $T = Ce^{\rho^2 kt}$ *satisfies the equation* $T' = \rho^2 kT$, *where* C, *and* ρ *are arbitrary constants. Use the physics of the problem to show that if* u *is not constantly zero, then* $k < 0$. *[Hint: Consider* $\lim_{t\to\infty} u(x,t)$*.]* $\Diamond$

Solution:

End of Solution

Problem 47. *Show that* $X = A\sin(px) + B\cos(px)$ *satisfies the equation* $X'' = -p^2 X$*, where* A *and* B *are arbitrary constants. Use the boundary conditions* $u(0,t) = u(1,t) = 0$*,* $\forall\, t \geq 0$ *to show that* $B = 0$ *and* $A \sin p = 0$*. Conclude that if* u *is not constantly zero, then* $p = n\pi$*, where* n *is any integer.* $\Diamond$

Solution:

End of Solution

Problem 48. *Show that if* u_1 *and* u_2 *satisfy the equations* $\rho^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ *and* $u(0,t) = u(1,t) = 0, \forall\, t \geq 0$ *then* $u = A_1 u_1 + A_2 u_2$ *satisfy these as well, where* A_1 *and* A_2 *are arbitrary constants.* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 49. *Let n and m be positive integers. Show*

$$\int_{x=0}^{1} \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x = \begin{cases} 0 & \textit{if } n \neq m \\ \frac{1}{2} & \textit{if } n = m \end{cases}. \qquad \Diamond$$

<u>SOLUTION:</u>

<u>$n = m$</u>

$$\begin{aligned} \int_{x=0}^{1} \sin^2(n\pi x)\,\mathrm{d}x &= \frac{1}{2}\int_{x=0}^{1} (1-\cos(n\pi x))\,\mathrm{d}x \\ &= \frac{1}{2}\left[x - \sin(n\pi x)\right]_{x=0}^{1} \\ &= \frac{1}{2}. \end{aligned}$$

<u>$n \neq m$</u>

$$\begin{aligned} \int_{x=0}^{1} \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x &= \frac{1}{2}\int_{x=0}^{1} \cos((n+m)\pi x) - \cos((n-m)\pi x)\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{1}{(n+m)\pi}\sin((n+m)\pi x) - \frac{1}{(n-m)\pi}\sin((n-m)\pi x)\right]_{x=0}^{1} \\ &= 0. \end{aligned}$$

<u>END OF SOLUTION</u>

Problem 50. *Let n be a positive integer. Show that if*

$$f(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right|$$

then

$$\int_{x=0}^{1} f(x) \sin(n\pi x)\, dx = \frac{2}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

and show that the Fourier sine series of f is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin(n\pi x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x)\,. \qquad \Diamond$$

SOLUTION:

Observe that

$$f(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right| = \begin{cases} -x+1 & \textit{if } x \geq 1/2 \\ x & \textit{if } x < 1/2 \end{cases}$$

so that

$$\int_0^1 f(x) \sin(n\pi x)\, \mathrm{d}x = \int_0^{1/2} f(x) \sin(n\pi x)\, \mathrm{d}x + \int_{1/2}^1 f(x) \sin(n\pi x)\, \mathrm{d}x$$

Taking these last two integrals one at a time we see that:

$$\int_0^{1/2} f(x) \sin(n\pi x)\, \mathrm{d}x = \int_0^{1/2} x \sin(n\pi x)\, \mathrm{d}x.$$

Integrating by parts we take $u = x$ and $\mathrm{d}v = \sin(n\pi x)\, \mathrm{d}x$ so that $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{-\cos(n\pi x)}{n\pi}$. Thus

$$\begin{aligned} \int_0^{1/2} f(x) \sin(n\pi x)\, \mathrm{d}x &= -\frac{x}{n\pi} \cos(n\pi x)\Big|_0^{1/2} + \int_0^{1/2} \frac{\cos(n\pi x)}{n\pi}\, \mathrm{d}x \\ &= \frac{1}{(n\pi)^2} \sin n\pi x \Big|_0^{1/2} \\ &= \frac{\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2} \end{aligned}$$

So that

$$\int_0^{1/2} f(x) \sin(n\pi x)\, \mathrm{d}x = \begin{cases} 1 & n \equiv 1 \mod 4 \\ -1 & n \equiv 3 \mod 4\,. \\ 0 & \textit{otherwise} \end{cases} \qquad (3.2)$$

The second integral is

$$\int_{1/2}^{1} f(x)\sin(n\pi x)\,\mathrm{d}x = \int_{1/2}^{1} (1-x)\sin(n\pi x)\,\mathrm{d}x.$$

Integrating by parts again gives

$$\begin{aligned} &= -\left.\frac{\sin(n\pi x)}{(n\pi)^2}\right|_{1/2}^{1} \\ &= -\frac{\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2} \end{aligned}$$

so that

$$\int_{1/2}^{1} f(x)\sin(n\pi x)\,\mathrm{d}x = \begin{cases} 1 & n \equiv 1 \mod 4 \\ -1 & n \equiv 3 \mod 4 \\ 0 & \textit{otherwise} \end{cases}. \tag{3.3}$$

Combining equation 3.2 and equation 3.3 we have

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin(n\pi x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left((2k+1)\pi x\right).$$

End of Solution

Problem 51. *Show that when* $x = \frac{1}{4}$

$$4\sum_{k=0}^{\infty}(-1)^{k+1}\sin\left(\left(2k+1\right)\pi x\right) = 4\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}+\cdots\right).$$

◊

Solution:

End of Solution

Problem 52. *Let n and m be positive integers. Show*

$$\int_{x=0}^{1} \cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2} & \text{if } n = m \end{cases}.$$

◊

SOLUTION:

END OF SOLUTION

Problem 53. *Use the result of Problem 52 to show that if*

$$f(x) = \sum_{n=1}^{\infty} B_n \cos(n\pi x)$$

on $[0, 1]$*, then*

$$B_m = 2 \int_{x=0}^{1} f(x) \cos(m\pi x) \, \mathrm{d}x.$$ $\lozenge$

<u>SOLUTION:</u>

<u>END OF SOLUTION</u>

Problem 54. *Apply the result of Problem 53 to show that the Fourier cosine series of $f(x) = x - \frac{1}{2}$ on $[0, 1]$ is given by*

$$\frac{-4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left((2k+1)\pi x\right).$$

Let $C(x, N) = \frac{-4}{\pi^2} \sum_{k=0}^{N} \frac{1}{(2k+1)^2} \cos\left((2k+1)\pi x\right)$ and plot $C(x, N)$ for $N = 1, 2, 5, 50$ $x \in [0, 1]$. How does this compare to the function $f(x) = x - \frac{1}{2}$ on $[0, 1]$? What if you plot it for $x \in [0, 2]$? $\Diamond$

Solution:

End of Solution

Problem 55.

(a) *Differentiate the series*

$$\frac{-4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\left((2k+1)\pi x\right)$$

term by term and plot various partial sums for that series on $[0,1]$. *How does this compare to the derivative of* $f(x) = x - \frac{1}{2}$ *on that interval?*

(b) *Differentiate the series you obtained in part a and plot various partial sums of that on* $[0,1]$. *How does this compare to the second derivative of* $f(x) = x - \frac{1}{2}$ *on that interval?* $\Diamond$

SOLUTION:

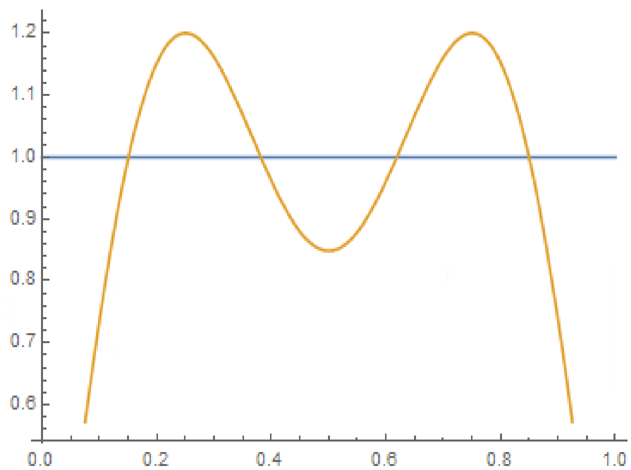
(a) $n = 1$

In[102]:= `n = 1;`

$$F[x\_] = \frac{4}{\pi}\sum_{k=0}^{n}\frac{1}{(2\,k+1)}\,\mathrm{Sin}[(2\,k+1)\,\pi\,x];$$

`Plot[{1, F[x]}, {x, 0, 1}]`

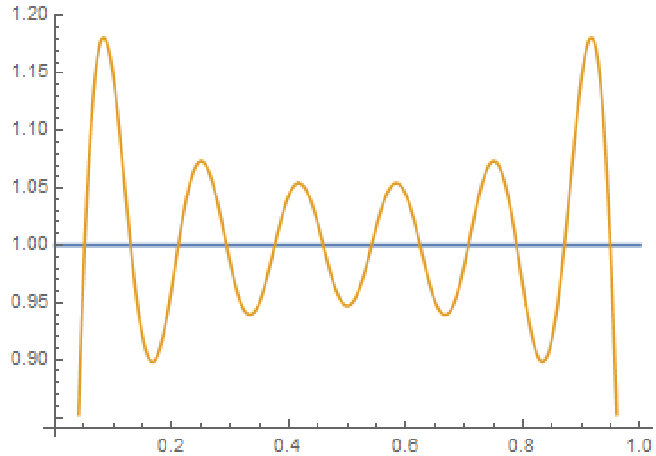
Out[104]=



$n = 5$

In[105]:=
```
n = 5;
```
$$\texttt{F[x\_]} = \frac{4}{\pi}\sum_{k=0}^{n}\frac{1}{(2\,k+1)}\ \texttt{Sin[(2 k + 1)}\ \pi\ \texttt{x];}$$
```
Plot[{1, F[x]}, {x, 0, 1}]
```

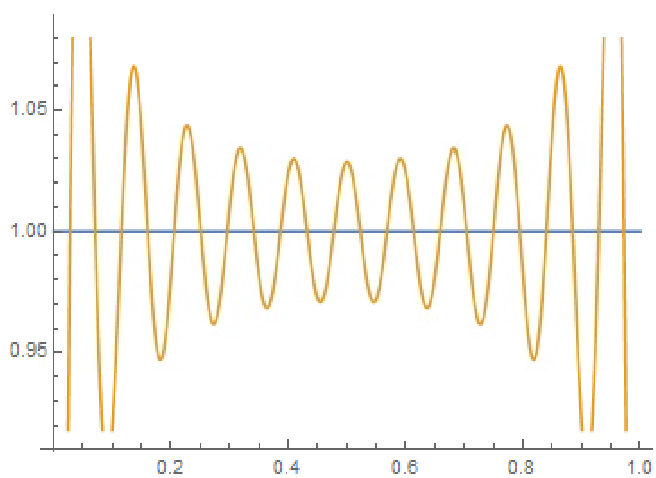
Out[107]=



$n = 10$

In[108]:=
```
n = 10;
```
$$\texttt{F[x\_]} = \frac{4}{\pi}\sum_{k=0}^{n}\frac{1}{(2\,k+1)}\ \texttt{Sin[(2 k + 1)}\ \pi\ \texttt{x];}$$
```
Plot[{1, F[x]}, {x, 0, 1}]
```

Out[110]=



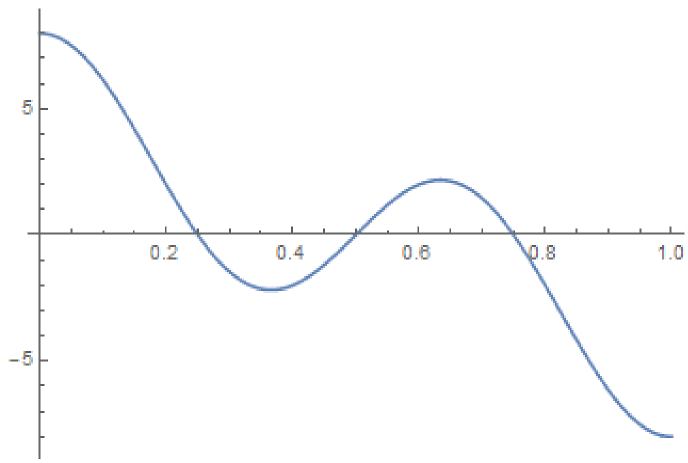
(b) $n = 1$

In[117]:= `n = 1;`

$$F[x\_] = 4\sum_{k=0}^{n} \mathrm{Cos}[(2\,k+1)\,\pi\,x];$$

`Plot[{F[x]}, {x, 0, 1}]`

Out[119]=



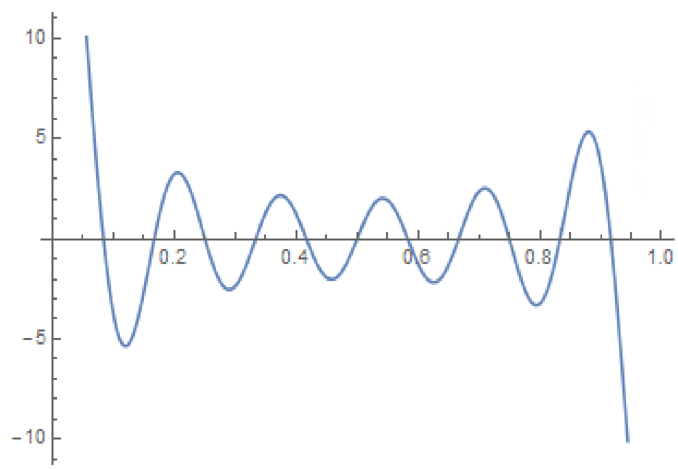
$n = 5$

In[114]:= `n = 5;`

$$F[x\_] = 4\sum_{k=0}^{n} \mathrm{Cos}[(2\,k+1)\,\pi\,x];$$

`Plot[{F[x]}, {x, 0, 1}]`

Out[116]=



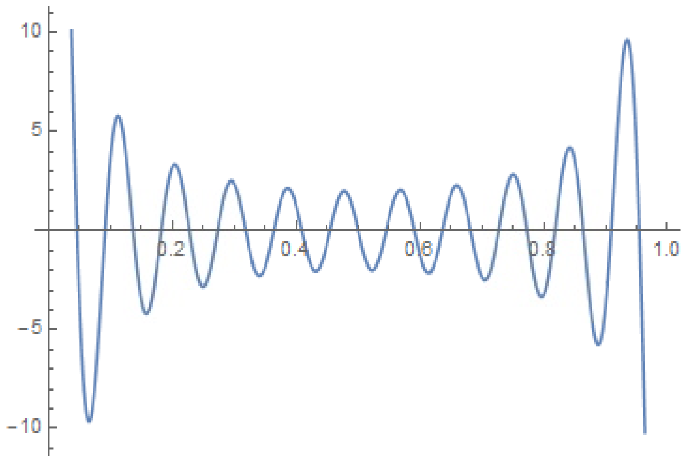
$n = 10$

In[111]:= `n = 10;`

$$F[x\_] = 4\sum_{k=0}^{n} \text{Cos}[(2\,k+1)\,\pi\,x];$$

```
Plot[{F[x]}, {x, 0, 1}]
```

Out[113]=



End of Solution

Part III

In Which We Find (Some) Answers

Chapter 4

Convergence of Sequences and Series

Problem 56. *Let a and b be real numbers with $b > 0$. Prove $|a| < b$ if and only if $-b < a < b$. Notice that this can be extended to $|a| \leq b$ if and only if $-b \leq a \leq b$.*

SOLUTION:

Recall that by definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Claim: *If $|a| < b$ then $-b < a < b$.*

Proof:

There are two cases:

Case 1: $a \geq 0$

In this case, since $b > 0$, we have $-b < \underbrace{a}_{=|a|} \leq b$.

Case 2: $a < 0$

In this case, $-a > 0$ so

$$-b < \underbrace{-a}_{=|a|} < b$$

and therefore

$$b > a > -b$$

or

$$-b < a < b.$$

■

Claim: *If* $-b < a < b$ *then* $|a| < b$.
Proof:

■

End of Solution

$\Diamond$

Problem 57. *Use the definition of convergence to zero to prove the following.*

(a) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Solution:

Let $\varepsilon > 0$ *be given. Take* $N > \frac{1}{\sqrt{\varepsilon}}$. *Then for every* $n > N$

$$\begin{aligned} \left|\frac{1}{n^2}\right| &= \frac{1}{n^2} \\ &< \frac{1}{N^2} \\ &< \frac{1}{(\sqrt{\varepsilon})^2} \\ &= \varepsilon. \end{aligned}$$

End of Solution

(b) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ◊

Solution:

Let $\varepsilon > 0$ *be given. Take* $N > \frac{1}{\varepsilon^2}$. *Then for every* $n > N$

$$\begin{aligned} \left|\frac{1}{\sqrt{n}}\right| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{N}} \\ &< \frac{1}{(\sqrt{\varepsilon^2})} \\ &= \varepsilon. \end{aligned}$$

End of Solution

Problem 58. *Use the definition of convergence to zero to prove*

$$\lim_{n\to\infty} \frac{n^2+4n+1}{n^3} = 0. \qquad \Diamond$$

Solution:

End of Solution

Problem 59. *Let b be a nonzero real number with $|b| < 1$ and let $\varepsilon > 0$.*

(a) *Solve the inequality $|b|^n < \varepsilon$ for n*

SOLUTION:

FIRST SOLUTION:

Since the natural logarithm is a strictly increasing function we know that if $x < y$ then $\log x < \log y$. Thus if

$$|b|^n < \varepsilon$$

then

Therefore

$$n \log |b| < \log \varepsilon.$$

Since $|b| < 1$, $\log |b| < 0$ so

$$n > \frac{\log \varepsilon}{\log |b|}.$$

SECOND SOLUTION:

Since $|b| < 1$ the $\log_{|b|}(x)$ is a strictly decreasing function so if $x < y$ then $\log x > \log y$. Thus if

$$|b|^n < \varepsilon$$

then

Therefore

$$n > \log_{|b|} \varepsilon = \frac{\log \varepsilon}{\log |b|}.$$

END OF SOLUTION

(b) *Use part (a) to prove $\lim_{n\to\infty} b^n = 0$.* ◊

SOLUTION:

Let $\varepsilon > 0$ be given. Take $N > \frac{\log \varepsilon}{\log |b|}$. Then for all $n > N$

$$n > \frac{\log(\varepsilon)}{\log |b|}$$

$$n \log |b| < \log(\varepsilon)$$

and since the exponential is also a strictly increasing function

$$|b|^n < \varepsilon.$$

END OF SOLUTION

Problem 60. *Negate the definition of* $\lim_{n\to\infty} s_n = 0$ *to provide a formal definition for* $\lim_{n\to\infty} s_n \neq 0$. $\Diamond$

Solution:

If $\exists\ N \in \mathbb{R}$ *such that* $\forall\ \varepsilon > 0$, $\exists\ n > N$ *such that*

$$|s_n| > \varepsilon.$$

End of Solution

Problem 61. *Use the definition to prove* $\lim_{n\to\infty} \frac{n}{n+100} \neq 0$. $\Diamond$
Solution:

End of Solution

Problem 62. *Let $b > 0$. Use the definition to prove $\lim_{n\to\infty} b^{\left(\frac{1}{n}\right)} = 1$. [Hint: You will probably need to separate this into two cases: $0 < b < 1$ and $b \geq 1$.]* $\Diamond$

SOLUTION:

Case 1: Suppose $b = 1$. Let $\epsilon > 0$ and let N be any real number. If $n > N$, then

$$\left|b^{1/n} - 1\right| = |1 - 1| = 0 < \epsilon.$$

Case 2: Suppose $b > 1$. Let $\epsilon > 0$ and let $N = \frac{\log(b)}{\log(1+\epsilon)}$. If $n > N$, then

$$\begin{aligned}
\frac{1}{n} < \frac{1}{N} = \frac{\log(1+\epsilon)}{\log(b)} &\implies \frac{1}{n}\log(b) < \log(1+\epsilon) \\
&\implies \log(b^{1/n}) < \log(1+\epsilon) \\
&\implies b^{1/n} < 1 + \epsilon \\
&\implies b^{1/n} - 1 = \left|b^{1/n} - 1\right| < \epsilon.
\end{aligned}$$

Case 3.1: Suppose $0 < b < 1$. Let $0 < \epsilon < 1$ and let $N = \frac{\log(b)}{\log(1-\epsilon)}$. If $n > N$, then

$$\begin{aligned}
\frac{1}{n} < \frac{1}{N} = \frac{\log(1-\epsilon)}{\log(b)} &\implies \frac{1}{n}\log(b) > \log(1-\epsilon) \\
&\implies \log(b^{1/n}) > \log(1-\epsilon) \ \textit{(since } \log(b) < 0 \\
&\implies b^{1/n} > 1 - \epsilon \\
&\implies b^{1/n} - 1 > -\epsilon \\
&\implies 1 - b^{1/n} = \left|b^{1/n} - 1\right| < \epsilon.
\end{aligned}$$

Case 3.2: Suppose $0 < b < 1$. Let $\epsilon \geq 1$ and let N be any real number. If $n > N$, then

$$\begin{aligned}
b > 0 &\implies b^{1/n} > 0 \\
&\implies -b^{1/n} < 0 \\
&\implies 1 - b^{1/n} < 1 \leq \epsilon \\
&\implies \left|b^{1/n} - 1\right| < \epsilon.
\end{aligned}$$

Therefore we can conclude that $\lim_{n\to\infty} b^{1/n} = 1$.

END OF SOLUTION

Problem 63.

(a) *Provide a rigorous definition for* $\lim_{n\to\infty} s_n \neq s$.

SOLUTION:

Let $(s_n)_{n=1}^{\infty}$ *be a sequence. We say that* s_n *does* **not** *converge to* s *(*$(s_n) \not\to s$*) if there is a real number,* ε*, such that for every* $N \in \mathbb{R}$ *there is an integer* n*,* $n \in \mathbb{N}$*, such that*

$$|s_n - s| > \varepsilon.$$

END OF SOLUTION

(b) *Use your definition to show that for any real number* a*,* $\lim_{n\to\infty} \left((-1)^n\right) \neq a$*. [Hint: Choose* $\varepsilon = 1$ *and use the fact that* $\left|a - (-1)^n\right| < 1$ *is equivalent to* $(-1)^n - 1 < a < (-1)^n + 1$ *to show that no choice of* N *will work for this* ε*.]*

SOLUTION:

Proof by Contradiction: *Suppose that* $\lim\limits_{n\to\infty} (-1)^n = a$*.*

In that case for every $\varepsilon > 0$ *there is an* $N \in \mathbb{R}$ *such that if* $n > N$

$$|a - (-1)^n| < \varepsilon$$

which is equivalent to

$$-1 < a - (-1)^n < 1$$

or

$$(-1)^n - 1 < a < (-1)^n + 1.$$

There is at least one value of $n > N$ *which is even and one which is odd. In both cases this last inequality must be satisfied.*

Case 1: n *is even. In this case*

$$0 < a < 2.$$

So a *is positive.*

Case 2: n *is odd. In this case*

$$-2 < a < 0.$$

So a *is negative.*

Since both cases must be true and a *cannot be* **simultaneously** *positive and negative we have a contradiction. Therefore* $(-1)^n$ *diverges.*

END OF SOLUTION

$\Diamond$

Problem 64. *Let* $(c)_{n=1}^{\infty} = (c, c, c, \ldots)$ *be a constant sequence. Show that* $\lim_{n\to\infty} c = c$. ▲

Solution:

Let $\varepsilon > 0$ *be given. Take* $N = 1$. *Then for every integer* $n > N$

$$|c - c| = 0 < \varepsilon.$$

Therefore $\lim\limits_{n\to\infty} c = c$.

End of Solution

Lemma 1.

(a) Triangle Inequality *Let a and b be real numbers. Then*

$$|a+b| \leq |a| + |b| \, .$$

(b) Reverse Triangle Inequality *Let a and b be real numbers. Then*

$$|a| - |b| \leq |a-b|$$

▲

Problem 65.

(a) *Prove Lemma 1. [Hint: For the Reverse Triangle Inequality, consider $|a| = |a-b+b|$.]*

(b) *Show $||a| - |b|| \leq |a-b|$. [Hint: You want to show $|a| - |b| \leq |a-b|$ and $-(|a| - |b|) \leq |a-b|$.]* ♢

Solution:

Part (a) of the Lemma: The Triangle Inequality

$$|a+b| \leq |a| + |b|$$

Proof:

Observe that

$$- |a| \leq a \leq |a|$$

and

$$- |b| \leq b \leq |b| \, .$$

Adding these gives

$$-(|a| + |b|) \leq a + b \leq |a| + |b| \, .$$

Therefore

$$|a+b| \leq |a| + |b| \, .$$

■

Part (b) of the Lemma: The Reverse Triangle Inequality

$$|a| - |b| \leq |a-b|$$

Proof:

Observe that

$$|a| = |a-b+b| \, .$$

So, by part (a) above:

$$|a| \leq |a-b| + |b|\,.$$

Therefore

$$|a| - |b| \leq |a-b|\,.$$

■

Part (b) of the Problem.
Proof:
To Show: $||a|-|b|| \leq |a-b|$

From part (b) of the lemma (The Reverse Triangle Inequality), we have

$$|a| - |b| \leq |a-b|\,.$$

Also from part (b) we have

$$|b| - |a| \leq |b-a|\,.$$

Therefore

$$|a| - |b| \geq -|b-a| = -|a-b|\,.$$

Combining these gives

$$-|a-b| \leq |a| - |b| \leq |a-b|$$

which is equivalent to

$$||a|-|b|| \leq |a-b|\,.$$

■

<u>End of Solution</u>

Theorem 5. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$*, then* $\lim_{n\to\infty} (a_n + b_n) = a + b$. ▲

Problem 66. *Prove Theorem 5.* ◊

Solution:

Let $\varepsilon > 0$ *be given.*

Since $\lim_{n\to\infty} a_n = a$ *there is an* $N_1 \in \mathbb{R}$ *such that* $n > N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$.

Since $\lim_{n\to\infty} b_n = b$ *there is an* $N_2 \in \mathbb{R}$ *such that* $n > N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$.

Take $N > \max(N_1, N_2)$*. Then if* $n > N$ *we have*

$$\begin{aligned}
|(a_n - b_n) - (a - b)| &= |(a_n - a) + (b - b_n)| \\
&< |(a_n - a)| + |(b - b_n)| \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore $\lim_{n\to\infty} (a_n + b_n) = a + b$.

End of Solution

Lemma 2. (A convergent sequence is bounded.) *If* $\lim_{n\to\infty} a_n = a$, *then there exists* $B > 0$ *such that* $|a_n| \leq B$ *for all* n. ▲

Problem 67. *Prove Lemma 2. [Hint: We know that there exists* N *such that if* $n > N$*, then* $|a_n - a| < 1$*. Let* $B = \max\left(|a_1|, |a_2|, \ldots, \left|a_{\lceil N \rceil}\right|, |a| + 1\right)$*, where* $\lceil N \rceil$ *represents the smallest integer greater than or equal to* N*. Also, notice that this is not a convergence proof so it is not safe to think of* N *as a large number.*[1]*]* ◊

SOLUTION:

Since $|a_n - a|$ *approaches zero as* n *increases there is an integer* $N \in \mathbb{N}$ *such if* $n > N$ *then*

$$|a - a_n| < 1.$$

Let $B = \max(|a_1|, \ldots, |a_N|, |a| + 1)$*. There are two cases:*

Case 1: $1 \leq n \leq N$

In this case $|a_n| \leq \max(|a_1|, \ldots, |a_N|) \leq \max(|a_1|, \ldots, |a_N|, |a| + 1)$.

Case 2: $n > N$

In this case we have

$$|a_n| - |a| < |a_n - a|$$

by the Reverse Triangle Inequality. Thus

$$|a_n| - |a| < 1$$

so

$$|a_n| < |a| + 1 < B.$$

Therefore a convergent sequence is bounded.

END OF SOLUTION

[1]Actually, this is a dangerous habit to fall into even in convergence proofs.

Theorem 6. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$*, then* $\lim_{n\to\infty} (a_n b_n) = ab$. ▲

Problem 68. *Prove Theorem 6.* ◊

SOLUTION:

Let $\varepsilon > 0$ *be given. Take* B *to be an upper bound of* $|a_n|$ *for* $n = 1, 2, \ldots,$ *and take* $\delta < \min\left(\frac{\varepsilon}{2B}, \frac{\varepsilon}{2(|b|+1)}\right)$. *Then* $\exists N \in \mathbb{R}$ *such that* $\forall n > N, |a - a_n| < \delta$ *and* $|b - b_n| < \delta$.

Therefore

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&\le |a_n| \, |b_n - b| + |b| \, |a_n - a| \\
&\le |a_n| \, |b_n - b| + (|b| + 1) \, |a_n - a| \\
&\le B \, |b_n - b| + (|b| + 1) \, |a_n - a| \\
&\le \frac{B\varepsilon}{2B} + \frac{\varepsilon(|b| + 1)}{2(|b| + 1)} \\
&\le \varepsilon.
\end{aligned}$$

Therefore $\lim_{n\to\infty} (a_n b_n) = ab$.

END OF SOLUTION

Corollary 1. (Corollary to Theorem 6.) *If* $\lim_{n\to\infty} a_n = a$ *and* $c \in \mathbb{R}$, *then* $\lim_{n\to\infty} c \cdot a_n = c \cdot a$. ◊

Problem 69. *Prove the above corollary to Theorem 6.* ◊

Solution:

This follows immediately from Theorem 6 and Problem 64.

End of Solution

Theorem 7. *Suppose* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Also suppose* $b \neq 0$ *and* $b_n \neq 0, \forall\, n$. *Then* $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$. ▲

Problem 70. *Prove Theorem 7.* ◊

SOLUTION:

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n}{b_n} - \frac{a_n}{b} + \frac{a_n}{b} - \frac{a}{b}\right| \\ &\leq \left|\frac{a_n}{b_n} - \frac{a_n}{b}\right| + \left|\frac{a_n}{b} - \frac{a}{b}\right| \\ &\leq |a_n|\left|\frac{1}{b_n} - \frac{1}{b}\right| + \left|\frac{1}{b}\right| |a_n - a|\end{aligned}$$

END OF SOLUTION

Problem 71. *Identify all of the theorems implicitly used to show that*

$$\lim_{n\to\infty}\frac{3n^3-100n+1}{5n^3+4n^2-7}=\lim_{n\to\infty}\frac{n^3\left(3-\frac{100}{n^2}+\frac{1}{n^3}\right)}{n^3\left(5+\frac{4}{n}-\frac{7}{n^3}\right)}=\frac{3}{5}.$$

Notice that this presumes that all of the individual limits exist. This will become evident as the limit is decomposed. ◊

Solution:

End of Solution

Theorem 8. (Squeeze Theorem for Sequences) *Let* $(r_n), (s_n),$ *and* (t_n) *be sequences of real numbers with* $r_n \leq s_n \leq t_n, \forall$ *positive integers* n*. Suppose* $\lim_{n\to\infty} r_n = s = \lim_{n\to\infty} t_n$*. Then* (s_n) *must converge and* $\lim_{n\to\infty} s_n = s$*.* ▲

Problem 72. *Prove Theorem 8. [Hint: This is probably a place where you would want to use* $s - \varepsilon < s_n < s + \varepsilon$ *instead of* $|s_n - s| < \varepsilon$*.]* ◊

Solution:

Let $\varepsilon > 0$ *be given. Since* $r_n \to s$ $\exists N_1 \in \mathbb{R}$ *such that* $\forall n > N_1$ $r_n \in (s-\varepsilon, s+\varepsilon)$*.*

Since $t_n \to s$ $\exists N_2 \in \mathbb{R}$ *such that* $\forall n > N_2$ $t_n \in (s-\varepsilon, s+\varepsilon)$*.*

Thus

$$s - \varepsilon < r_n \leq s_n$$

and

$$s_n \leq t_n < s + \varepsilon.$$

Therefore $s - \varepsilon < s_n < s + \varepsilon$*.*

Therefore $\lim\limits_{n\to\infty} s_n = s$*.*

End of Solution

Problem 73. *Show that* $(n)_{n=1}^{\infty}$ *diverges to infinity.* $\Diamond$

SOLUTION:

Take $\varepsilon = 1/2$ *and let* $N \in \mathbb{R}$ *be given. Then there is an integer* n *such that* $n > N$ *and* $n > 1 > 1/2$.

Therefore $(n)_{n=1}^{\infty}$ *diverges to infinity.*

END OF SOLUTION

Problem 74. *Show that if* $(a_n)_{n=1}^{\infty}$ *diverges to infinity then* $(a_n)_{n=1}^{\infty}$ *diverges.* ♢

Solution:

First Proof

Proof by Contradiction: *Suppose that* $(a_n)_{n=1}^{\infty}$ *diverges to infinity and that it converges.*

If the sequence converges then it is bounded, by Lemma 2. But if it is bounded then it cannot diverge to infinity. Thus we have a contradiction.

Therefore if a sequence diverges to infinity, then it diverges.

Second Proof

Let $\varepsilon > 0$, *and* $r \in \mathbb{R}$ *be fixed real numbers. Since* $(a_n)_{n=1}^{\infty}$ *diverges to infinity there is an* $N \in \mathbb{R}$ *such that for all* $n > N$, $a_n > 0$ *and* $a_n \geq \varepsilon + |r|$. *Therefore*

$$|a_n - r| > |a_n| - |r| = a_n - |r| > \varepsilon.$$

Therefore $(a_n)_{n=1}^{\infty}$ *diverges.*

End of Solution

Problem 75. *Suppose* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} b_n = \infty$.

(a) *Show that* $\lim_{n\to\infty} a_n + b_n = \infty$

(b) *Show that* $\lim_{n\to\infty} a_n b_n = \infty$

(c) *Is it true that* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$? *Explain.* ◊

Solution:

End of Solution

Problem 76. *Suppose* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} b_n = -\infty$ *and* $\alpha \in \mathbb{R}$. *Prove or give a counterexample:*

(a) $\lim_{n\to\infty} a_n + b_n = \infty$

SOLUTION:

This statement is false.

COUNTEREXAMPLE: *Take* $a_n = n$, *and* $b_n = -n$. *Then* $\lim_{n\to\infty} a_n + b_n = -\infty$.

END OF SOLUTION

(b) $\lim_{n\to\infty} a_n b_n = -\infty$

SOLUTION:

This statement is true.

Proof:

Let $r \in \mathbb{R}, r > 1$ *be given. Since* $a_n \to \infty$ *there is a real number* N_1 *such that for every* $n > N_1$, $a_n > |r|$.

Since $b_n \to -\infty$ *there is a real number* N_2 *such that for every* $n > N_2$, $b_n < -|r|$. *Take* $N > \max(N_1, N_2)$. *Then* $\forall n > N$,

$$\begin{aligned} a_n b_n &< |r| \left(-|r|\right. \\ &= -\left|r^2\right| \\ &< -|r|. \end{aligned}$$

Therefore $a_n b_n \to -\infty$. ■

END OF SOLUTION

(c) $\lim_{n\to\infty} \alpha a_n = \infty$

SOLUTION:

This statement is false.

COUNTEREXAMPLE: *Take* $\alpha < 0$. *Then* $\lim_{n\to\infty} \alpha a_n = -\infty$

END OF SOLUTION

(d) $\lim_{n\to\infty} \alpha b_n = -\infty$ ◊

SOLUTION:

This statement is false.

COUNTEREXAMPLE: *Take* $\alpha < 0$. *Then* $\lim_{n\to\infty} \alpha a_n = \infty$

END OF SOLUTION

Problem 77. *Show that each of the following sequences diverge.*

(a) $\left((-1)^n\right)_{n=1}^{\infty}$

(b) $\left((-1)^n\, n\right)_{n=1}^{\infty}$

(c) $a_n = \begin{cases} 1 & \text{if } n = 2^p \text{ for some } p \in \mathbb{N} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$ $\Diamond$

Solution:

End of Solution

Problem 78. *Suppose that $(a_n)_{n=1}^\infty$ diverges but not to infinity and that α is a real number. What conditions on α will guarantee that:*

(a) $(\alpha a_n)_{n=1}^\infty$ *converges?*

(b) $(\alpha a_n)_{n=1}^\infty$ *diverges?*

◊

SOLUTION:

There are two cases: $\alpha \neq 0$ and $\alpha = 0$.

Case 1: $\alpha \neq 0$ *Since $(a_n)_{n=1}^\infty$ diverges there is an ε_1 such that $\forall\ r_1 \in \mathbb{R}$, and $N \in \mathbb{R}\ \exists\ n > N$ such that*

$$|a_n - r_1| > \varepsilon_1.$$

Take $\varepsilon = |\alpha|\,\varepsilon_1$ and $r = r_1\alpha$. Then $\forall\ N \in \mathbb{R}$ such that

$$\begin{aligned} |a_n - r_1| &> \varepsilon_1 \\ \left|a_n - \frac{r}{\alpha}\right| &> \frac{\varepsilon}{|\alpha|} \\ \frac{1}{|\alpha|}\,|\alpha a_n - r| &> \frac{\varepsilon}{|\alpha|}. \end{aligned}$$

Therefore $\forall\ r \in \mathbb{R}$, and $N \in \mathbb{R}\ \exists\ n > N$ such that

$$|\alpha a_n - r| > \varepsilon$$

so $(\alpha a_n)_{n=1}^\infty$ diverges.

Case 2: $|\alpha| = 0$ *This is the special case of Problem* **??** *where $c = 0$.*

END OF SOLUTION

Problem 79. *Show that if $|r| > 1$ then $(r^n)_{n=1}^{\infty}$ diverges. Will it diverge to infinity?* ◊

SOLUTION:

Proof 1: *Let $N \in \mathbb{R}$ be given. Take $n > \frac{\ln(N)}{\ln(|r|)}$. Then*

$$\begin{aligned} |r|^n &> \left| r^{\frac{\ln(N)}{\ln(|r|)}} \right| \\ &= N. \end{aligned}$$

Therefore $|r|^n$ diverges to infinity.

Case 1, $r > 1$: *In this case $|r| = r$ so r^n also diverges to infinity.*

Case 2, $r < -1$: *In this case $|r| = -r$ so r^n will be positive when n is even, and negative when n is odd. Therefore r^n does not diverge to infinity.*

Proof 2: *Since $|r| > 1$, $|r| = 1 + \alpha$ for some $\alpha > 0$. Therefore*

$$\begin{aligned} |r|^n &= 1 + \alpha r + \frac{\alpha}{2} r^2 + \dots \\ &> 1 + \alpha r. \end{aligned}$$

Let $N \in \mathbb{R}^+$ be given. Then $\forall\ n > \frac{N^{1/n}-1}{\alpha}$

$$\begin{aligned} |r|^n &> |1 + n\alpha|^n \\ &= (1 + n\alpha)^n \\ &\geq \left(1 + \left(\frac{N^{1/n} - 1}{\alpha}\right)\alpha\right)^n \\ &= (1 + N^{1/n} - 1)^n \\ &= N. \end{aligned}$$

Therefore $|r|^n$ diverges to infinity.

Case 1, $r > 1$: *In this case $|r| = r$ so r^n also diverges to infinity.*

Case 2, $r < -1$: *In this case $|r| = -r$ so r^n will be positive when n is even, and negative when n is odd. Therefore r^n does not diverge to infinity.*

END OF SOLUTION

Problem 80. *Prove that if* $\lim_{n\to\infty} s_n = s$ *then* $\lim_{n\to\infty} |s_n| = |s|$. *Prove that the converse is true when* $s = 0$, *but it is not necessarily true otherwise.* $\Diamond$

Solution:

- *To Show: if* $\lim_{n\to\infty} s_n = s$ *then* $\lim_{n\to\infty} |s_n| = |s|$.

Proof:

Let $\varepsilon > 0$ *be given, and suppose* $s_n \to s$. *Then by definition* $\exists\, N \in \mathbb{R}$ *such that* $\forall\, n > N$ $|s_n - s| < \varepsilon$.

By Problem 65, part (b) we have

$$||s_n| - |(|\,s)| < |s_n - s| < \varepsilon.$$

Therefore $|s_n| \to |s|$. $\blacksquare$

- *To Show: If* $|s_n| \to 0$ *then* $s_n \to 0$.

Proof:

Let $\varepsilon > 0$ *be given Assume that* $|s_n| \to 0$. *Then* $\exists\, N \in \mathbb{R}$ *such that* $\forall\, n > N$ $|s_n| = ||s_n|| < \varepsilon$. *Therefore* $s_n \to 0$. $\blacksquare$

- *Counterexample: Consider* $s_n = -1 + \frac{1}{n}$, $n = 1, 2, 3, \ldots$.

  Clearly $s_n \to -1$ *while* $|s_n| = 1 - \frac{1}{n} \to 1 \neq -1$.

End of Solution

Problem 81.

(a) *Let (s_n) and (t_n) be sequences with $s_n \leq t_n, \forall n$. Suppose $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Prove $s \leq t$. [Hint: Assume for contradiction, that $s > t$ and use the definition of convergence with $\varepsilon = \frac{s-t}{2}$ to produce an n with $s_n > t_n$.]* ◊

SOLUTION:

Proof:

BY CONTRADICTION: *Assume that $s > t$.*

Since $s_n \to s$ there is a real number N_1 such that $\forall n > N_1$:

$$s_n \in \left(s - \frac{s-t}{2}, s + \frac{s-t}{2}\right)$$

or

$$s_n \in \left(\frac{s+t}{2}, \frac{3s-t}{2}\right).$$

Therefore

$$s_n > \frac{s+t}{2}.$$

Similarly, $t_n \in \left(\frac{s-3t}{2}, \frac{s+t}{2}\right)$ which implies that

$$t_n < \frac{s+t}{2}.$$

Therefore

$$t_n < \frac{s+t}{2} < s_n$$

which contradicts our assumption that $s_n \leq t_n, \forall n$. Therefore $s \leq t$. ■

END OF SOLUTION

(b) *Prove that if a sequence converges, then its limit is unique. That is, prove that if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = t$, then $s = t$.* ◊

SOLUTION:

Proof:

Suppose $a_n \to s$ and $a_n \to t$. Take $s_n = a_n$ and $t_n = a_n$.

Since $s_n \leq t_n$, $s_n \to s$, and $t_n \to t$, we see by part (a) above that:

$$s \leq t.$$

Also since $t_n \leq s_n$, $t_n \to t$, and $s_n \to s$, we see by part (a) above that:

$$t \leq s.$$

Since $s \leq t$ and $t \leq s$ we conclude that $s = t$. ■

END OF SOLUTION

Problem 82. *Prove that if the sequence* (s_n) *is bounded then* $\lim_{n\to\infty}\left(\frac{s_n}{n}\right) = 0$. $\Diamond$

Solution:

Let $\varepsilon > 0$ *be given. Suppose* $|s_n| < B \neq 0, \forall\ n \in \mathbb{N}$. *Take* $N > \frac{B}{\varepsilon}$. *Then*

$$\left|\frac{s_n}{n}\right| \leq \frac{B}{n} \leq \frac{B}{N} = \frac{B}{B/\varepsilon} = \varepsilon.$$

End of Solution

Problem 83.

(a) *Prove that if $x \neq 1$, then*

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

$\Diamond$

Solution:

Observe that

$$1 - x^{n+1} = (1 + x + x^2 + \cdots + x^n)(1 - x).$$

Therefore, when $x \neq 1$ we have

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

End of Solution

(b) *Use (a) to prove that if $|x| < 1$, then $\lim_{n\to\infty}\left(\sum_{j=0}^{n} x^j\right) = \frac{1}{1-x}$.* $\Diamond$

Solution:

We will show that $\lim\limits_{n\to\infty}\left|\sum\limits_{j=0}^{n} x^j - \dfrac{1}{1-x}\right| = 0$, *from which the conclusion follows immediately.*

Let $\varepsilon > 0$ be given. Observe that x is fixed, and that we are assuming that $|x| < 1$. Take $N \in \mathbb{R}$, such that $\left|x^{n+1}\right| < \varepsilon\,|1 - x|$. Then

$$\begin{aligned}
\left|\sum_{j=0}^{n} x^j - \frac{1}{1-x}\right| &= \left|\frac{1 - x^{n+1}}{1 - x} - \frac{1}{1-x}\right| \\
&= \frac{\left|x^{n+1}\right|}{|1 - x|} \\
&< \frac{\varepsilon\,|1 - x|}{|1 - x|} \\
&= \varepsilon.
\end{aligned}$$

Therefore $\lim\limits_{n\to\infty}\left|\sum\limits_{j=0}^{n} x^j - \dfrac{1}{1-x}\right| = 0.$

End of Solution

Problem 84. *Prove*

$$\lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k}{b_0 + b_1 n + b_2 n^2 + \cdots + b_k n^k} = \frac{a_k}{b_k},$$

provided $b_k \neq 0$. *[Notice that since a polynomial only has finitely many roots, then the denominator will be non-zero when* n *is sufficiently large.]* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 85. *Prove that if* $\lim_{n\to\infty} s_n = s$ *and* $\lim_{n\to\infty}(s_n - t_n) = 0$, *then* $\lim_{n\to\infty} t_n = s$. $\Diamond$

SOLUTION:

Let $\varepsilon > 0$ *be given.*

Since $\lim\limits_{n\to\infty} s_n - t_n = 0$, $\exists N_1 \in \mathbb{R}$ *such that* $n > N_1 \;\Rightarrow\; |s_n - t_n| < \frac{\varepsilon}{2}$.

Since $\lim\limits_{n\to\infty} s_n = s$, $\exists N_2 \in \mathbb{R}$ *such that* $n > N_2 \;\Rightarrow\; |s_n - s| < \frac{\varepsilon}{2}$.

Take $N = \max(N_1, N_2)$. *Then for every* $n > N$,

$$\begin{aligned} |s - t_n| &= |s - s_n + s_n - t_n| \\ &\le |s - s_n| + |s_n - t_n| \\ &= \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Therefore $\lim_{n\to\infty} t_n = s$.

END OF SOLUTION

SOLUTION:

Alternative Solution *Since* (a_n) *and* $(s_n - t_n)$ *both converge we have*

$$\begin{aligned} \lim_{n\to\infty} t_n &= \lim_{n\to\infty}[s_n - (s_n - t_n)] \\ &= \lim_{n\to\infty} s_n - \lim_{n\to\infty}(s_n - t_n) \\ &= s - 0 \\ &= s. \end{aligned}$$

END OF SOLUTION

Problem 86.

(a) *Prove that if* $\lim_{n\to\infty} s_n = s$ *and* $s < t$*, then there exists a real number* N *such that if* $n > N$ *then* $s_n < t$.

Solution:

Since $s_n \to s$ $\exists N \in \mathbb{R}$ *such that* $n > N \Rightarrow |s_n - s| < t - s$. *Therefore*

$$\begin{aligned} s - t < s_n - s &< t - s \\ 2s - t < \quad s_n \quad &< t. \end{aligned}$$

In particular

$$s_n < t.$$

End of Solution

(b) *Prove that if* $\lim_{n\to\infty} s_n = s$ *and* $r < s$*, then there exists a real number* M *such that if* $n > M$ *then* $r < s_n$. $\Diamond$

Solution:

By Corollary 1, if $s_n \to s$ *then* $-s_n \to -s$. *If* $r < s$ *then* $-s < -r$. *Thus* $(-s_n)_{n=0}^{\infty}$, $-s$*, and* $-r$ *satisfy all of the conditions of part (a) of this problem. Therefore*

$$\begin{aligned} -s_n &< -r \text{ or} \\ r &< s_n. \end{aligned}$$

End of Solution

Problem 87. *Suppose* (s_n) *is a sequence of positive numbers such that*

$$\lim_{n\to\infty}\left(\frac{s_{n+1}}{s_n}\right) = L.$$

(a) *Prove that if* $L < 1$, *then* $\lim_{n\to\infty} s_n = 0$. *[Hint: Choose* R *with* $L < R < 1$. *By the previous problem,* $\exists\ N$ *such that if* $n > N$, *then* $\frac{s_{n+1}}{s_n} < R$. *Let* $n_0 > N$ *be fixed and show* $s_{n_0+k} < R^k s_{n_0}$. *Conclude that* $\lim_{k\to\infty} s_{n_0+k} = 0$ *and let* $n = n_0 + k$.*]*

Solution:

Choose R *such that* $|L| < R < 1$. *Then by problem 86* $\exists\ N \in \mathbb{R}$ *such that* $n > N \ \Rightarrow\ \frac{s_{n+1}}{s_n} < R$. *Take* $n_0 > N$.

Claim 1: $\forall\ k \in \mathbb{N},\ \ s_{n_0+k} < R^k s_{n_0}$.

Proof of Claim 1: *(Proof by Induction on* k.*)*

Base case $(k = 1)$: *Since* $n_0 > N$,

$$\frac{s_{n_0+1}}{s_{n_0}} < R \text{ or}$$
$$s_{n_0+1} < R s_{n_0}.$$

Induction Hypothesis: $s_{n_0+(k-1)} < R^{k-1} s_{n_0}$

$$\frac{s_{n_0+k}}{s_{n_0+(k-1)}} < R$$
$$s_{n_0+k} < R s_{n_0+(k-1)}.$$

Therefore

$$s_{n_0+k} < R^k s_{n_0}.$$

■*(Claim 1)*

Claim 2: $\lim\limits_{n\to\infty} s_n = 0$.

Proof of Claim 2:

Let $\varepsilon > 0$ *be given. Since* $R < 1$ *we know from problem 59 that* $R^k \to 0$. *Take* K *such that* $k > K \ \Rightarrow\ R^k < \frac{\varepsilon}{s_{n_0}}$, *and take* $N = n_0 + k$. *Then for every* $n > N$

$$\begin{aligned} |s_n| &< R^k s_{n_0} \\ &< \frac{\varepsilon}{s_{n_0}} s_{n_0} \\ &< \varepsilon. \end{aligned}$$

Therefore $\lim\limits_{n\to\infty} s_n = 0$.

End of Solution

(b) *Let c be a positive real number. Prove* $\lim_{n\to\infty}\left(\frac{c^n}{n!}\right)=0.$ $\lozenge$

Solution:

Take $s_n = \frac{c^n}{n!}$ *and observe that*

$$\frac{s_{n+1}}{s_n} = \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \frac{c}{n+1} \to 0.$$

Therefore by part (a) $s_n = \frac{c^n}{n!} \to 0.$

End of Solution

Chapter 5

Convergence of the Taylor Series: A “Tayl” of Three Remainders

Lemma 3. [Triangle Inequality for Integrals] *If f and $|f|$ are integrable functions and $a \leq b$, then*

$$\left|\int_{t=a}^{b} f(t)\,\mathrm{d}t\right| \leq \int_{t=a}^{b} |f(t)|\,\mathrm{d}t. \qquad \blacktriangle$$

Problem 88. *Prove Lemma 3. [Hint: $-|f(t)| \leq f(t) \leq |f(t)|$.]* ◊

SOLUTION:

Since $f(t) \in \mathbb{R}$, we see that

$$-|f(t)| < f(t) \qquad < |f(t)|.$$

Therefore

$$-\int_{t=a}^{x} |f(t)|\,\mathrm{d}t < \int_{t=a}^{x} f(t)\,\mathrm{d}t < \int_{t=a}^{x} |f(t)|\,\mathrm{d}t$$

which is equivalent to

$$\left|\int_{t=a}^{x} f(t)\,\mathrm{d}t\right| < \int_{t=a}^{x} |f(t)|\,\mathrm{d}t.$$

END OF SOLUTION

Theorem 9. *If there exists a real number B such that $|f^{(n+1)}(t)| \leq B$ for all nonnegative integers n and for all t on an interval containing a and x, then*

$$\lim_{n\to\infty}\left(\frac{1}{n!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t\right) = 0$$

and so

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad \blacktriangle$$

Problem 89. *Prove Theorem 9. [Hint: You might want to use Problem 87 of Chapter 4. Also there are two cases to consider: $a < x$ and $x < a$ (the case $x = a$ is trivial). You will find that this is true in general. This is why we will often indicate that t is between a and x as in the theorem. In the case $x < a$, notice that*

$$\begin{aligned}\left|\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t\right| &= \left|(-1)^{n+1}\int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n\,\mathrm{d}t\right| \\ &= \left|\int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n\,\mathrm{d}t\right|.]\end{aligned} \qquad \Diamond$$

Solution:

Case 1: $x > a$

Since $x - t \geq 0$ we see that:

$$\begin{aligned}\left|\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t\right| &\leq \frac{1}{n!}\int_a^x \left|f^{(n+1)}(t)\right|\,\mathrm{d}t \\ &\leq \frac{B}{n!}\int_a^x (x-t)^n\,\mathrm{d}t \\ &= \left.\frac{-B(x-t)^{n+1}}{(n+1)!}\right|_a^x \\ &= \frac{B(x-a)^{n+1}}{(n+1)!}\end{aligned}$$

By part b) of Problem 87 $\lim_{n\to\infty}\frac{B(x-a)^{n+1}}{(n+1)!} \to 0.$

Case 2: $x < a$

This time $x - t \leq 0$ *so:*

$$\begin{aligned}
\left|\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t\right| &\leq \frac{1}{n!}\left|-\int_x^a f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t\right| \\
&= \frac{1}{n!}\left|\int_x^a f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t\right| \\
&\leq \frac{1}{n!}\int_x^a \left|f^{(n+1)}(t)\right| \left|(t-x)^n\right| \,\mathrm{d}t \\
&\leq \frac{B}{n!}\int_x^a (t-x)^n \,\mathrm{d}t \\
&= \left.\frac{B(t-x)^{n+1}}{(n+1)!}\right|_x^a \\
&= \frac{B(a-x)^{n+1}}{(n+1)!}
\end{aligned}$$

and again, by part b) of Problem 87 $\lim\limits_{n\to\infty} \dfrac{B(a-x)^{n+1}}{(n+1)!} \to 0.$

End of Solution

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Problem 90. *Use Theorem 9 to prove that for any real number x*

a) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Solution:

Since all of the derivatives of $\sin(x)$ are bounded above by 1 the result follows. The series converges for all $x \in \mathbb{R}$.

End of Solution

b) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Solution:

Same as part a).

End of Solution

c) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution:

On any interval $[a, b]$ the derivative of e^x is bounded above by e^b the result follows on $[a, b]$. Since a and b are arbitrary the result follows on $\mathbb{R}$.

End of Solution

Problem 91. *Recall that if $f(x) = e^x$ then $f'(x) = e^x$. Use this along with the Taylor series expansion of e^x about a to show that*

$$e^{a+b} = e^a e^b. \qquad \lozenge$$

Solution:

Let $x = a + b$. Then the Taylor Series of e^x expanded about a is:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n \\ &= e^a \sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n \\ &= e^a \sum_{n=0}^{\infty} \frac{b^n}{n!} \end{aligned}$$

and since $x = a + b$ we see that

$$e^{a+b} = e^a e^b.$$

End of Solution

Theorem 10. (Lagrange's Form of the Remainder) *Suppose f is a function such that $f^{(n+1)}(t)$ is continuous on an interval containing a and x. Then*

$$f(x)-\left(\sum_{j=0}^{n}\frac{f^{(j)}(a)}{j!}(x-a)^j\right)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is some number between a and x. ▲

Problem 92. *Prove Theorem 10 for the case where $x<a$. [Hint: Note that*

$$\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t=(-1)^{n+1}\int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n\,\mathrm{d}t.$$

Use the same argument on this integral. It will work out in the end. Really! You just need to keep track of all *of the negatives.]* ◊

Solution:

There are two cases:

Case 1 $x\geq a$: *Let* $m=\min_{t\in[a,x]}\left(f^{(n+1)}(t)\right)$ *and* $M=\max_{t\in[a,x]}\left(f^{(n+1)}(t)\right)$. *(Comment: We are assuming that $f^{(n+1)}(t)$ is continuous on the interval $[a,x]$.) Then $\forall t\in[a,x]$*

$$\begin{aligned}
m\leq{}& f^{(n+1)}(t)\leq M\\
m(x-t)^n\leq{}& f^{(n+1)}(t)(x-t)^n\leq M(x-t)^n\\
\int_{t=a}^{x} m(x-t)^n\,\mathrm{d}t\leq{}& \int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t\leq \int_{t=a}^{x} M(x-t)^n\,\mathrm{d}t\\
m\left[\frac{-(x-t)^{n+1}}{n+1}\right]_{t=a}^{x}\leq{}& \int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t\leq M\left[\frac{-(x-t)^{n+1}}{n+1}\right]_{t=a}^{x}\\
m\frac{(x-a)^{n+1}}{n+1}\leq{}& \int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t\leq M\frac{(x-a)^{n+1}}{n+1}\\
m\leq{}& \frac{\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t}{\frac{(x-a)^{n+1}}{n+1}}\leq M.
\end{aligned}$$

Thus, by the Intermediate Value Theorem, there is a number $c\in[a,x]$ such that

$$f^{(n+1)}(c)=\frac{\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t}{\frac{(x-a)^{n+1}}{n+1}}$$

or

$$\frac{1}{(n+1)!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t=\frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1}$$

Case 2 $a \geq x$: *Let* $m = \min_{t\in[x,a]}\left(f^{(n+1)}(t)\right)$ *and* $M = \max_{t\in[x,a]}\left(f^{(n+1)}(t)\right)$. *As before,*

$$\begin{aligned}
m \leq \quad & f^{(n+1)}(t) \leq & M \\
m(t-x)^n \leq \quad & f^{(n+1)}(t)(t-x)^n \leq & M(t-x)^n \\
\int_{t=x}^{a} m(t-x)^n \, \mathrm{dt} \leq \quad & \int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n \, \mathrm{dt} \leq & \int_{t=x}^{a} M(t-x)^n \, \mathrm{dt} \\
m\frac{(t-x)^{n+1}}{n+1}\bigg|_{t=a}^{x} \leq \quad & \int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n \, \mathrm{dt} \leq & M\frac{(t-x)^{n+1}}{n+1}\bigg|_{t=a}^{x} \\
m\frac{(a-x)^{n+1}}{n+1} \leq \quad & \int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n \, \mathrm{dt} \leq & M\frac{(a-x)^{n+1}}{n+1} \\
m \leq \quad & \frac{\int_{t=x}^{a} f^{(n+1)}(t)(t-x)^n \, \mathrm{dt}}{\frac{(a-x)^{n+1}}{n+1}} \leq & M. \\
m \leq \quad & \frac{\left[(-1)^{n+1}\right]\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{dt}}{\left[(-1)^{n+1}\right]\frac{(x-a)^{n+1}}{n+1}} \leq & M. \\
m \leq \quad & \frac{\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{dt}}{\frac{(x-a)^{n+1}}{n+1}} \leq & M.
\end{aligned}$$

Thus in either case we have, by the Intermediate Value Theorem, a number $c \in [a, x]$ *such that*

$$f^{(n+1)}(c) = \frac{\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{dt}}{\frac{(x-a)^{n+1}}{n+1}}$$

or

$$\frac{1}{(n+1)!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{dt} = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

End of Solution

Problem 93. *This problem investigates the Taylor series representation*

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

(a) *Use the fact that* $\frac{1-(-x)^{n+1}}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-x)^n$ *to compute the remainder*

$\frac{1}{1+x} - \left(1 - x + x^2 - x^3 + \cdots + (-x)^n\right)$.

Specifically, compute this remainder when $x = 1$ *and conclude that the Taylor Series does not converge to* $\frac{1}{1+x}$ *when* $x = 1$.

Solution:

Let $R(n)$ *be the nth order remainder. Then*

$$R(n) = \begin{cases} -1/2 & \text{when } n \text{ is even} \\ 1/2 & \text{when } n \text{ is odd.} \end{cases}$$

End of Solution

(b) *Compare the remainder in part a with the Lagrange form of the remainder to determine what c is when* $x = 1$.

Solution:

When $x = 1$ *the Lagrange form of the remainder is* $R(n) = \frac{(-1)^{n+1}}{(1+c)^{n+2}}$. *When n is even we have*

$$\frac{(-1)^{n+1}}{(1+c)^{n+2}} = -1/2$$

so that

$$c = 2^{\frac{1}{n+2}} - 1.$$

When n is odd we get the same.

End of Solution

(c) *Consider the following argument: If* $f(x) = \frac{1}{1+x}$, *then*

$$f^{(n+1)}(c) = \frac{(-1)^{n+1}(n+1)!}{(1+c)^{n+2}}$$

so the Lagrange form of the remainder when $x = 1$ *is given by*

$$\frac{(-1)^{n+1}(n+1)!}{(n+1)!(1+c)^{n+2}} = \frac{(-1)^{n+1}}{(1+c)^{n+2}}$$

where $c \in [0, 1]$. *It can be seen in part b that* $c \neq 0$. *Thus* $1 + c > 1$ *and so by Problem 59 of Chapter 4, the Lagrange remainder converges to 0 as* $n \to \infty$.

This argument would suggest that the Taylor series converges to $\frac{1}{1+x}$ *for* $x = 1$. *However, we know from part (a) that this is incorrect. What is* *wrong with the argument?* ◊

SOLUTION:

Problem #59 of Chapter 4 reuires that $|b| < 0$ *and that* b *is fixed. For this problem we have*

$$b = \frac{(-1)^{n+1}}{(1+c)^{n+2}}$$

but since c *is not fixed, neither is* b*.*

Therefore problem #59 does not apply.

END OF SOLUTION

Problem 94. *Show that if* $-\frac{1}{2} \leq x \leq c \leq 0$*, then* $|\frac{x}{1+c}| \leq 1$ *and modify the above proof to show that the binomial series converges to* $\sqrt{1+x}$ *for* $x \in \left[-\frac{1}{2}, 0\right]$. $\Diamond$

Solution:

End of Solution

Problem 95.

(a) *Compute the Lagrange form of the remainder for the Maclaurin series for* $\ln(1+x)$.

<u>Solution:</u>

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

so the Lagrange form of the remainder is:

$$\begin{aligned}
\mathcal{L}(f(x)) &= \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \\
&= \frac{(-1)^n n!}{(n+1)!}\frac{x^{n+1}}{(1+c)^{n+1}} \\
&= \frac{(-1)^n}{(n+1)}\frac{x^{n+1}}{(1+c)^{n+1}}
\end{aligned}$$

<u>End of Solution</u>

(b) *Show that when* $x = 1$, *the Lagrange form of the remainder converges to* 0 *and so the equation* $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ *is actually correct.* $\Diamond$

<u>Solution:</u>

When $x = 1$ *the Lagrange form of the remainder of* $f(x) = \ln(1+x)$ *is* $\mathcal{L}(f(x)) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ *so*

$$\begin{aligned}
|\mathcal{L}(f(1))| &= \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| \\
&= \left|\frac{(-1)^n n!}{(n+1)!(1+c)^{n+1}}\right| \\
&\leq \frac{1}{n+1}.
\end{aligned}$$

Therefore $\mathcal{L}(f(1)) \to 0$ *as* $n \to \infty$.

Therefore $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

<u>End of Solution</u>

Theorem 11. (Cauchy's Form of the Remainer in Taylor Series) *Suppose f is a function such that $f^{(n+1)}(t)$ is continuous on an interval containing a and x. Then*

$$f(x) - \left(\sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j\right) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

where c is some number between a and x. ▲

Problem 96. *Prove Theorem 11 using an argument similar to the one used in the proof of Theorem 10. Don't forget there are two cases to consider.* ♢

Solution:

End of Solution

Problem 97. *Suppose* $-1 < x \leq c \leq 0$ *and consider the function* $g(c) = \frac{c-x}{1+c}$*. Show that on* $[x, 0]$*,* g *is increasing and use this to conclude that for* $-1 < x \leq c \leq 0$,

$$\frac{c-x}{1+c} \leq |x|.$$

Use this fact to finish the proof that the binomial series converges to $\sqrt{1+x}$ *for* $-1 < x < 0$. ♢

Solution:

End of Solution

Problem 98. *Find the Integral form, Lagrange form, and Cauchy form of the remainder for Taylor series for the following functions expanded about the given values of a.*

(a) $f(x) = e^x, \quad a = 0$

Solution:

End of Solution

(b) $f(x) = \sqrt{x}$, $a = 1$

Solution:

End of Solution

(c) $f(x) = (1+x)^\alpha, \quad a = 0$

Solution:

End of Solution

(d) $f(x) = \frac{1}{x}$, $a = 3$

Solution:

End of Solution

(e) $f(x) = \ln x, \;\; a = 2$

Solution:

End of Solution

(f) $f(x) = \cos x, a = \frac{\pi}{2}$

Solution:

End of Solution

◊

Chapter 6

Continuity: What It Isn't and What It Is

Problem 99.

(a) *Given* $f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos(a^n \pi x)$, *what is the smallest value of* a *for which* f *satisfies Weierstrass' criterion to be continuous and nowhere differentiable.*

(b) *Let* $f(x, N) = \sum_{n=0}^{N} \left(\frac{1}{2}\right)^n \cos(13^n \pi x)$ *and use a computer algebra system to plot* $f(x, N)$ *for* $N = 0, 1, 2, 3, 4, 10$ *and* $x \in [0, 1]$.

(c) *Plot* $f(x, 10)$ *for* $x \in [0, c]$, *where* $c = 0.1, 0.01, 0.001, 0.0001, 0.00001$. *Based upon what you see in parts b and c, why would we describe the function to be somewhat "fractal" in nature?* ♢

<u>SOLUTION:</u>

<u>END OF SOLUTION</u>

Problem 100. *Use the definition of continuity to show that if m and b are fixed (but unspecified) real numbers then the function*

$$f(x) = mx + b$$

is continuous at every real number a. ◊

SOLUTION:

Let $\varepsilon > 0$ and $a \in \mathbb{R}$ be given. There are two cases:

Case 1 $m = 0$: *Take $\delta = \varepsilon$. Then $\forall x$ such that $|x-a| < \delta$ we see that $|f(x) - f(a)| = b - b = 0 < \varepsilon$.*

Case 2 $m \neq 0$: *Take $\delta < \frac{\varepsilon}{|m|}$. Then $\forall x$ such that $|x-a| < \delta$ we see that $|f(x) - f(a)| = |(mx - b) - (ma - b)| = |m(x-a)| < |m|\,\delta = |m|\,\frac{\varepsilon}{|m|} = \varepsilon$.*

END OF SOLUTION

Problem 101.

(a) *Given a particular $\varepsilon > 0$ in the definition of continuity, show that if a particular $\delta_0 > 0$ satisfies the definition, then any δ with $0 < \delta < \delta_0$ will also work for this ε.*

Solution:

Suppose that $|x - a| < \delta_0 \;\Rightarrow\; |f(x) - f(a)| < \varepsilon$. Then $|x - a| < \delta < \delta_0 \;\Rightarrow\; |f(x) - f(a)| < \varepsilon$ also.

End of Solution

(b) *Show that if a δ can be found to satisfy the conditions of the definition of continuity for a particular $\varepsilon_0 > 0$, then this δ will also work for any ε with $0 < \varepsilon_0 < \varepsilon$.* $\Diamond$

Solution:

Suppose that $|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \varepsilon_0$. Then $|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \varepsilon < \varepsilon_0$ also.

End of Solution

Problem 102. *Use the definition of continuity to show that*

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

is continuous at $a = 0$. ◊

Solution:

Le $\varepsilon > 0$ be given. The for every $x \in \mathbb{R}$ such that $|x| < \delta$

$$\begin{aligned} |f(x) - f(0)| &= \left|\pm\sqrt{x}\right| \\ &= \sqrt{x} \\ &< \sqrt{\delta} \\ &= \sqrt{\varepsilon^2} \\ &= \varepsilon. \end{aligned}$$

End of Solution

Problem 103. *Use the definition of continuity to show that* $f(x) = \sqrt{x}$ *is continuous at* $a = 0$*. How is this problem different from problem 102? How is it similar?* $\Diamond$

Solution:

End of Solution

Problem 104. *Use the definition of continuity to show that* $f(x) = \sqrt{x}$ *is continuous at any positive real number* a. $\Diamond$

Solution:

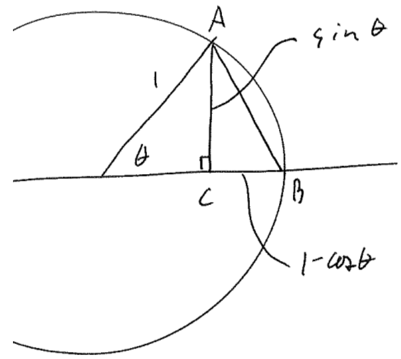
End of Solution

Problem 105.

(a) *Use a unit circle to show that for $0 \le \theta < \frac{\pi}{2}$, sin $\theta \le \theta$ and $1 - \cos\theta \le \theta$ and conclude $|\sin\theta| \le |\theta|$ and $|1-\cos\theta| \le |\theta|$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.*

SOLUTION:

From the diagram



we see that when $0 < \theta < \frac{\pi}{2}$, $1 - \cos(\theta) \le \overline{AB} \le \theta$.

Moreover since $\cos(x)$ *is an even function* $1 - \cos(\theta) > 0$ *we see that*

$$|1 - \cos\theta| < \theta$$

for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

Also on $0 < \theta < \frac{\pi}{2}$ $\sin(\theta)$ *is positive so we have*

$$-\theta < \sin(\theta) \le \overline{AB} \le \theta$$

and since $\sin(\theta)$ *is an odd function we see that*

$$\begin{aligned} -\theta &< \sin(\theta) \le \theta \\ \theta &> -\sin(\theta) \ge \theta \\ \theta &> \sin(-\theta) \ge \theta \end{aligned}$$

Therefore on the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$-\theta < \sin(\theta) < \theta$$

or

$$|\sin(\theta)| < \theta.$$

END OF SOLUTION

(b) *Use the definition of continuity to prove that* $f(x) = \sin x$ *is continuous at any point* a. *[Hint:* $\sin x = \sin(x - a + a)$.*]*

SOLUTION:

Let $\varepsilon > 0$ *be given. Take* $\delta < \min\left(\frac{\varepsilon}{|\cos(a)|+|\sin(a)|}, \frac{\pi}{2}\right)$. *Then* $\forall$ x *such that* $|x-a| < \delta$ *we have*

$$\begin{aligned}
|f(x) - f(a)| &= |\sin(x) - \sin(a)| \\
&= |\sin(x - a + a) - \sin(a)| \\
&= |\sin(x-a)\cos(a) + \cos(x-a)\sin(a) - \sin(a)| \\
&= |\sin(x-a)\cos(a) + \sin(a)\left(\cos(x-a) - 1\right)| \\
&\leq |\sin(x-a)\cos(a)| + |\sin(a)\left(\cos(x-a) - 1\right)| \\
&= |\cos(a)|\,|\sin(x-a)| + |\sin(a)|\,|(\cos(x-a) - 1)| \\
&\leq |\cos(a)|\,|x-a| + |\sin(a)|\,|x-a| \\
&= |x-a|\left(|\cos(a)| + |\sin(a)|\right) \\
&= \varepsilon.
\end{aligned}$$

END OF SOLUTION

$\Diamond$

Problem 106.

(a) *Use the definition of continuity to show that $f(x) = e^x$ is continuous at $a = 0$.*

SOLUTION:

Let $\varepsilon > 0$ be given. There are two cases:

Case 1: $\varepsilon \geq 1$

Take $\delta < \log 2$. Then $\forall\ x$ such that $|x| < \delta$ we have

$$\begin{array}{rcccl} -\log(2) < & x & < \log(2) \\ 1/2 & < & e^x & < & 2 \\ -\varepsilon < -1/2 & < & e^x - 1 & < & 1 \ < \varepsilon. \end{array}$$

Therefore $|e^x - 1| < \varepsilon$ when $\varepsilon \geq 1$.

Case 2: $\varepsilon < 1$

Take $\delta = \min\left(\log(1+\varepsilon), -\log(1-\varepsilon)\right)$. Then $\forall\ x$ such that $|x| < \delta$ we see that:

$$\begin{array}{rcccl} \log(1-\varepsilon) & < & x & < & \log(1+\varepsilon) \\ 1-\varepsilon & < & e^x & < & 1+\varepsilon \\ -\varepsilon & < & e^x - 1 & < & \varepsilon \end{array}$$

Therefore $|e^x - 1| < \varepsilon$ when $\varepsilon < 1$.

Therefore $f(x) = e^x$ is continuous at $x = 0$

END OF SOLUTION

(b) *Show that $f(x) = e^x$ is continuous at any point a. [Hint: Rewrite $e^x - e^a$ as $e^{a+(x-a)} - e^a$ and use what you proved in part a.]* $\Diamond$

SOLUTION:

Let $\varepsilon > 0$ be given. Take $y = x - a$. Since $f(y) = e^y$ is continuous at $y = 0$ by part a) there is a $\delta_1 > 0$ such that if $|y| < \delta_1$ then:

$$|e^y - 1| < \frac{\varepsilon}{e^a}.$$

Take $\delta = \delta_1$. Then $\forall\ y = x - a$ such that $|y| = |x - a| < \delta$ we have:

$$\begin{aligned} |e^a(e^y - 1)| &< \left|e^a \frac{\varepsilon}{e^a}\right| \\ |e^x - e^a)| &< \varepsilon \end{aligned}$$

Therefore $f(x) = e^x$ is continuous at every real number.

END OF SOLUTION

Problem 107. *Use the definition of continuity to show that*

$$f(x) = \begin{cases} x\,\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at 0. ◊

SOLUTION:

END OF SOLUTION

Problem 108.

(a) *Use the definition of continuity to show that the function*

$$D(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is continuous at 0.

SOLUTION:

Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon$. There are two cases:

Case 1, x is rational: *In this case* $|x| < \delta \Rightarrow |Dx - D(0)| = |x| < \delta = \varepsilon$.

Case 2, x is irrational *In this case* $|x| < \delta \Rightarrow |Dx - D(0)| = 0 < \varepsilon$.

END OF SOLUTION

(b) *Let $a \neq 0$. Use the definition of continuity to show that D is not continuous at a. [Hint: You might want to break this up into two cases where a is rational or irrational. Show that no choice of $\delta > 0$ will work for $\varepsilon = |a|$. Note that Theorem 2 of Chapter 1 will probably help here.]*

SOLUTION:

Again there are two cases:

Case 1, $a \in \mathbb{Q}$: *Let* $\varepsilon = \frac{|a|}{2}$. *Then* $\forall\ \delta > 0 \exists \bar{x} \in (a - \delta, a + \delta)$ *such that* $\bar{x} \in \mathbb{R} - \mathbb{Q}$, *and*

$$\begin{aligned} |D(x) - D(a)| &= |D(a)| \\ &= |a| \\ &> \frac{|a|}{2} \\ &= \varepsilon. \end{aligned}$$

Case 2, $a \in \mathbb{R} - \mathbb{Q}$: *In this case* $\forall \delta > 0, \exists \bar{x} \in \left(a - \frac{|a|}{2}, a + \frac{|a|}{2}\right)$ *such that* $\bar{x} \in \mathbb{Q}$. *Therefore*

$$\begin{aligned} |D(\bar{x}) - D(a)| &= |D(\bar{x})| \\ &= |\bar{x}| \\ &\geq \left|a - \frac{|a|}{2}\right| \\ &= \frac{|a|}{2}. \end{aligned}$$

END OF SOLUTION

◊

Theorem 12. *The function f is continuous at a if and only if f satisfies the following property:*

$$\forall \textit{ sequences } (x_n)\,, \textit{ if } \lim_{n\to\infty} x_n = a \textit{ then } \lim_{n\to\infty} f(x_n) = f(a). \qquad \blacktriangle$$

Problem 109. *Use Theorem 12 to show that*

$$f(x) = \begin{cases} \frac{|x|}{x}, & \textit{if } x \neq 0 \\ a, & \textit{if } x = 0 \end{cases}$$

is not continuous at 0*, no matter what value a is.* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 110. *Use Theorem 12 to show that*

$$D(x) = \begin{cases} x, & \textit{if } x \textit{ is rational} \\ 0, & \textit{if } x \textit{ is irrational} \end{cases}$$

is not continuous at $a \neq 0$. ◊

SOLUTION:

There are two cases:

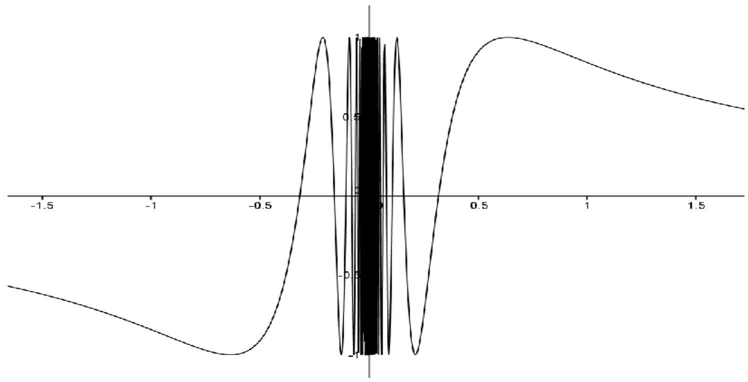
Case 1: *a is rational.*
In this case choose $x_n \in \mathbb{R} - \mathbb{Q}$ such that $x_n \in (a, a + 1/n)$. Then $x_n \to a \neq 0$. But $D(x_n) = 0$, $\forall\ n \in \mathbb{N}$ so that $D(x_n) \to 0 \neq a$. Therefore $D(x)$ is not continuous at a when a is rational.

Case 2: *a is irrational. In this case choose $x_n \in \mathbb{Q}$ such that $x_n \in (a, a+1/n)$. Then $x_n \to a$, but $\lim_{n\to\infty} D(x_n) = \lim_{n\to\infty} x_n = a \neq 0 = D(a)$. Therefore $D(x)$ is not continuous at a when a is irrational.*

Therefore $D(x)$ is not continuous at $a \neq 0$.

END OF SOLUTION

Problem 111. *The function $T(x) = \sin\left(\frac{1}{x}\right)$ is often called the topologist's sine curve. Whereas $\sin x$ has roots at $n\pi$, $n \in \mathbb{Z}$ and oscillates infinitely often as $x \to \pm\infty$, T has roots at $\frac{1}{n\pi}$, $n \in \mathbb{Z}$, $n \neq 0$, and oscillates infinitely often as x approaches zero. A rendition of the graph follows.*



Notice that T is not even defined at $x = 0$. We can extend T to be defined at 0 by simply choosing a value for $T(0)$:

$$T(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ b, & \text{if } x = 0 \end{cases}.$$

Use Theorem 12 to show that T is not continuous at 0, no matter what value is chosen for b.

SOLUTION:

Case 1: $b \neq 1$

Consider the sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \left(\frac{\pi(4n+1)}{2}\right)$. Clearly $a_n \to 0$. However $T(a_n) = \sin\left(\frac{2}{\pi(4n+1)}\right) = 1$. Therefore by Theorem 12 $T(x)$ is not continuous at zero.

Case 2: $b = 1$

Consider the sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \left(\frac{\pi(4n-1)}{2}\right)$. Clearly $a_n \to 0$. However $T(a_n) = \sin\left(\frac{2}{\pi(4n+1)}\right) = -1$. Therefore by Theorem 12 $T(x)$ is not continuous at zero.

END OF SOLUTION

Problem 112. *Turn the ideas of the previous two paragraphs into a formal proof of Theorem 12.* ◊

Solution:

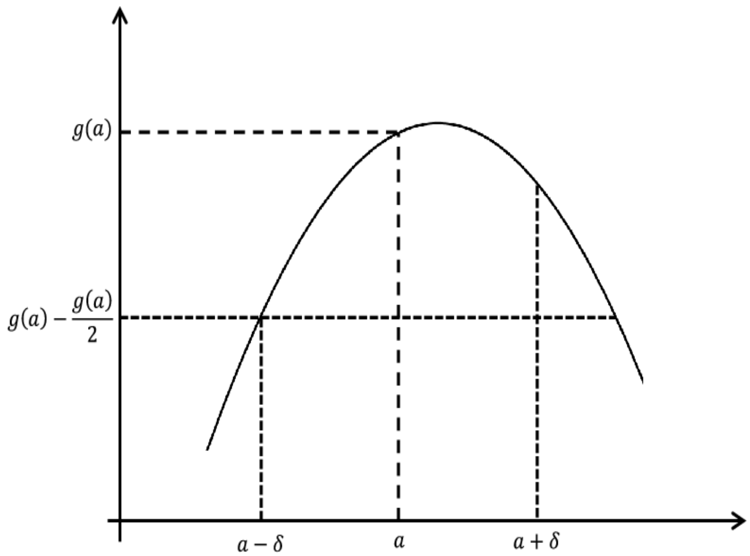
End of Solution

Problem 113. *Use Theorem 12 to show that if f and g are continuous at a, then $f \cdot g$ is continuous at a.* $\Diamond$
SOLUTION:

END OF SOLUTION

Lemma 4. *If g is continuous at a and $g(a) \neq 0$, then there exists $\delta > 0$ such that $g(x) \neq 0$ for all $x \in (a - \delta, a + \delta)$.* ▲

Problem 114. *Prove Lemma 4. [Hint: Consider the case where $g(a) > 0$. Use the definition with $\varepsilon = \frac{g(a)}{2}$. The picture is below; make it formal.*



For the case $g(a) < 0$, consider the function $-g$.] ◊

SOLUTION:

END OF SOLUTION

Problem 115. *Use Theorem 12, to prove that if f and g are continuous at a and $g(a) \neq 0$, then f/g is continuous at a.* $\Diamond$

SOLUTION:

END OF SOLUTION

Theorem 13. *Suppose f is continuous at a and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a. [Note that $(g \circ f)(x) = g(f(x))$.]* ▲

Problem 116. *Prove Theorem 13*

(a) *Using the definition of continuity.*

(b) *Using Theorem 12.* ◊

Solution:

End of Solution

Problem 117. *Show that each of the following is a continuous function at every point in its domain.*

1. *Any polynomial.*
2. *Any rational function. (A rational function is defined to be a ratio of polynomial.*
3. $\cos x$.
4. *The other trig functions:* $\tan(x)$, $\cot(x)$, $\sec(x)$, *and* $\csc(x)$. $\Diamond$

Solution:

End of Solution

Problem 118. *What allows us to conclude that $f(x) = \sin(e^x)$ is continuous at any point a without referring back to the definition of continuity?* $\Diamond$

Solution:

End of Solution

Problem 119. *Compute the following limits. Be sure to point out how continuity is involved.*

(a) $\lim_{n\to\infty} \sin\left(\dfrac{n\pi}{2n+1}\right)$

(b) $\lim_{n\to\infty} \sqrt{\dfrac{n}{n^2+1}}$

(c) $\lim_{n\to\infty} e^{(sin\ (1/n))}$

$\Diamond$

Solution:

End of Solution

Problem 120. *Use the definition of a limit to verify that*

$$\lim_{x\to a} \frac{x^2-a^2}{x-a} = 2a. \qquad \Diamond$$

SOLUTION:
Let $\varepsilon > 0$ *be given. Take* $\delta = \varepsilon$*. Then* $\forall x \neq a$ *such that* $|x-a| < \delta$ *we see that*

$$\begin{aligned}\left|\frac{x^2-a^2}{x-a} - 2a\right| &= |x+a-2a| \\ &= |x-a| \\ &< \delta \\ &= \varepsilon.\end{aligned}$$

Therefore $\lim_{x\to a} \frac{x^2-a^2}{x-a} = 2a.$
END OF SOLUTION

Problem 121. *Use the definition of a limit to verify each of the following limits.*

(a) $\lim_{x\to 1} \dfrac{x^3-1}{x-1} = 3$

Hint:

$$\begin{aligned}\left|\frac{x^3-1}{x-1} - 3\right| &= \left|x^2+x+1-3\right| \\ &\leq \left|x^2-1\right| + |x-1| \\ &= \left|(x-1+1)^2-1\right| + |x-1| \\ &= \left|(x-1)^2+2(x-1)\right| + |x-1| \\ &\leq |x-1|^2 + 3\,|x-1|\,.\end{aligned}$$

SOLUTION:
Let $\varepsilon > 0$ *be given. Take* $\delta < \min\left(\sqrt{\frac{\varepsilon}{6}}, \frac{\varepsilon}{3}\right)$. *Then for every* $x \neq a$ *such that* $|x-a| < \delta$ *we see that*

$$\begin{aligned}\left|\frac{x^3-1}{x-1} - 3\right| &= \left|x^2+x+1-3\right| \\ &\leq \left|x^2+x-2\right| \\ &\leq \left|x^2-1+x-1\right| \\ &\leq \left|x^2-1\right| + |x-1| \\ &= \left|(x-1+1)^2-1\right| + |x-1| \\ &= \left|(x-1)^2-2(x-1)\right| + |x-1| \\ &\leq |x-1|^2 + 3\,|x-1| \\ &\leq |x-1|^2 + 3\,|x-1| \\ &\leq \left(\sqrt{\varepsilon/2}\right)^2 + 3\left(\frac{\varepsilon}{6}\right) \\ &\leq \varepsilon.\end{aligned}$$

END OF SOLUTION

(b) $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} = 1/2$

Hint:

$$\begin{aligned}
\left|\frac{\sqrt{x}-1}{x-1}-\frac{1}{2}\right| &= \left|\frac{1}{\sqrt{x}+1}-\frac{1}{2}\right| \\
&= \left|\frac{2-(\sqrt{x}+1)}{2(\sqrt{x}+1)}\right| \\
&= \left|\frac{1-x}{2(1+\sqrt{x})^2}\right| \\
&\leq \frac{1}{2}|x-1|.
\end{aligned}$$

SOLUTION:

Let $\varepsilon > 0$ be given. Take $\delta < 2\varepsilon$. Then for all $x \neq 1$ such that $|x-a| < \delta$ we see that

$$\begin{aligned}
\left|\frac{\sqrt{x}-1}{x-1}-\frac{1}{2}\right| &= \left|\frac{1}{\sqrt{x}+1}-\frac{1}{2}\right| \\
&= \left|\frac{2-(\sqrt{x}+1)}{2(\sqrt{x}+1)}\right| \\
&= \left|\frac{1-x}{2(1+\sqrt{x})^2}\right| \\
&\leq \frac{1}{2}|x-1| \\
&\leq \frac{1}{2}\delta \\
&\leq \frac{1}{2}(2\varepsilon) \\
&\leq \varepsilon.
\end{aligned}$$

END OF SOLUTION $\Diamond$

Theorem 14. *Suppose* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *then*

(a) $\lim_{x\to a} (f(x) + g(x)) = L + M$

(b) $\lim_{x\to a} (f(x) \cdot g(x)) = L \cdot M$

(c) $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = L/M$ *provided* $M \neq 0$ *and* $g(x) \neq 0$, *for* x *sufficiently close to* a *(but not equal to* a*).* ▲

Problem 122. *Prove parts (b) and (c) of Theorem 14.* ◊
SOLUTION:

Part (a):

Part (b):

END OF SOLUTION

Theorem 15. (Squeeze Theorem for functions) *Suppose* $f(x) \leq g(x) \leq h(x)$, *for* x *sufficiently close to* a *(but not equal to* a*). If* $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$, *then* $\lim_{x \to a} g(x) = L$ *also.* ▲

Problem 123. *Prove Theorem 15. [Hint: Use the Squeeze Theorem for sequences (Theorem 8) from Chapter 4.]* ◊

SOLUTION:

Suppose $x_n \to a$. *Then for* $n \in \mathbb{N}$ *sufficiently large,*

$$f(x_n) < g(x_n) < h(x_n).$$

Therefore, by the Squeeze Theorem for sequences 8 $g(x_n) \to L$. *Since* (x_n) *was arbitrary we have*

$$\lim_{x \to a} g(x) = L.$$

END OF SOLUTION

Problem 124. *Use the fact that*

$$\textit{area}(\Delta OAC) < \textit{area}(\textit{sector } OAC) < \textit{area}(\Delta OAB)$$

to show that if $0 < x < \pi/2$*, then* $\cos x < \sin x/x < 1$*. Use the fact that all of these functions are even to extend the inequality for* $-\pi/2 < x < 0$ *and use the Squeeze Theorem to show* $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. $\Diamond$

Solution:

End of Solution

Theorem 16. (Differentiability Implies Continuity) *If f is differentiable at a point c then f is continuous at c as well.* ▲

Problem 125. *Prove Theorem 16* ◊

SOLUTION:

First Proof:

Let $\varepsilon > 0$ be given.

Case 1: $f'(c) = 0$.

There is a $\delta_1 > 0$ such that if $|x - c| < \delta_1$ then

$$\left|\frac{f(x) - f(c)}{x - c}\right| < \sqrt{\varepsilon},$$

because $f(x)$ is differentiable, and there is a $\delta_2 > 0$ such that if $|x - c| < \delta_2$ then

$$|x - c| < \sqrt{\varepsilon}.$$

Take $\delta = \min(\delta_1, \delta_2)$. Then $\forall$ x such that $|x - c| < \delta$ we see that

$$\begin{aligned} |f(x) - f(c)| &= \left|\frac{f(x) - f(c)}{(x - c)}(x - c)\right| \\ &= \left|\frac{f(x) - f(c)}{x - c}\right| |x - c| \\ &< \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} \\ &= \varepsilon. \end{aligned}$$

Case 2: $f'(c) \neq 0$. *There is a $\delta_1 > 0$ such that if $|x - c| < \delta_1$ then*

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \sqrt{\frac{\varepsilon}{2}},$$

because $f(x)$ is differentiable, and there is a $\delta_2 > 0$ such that if $|x - c| < \delta_2$ then

$$|x - c| < \frac{\varepsilon}{2f'(c)}.$$

Take $\delta = \min\left(\delta_1, \delta_2, \sqrt{\frac{\varepsilon}{2}}\right)$. Then $\forall$ x such that $|x - c| < \delta$ we see that

$$\begin{aligned} |f(x) - f(c)| &= \left|\frac{f(x) - f(c)}{(x - c)}(x - c)\right| \\ &= \left|\frac{f(x) - f(c)}{(x - c)} - f'(c) + f'(c)\right| |(x - c)| \\ &\leq \left|\frac{f(x) - f(c)}{(x - c)} - f'(c)\right| |x - c| + |f'(c)| \, |(x - c)| \\ &\leq \sqrt{\frac{\varepsilon}{2}} \cdot \sqrt{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Second Proof: *Let $\varepsilon > 0$ be given. Let $\delta < \frac{\varepsilon}{|f'(a)|+\varepsilon}$. Then for every x such that $0 < |x-a| < \delta$ we have*

$$\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \varepsilon$$
$$|f(x)-f(a)-(x-a)f'(a)| < \varepsilon\,|x-a.|$$

But the Reverse Triangle Inequality we have

$$\begin{aligned}|f(x)-f(a)| - |(x-a)f'(a)| &< \varepsilon\,|x-a|\\ |f(x)-f(a)| &< \varepsilon\,|x-a| + |(x-a)f'(a)|\\ &< |x-a|\,(\varepsilon + |f'(a)|)\\ &< \varepsilon.\end{aligned}$$

Third Proof: *Observe that*

$$\begin{aligned}f(x)-f(c) &= (f(x)-f(c))\cdot\frac{(x-c)}{(x-c)}\\ &= \frac{f(x)-f(c)}{x-c}\cdot(x-c).\end{aligned}$$

Since $f(x)$ is differentiable at $x=c$, $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists. Also $\lim_{x\to c}(x-c) =$ 0, so

$$\begin{aligned}\lim_{x\to c}(f(x)-f(c)) &= \lim_{x\to 0}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)\\ &= f'(c)\cdot 0\\ &= 0.\end{aligned}$$

Therefore $f(x)$ is continuous at $x=c$.
END OF SOLUTION

Problem 126. *Show that* $f'(c) \geq 0$ *and conclude that* $f'(c) = 0$. $\Diamond$
SOLUTION:

END OF SOLUTION

Problem 127. *Show that if $f(c) \leq f(x)$ for all x in some interval (a, b) then $f'(c) = 0$ too.* $\Diamond$

SOLUTION:

Let $g(x) = -f(x)$. Then $\forall\, x \in (a, b)$, $g(c) \geq g(x)$. By the previous problem $g'(c) = 0$ so $f'(c) = -g'(c) = 0$.

END OF SOLUTION

Problem 128. *Prove the Mean Value Theorem.*
Solution:

End of Solution

Problem 129. *Show that if $f'(x) < 0$ for every x in the interval (a, b) then f is decreasing on (a, b).* $\Diamond$

Solution:

End of Solution

Problem 130. *Prove Corollary* **??**. ◊

SOLUTION:

END OF SOLUTION

Problem 131. *Show that if f is differentiable on some interval (a, b) and that $f'(c) < 0$ for some $c \in (a, b)$. Show that there is an interval, $I \subset (a, b)$, containing c such that for every x, y in I where $x \leq y$, $f(x) \leq f(y)$.* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 132. *Use the definition of continuity to prove that the constant function $g(x) = c$ is continuous at any point a.* $\Diamond$

Solution:

Let $\varepsilon > 0$ be given. Let $a \in \mathbb{R}$ be any real number. Take $\delta > 0$. Then for every $x \in \mathbb{R}$ such that

$$|x - c| < \delta$$

we see that

$$|f(x) - f(a)| = |c - c| = 0 < \varepsilon.$$

Therefore $f(x)$ is continuous at every real number a.

End of Solution

Problem 133.

(a) *Use the definition of continuity to prove that* $\ln x$ *is continuous at* 1. *[Hint: You may want to use the fact* $|\ln x| < \varepsilon \Leftrightarrow -\varepsilon < \ln x < \varepsilon$ *to find a* δ.*]*

(b) *Use part (a) to prove that* $\ln x$ *is continuous at any positive real number* a. *[Hint:* $\ln(x) = \ln(x/a) + \ln(a)$. *This is a combinabtion of functions which are continuous at* a. *Be sure to explain how you know that* $\ln(x/a)$ *is continuous at* a.*]* ◊

SOLUTION:

END OF SOLUTION

Problem 134. *Write a formal definition of the statement f is not continuous at a, and use it to prove that the function* $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ *is not continuous at $a = 1$.* $\Diamond$

SOLUTION:

END OF SOLUTION

Chapter 7

Intermediate and Extreme Values

Problem 135. *Let $(x_n), (y_n)$ be sequences as in the NIP. Show that for all $n, m \in \mathbb{N}$, $x_n \leq y_m$.* $\Diamond$

SOLUTION:

First Proof

One of the assumptions of the NIP is that when $n = m$, $x_n \leq y_n$ so we need only consider the case $n \neq m$. There are two cases:

$n < m$: *In this case we see that*

$$x_n \leq y_n \leq y_m.$$

$n > m$: *In this case we see that*

$$x_n \leq x_m \leq y_m.$$

Second Proof:

From the NIP, there is a unique real number, c, such that $c \in [x_n, y_n]$, $\forall n \in \mathbb{N}$. Therefore $x_n \leq c$, $\forall n \in \mathbb{N}$, and $c \leq y_n$, $\forall n \in \mathbb{N}$. Thus $x_n \leq x_m$ $\forall n, m \in \mathbb{N}$.

END OF SOLUTION

Problem 136. **(a)** *Find two sequences of rational numbers* (x_n) *and* (y_n) *which satisfy properties 1-4 of the NIP and such that there is no rational number c satisfying the conclusion of the NIP. [Hint: Consider the decimal expansion of an irrational number.]*

(b) *Find two sequences of rational numbers* (x_n) *and* (y_n) *which satisfy properties 1-4 of the NIP and such that there is a rational number c satisfying the conclusion of the NIP.* ◊

Solution:

End of Solution

Theorem 17. *Suppose that we have two sequences* (x_n) *and* (y_n) *satisfying all of the assumptions of the Nested Interval Property. If* c *is the unique number such that* $x_n \leq c \leq y_n$ *for all* n*, then* $\lim_{n\to\infty} x_n = c$ *and* $\lim_{n\to\infty} y_n = c$. ▲

Problem 137. *Prove Theorem 17.* ◊

SOLUTION:

END OF SOLUTION

Theorem 18. *Suppose* $a \in \mathbb{R}$, $a \geq 0$. *There exists a real number* $c \geq 0$ *such that* $c^2 = a$. ▲

Problem 138. *Turn the above outline into a formal proof of Theorem 18.* ◊
Solution:

End of Solution

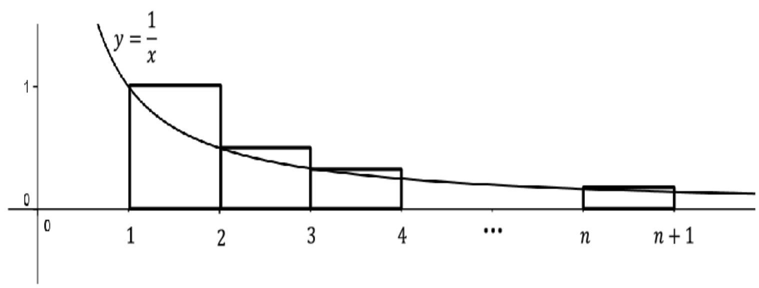
Problem 139. *The purpose of this problem is to show that*

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\ln(n+1)\right)$$

exists.

(a) *Let* $x_n = \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\ln(n+1)$. *Use the following diagram to show*

$$x_1 \le x_2 \le x_3 \le \cdots$$



SOLUTION:

From the diagram it is clear that x_n *is the sum of the areas of the upper right-hand corners of each of the shown rectangles. That is, if* p_n *is the upper right-hand corner of the nth rectangle then*

$$x_n = \sum_{k=1}^{n} p_k < \sum_{k=1}^{n+1} p_k < x_{n+1}.$$

Therefore

$$x_1 \le x_2 \le x_3 \le \cdots.$$

END OF SOLUTION

(b) *Let* $z_n = \ln(n+1) - \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$. *Use a similar diagram to show that* $z_1 \le z_2 \le z_3 \le \cdots$.

SOLUTION:

Computing the height of each rectangle from the right endpoint and setting q_n *to be the area of the region between the curve and the top of the nth rectangle we get*

$$z_n = \sum_{k=1}^{n} q_k < \sum_{k=1}^{n+1} q_k < z_{n+1}.$$

END OF SOLUTION

(c) *Let $y_n = 1 - z_n$. Show that (x_n) and (y_n) satisfy the hypotheses of the nested interval property and use the NIP to conclude that there is a real number γ such that $x_n \leq \gamma \leq y_n$ for all n.*

SOLUTION:

Observe that

$$\begin{aligned} y_n - x_n &= 1 - z_n - x_n \\ &= 1 - \ln(n+1) + (1/2 + 1/3 + \ldots + 1/n + 1/(n+1)) \\ &\quad - (1 + 1/2 + 1/3 + \ldots + 1/n) + \ln(n+1) \\ &= 1/(n+1). \end{aligned}$$

Since $y_n - x_n > 0$ we have $y_n > x_n \,\forall\, n \in \mathbb{N}$.

Since $|y_n - x_n| = 1/(n+1)$ we have $\lim\limits_{n\to\infty} |y_n - x_n| = 0$.

Therefore (x_n) and (y_n) satisfy the hypotheses of the nested interval property.

Therefore (x_n) and (y_n) satisfy the hypotheses of the nested interval property

END OF SOLUTION

(d) *Conclude that $\lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln(n+1)\right) = \gamma$.*

SOLUTION:

Therefore $\lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln(n+1)\right) = \gamma$. $\Diamond$

END OF SOLUTION

Problem 140. *Use the fact that* $x_n \leq \gamma \leq y_n$ *for all* n *to approximate* γ *to three decimal places.* $\Diamond$

SOLUTION:

$\gamma \approx 0.577215$

END OF SOLUTION

Problem 141.

(a) *Use the fact that for large* n*,* $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \approx ln(n+1)+\gamma$ *to determine approximately how large* n *must be to make*

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \geq 100.$$

SOLUTION:
Since

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \ln(n+1) \approx +\gamma$$

we have

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \approx \ln(n+1)+\gamma$$

so we need

$$\begin{aligned} \ln(n+1)+\gamma &\geq 100 \\ \ln(n+1) &\geq 100-\gamma \\ n &\geq e^{100}-1 \end{aligned}$$

or

$$n \geq 2.688 \times 10^{43}$$

END OF SOLUTION

(b) *Suppose we have a supercomputer which can add 10 trillion terms of the harmonic series per second. Approximately how many earth lifetimes would it take for this computer to sum the harmonic series until it surpasses 100?*

$\Diamond$

SOLUTION:
Assuming the earth is approximately six billion (10^9) *years old*
END OF SOLUTION

Theorem 19. (Intermediate Value Theorem) *Suppose $f(x)$ is continuous on $[a, b]$ and v is any real number between $f(a)$ and $f(b)$. Then there exists a real number $c \in [\,a, b]$ such that $f(c) = v$.* ▲

Problem 142. *Turn the ideas of the previous paragraphs into a formal proof of the IVT for the case $f(a) \leq v \leq f(b)$.* ♢

SOLUTION:

Inductive Construction: *Let $x_1 = a$, and $y_1 = b$ so that $x_1 \leq y_1$ and $f(x_1) \leq v \leq f(y_1)$. Then $[x_1, y_1]$ is our first intervals.*

Assume $[x_n, y_n]$ and $[f(x_n), f(y_n)]$ have been constructed as above. To construct $[x_{n+1}, y_{n+1}]$ let m be the midpoint of $[x_n, y_n]$.

If $f(m) \leq v$ take $x_{n+1} = m$ and $y_{n+1} = y_n$.

If $f(m) \geq v$ take $x_{n+1} = x_n$ and $y_{n+1} = m$.

In either case we have $x_{n+1} \leq y_{n+1}$ and $f(x_{n+1}) \leq v \leq f(y_{n+1})$.

By construction, $\{[x_n, y_n]\}_{n=1}^{\infty}$ is a set of Nested Intervals. Therefore there is a unique $c \in [x_n, y_n], \forall\, n \in \mathbb{N}$.

Claim: $f(c) = v$.

Proof of Claim: *(by Contradiction)*

Case 1: $f(c) > v$ *In this case, by problem 86, part b, there is a real number N such that if $n > N$ then $f(x_n) > v$. But by construction, $f(x_n) \leq v$.*

This is a contradiction.

Case 2: $f(c) < v$ *As in part a, we obtain the contradiction $f(y_n) < v$. (Use Problem 86, part a.*

Therefore $f(c) = v$.

END OF SOLUTION

Problem 143. *We can modify the proof of the case $f(a) \le v \le f(b)$ into a proof of the IVT for the case $f(a) \ge v \ge f(b)$. However, there is a sneakier way to prove this case by applying the IVT to the function $-f$. Do this to prove the IVT for the case $f(a) \ge v \ge f(b)$.* ♢

SOLUTION:

Suppose $f(a) \ge v \ge f(b)$. Then $-f(a) \le -v \le -f(b)$

Define $h(x) = -f(x)$, so that

$$h(a) \le -v \le h(b)$$

By the previous problem $\exists\ c \in [a, b]$ such that

$$h(c) = -v = -f(c).$$

Therefore

$$f(c) = v.$$

END OF SOLUTION

Problem 144. *Use the IVT to prove that any polynomial of odd degree must have a real root.* ◊

SOLUTION:

END OF SOLUTION

Problem 145. *Suppose* $\lim_{n\to\infty} x_n = c$. *Prove that* $\lim_{k\to\infty} x_{n_k} = c$ *for any subsequence* (x_{n_k}) *of* (x_n). *[Hint:* $n_k \geq k$.*]* $\Diamond$

SOLUTION:

Claim: *If* $(x_{n_k})_{k=1}^{\infty}$ *is a subsequence of* $(x_n)_{n=1}^{\infty}$ *then* $n_k \geq k$.

Proof:

by Induction:

Base case: $k = 1$. *Since* n_k *is a positive integer, clearly* $n_k \geq 1$.

Induction Hypothesis: $n_k \geq k$. *Since* n_k *is a strictly* increasing *sequence of integers we have* $n_{k+1} \geq n_k + 1 \geq k + 1$. ■

Let $\varepsilon > 0$ *be given and suppose* $x_k \to c$ *as* $k \to \infty$. *Then* $\exists\ N \in \mathbb{R}$ *such that* $\forall\ k > N, |x_k - c| < \varepsilon$. *Let* $(x_{n_k})_{k=1}^{\infty}$ *be a subsequence of* (x_k). *Since* $x_{n_k} \geq x_k$ *we see that* $\forall\ k > N$, $n_k \geq k > N$. *Therefore* $|x_{n_k} - c| < \varepsilon$.

END OF SOLUTION

Theorem 20. The Bolzano-Weierstrass Theorem *Let* (x_n) *be a sequence of real numbers such that* $x_n \in [a, b]$, $\forall\ n$. *Then there exists* $c \in [a, b]$ *and a subsequence* (x_{n_k}), *such that* $\lim_{k\to\infty} x_{n_k} = c$. ▲

Problem 146. *Turn the ideas of the above outline into a formal proof of the Bolzano-Weierstrass Theorem.* ◊

SOLUTION:

END OF SOLUTION

Theorem 21. *A continuous function defined on a closed, bounded interval must be bounded. That is, let f be a continuous function defined on $[a, b]$. Then there exists a positive real number B such that $|f(x)| \le B$ for all $x \in [a, b]$.* ▲

Problem 147. *Use the Bolzano-Weierstrass Theorem to complete the proof of Theorem 21.* ◊

SOLUTION:

END OF SOLUTION

Theorem 22. (The Least Upper Bound Property (LUBP)) *Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Then S has a supremum.* ▲

Problem 148. *Complete the above ideas to provide a formal proof of Theorem 22.* ◊

Solution:

End of Solution

Corollary 2. *Let* (x_n) *be a bounded, increasing sequence of real numbers. That is,* $x_1 \leq x_2 \leq x_3 \leq \cdots$ *. Then* (x_n) *converges to some real number* c. ▲

Problem 149. *Prove Corollary 2.*

[Hint: Let $c = \sup\{x_n | \, n = 1, 2, 3, \ldots\}$*. To show that* $\lim_{n\to\infty} x_n = c$*, let* $\epsilon > 0$*. Note that* $c - \epsilon$ *is not an upper bound. You take it from here!]* ◊

SOLUTION:

Let $c = \sup\{x_n \| n \in \mathbb{N}\}$.

Claim: $x_n \to c$.

Proof:

Since $c-\varepsilon < c$, $c-\varepsilon$ *is not an upper bound of* $\{x_n\}_{n=1}^{\infty}$. *Therefore* $\exists\ N \in \mathbb{N} \subset \mathbb{R}$ *such that*

$$x_N > c - \varepsilon.$$

If $m > n$ *then* $x_m > x_N$ *so*

$$\begin{aligned} x_m &> c - \varepsilon \\ x_m - c &> -\varepsilon \\ c - x_m &< \varepsilon. \end{aligned}$$

Since $x_m \leq c$, $c - x_m > 0$ *so*

$$|c - x_m| < \varepsilon, \quad \forall\ m > N.$$

Therefore $x_m \to c$.

■

END OF SOLUTION

Problem 150. *Consider the following curious expression* $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}$. *We will use Corollary 2 to show that this actually converges to some real number. After we know it converges we can actually compute what it is. Of course to do so, we need to define things a bit more precisely. With this in mind consider the following sequence* (x_n) *defined as follows:*

$$x_1 = \sqrt{2}$$
$$x_{n+1} = \sqrt{2 + x_n}.$$

(a) *Use induction to show that* $x_n < 2$ *for* $n = 1, 2, 3, \ldots$.
SOLUTION:

First Solution: *The proof will be by induction.*
Initial Case: $x_1 = \sqrt{2} < 2$.
Induction Step: *Suppose* $x_n < 2$. *Then*

$$\begin{aligned} x_{n+1} &= \sqrt{2 + x_n} \\ &< \sqrt{2+2} \\ &= 2. \end{aligned}$$

Second Solution: *The proof will be by induction.*
Initial Case: $x_1 = \sqrt{2} < 2$.
Induction Step: *Suppose* $x_n < 2$. *Then*

$$\begin{aligned} x_n &< 2 \\ x_n + 2 &< 4 \\ \sqrt{x_n + 2} &< 2 \end{aligned}$$

END OF SOLUTION

(b) *Use the result from (a) to show that* $x_n < x_{n+1}$ *for* $n = 1, 2, 3, \ldots$.
SOLUTION:

First Solution *The proof will be by induction.*
Initial Case: $x_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = x_1$
Induction Step: *Suppose* $x_{n+1} > x_n$. *Then Observe that:*

$$x_{n+1} = \sqrt{2 + x_n}$$

so that, by part (a)

$$\begin{aligned} x_{n+1} &> \sqrt{x_n + x_n} \\ &> \sqrt{2} \cdot \sqrt{x_n} \\ &> \sqrt{x_n} \cdot \sqrt{x_n} \end{aligned}$$

so that

$$x_{n+1} > x_n.$$

Second Solution *The proof will be by induction.*

Initial Case: $x_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = x_1$

Induction Step: *Suppose* $x_{n+1} > x_n$. *Then*

$$\begin{aligned} x_n &< x_{n+1} \\ x_n + 2 &< x_{n+1} + 2 \\ \sqrt{x_n + 2} &< \sqrt{x_{n+1} + 2} \\ x_{n+1} &< x_{n+2} \end{aligned}$$

END OF SOLUTION

(c) *From Corollary 2, we have that* (x_n) *must converge to some number* c. *Use the fact that* (x_{n+1}) *must converge to* c *as well to compute what* c *must be.*

$\diamondsuit$

SOLUTION:

We know that $x_{n+1} = \sqrt{2 + x_n}$. *Taking the limit of both sides gives*

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n}$$

or

$$c = \sqrt{2 + c}.$$

Solving this quadratic gives $c = 2$ *and* $c = -1$. *Of these, only the first makes sense because* $x_n > 0, \ \forall\, n \in \mathbb{N}$.

END OF SOLUTION

Theorem 23. (Extreme Value Theorem (EVT)) *Suppose f is continuous on $[a,b]$. Then there exists $c,d \in [a,b]$ such that $f(d) \leq f(x) \leq f(c)$, for all $x \in [a,b]$.* ▲

Problem 151. *Formalize the above ideas into a proof of Theorem 23.* ◊
SOLUTION:

END OF SOLUTION

Problem 152. *Use the Bolzano-Weierstrass Theorem to prove the NIP. That is, assume that the Bolzano-Weierstrass Theorem holds and suppose we have two sequences of real numbers,* (x_n) *and* (y_n), *satisfying:*

1. $x_1 \leq x_2 \leq x_3 \leq \ldots$
2. $y_1 \geq y_2 \geq y_3 \geq \ldots$
3. $\forall\ n, x_n \leq y_n$
4. $\lim_{n\to\infty} (y_n - x_n) = 0.$

Prove that there is a real number c *such that* $x_n \leq c \leq y_n$, *for all* n. *(The* c *will, of necessity, be unique, but don't worry about that.)* ♢

Solution:

End of Solution

Problem 153. *Find a bounded sequence of rational numbers such that no subsequence of it converges to a rational number.* $\Diamond$

SOLUTION:

Suppose the decimal representation of some irrational number between zero and one is given by: $0.d_1d_2d_3d_4d_5\ldots$. *Then* $(a_n = 0.d_1d_2\cdots d_n)_{n=1}^{\infty}$ *is such a sequence.*

END OF SOLUTION

Problem 154. *Use the Least Upper Bound Property to prove the Nested Interval Property. That is, assume that every nonempty subset of the real numbers which is bounded above has a least upper bound; and suppose that we have two sequences of real numbers* (x_n) *and* (y_n), *satisfying:*

1. $x_1 \leq x_2 \leq x_3 \leq \ldots$
2. $y_1 \geq y_2 \geq y_3 \geq \ldots$
3. $\forall\ n, x_n \leq y_n$
4. $\lim_{n\to\infty} (y_n - x_n) = 0.$

Prove that there exists a real number c *such that* $x_n \leq c \leq y_n$, *for all* n. *(Again, the* c *will, of necessity, be unique, but don't worry about that.) [Hint: Corollary 2 might work well here.]* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 155. *Since the LUBP is equivalent to the NIP it does not hold for the rational number system. Demonstrate this by finding a nonempty set of rational numbers which is bounded above, but whose supremum is an irrational number.* ◊

Solution:

Suppose the decimal representation of some irrational number between zero and one is given by: $0.d_1d_2d_3d_4d_5\ldots$. *Then* $(a_n = 0.d_1d_2\cdots d_n)_{n=1}^{\infty}$ *is such a sequence.*

End of Solution

Theorem 24. (Archimedean Property of $\mathbb{R}$) *Given any positive real numbers a and b, there exists a positive integer n, such that $na > b$.* ▲

Problem 156. *Prove Theorem 24. [Hint: Assume that there are positive real numbers a and b, such that $na \leq b$ $\forall n \in \mathbb{N}$. Then $\mathbb{N}$ would be bounded above by b/a. Let $s = \sup(\mathbb{N})$ and consider $s - 1$.]* ◊

SOLUTION:

Proof:

Our proof will be by contradiction. Suppose $\exists\ a, b \in \mathbb{R}^+$ such that

$$na \leq b,\ \forall\ n \in \mathbb{N}.$$

Then

$$n \leq \frac{b}{a}\ \forall\ n \in \mathbb{N}.$$

In other words the set of positive integers is bounded above. Therefore (by B-W) $\mathbb{N}$ has a least upper bound. Let $s = \mathrm{lub}(\mathbb{N})$ and observe that there is an integer, k, in the interval $(s-1, s]$. Since $k \in \mathbb{N}$ we see that $k+1 \in \mathbb{N}$ as well. But $k+1 > s$ which is a contradiction since s is an upper bound on $\mathbb{N}$.

Therefore given any positive real numbers a and b, there exists a positive integer n, such that $na > b$. ■

END OF SOLUTION

Problem 157. *Does $\mathbb{Q}$ satisfy the Archimedean Property and what does this have to do with the question of taking the Archimedean Property as an axiom of completeness?* $\Diamond$

Solution:

End of Solution

Problem 158. *Mimic the definitions of an upper bound of a set and the least upper bound (supremum) of a set to give definitions for a lower bound of a set and the greatest lower bound (infimum) of a set. Note: The infimum of a set S is denoted by* $\inf(S)$. $\Diamond$

Solution:

End of Solution

Problem 159. *Find the least upper bound (supremum) and greatest lower bound (infimum) of the following sets of real numbers, if they exist. (If one does not exist then say so.)*

(a) $S = \{\frac{1}{n} \,|\, n = 1, 2, 3, \ldots\}$

(b) $T = \{r \,|\, r \text{ is rational and } r^2 < 2\}$

(c) $(-\infty, 0) \cup (1, \infty)$

(d) $R = \{\frac{(-1)^n}{n} \,|\, n = 1, 2, 3, \ldots\}$

(e) $(2, 3\pi] \cap \mathbb{Q}$

(f) *The empty set* $\emptyset$ ◊

SOLUTION:

END OF SOLUTION

Problem 160. *Let $S \subseteq \mathbb{R}$ and let $T = \{-x|\, x \in S\}$.*

(a) *Prove that b is an upper bound of S if and only if $-b$ is a lower bound of T.*

SOLUTION:

Observe that $s \in S \;\Rightarrow\; -s \in T$.

Thus if

$$b \geq s, \;\forall\, s \in S$$

then

$$-b \leq -s, \;\forall\, s \in S.$$

Therefore

$$-b \leq t, \;\forall\, t \in T.$$

Therefore $-b$ is a lower bound of T.

END OF SOLUTION

(b) *Prove that $b = \sup S$ if and only if $-b = \inf T$.* $\Diamond$

Chapter 8

Back to Power Series

Problem 161. *Find the flaw in the following "proof" that f is also continuous at a.*

Suppose $f_1, f_2, f_3, f_4 \ldots$ are all continuous at a and that $\sum_{n=1}^{\infty} f_n = f$. Let $\varepsilon > 0$. Since f_n is continuous at a, we can choose $\delta_n > 0$ such that if $|x-a| < \delta_n$, then $|f_n(x) - f_n(a)| < \frac{\varepsilon}{2^n}$. Let $\delta = \inf(\delta_1, \delta_2, \delta_3, \ldots)$. If $|x - a| < \delta$ then

$$\begin{aligned}
|f(x) - f(a)| &= \left|\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(a)\right| \\
&= \left|\sum_{n=1}^{\infty} (f_n(x) - f_n(a))\right| \\
&\leq \sum_{n=1}^{\infty} |f_n(x) - f_n(a)| \\
&\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\
&\leq \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= \varepsilon.
\end{aligned}$$

Thus f is continuous at a. $\Diamond$

SOLUTION:

The flaw is that $\delta = \inf(\delta_1, \delta_2, \delta_3, \ldots)$ might be zero. In that case then $x = a$ and all we proven is that $|f(a) - f(a)| < \varepsilon$, which is true but does not show continuity.

END OF SOLUTION

Problem 162. *Let* $0 < b < 1$ *and consider the sequence of functions* (f_n) *defined on* $[0, b]$ *by* $f_n(x) = x^n$*. Use the definition to show that* $f_n \xrightarrow{unif} 0$ *on* $[0, b]$*. [Hint:* $|x^n - 0| = x^n \leq b^n$*.]* $\Diamond$

Solution:

End of Solution

Theorem 25. *Consider a sequence of functions (f_n) which are all continuous on an interval I. Suppose $f_n \xrightarrow{unif} f$ on I. Then f must be continuous on I.* ▲

Problem 163. *Provide a formal proof of Theorem 25 based on the above ideas.* ◊

Solution:

Take $N \in \mathbb{R}$ such that $\forall\ n > N$ $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$, $\forall\ x \in I$.

Let $n > N$ be fixed. Since f_n is continuous $\exists\ \delta > 0$ such that if $|x - a| < \delta$ then $|f_n(x) - f_n(a)| \le \frac{\varepsilon}{3}$. Let $a \in I$ and suppose that $|x - a| < \delta$. Then

$$\begin{aligned}
|f(x) - f_n(x)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(x)| \\
&\le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(x)|. \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\le \varepsilon.
\end{aligned}$$

Therefore $f(x)$ is continuous at every $a \in I$.

End of Solution

Problem 164. *Consider the sequence of functions* (f_n) *defined on* $[0,1]$ *by* $f_n(x) = x^n$. *Show that the sequence converges to the function*

$$f(x) = \begin{cases} 0 & \textit{if } x \in [0,1) \\ 1 & \textit{if } x = 1 \end{cases}$$

pointwise on $[0,1]$, *but not uniformly on* $[0,1]$. $\Diamond$

Solution:

There are two cases:

Case 1, $x = 1$**:** *In this case* $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1 = f(x)$.

Case 2, $0 \leq x < 1$**:** *In this case there is a* b *such that* $x < b < 1$. *Since* $\lim_{n\to\infty} b^n = 0$ *it follows from the comparison test that* $\lim_{n\to\infty} x^n = 0$ *also.*

Therefore $f_n \overset{ptwise}{\longrightarrow} f$ *on* I.

To show that the convergence is not uniform take $\varepsilon = 1/2$, *take* $\delta > 0$, *and consider* $a = 1$. *For every* $x \in (\delta, 1]$, $\lim_{n\to\infty} abs f_n(a) - f_n(x) = 1 > 1/2 = \varepsilon$.

End of Solution

Theorem 26. *Suppose f_n and f are integrable and $f_n \overset{unif}{\longrightarrow} f$ on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_{x=a}^{b} f_n(x)\,\mathrm{d}x = \int_{x=a}^{b} f(x)\,\mathrm{d}x. \qquad \blacktriangle$$

Problem 165. *Prove Theorem 26. [Hint: For $\varepsilon > 0$, we need to make $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$, for all $x \in [a, b]$.]* $\Diamond$

Solution:

Observe that $(s_n)_{n=1}^{\infty} = \left(\int_a^b f_n(x)\,\mathrm{d}x\right)_{n=1}^{\infty}$ is a sequence.

Let $\varepsilon > 0$ be given and take n sufficiently large that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}.$$

Then

$$\begin{aligned} \left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| &= \left| \int_a^b f_n(x) - f(x)\,\mathrm{d}x \right| \\ &\le \int_a^b |f_n(x) - f(x)|\,\mathrm{d}x \\ &\le \int_a^b \frac{\varepsilon}{b-a}\,\mathrm{d}x \\ &= \frac{\varepsilon}{b-a}(b-a) \\ &= \varepsilon. \end{aligned}$$

End of Solution

Problem 166. *Consider the sequence of functions* (f_n) *given by*

$$f_n(x) = \begin{cases} n & \text{if } x \in \left(0, \frac{1}{n}\right) \\ 0 & \text{otherwise} \end{cases}.$$

(a) *Show that* $f_n \overset{ptwise}{\longrightarrow} 0$ *on* $[0, 1]$*, but* $\lim_{n\to\infty} \displaystyle\int_{x=0}^{1} f_n(x)\,\mathrm{d}x \neq \int_{x=0}^{1} 0\,\mathrm{d}x$

(b) *Can the convergence be uniform? Explain.*

$\Diamond$

Solution:

End of Solution

Corollary 3. *If* $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly*[1] *to* f *on an interval containing* 0 *and* x *then* $\int_{t=0}^{x} f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty}\left(\frac{a_n}{n+1}x^{n+1}\right)$. ▲

Problem 167. *Prove Corollary 3. [Hint: Remember that*

$$\sum_{n=0}^{\infty} f_n(x) = \lim_{N\to\infty}\sum_{n=0}^{N} f_n(x).]$$

◊

Solution:

End of Solution

[1] Notice that we must explicitly assume uniform convergence. This is because we have not yet proved that power series actually do converge uniformly.

Theorem 27. *Suppose for every $n \in \mathbb{N}$ f_n is differentiable, f_n' is continuous, $f_n \overset{ptwise}{\longrightarrow} f$, and $f_n' \overset{unif}{\longrightarrow} g$ on an interval, I. Then f is differentiable and $f' = g$ on I.* ▲

Problem 168. *Prove Theorem 27. [Hint: Let a be an arbitrary fixed point in I and let $x \in I$. By the fundamental theorem of calculus, we have*

$$\int_{t=a}^{x} f_n'(t)\,\mathrm{d}t = f_n(x) - f_n(a).$$

Take the limit of both sides and differentiate with respect to x.] ◊

SOLUTION:

Since $f' \overset{unif}{\longrightarrow} g$ we have, by theorem 26

$$\lim_{n\to\infty} \int_{t=a}^{x} f'(t)dt = \int_{t=a}^{x} g(t)\,\mathrm{d}t$$

or

$$\lim_{n\to\infty} (f_n(x) - f_n(a)) = \int_{t=a}^{x} g(t)\,\mathrm{d}t$$

and finally,

$$f(x) - f(a) = \int_{t=a}^{x} g(t)\,\mathrm{d}t.$$

Since the right side of this last equation is differentiable, so is the left side. Therefore $f'(x)$ exists.

Differentiating gives

$$f'(x) = g(x).$$

END OF SOLUTION

Corollary 4. *If* $\sum_{n=0}^{\infty} a_n x^n$ *converges pointwise to* f *on an interval containing* 0 *and* x *and* $\sum_{n=1}^{\infty} a_n n x^{n-1}$ *converges uniformly on an interval containing* 0 *and* x*, then* $f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$. ▲

Problem 169. *Prove Corollary 4.* ◊

Solution:

End of Solution

Theorem 28. *Suppose (s_n) is a sequence of real numbers which converges to s. Then (s_n) is a Cauchy sequence.* ▲

Problem 170. *Prove Theorem 28. [Hint: $|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n|$.]* ◊

SOLUTION:

Let $\varepsilon > 0$ be given.
Since $s_n \overset{ptwise}{\longrightarrow} s$, $\exists\, n \in \mathbb{N}$ such that $\forall\, m > n\, |s_m - s| < \frac{\varepsilon}{2}$
Thus

$$\begin{aligned} |s_m - s_n| = |s_m - s + s - s_n| &\leq |s_m - s| + |s - s_n| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

END OF SOLUTION

Lemma 5. (A Cauchy sequence is bounded) *Suppose* (s_n) *is a Cauchy sequence. Then there exists* $B > 0$ *such that* $|s_n| \leq B$ *for all* n. ▲

Problem 171. *Prove Lemma 5. [Hint: This is similar to problem 67 of Chapter 4. There exists* N *such that if* $m, n > N$ *then* $|s_n - s_m| < 1$*. Choose a fixed* $m > N$ *and let* $B = \max\left(|s_1|, |s_2|, \ldots, \left|s_{\lceil N \rceil}\right|, |s_m| + 1\right)$*.]* ◊

Solution:

End of Solution

Theorem 29. (Cauchy sequences converge) *Suppose (s_n) is a Cauchy sequence of real numbers. There exists a real number s such that $\lim_{n\to\infty} s_n = s$.* ▲

Problem 172. *Provide a formal proof of Theorem 29.* ◊

SOLUTION:

END OF SOLUTION

Theorem 30. *Suppose every Cauchy sequence converges. Then the Nested Interval Property is true.* ▲

Problem 173. *Prove Theorem 30. [Hint: If we start with two sequences* (x_n) *and* (y_n), *satisfying all of the conditions of the NIP, you should be able to show that these are both Cauchy sequences.]* ◊

Solution:

End of Solution

Problem 174. *Since the convergence of Cauchy sequences can be taken as the completeness axiom for the real number system, it does not hold for the rational number system. Give an example of a Cauchy sequence of rational numbers which does not converge to a rational number.* $\Diamond$

Solution:

End of Solution

Theorem 31. (Cauchy Criterion) *The series $\sum_{k=0}^{\infty} a_k$ converges if and only if $\forall \varepsilon > 0$, $\exists N$ such that if $m > n > N$ then $|\sum_{k=n+1}^{m} a_k| < \varepsilon$.* ▲

Problem 175. *Prove the Cauchy criterion.* ◊

Solution:

Suppose $\sum_{n=0}^{\infty} s_n = s$. Then, by definition, the sequence $(\sum_{k=0}^{n} s_k)_{n=0}^{\infty}$ converges to s also.

Therefore the sequence $(\sum_{k=0}^{n} s_k)_{n=0}^{\infty}$ is Cauchy.

Let $\varepsilon > 0$ be given. Then for m, n sufficiently large and WLOG $m \geq n$ we have

$$\begin{aligned} \varepsilon &> \left| \sum_{k=0}^{m} s_k - \sum_{k=0}^{n} s_k \right| \\ &= \left| \sum_{k=n+1}^{m} s_k \right| \end{aligned}$$

End of Solution

Problem 176. (The nth Term Test) *Show that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.* ◊

Solution:

Apply the Cauchy Criterion with $m = n + 1$.

End of Solution

Problem 177. (The Strong Cauchy Criterion) *Show that* $\sum a_n$ *converges if and only if* $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_n = 0$. *[Hint: The hardest part of this problem is recognizing that it is really about the limit of a sequence as in Chapter 4.]* ◊

Solution:

Case 1: *Assume that* $\sum a_n = c$.

Let $\varepsilon > 0$ *be given.*

Since $\sum_{k=1}^{\infty} a_k = c$ *we see that there is a real number* N *such that if* $n > N$ *then:*

$$\left| c - \sum_{k=1}^{n} a_k \right| < \varepsilon.$$

Therefore if $n > N$

$$\left| c - \sum_{k=1}^{n} a_k - \sum_{k=n+1}^{\infty} a_k + \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$$

$$\left| \underbrace{c - \sum_{k=1}^{\infty} a_k}_{=0} + \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$$

$$\left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$$

Therefore $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_n = 0$.

Case 2: *Assume* $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_n = 0$. *Let* $\varepsilon > 0$ *be given.*

There is an $N \in \mathbb{R}$ *such that* $\forall\ n > N$ *we have* $\sum_{k=n+1}^{\infty} a_n < \frac{\varepsilon}{2}$. *Let* $m > n > N$. *Then*

$$\sum_{k=n+1}^{m} a_k = \sum_{k=n+1}^{\infty} a_k - \sum_{k=m+1}^{\infty} a_k$$

so that

$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k\right| &= \left|\sum_{k=n+1}^{\infty} a_k - \sum_{k=m+1}^{\infty} a_k\right| \\ \left|\sum_{k=n+1}^{m} a_k\right| &\leq \left|\sum_{k=n+1}^{\infty} a_k\right| + \left|\sum_{k=m+1}^{\infty} a_k\right| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.\end{aligned}$$

Therefore

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \varepsilon.$$

Therefore $\displaystyle\sum_{k=n+1}^{\infty} a_n$ *satisfies the Cauchy Criterion.*

Therefore $\displaystyle\sum_{k=n+1}^{\infty} a_n$ *converges.*

End of Solution

Theorem 32. *Suppose $|a_n| \leq b_n$ for all n. If $\sum b_n$ converges then $\sum a_n$ also converges.* ▲

Problem 178. *Prove Theorem 32. [Hint: Use the Cauchy criterion with the fact that $\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k|$.]* ◊

SOLUTION:

Let $\varepsilon > 0$ be given. Since $b_n \geq 0$ and $\sum_{k=1}^{\infty} b_k$ is Cauchy $\exists\, N \in \mathbb{R}$ such that $\forall\, m > n > N$

$$\sum_{k=n+1}^{m} b_k = \left|\sum_{k=n+1}^{m} b_k\right| < \varepsilon.$$

Therefore

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} b_k < \varepsilon.$$

Therefore $\sum_{k=1}^{\infty} a_k$ is Cauchy.
Therefore $\sum_{k=1}^{\infty} a_k$ converges.

END OF SOLUTION

Corollary 5. *If* $\sum a_n$ *converges absolutely, then* $\sum a_n$ *converges.* ▲

Problem 179. *Show that Corollary 5 is a direct consequence of Theorem 32.* ◊

SOLUTION:

END OF SOLUTION

Problem 180. *If* $\sum_{n=0}^{\infty} |a_n| = s$*, then does it follow that* $s = |\sum_{n=0}^{\infty} a_n|$*? Justify your answer. What can be said?* $\Diamond$

SOLUTION:

END OF SOLUTION

Theorem 33. *Suppose $\sum a_n$ converges absolutely and let $s = \sum_{n=0}^{\infty} a_n$. Then any rearrangement of $\sum a_n$ must converge to s.* ▲

Problem 181. *Fill in the details and provide a formal proof of Theorem 33.* ◊

SOLUTION:

Special Case: Assume $a_n \geq 0 \; \forall \, n \in \mathbb{N}$.

Let $p(n)$ be a permutation of $\mathbb{N}$. Then $\sum_n^{\infty} a_{p(n)}$ is a rearrangement of $\sum_n^{\infty} a_n$. Since $a_n \geq 0$ and $\sum_n^{\infty} a_n$ converges, say to A, we see that for every $k \in \mathbb{N}$,

$$0 \leq \sum_{n=1}^{k} a_n \leq A$$

and

$$0 \leq \sum_{n=1}^{k} a_{p(n)} \leq A.$$

Since the sequence $\left(\sum_{n=1}^{k} a_{p(n)}\right)_k$ *is bounded, and increasing it converges, say to B.*

Claim: $B \leq A$.

Proof (by Contradiction) of Claim: *Suppose $B > A$. Then there is a positive integer, m, such that*

$$\sum_{n=1}^{m} a_{p(n)} > A.$$

Take $P = \max_{0 \leq n \leq m} \{p(n)\}$.

Then

$$\sum_{n=1}^{p} a_n \geq \sum_{n=1}^{p} a_{p(n)} > A \qquad \Rightarrow\Leftarrow$$

Therefore $B \leq A$. ■ *(Claim.)*

Since $\sum_{n=1}^{p} a_n$ is a rearrangement of $\sum_{n=1}^{p} a_{p(n)}$ we also have $A \leq B$. Therefore $A = B$. ■ *(Special Case.)*

General Case:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{|a_n| + a_n}{2} - \frac{|a_n| - a_n}{2} \right)$$

Since $\frac{|a_n|+a_n}{2} \leq |a_n|$ and $\frac{|a_n|-a_n}{2} \leq |a_n|$, both $\sum_{n=1}^{\infty} \frac{|a_n|+a_n}{2}$ and $\sum_{n=1}^{\infty} \frac{|a_n|-a_n}{2}$ converge, by the Comparison Test. Therefore

$$\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{\infty} \frac{|a_n| + a_n}{2} - \sum_{n=1}^{\infty} \frac{|a_n| - a_n}{2} \right).$$

Recall that $p(n)$ is a permutaion of $\mathbb{N}$. Since both of the above series are positive, by our <u>*Special Case*</u>

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \left(\sum_{n=1}^{\infty} \frac{\left|a_{p(n)}\right| + a_{p(n)}}{2} - \sum_{n=1}^{\infty} \frac{\left|a_{p(n)}\right| - a_{p(n)}}{2}\right). \\ &= \left(\sum_{n=1}^{\infty} \frac{\left|a_{p(n)}\right| + a_{p(n)}}{2} - \frac{\left|a_{p(n)}\right| - a_{p(n)}}{2}\right). \\ &= \sum_{n=1}^{\infty} a_{p(n)}.\end{aligned}$$

<u>End of Solution</u>

Theorem 34. *Suppose* $\sum_{n=0}^{\infty} a_n c^n$ *converges for some nonzero real number* c. *Then* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for all* x *such that* $|x| < |c|$. ▲

Problem 182. *Prove Theorem 34.* ◊

Solution:

By the nth Term Test $a_n c^n \to 0$ *as* $n \to 0$. *Therefore* $|a_n c^n| < B$ *for some real number,* B. *Therefore*

$$\begin{aligned} |a_n x^n| &= \left| a_n c^n \left(\frac{x}{c}\right)^n \right| \\ &= |a_n c^n| \left|\frac{x}{c}\right|^n \\ &\leq B \left|\frac{x}{c}\right|^n. \end{aligned}$$

Since $\left|\frac{x}{c}\right| < 1$, $\sum_{n=0}^{\infty} B \left|\frac{x}{c}\right|^n$ *is a convergent geometric series. Therefore, by the Comparison Test* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely. Therefore* $\sum_{n=0}^{\infty} a_n x^n$ *converges.*

End of Solution

Corollary 6. *Suppose* $\sum_{n=0}^{\infty} a_n c^n$ *diverges for some real number* c. *Then* $\sum_{n=0}^{\infty} a_n x^n$ *diverges for all* x *such that* $|x| > |c|$. ▲

Problem 183. *Prove Corollary 6.* ◊

Solution:

The proof is by contradiction. Suppose for some real number $|x| > |c|$ *the series converges. Then by theorem 34 the series* $\sum_{n=0}^{\infty} a_n c^n$ *converges, which contradicts the assumption that* $\sum_{n=0}^{\infty} a_n c^n$ *diverges.*

Therefore Corollary 6 is true.

End of Solution

Theorem 35. (The Weierstrass-M Test) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $S \subseteq \mathbb{R}$ and suppose that $(M_n)_{n=1}^{\infty}$ is a sequence of nonnegative real numbers such that*

$$|f_n(x)| \leq M_n, \ \forall x \in S, \ n = 1, 2, 3, \ldots .$$

If $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S to some function (which we will denote by $f(x)$. ▲

Problem 184. *Use the ideas above to provide a proof of Theorem 35.* ◊

SOLUTION:

END OF SOLUTION

Problem 185.

(a) *Referring back to Part* **??**, *show that the Fourier series*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left((2k+1)\pi x\right)$$

converges uniformly on $\mathbb{R}$.

(b) *Does its differentiated series converge uniformly on* $\mathbb{R}$? *Explain.* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 186. *Observe that for all* $x \in [-1,1]$ $|x| \leq 1$. *Identify which of the following series converges pointwise and which converges uniformly on the interval* $[-1,1]$. *In every case identify the limit function.*

(a) $\displaystyle\sum_{n=1}^{\infty}\left(x^n - x^{n-1}\right)$ *(b)* $\displaystyle\sum_{n=1}^{\infty}\frac{\left(x^n - x^{n-1}\right)}{n}$ *(c)* $\displaystyle\sum_{n=1}^{\infty}\frac{\left(x^n - x^{n-1}\right)}{n^2}$ $\Diamond$

SOLUTION:

END OF SOLUTION

Theorem 36. *Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r (where r could be ∞ as well). Let b be any nonnegative real number with $b < r$. Then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-b, b]$.* ▲

Problem 187. *Prove Theorem 36. [Hint: We know that $\sum_{n=0}^{\infty} |a_n b^n|$ converges. This should be all set for the Weierstrass-M test.]* ◊

SOLUTION:

END OF SOLUTION

Problem 188. *Show that* $\sum_{n=1}^{\infty} nx^{n-1}$ *converges for* $|x| < 1$*. [Hint: We know that* $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$*. Differentiate both sides and take the limit as* n *approaches infinity.]* $\Diamond$

Solution:

End of Solution

Theorem 37. *Suppose* $\sum_{n=0}^{\infty} a_n x^n$ *has a radius of convergence* r *and let* $|x| < r$. *Then* $\sum_{n=1}^{\infty} a_n n x^{n-1}$ *converges.* ▲

Problem 189. *Prove Theorem 37. [Hint: Let* b *be a number with* $|x| < b < r$ *and consider* $\left| a_n n x^{n-1} \right| = \left| a_n b^n \cdot \frac{1}{b} \cdot n \left(\frac{x}{b}\right)^{n-1} \right|$. *You should be able to use the Comparison Test and Problem 188.]* ◊

Solution:

End of Solution

Problem 190. *Suppose the power series* $\sum a_n x^n$ *has radius of convergence* r *and the series* $\sum a_n r^n$ *converges absolutely. Then* $\sum a_n x^n$ *converges uniformly on* $[-r, r]$. *[Hint: For* $|x| \leq r$, $|a_n x^n| \leq |a_n r^n|$*.]* ♢

SOLUTION:

END OF SOLUTION

Lemma 6. Abel's Partial Summation Formula *Let*

$$a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$$

be real numbers and let $A_m = \sum_{k=1}^{m} a_k$. *Then*

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum_{j=1}^{n-1} A_j \left(b_j - b_{j+1} \right) + A_n b_n. \qquad \blacktriangle$$

Problem 191. *Prove Lemma 6. [Hint: For* $j > 1$, $a_j = A_j - A_{j-1}$.*]* ◊

SOLUTION:

END OF SOLUTION

Lemma 7. *Let* $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ *be real numbers with* $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$ *and let* $A_m = \sum_{k=1}^{m} a_k$. *Suppose* $|A_m| \leq B$ *for all* m. *Then* $|\sum_{j=1}^{n} a_j b_j| \leq B \cdot b_1$ ▲

Problem 192. *Prove Lemma 7.* ◊

SOLUTION:

END OF SOLUTION

SOLUTION:

END OF SOLUTION

Problem 193. *Prove Theorem* **??**. *[Hint: Let* $\epsilon > 0$*. Since* $\sum_{n=0}^{\infty} a_n r^n$ *converges then by the Cauchy Criterion, there exists* N *such that if* $m > n > N$ *then* $\left|\sum_{k=n+1}^{m} a_k r^k\right| < \frac{\epsilon}{2}$ *Let* $0 \leq x \leq r$*. By Lemma 7,*

$$\left|\sum_{k=n+1}^{m} a_k x^k\right| = \left|\sum_{k=n+1}^{m} a_k r^k \left(\frac{x}{r}\right)^k\right| \leq \left(\frac{\epsilon}{2}\right)\left(\frac{x}{r}\right)^{n+1} \leq \frac{\epsilon}{2}.$$

Thus for $0 \leq x \leq r$*,* $n > N$*,*

$$\left|\sum_{k=n+1}^{\infty} a_k x^k\right| = \lim_{n\to\infty}\left|\sum_{k=n+1}^{m} a_k x^k\right| \leq \frac{\epsilon}{2} < \epsilon.]$$

◊

Solution:

End of Solution

Problem 194. *Prove Corollary* **??**. *[Hint: Consider* $\sum a_n (-x)^n$.*]* $\Diamond$

The wrong problem is given here.

Solution:

Proof:

Case 1 ($\Longrightarrow$)**:** *Assume* a *is a limit point of* S. *Then there is a sequence,* $(a_n)_{n=1}^{\infty}$ *such that* $a_n \to a$ *and* $a_n \in S - \{a\}$. *Therefore for every* $\varepsilon > 0$, $\exists\, N \in \mathbb{R}$ *such that* $\forall n > N$

$$a_n \in (a - \varepsilon, a + \varepsilon).$$

Therefore $a_n \in (a - \varepsilon, a + \varepsilon) \bigcap S - \{a\}$. ■

Case 2($\Longleftarrow$)**:** *Assume* $\forall \varepsilon > 0$,

$$(a - \varepsilon, a + \varepsilon) \bigcap S - \{a\} \neq \emptyset$$

Consider the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$. *By assumption* $\forall n \in \mathbb{N}$ *there is at least on* $a_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \bigcap S - \{A\}$. *Since* $a - \frac{1}{n} \to a$, *and* $a + \frac{1}{n} \to a$, *we see that* $a_n \to a$ *by the Squeeze Theorem.* ■

■

End of Solution

Chapter 9

Back to the Real Numbers

Problem 195. *Let $S \subseteq \mathbb{R}$ and let a be a real number. Prove that a is a limit point of S if and only if for every $\varepsilon > 0$ the intersection*

$$(a - \varepsilon, a + \varepsilon) \cap S - \{a\} \neq \emptyset. \qquad \Diamond$$

SOLUTION:

END OF SOLUTION

Problem 196. *Determine the derived set, S', of each of the following sets.*

(a) $S = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\right\}$

(b) $S = \left\{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\right\}$

(c) $S = (0, 1]$

(d) $S = [0, 1/2) \cup (1/2, 1]$

(e) $S = \mathbb{Q}$

(f) $S = \mathbb{R} - \mathbb{Q}$

(g) $S = \mathbb{Z}$

(h) *Any finite set S.*

$\Diamond$

SOLUTION:

END OF SOLUTION

Problem 197. *Let $S \subseteq \mathbb{R}$.*

(a) *Prove that $(S')' \subseteq S'$.*

(b) *Give an example where these two sets are equal.*

(c) *Give an example where these two sets are not equal.*

$\Diamond$

SOLUTION:

END OF SOLUTION

Problem 198. *Suppose* $\lim_{n\to\infty} |b_n - a_n| > 0$. *Show that there are at least two points, c and d, such that $c \in [a_n, b_n]$ and $d \in [a_n, b_n]$ for all $n \in \mathbb{N}$.*

Solution:

End of Solution

Problem 199. *Show that each of the following sets is countable.*

(a) $\{2, 3, 4, 5, \ldots\} = \{n\}_{n=2}^{\infty}$

(b) $\{0, 1, 2, 3, \ldots\} = \{n\}_{n=0}^{\infty}$

(c) $\left\{1, 4, 9, 16, \ldots, n^2, \ldots\right\} = \left\{n^2\right\}_{n=1}^{\infty}$

(d) *The set of prime numbers*

(e) $\mathbb{Z}$

SOLUTION:

$$f(n) = \begin{cases} 2n & \textit{if } n > 0 \\ |2n| + 1 & \textit{if } n \leq 0 \end{cases}$$

$\Diamond$

SOLUTION:

END OF SOLUTION

Problem 200. *Let $\{A_i\}$ be a collection of countable sets. Show that each of the following sets is also countable:*

(a) *Any subset of A_1.*

SOLUTION:

Let $A_1 = \{a_n\}_{n=1}^{\infty}$ and $T \subset A_1$. Let $t_1 = a_{n_1}$ where n_1 is the smallest subscript appearing in T, $t_2 = a_{n_2}$ the second smallest, and in general $t_k = a_{n_k}$ is the kth smallest subscript appearing in T. Then the function

$$f(t_k) = k$$

is clearly one to one and onto.

END OF SOLUTION

(b) $A_1 \cup A_2$

SOLUTION:

Since A_1 and A_2 are countable there are bijections $f|\mathbb{N} \to A_1$ and $g|\mathbb{N} \to A_2$. Define $h|\mathbb{N} \to A_1 \cup A_2$ by

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ g\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

END OF SOLUTION

(c) $A_1 \cup A_2 \cup A_3$

SOLUTION:

By part (b) the set $A_1 \cup A_2$ is countable. Thus, also by part (b) the set $A_1 \cup A_2 \cup A_3$ is also countable.

END OF SOLUTION

(d) $\displaystyle\bigcup_{i=1}^{n} A_i$

SOLUTION:

Proof by Induction:

Base Case: *A_1 is countable by assumption.*

Induction Hypothesis: $\displaystyle\bigcup_{i=1}^{n-1} A_i$ **is countable.**

Then by part (b) of this problem $\displaystyle\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n-1} A_i\right) \bigcup A_n$ is also countable.

END OF SOLUTION

(e) $\displaystyle\bigcup_{i=1}^{\infty} A_i$

SOLUTION:

Let $T_n = \frac{n(n+1)}{2}$ *be the nth Triangular Number. Observe that* $T_n + (n+1) = T_{n+1}$ *so that every non-negative*[1] *integer can be represented uniquely as* $k = T_n + m$ *where* T_n *is the largest Triangular Number less than or equal to* k *and* $0 \leq m \leq n$.

We label the elements of A_n *as follows:*

$$A_n = \{a_{nn}, a_{nn+1}, a_{nn+2}, \ldots\},$$

and define $\mu : \bigcup_{i=1}^{\infty} A_i \to \mathbb{N} \bigcup \{0\}$ *by*

$$\mu(a_{nm}) = T_m + n.$$

END OF SOLUTION

$\Diamond$

[1]Note that $T_0 = 0$.

Theorem 38. *Show that $\mathbb{Q}$ is countable.* ▲

Problem 201. *Prove Theorem 38.* $\Diamond$
SOLUTION:

END OF SOLUTION

Problem 202. *Prove Corollary* **??**. *[Hint: If we had only finitely many rationals to deal with this would be easy. Let* $\{r_1, r_2, \ldots, r_k\}$ *be these rational numbers and take* $a_n = r_n - \frac{\varepsilon}{2k}$ *and* $b_n = r_n + \frac{\varepsilon}{2k}$*. Then for all* $n = 1, \ldots, k$ $r_n \in [a_n, b_n]$ *and*

$$\sum_{n=1}^{k} b_n - a_n = \sum_{n=1}^{k} \frac{\varepsilon}{k} = \varepsilon.$$

The difficulty is, how do we move from the finite to the infinite case?]
SOLUTION:

END OF SOLUTION

Problem 203.

(a) *Let (a, b) and (c, d) be two open intervals of real numbers. Show that these two sets have the same cardinality by constructing a one-to-one onto function between them. [Hint: A linear function should do the trick.]*
SOLUTION:

$$f(x) = \left(\frac{c-d}{a-b}\right)x + \frac{ad-bc}{a-b}$$

or

$$f(x) = \left(\frac{c-d}{a-b}\right)(x-b) + c$$

END OF SOLUTION

(b) *Show that any open interval of real numbers has the same cardinality as $\mathbb{R}$. [Hint: Consider the interval $(-\pi/2, \pi/2)$.]*
SOLUTION:
Let (a, b) be any open interval. By part (a) there is a mapping $\phi : (a, b) \to (-\pi/2, \pi/2)$.
Let $\psi : (-\pi/2, \pi/2) \to \mathbb{R}$ be the mapping:

$$\psi(x) = \tan(x).$$

Then the composition $(\psi \circ \phi)$ maps (a, b) to $\mathbb{R}$.
END OF SOLUTION

(c) *Show that $(0, 1]$ and $(0, 1)$ have the same cardinality. [Hint: Note that $\{1, 1/2, 1/3, \ldots\}$ and $\{1/2, 1/3, \ldots\}$ have the same cardinality.]*
SOLUTION:
Define $\phi : (0, 1] \to (0, 1)$ by

$$\phi(x) = \begin{cases} x; & \text{if } x \in (0,1) \text{ and } x \neq \frac{1}{n}, \forall\, n \in \mathbb{N} \\ \frac{1}{n+1}; & \text{if } x = \frac{1}{n}, \text{ for some } n \in \mathbb{N} \end{cases}$$

END OF SOLUTION

(d) *Show that $[0, 1]$ and $(0, 1)$ have the same cardinality.*
SOLUTION:
Define $\phi : [0, 1] \to (0, 1)$ by

$$\phi(x) = \begin{cases} x; & \text{if } x \in (0,1) \text{ and } x \neq \frac{1}{n}, \forall\, n \in \mathbb{N} \\ \frac{1}{2}; & \text{if } x = 0 \\ \frac{1}{n+2}; & \text{if } x = \frac{1}{n}, \text{ for some } n \in \mathbb{N} \end{cases}$$

END OF SOLUTION

◊

Problem 204. *Consider the sequence* $(0.9, 0.99, 0.999, \ldots)$. *Determine that this sequence converges and, in fact, it converges to* 1. *This suggests that* $0.999\ldots = 1$. $\diamondsuit$

Solution:

End of Solution

Problem 205. *Prove: If* $|S| = n$*, then* $|P(S)| = 2^n$*. [Hint: Let* $S = a_1, a_2, \ldots, a_n$*. Consider the following correspondence between the elements of* $P(S)$ *and the set* T *of all* n*-tuples of yes (Y) or no (N):*

$$\begin{aligned}
\{\} &\leftrightarrow \{N, N, N, \ldots, N\} \\
\{a_1\} &\leftrightarrow \{Y, N, N, \ldots, N\} \\
\{a_2\} &\leftrightarrow \{N, Y, N, \ldots, N\} \\
&\vdots \\
S &\leftrightarrow \{Y, Y, Y, \ldots, Y\}
\end{aligned}$$

How many elements are in T*?]* $\Diamond$

SOLUTION:

END OF SOLUTION

Theorem 39. (Cantor's Theorem) *Let S be any set. Then there is no one-to-one correspondence between S and $P(S)$, the set of all subsets of S.* ▲

Problem 206. *Prove Cantor's Theorem (Theorem 39). [Hint: Assume for contradiction, that there is a one-to-one correspondence $f : S \to P(S)$. Consider $A = \{x \in S | x \notin f(x)\}$. Since f is onto, then there is $a \in A$ such that $A = f(a)$. Is $a \in A$ or is $a \notin A$?]* ◊

SOLUTION:

END OF SOLUTION

Problem 207. *Let* $U_n = \{(a_1, a_2, a_3, \ldots) \mid a_j \in \{0, 1\}$ *and* $a_{n+1} = a_{n+2} = \cdots = 0\}$. *Show that for each* n, U_n *is finite and use this to conclude that* U *is countably infinite.* ◊

SOLUTION:

END OF SOLUTION

Problem 208. *Let S be an infinite set. Prove that S contains a countably infinite subset.* $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 209. *Suppose X is an uncountable set and $Y \subset X$ is countably infinite. Prove that X and $X - Y$ have the same cardinality. [Hint: Let $Y = Y_0$. By the previous problem, $X - Y_0$ is an infinite set, so it contains a countably infinite set Y_1. Likewise $X - (Y_0 \cup Y_1)$ is infinite so it contains an infinite set Y_2. Again, $X - (Y_0 \cup Y_1 \cup Y_2)$ is an infinite set so it contains an infinite set Y_3, etc. For $n = 1, 2, 3, \ldots$, let $f_n : Y_{n-1} \to Y_n$ be a one-to-one correspondence and define $f : X \to X - Y$ by*

$$\begin{cases} f(x) = f_n(x), & \text{if } x \in Y_n, n = 0, 1, 2, \ldots \\ f(x) = x, & \text{if } x \in X - (\cup_{n=0}^{\infty} Y_n) \end{cases}.$$

Show that f is one-to-one and onto.] $\Diamond$

SOLUTION:

END OF SOLUTION

Epilogue

On the Nature of Numbers: A Dialogue (with Apologies to Galileo)

Problem 210. *Show that* $0 \neq 1$. *[Hint: Show that if* $x \neq 0$, *then* $0 \cdot x \neq x$.*]* $\diamondsuit$
SOLUTION:

END OF SOLUTION

Problem 211. *Consider the set of ordered pairs of integers:* $\{(x, y)|s, y \in \mathbb{Z}\}$, *and define addition and multiplication as follows:*

Addition: $(a, b) + (c, d) = (ad + bc, bd)$

Multiplication: $(a, b) \cdot (c, d) = (ac, bd)$.

(a) *If we add the convention that*

$$(ab, ad) = (b, d)$$

show that this set with these operations forms a number field.

(b) *Which number field is this?* ◊

SOLUTION:

END OF SOLUTION

Problem 212. *Consider the set of ordered pairs of real numbers,* $\{(x, y)|x, y \in \mathbb{R}\}$, *and define addition and multiplication as follows:*

Addition: $(a, b) + (c, d) = (a + c, b + d)$

Multiplication: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

(a) *Show that this set with these operations forms a number field.*

(b) *Which number field is this?* ◊

SOLUTION:

END OF SOLUTION

Building the Real Numbers

Problem 213.

(a) *Prove that the following must hold in* **any** *linearly ordered number field.*

1. $0 < x$ *if and only if* $-x < 0$.
2. *If* $x < y$ *and* $z < 0$ *then* $y \cdot z < x \cdot z$.
3. *For all* $x \neq 0$, $0 < x^2$.
4. $0 < 1$.

(b) *Show that the set of complex numbers* ($\mathbb{C}$) *is not a linearly ordered field.*

SOLUTION:

END OF SOLUTION

Problem 214. *Define addition on infinite decimals in a manner that is closed. [Hint: Find an appropriate "carry" operation for our definition.]* ◊

SOLUTION:

END OF SOLUTION

Problem 215. *Show that:*

a) $x \equiv x$

b) $x \equiv y \;\Rightarrow\; y \equiv x$

c) $x \equiv y$ *and* $y \equiv z \;\Rightarrow\; x \equiv z$ $\Diamond$

SOLUTION:

END OF SOLUTION

Problem 216. *Let x and y be real numbers in $\mathbb{Q}$ (that is, let them be sets of equivalent Cauchy sequences). If (s_n) and (t_n) are in x and (σ_n) and (τ_n) are in y then*

$$(s_n + t_n)_{n=1}^{\infty} \equiv (\sigma_n + \tau_n)_{n=1}^{\infty}. \qquad \Diamond$$

SOLUTION:

END OF SOLUTION

Problem 217. *Identify the set of equivalent Cauchy sequences,* 1^*, *such that*

$$1^* \cdot x = x.$$

$\diamond$

Solution:

End of Solution

Problem 218. *Let x, y, and z be real numbers (equivalent sets of Cauchy sequences). Show that with addition and multiplication defined as above we have:*

a) $x + y = y + x$

b) $(x + y) + z = x + (y + z)$

c) $x \cdot y = y \cdot x$

d) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

e) $x \cdot (y + z) = x \cdot y + x \cdot z$ $\Diamond$

SOLUTION:

END OF SOLUTION

Theorem 40. *Let α and β be cuts. Then $\alpha < \beta$ if and only if $\alpha \subset \beta$.* ▲

Problem 219. *Prove Theorem 40 and use this to conclude that if α and β are cuts then exactly one of the following is true:*

1. $\alpha = \beta$.
2. $\alpha < \beta$.
3. $\beta < \alpha$.

SOLUTION:

END OF SOLUTION

Problem 220. *Show that if α and β are cuts then $\alpha \cdot \beta$ is also a cut.*
SOLUTION:

END OF SOLUTION

Lemma 8. *Let β be a cut, y and z be positive rational numbers not in β with $y < z$, and let $\varepsilon > 0$ be any rational number. Then there exist positive rational numbers r and s with $r \in \beta$, and $s \notin \beta$, such that $s < z$, and $s - r < \varepsilon$.* ▲

Problem 221. *Prove Lemma 8. [Hint: Since β is a cut there exists $r_1 \in \beta$. Let $s_1 = y \notin \beta$. We know that $r_1 < s_1 < z$. Consider the midpoint $\frac{s_1+r_1}{2}$. If this is in β then relabel it as r_2 and relabel s_1 as s_2. If it is not in β then relabel it as s_2 and relabel r_1 as r_2, etc.]* ♢

SOLUTION:

END OF SOLUTION

Problem 222. *Let α and β be cuts with $\beta < \alpha$. Prove that $\beta + (\alpha - \beta) = \alpha$. [Hint: It is pretty straightforward to show that $\beta + (\alpha - \beta) \subseteq \alpha$. To show that $\alpha \subseteq \beta + (\alpha - \beta)$, we let $x \in \alpha$. Since $\beta < \alpha$, we have $y \in \alpha$ with $y \notin \beta$. We can assume without loss of generality that $x < y$. (Why?) Choose $z \in \alpha$ with $y < z$. By the Lemma 8, there exists positive rational numbers r and s with $r \in \beta$, $s \in \beta$, $s < z$, and $s - r < z - x$. Show that $x < r + (z - s)$.]* ◊

SOLUTION:

END OF SOLUTION