A Contextual Introduction to Real Analysis: How We Got From There To Here

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Acknowledgements

Mactutor While we have tried to tell the story of the development of Real Analysis as completely as possible, our overriding goal was always to teach mathematics, not history. Thus we have necessarily left the history incomplete.

The interested student can fill in the gaps we have left by making use of the extensive resources that can be found at the MacTutor history of mathematics repository.

All of the portraits of mathematicians used in this text have been taken from MacTutor.

MacTutor was created, and is maintained by Professor Edmund Robertson (Emeritus), and Professor John O'Connor (Emeritus), both of the University of St. Andrews in Scotland.

To the Instructor

The irony of this section is that it exists to tell you that this book was not written for you; it was written for your students. After all, we don't need to teach you about Real Analysis. You already understand the subject. The purpose of this text is to help your students make sense of the formal definitions, theorems, and proofs that they will encounter in your course. We do this by immersing the student in the story of how what is usually called Calculus evolved into modern Real Analysis. Our hope and intention is that this will help the student to appreciate why their intuitive understanding of topics encountered in Calculus needs to be replaced by the formalism of Real Analysis.

The traditional approach to this topic (what we might call the "logical" story of Real Analysis), starts with a rigorous development of the real number system and uses this to provide rigorous definitions of limits, continuity, derivatives and integrals, and convergence of series; typically in that order. This is a perfectly legitimate story of Real Analysis and, logically, it makes the most sense. Indeed, this is a view of the subject that every mathematician–in–training should eventually attain. However, it is our contention that your students will appreciate the subject more, and hopefully retain it better, if they see how the subject developed from the intuitive notions of Leibniz, Newton and others in the seventeenth and eighteenth centuries to the more modern approach developed in the nineteenth and twentieth centuries. After all, they are coming at it from a viewpoint very similar to that of the mathematicians of the seventeenth and eighteenth centuries. Our goal is to bring them into the nineteenth and early twentieth centuries, mathematically speaking.

We hasten to add that this is not a history of analysis book. It is an introductory textbook on Real Analysis which uses the historical context of the subject to frame the concepts and to show why mathematicians felt the need to develop rigorous, non-intuitive definitions to replace their intuitive notions.

You will notice that most of the problems are embedded in the chapters, rather than lumped together at the end of each chapter. This is done to provide a context for the problems which, for the most part, are presented on an as–needed basis.

Thus the proofs of nearly all of the theorems appear as problems in the text. Of course, it would be very unfair to ask most students at this level to prove, say, the Bolzano–Weierstrass Theorem without some sort of guidance. So in each case we provide an outline of the proof and the subsequent problem will be to use the outline to develop a formal proof. Proof outlines will become less detailed as the students progress. We have found that this approach helps students develop their proof writing skills.

We state in the text, and we encourage you to emphasize to your students, that often they will use the results of problems as tools in subsequent problems.

Trained mathematicians do this naturally, but it is our experience that this is still somewhat foreign to students who are accustomed to simply "getting the problem done and forgetting about it."

The problems range from the fairly straightforward to the more challenging. Some of them require the use of a computer algebra system (for example, to plot partial sums of a power series). These tend to occur earlier in the book where we encourage the students to use technology to explore the wonders of series. A number of these problems can be done on a sufficiently advanced graphing calculator or even on Wolfram Alpha, so you should assure your students that they do not need to be super programmers to do this. Of course, this is up to you.

A testing strategy we have used successfully is to assign more time consuming problems as collected homework and to assign other problems as possible test questions. Students could then be given some subset of these (verbatim) as an in-class test. Not only does this make test creation more straightforward, but it allows the opportunity to ask questions that could not reasonably be asked otherwise in a timed setting. Our experience is that this does not make the tests "too easy," and there are worse things than having students study by working together on possible test questions beforehand. If you are shocked by the idea of giving students all of the possible test questions ahead of time, think of how much (re)learning you did studying the list of possible questions you *knew* would be asked on a qualifying exam.

In the end, use this book as you see fit. We believe your students will find it readable, as it is intended to be, and we are confident that it will help them to make sense out of the rigorous, non-intuitive definitions and theorems of Real Analysis and help them to develop their proof-writing skills.

If you have suggestions for improvement, comments or criticisms of our text please contact us at the email addresses below. We appreciate any feedback you can give us on this.

Thank you.

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Contents

Part I

In Which We Raise A Number Of Questions

Chapter 1

Prologue: Three Lessons Before We Begin

1.1 Lesson One

Get a pad of paper and write down the answer to this question: What is . . . No, really. We're serious. *Get a writing pad.* We'll wait.

We really are serious about this. Get a pad of paper!

Got it? Good. Now write down your answer to this question: What is a number? Don't think about it. Don't analyze it. Don't consider it. Just write down the best answer you can without thinking. You are the only person who ever needs to see what you've written.

Done? Good.

Now consider this: All of the objects listed below are "numbers" in a sense we will not make explicit here. How many of them does your definition include?

- 1
- -1
- 0
- $3/5$
- $\sqrt{2}$
- $\sqrt{-1}$
- i^i
- e^{5i}
- $4 + 3i - 2j + 6k$ (this is called a quaternion)
- $\mathrm{d}x$ (this is the differential you learned all about in Calculus)
- $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ (yes, matrices can be considered numbers).

Surely you included 1. Almost surely you included 3/5. But what about 0? -1? Does your definition include $\sqrt{2}$? Do you consider $\mathrm{d}x$ a number? Leibniz did. Any of the others? (And, yes, they really are all "numbers.")

The lesson in this little demonstration is this: You don't really have a clear notion of what we mean when we use the word "number." And this is fine. Not knowing is acceptable.

Sometimes it is even encouraged.

A principal goal of this course of study is to rectify this, at least a little bit. When the course is over you may or may not be able to give a better definition of the word "number" but you will have a deeper understanding of the real numbers at least. That is enough for now.

1.2 Lesson Two

Read and understand the following development of the **Quadratic Formula**.

Suppose $a \neq 0$. If

$$ax^2 + bx + c = 0 \tag{I.1}$$

then

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{I.2}$$

Now let

$$y^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \tag{I.3}$$

so that

$$y = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \tag{I.4}$$

and

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{I.5}$$

which is the **Quadratic Formula**.

Were you able to follow the argument? Probably the step from equation (I.1) to equation (I.2) presented no difficulties. But what about the next step? Do you see where equation (I.3) came from? If so, good for you. Most students, in fact most mathematicians, cannot make that step in their heads. But are you sure? Is there, perhaps, some small detail you've overlooked?

Check to see.

That is, let $x = y - \frac{b}{2a}$ in equation (I.2) and see if you can get equation (I.3). Do it on that handy pad of paper we told you to get out earlier. Do it now. We'll wait.

If you *still* haven't gotten out a pad of paper, give up now. You're going to fail this course. Seriously. Do you think we would spend so much time on this, that we would repeat it so many times, if it weren't important. ***GET OUT A PAD OF PAPER NOW!*** Last chance. You've been warned.

Done? Good.

Perhaps you haven't been able to fill in the details on your own. That's ok. Many people can't. If not then get help: from a classmate, a friend, your instructor, whomever. Unfortunately most people won't get help in this situation. Instead they will perceive this as "failure," hide it and berate themselves or the problem as "stupid." In short they will let their personal insecurities and demons overwhelm them. *Don't do this. Get help.* You are neither dumb nor incapable. There are a thousand reasons that on any given day you might not be able to solve this problem. But don't let a bad day interfere with the education you are here for. Get someone to help you over this hump. Later you will be able to help your helper in the same way. Really.

At this point we assume that you've successfully negotiated the transition from equation (I.2) to equation (I.5).

See? It really wasn't that bad after all. Just a lot of elementary algebra. Now that you've done it (or seen it done), it is easy to see that there really wasn't much there.

But this is the point! We left those computations out precisely because we knew that they were routine and that you could fill in the details. Moreover, filling in the details yourself gives you a little better insight into the computations. If we'd filled them in for you we would have robbed you of that insight. And we would have made this book longer than it needs to be. We don't want to do either of those things. If we fill in all of the details of every computation for you, you won't learn to have confidence in your ability to do them yourself. And this book will easily double in length.

So the lesson here is this: Keep that pad of paper handy whenever you are reading this (or any other) mathematics text. You will need it. Routine computations will often be skipped. But calling them routine and skipping them does not mean that they are unimportant. If they were truly unimportant we would leave them out entirely.

Moreover, routine does not mean obvious. Every step we took in the development of the **Quadratic Formula** was routine. But even routine computations need to be understood and the best way to understand them is to do them. This is the way to learn mathematics; it is the *only* way that really works. Don't deprive yourself of your mathematics education by skipping the most important parts.

If you didn't fill in those details then you are depriving yourself of the education you are here to obtain. This is sad. There is a good reason for putting these three lessons first. Stop wasting your time and intellect! Go do it now.

As you saw when you filled in the details of our development of the **Quadratic Formula** the substitution $x = y - \frac{b}{2a}$ was crucial because it turned

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

into

$$y^2 = k$$

where k depends only on a, b, and c. In the sixteenth century a similar technique was used by Ludovico Ferrari (1522-1565) to reduce the general cubic equation

$$ax^3 + bx^2 + cx + d = 0 \tag{I.6}$$

into the so-called **depressed cubic**

$$y^3 + py + q = 0$$

where p, and q depend only on a, b, c, and d.

The general **depressed cubic** had previously been solved by Tartaglia (1500–1557) so converting the general cubic into a **depressed cubic** provided a path for Ferrari to compute the **Cubic Formula**. (It's like the **Quadratic Formula** but bigger.)

It is not entirely clear why eliminating the quadratic term should be depressing, but there it is.

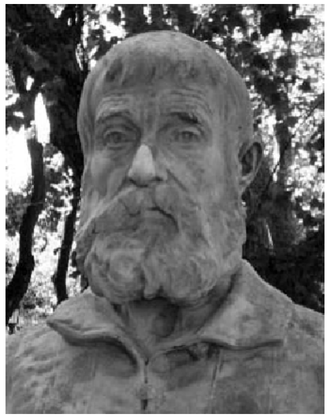

Figure 1.2.1 Tartaglia, "The Stutterer"

Ferrari also knew how to compute the general solution of the **depressed quartic** so when he and his teacher Girolomo Cardano (1501-1576) figured out how to depress a general quartic they had a complete solution of the general quartic as well.

Figure 1.2.2 Girolomo Cardano

Alas, their methods broke down entirely when they tried to solve the general quintic equation. Unfortunately the rest of this story belongs in a course on Abstract Algebra, not Real Analysis. But the lesson in this story applies to all of mathematics: Every problem solved is a new theorem which then becomes a tool for later use. Depressing a cubic would have been utterly useless had not Tartaglia had a solution of the depressed cubic in hand. The technique they used, with slight modifications, then allowed for a solution of the general quartic as well.

Keep this in mind as you proceed through this course and your mathematical education. Every problem you solve is really a theorem, a potential tool that you can use later. We have chosen the problems in this text deliberately with this in mind. Don't just solve the problems and move on. Just because you have solved a problem does not mean you should stop thinking about it. Keep thinking about the problems you've solved. Internalize them. Make the ideas your own so that when you need them later you will have them at hand to use.

Problem 1.2.3 The Quadratic Formula.

(a) Find M so that the substitution $x = y - M$ depresses equation (I.6), the general cubic equation. Then find p and q in terms of a, b, c, and d.

(b) Find K so that the substitution $x = y - K$ depresses the general quartic equation. Make sure you demonstrate how you obtained that value or why it works (if you guessed it).

(c) Find N so that the substitution $x = y - N$ depresses a polynomial of degree n. Make sure you demonstrate how you obtained that value or why it works (if you guessed it).

$\square$

Problem 1.2.4 Another Derivation of the Quadratic Formula. Here is yet another way to solve a quadratic equation. Read the development below with pencil and paper handy. Confirm all of the computations that are not completely transparent to you. Then use your notes to present the solution with *all* steps filled in.

Be sure you are clear on the purpose of this problem before you begin. This is not about solving the Quadratic Equation. You already know how to do that. Our purpose here is to give you practice filling in the skipped details of mathematical exposition. We've chosen this particular problem because it should be a comfortable setting for you, but this particular solution is probably outside of your previous experience.

Suppose that r_1 and r_2 are solutions of $ax^2 + bx + c = 0$. Without loss of generality suppose that $a > 0$. Suppose further that $r_1 \geq r_2$. Then

$$\begin{aligned} ax^2 + bx + c &= a(x - r_1)(x - r_2) \\ &= a\left[x^2 - (r_1 + r_2)x + (r_1 + r_2)^2 - (r_1 - r_2)^2 - 3r_1 r_2\right]. \end{aligned}$$

Therefore

$$r_1 + r_2 = -\frac{b}{a} \tag{I.7}$$

and

$$r_1 - r_2 = \sqrt{\left(\frac{b}{a}\right)^2 - \frac{4c}{a}}. \tag{I.8}$$

Equations (I.7) and (I.8) can be solved simultaneously to yield

$$\begin{aligned} r_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

$\square$

1.3 Lesson Three

In the hustle and bustle of a typical college semester, with a lot of demands on your time and very little time to think, it becomes very easy to see each problem you solve as a small, isolated victory and then move on to the next challenge. This is understandable. Each problem you solve *is* a small victory and you've every right to be proud of it. But it is not isolated and it is a mistake to think that it is.

In his book *How to Solve It* the mathematician and teacher George Polya

(1887–1985) gave four steps for problem solving. The steps may be paraphrased as

1. Understand the problem.
2. Formulate a plan.
3. Execute the plan.
4. Reflect on what you've done.

Figure 1.3.1 George Polya

This process is iterative. That is, once a plan is formulated and executed we often find that our plan was not up to the task. So we have to ask what went wrong, form a new plan and try again. This is the fourth step: Reflect on what you've done.

Almost everyone remembers this fourth step when their plan *doesn't* work. After all, you've got to try again so you have to ask what went wrong. But it is all too easy to neglect that crucial fourth step when the plan succeeds. In that case, flush with success we usually move on to the next problem and start over from scratch.

This is a mistake. Having solved a problem is no reason to stop thinking about it.

That fourth step is at least as important when we have succeeded as when we have failed. Each time you solve a problem stop and ask yourself a few questions:

- Are there any easy consequences that follow from the result?
- How does it fit into the broader scheme of other problems you have solved?
- How might it be used in the future?

Also, notice the structure of the problem. Some assumptions had to be made. What were they? Were they all necessary? That is, did your solution use everything that was assumed? If not, you may have something considerably more general than it at first appears to be. What is that more general statement? Even if you used all of the assumptions, was that really necessary? Can you solve a similar problem with weaker assumptions?

Take a moment to pack all of these questions (and their answers) away in your mind so that when you see something similar in the future you will be reminded of it. *Don't* solve any problem and then forget it and move on.

The nature of mathematics is cumulative. Remember, you are not here to accumulate grade points. You are here to learn and understand the concepts and methods of mathematics, to gain "mathematical maturity." Part of that maturation process is the accumulation of a body of facts (theorems), and techniques that can be used to prove new theorems (solve new problems).

This text has been written with the maturation process in mind. You will frequently find that the problems you solve today can be used to good effect in the ones you attempt tomorrow, but only if you remember them. So take a moment after you've solved each problem to think about how it fits into the patterns you already know. This is important enough to bear repeating: *A problem, once solved, becomes a tool for solving subsequent problems!*

The purpose of the following sequence of problems is to help you become accustomed to this notion (if you aren't already). It is a progression of results about prime numbers. As you probably recall, a prime number is any integer greater than 1 whose only factors are itself and 1. For example, 2, 3, 5, 7, 11 are prime, while 4, 6, 9 are not. A major result about prime numbers is the following:

Theorem 1.3.2 The Fundamental Theorem of Arithmetic. *Any integer greater than* 1 *is either prime or it is a product of prime numbers. Furthermore, this prime decomposition is unique up to the order of the factors.*

We will not prove this, but we will use it as a starting point to examine the following problems. As you do these problems, notice how subsequent problems make use of the previous results.

Notice that the notation $p\,|a$ simply means that the integer p divides the integer a with no remainder.

Problem 1.3.3 Fermat's Little Theorem, step 1. Let p be a prime number and a, b positive integers such that $p\,|(a \cdot b)$. Show that $p\,|a$ or $p\,|b$.

Hint. If $p\,|a$ then we are done. If not then notice that p is a prime factor of $a \cdot b$. What does the Fundamental Theorem of Arithmetic say about the prime factors of $a \cdot b$ compared to the prime factors of a and b? □

Problem 1.3.4 Fermat's Little Theorem, step 2. Let p be a prime number and let $a_1, a_2, \ldots, a_n$ be positive integers such that $p\,|\,(a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n)$.

Use Problem 1.3.3 to show that $p\,|a_k$ for some $k \in \{1, 2, 3, \ldots, n\}$.

Hint. Use induction on n and the result of the previous problem. □

Problem 1.3.5 Fermat's Little Theorem, step 3. Let p be a prime number and let k be an integer with $1 \leq k \leq p-1$.

Use Problem 1.3.4 to prove that $p\,\Big|\binom{p}{k}$, where $\binom{p}{k}$ is the binomial coefficient $\frac{p!}{k!(p-k)!}$.

Hint. We know $p\,|p\,!$, so $p\,|\binom{p}{k}k!(p-k)!$. How does the previous result apply? □

We now have all the machinery in place to prove one of the really cool theorems from number theory.

Theorem 1.3.6 Fermat's Little Theorem. *Let p be any prime number. Use Problem 1.3.5 to show that $p\,|(n^p - n)$ for all positive integers n.*

Problem 1.3.7 Fermat's Little Theorem, step 4. Prove Fermat's Little Theorem.

Hint. Use induction on n. To get from n to $n+1$, use the binomial theorem on $(n+1)^p$. □

Fermat's Little Theorem is the foundational basis for a number of results in number theory and encryption.

Problem 1.3.8 Recall what we said above: Having solved a problem is no reason to stop thinking about it. Reflect upon your proof of Fermat's Little Theorem by answering these questions.

(a) Will the result of Fermat's Little Theorem hold if p is not a prime number?

(b) If not, then which of the steps in the proof breaks down?

□

Chapter 2

Numbers, Real ($\mathbb{R}$) and Rational ($\mathbb{Q}$)

The set of real numbers (denoted, $\mathbb{R}$) is badly named. The real numbers are no more or less real — in the non-mathematical sense that they exist — than any other set of numbers, just like the set of rational numbers ($\mathbb{Q}$), the set of integers ($\mathbb{Z}$), or the set of natural numbers ($\mathbb{N}$). The name "real numbers" is (almost) an historical anomaly not unlike the name "Pythagorean Theorem" which was actually known and understood long before Pythagoras lived.

When Calculus was being invented in the 17th century, numbers were thoroughly understood, or so it was believed.

Some would say "re-invented." See [13], or [9].

They were, after all, just numbers. Combine them. We call that addition. If you add them repeatedly we call it multiplication. Subtraction and division were similarly understood.

It was (and still is) useful to visualize these things in a more concrete way. If we take a stick of length 2 and another of length 3 and lay them end-to-end we get a length of 5. This is addition. If we lay them end–to–end but at right angles then our two sticks are the length and width of a rectangle whose area is 6. This is multiplication.

Of course measuring lengths with whole numbers has limitations, but these are not hard to fix. If we have a length (stick) of length 1 and another of length 2, then we can find another whose length when compared to 1 is the same (has the same proportion) as 1 is to 2. That number of course, is 1/2.

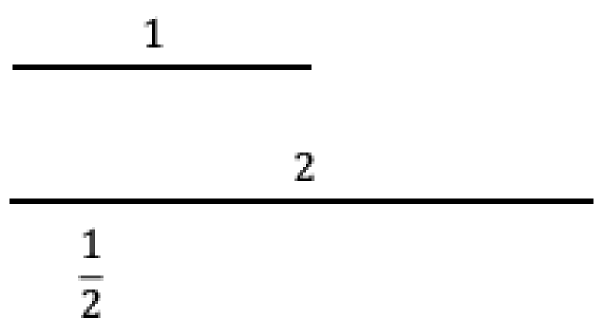

Figure 2.0.1

Notice how fraction notation reflects the operation of comparing 1 to 2. This comparison is usually referred to as the **ratio** of 1 to 2 so numbers of this sort are called **rational numbers.** The set of rational numbers is denoted $\mathbb{Q}$ for quotients. In grade school they were introduced to you as fractions. Once fractions are understood, this visualization using line segments (sticks) leads quite naturally to their representation with the rational number line.

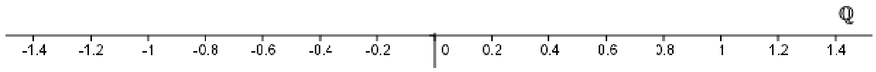

Figure 2.0.2 The Rational Number Line

This seems to work as a visualization because the **rational numbers** and the points on a line seem to share certain properties. Chief among these is that between any two points on the rational line there is another point, just as between any two rational numbers there is another rational number.

Problem 2.0.3 Let $a, b, c, d \in \mathbb{N}$ and find a rational number between a/b and c/d. $\square$

This is all very clean and satisfying until we examine it just a bit closer. Then it becomes quite mysterious. Consider again the rational numbers a/b and c/d. If we think of these as lengths we can ask, "Is there a third length, say α, such that we can divide a/b into M pieces, each of length α *and also* divide c/d into N pieces each of length α?" A few minutes thought should convince you that this is the same as the problem of finding a common denominator so $\alpha = \frac{1}{bd}$ will work nicely. (Confirm this yourself.)

You may be wondering what we're making all of this fuss about. *Obviously* this is *always* true. In fact the previous paragraph gives an outline of a very nice little proof of this. Here are the theorem and its proof presented formally.

Theorem 2.0.4 *Suppose a, b, c, and d are integers, with $b, d \neq 0$. There is a number $\alpha \in \mathbb{Q}$ such that $M\alpha = a/b$ and $N\alpha = c/d$ where M and N are also integers.*

Proof. To prove this theorem we will display α, M and N. It is your responsibility to confirm that these actually work. Here they are: $\alpha = 1/bd$, $M = ad$, and $N = cb$. $\blacksquare$

Problem 2.0.5 Confirm that α, M, and N as given in the proof of Theorem 2.0.4 satisfy the requirements of the theorem. $\square$

Theorem 2.0.4 suggests the following very deep and important question: Are there lengths which can *not* be expressed as the ratio of two integer lengths? The answer, of course, is yes. Otherwise we wouldn't have asked the question.

One of the best known examples of such a number is the circumference of a circle with diameter 1. This is the number usually denoted by π. But circles are extremely complex objects — they only seem simple because they are so familiar. Arising as it does from a circle, you would expect the number π to be very complex as well and this is true. In fact π is an exceptionally weird number for a variety of reasons. Let's start with something a little easier to think about.

Squares are simple. Two sets of parallel lines at right angles, all of the same length. What could be simpler? If we construct a square with sides having length 1 then its diagonal has length $\sqrt{2}$.

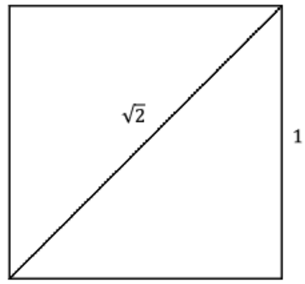

Figure 2.0.6 A construction of $\sqrt{2}$

This is a number which cannot be expressed as the ratio of two integers. That is, it is **irrational**. This has been known since ancient times, but it is still quite disconcerting when first encountered. It seems so counter–intuitive that the intellect rebels. "This can't be right," it says. "That's just crazy!"

Nevertheless it is true and we can prove it is true as follows.

What happens if we suppose that the square root of two *can* be expressed as a ratio of integers? We will show that this leads irrevocably to a conclusion that is manifestly not true.

Suppose $\sqrt{2} = a/b$ where a and b are integers. Suppose further that the fraction a/b is in lowest terms. *This assumption is crucial because if a/b is in lowest terms we know that at most only one of them is even.*

So

$$\frac{a}{b} = \sqrt{2}.$$

Squaring both sides gives:

$$a^2 = 2b^2.$$

Therefore a^2 is even. But if a^2 is even then a must be even also (why?). If a is even then $a = 2k$ for some integer k. Therefore

$$4k^2 = 2b^2 \textit{ or}$$
$$2k^2 = b^2.$$

Therefore b^2 is also even and so b must be even too. But this is impossible. We've just concluded that a and b are both even and this conclusion follows directly from our initial assumption that at most one of them could be even.

This is nonsense. Where is our error? It is not in any single step of our reasoning. That was all solid. Check it again to be sure.

Therefore our error must be in the initial assumption that $\sqrt{2}$ could be expressed as a fraction. That assumption must therefore be false. In other words, $\sqrt{2}$ cannot be so expressed.

Problem 2.0.7 Irrational Numbers. Show that each of the following numbers is irrational:

(a) $\sqrt{3}$

(b) $\sqrt{5}$

(c) $\sqrt[3]{2}$

(d) $i\ (= \sqrt{-1})$

(e) The square root of every positive integer which is not the square of an integer.

□

Problem 2.0.8 Determine if each of the following is always rational or always irrational. Justify your answers.

(a) The sum of two rational numbers.

(b) The sum of two irrational numbers.

(c) The sum of a rational and an irrational number.

□

Problem 2.0.9

(a) Decide if it is possible to have two rational numbers, a and b, such that a^b is irrational. If so, display an example of such a and b. If not, prove that it is not possible.

(b) Decide if it is possible to have two irrational numbers, a and b, such that a^b is rational. If so, display an example of such a and b. If not, prove that it is not possible.

Hint. The number $\left(\sqrt{2}\right)^{\sqrt{2}}$ must be either rational or irrational (why?). What can you conclude in each case?

□

The fact that $\sqrt{2}$ is not rational is cute and interesting but unless, like the Pythagoreans of ancient Greece, you have a strongly held religious conviction that all numbers are rational, it does not seem terribly important. On the other hand, the very existence of $\sqrt{2}$ raises some interesting questions. For example what can the symbol $4^{\sqrt{2}}$ possibly mean? If the exponent were a rational number, say m/n with $m, n \in \mathbb{Z}$, then clearly $4^{m/n} = \sqrt[n]{4^m}$. But since $\sqrt{2} \neq m/n$ for *any* integers m and n how do we interpret $4^{\sqrt{2}}$? Does it have any meaning at all? The more you think about this, the more puzzling the existence of irrational numbers becomes. Suppose for example we reconsider the construction of a line segment of length $\sqrt{2}$. It is clear that the construction works and that we really can build such a line segment. It exists.

Repeat the construction but this time let's put the base side on the rational line.

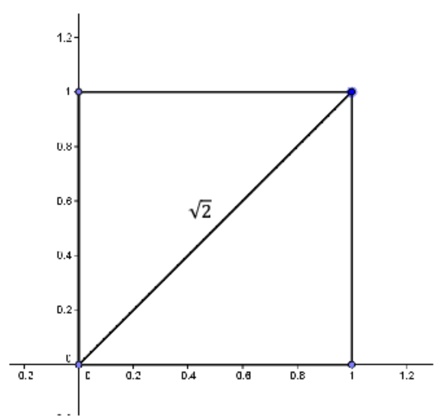

Figure 2.0.10

We know that the diagonal of this square is $\sqrt{2}$ as indicated. And we know that $\sqrt{2}$ is not a rational number.

Now leave the diagonal pinned at $(0, 0)$ but allow it to rotate down so that it coincides with the $x-$axis.

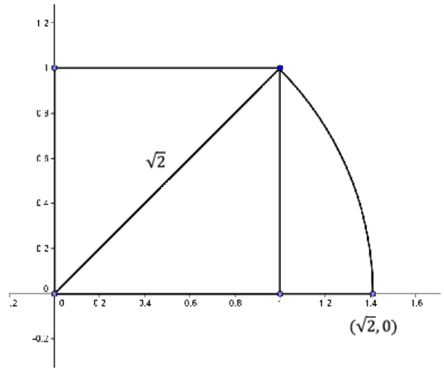

Figure 2.0.11

The end of our diagonal will trace out an arc of the circle with radius $\sqrt{2}$. When the diagonal coincides with the $x-$axis, its endpoint will obviously be the point $(\sqrt{2}, 0)$ as shown.

But wait! We're using the *rational* number line for our $x-$axis. That means the only points on the $x-$axis are those that correspond to rational numbers (fractions). But we know that $\sqrt{2}$ is not rational! Conclusion: There is no point $(\sqrt{2}, 0)$. It simply doesn't exist.

Put differently, there is a hole in the rational number line right where $\sqrt{2}$ should be.

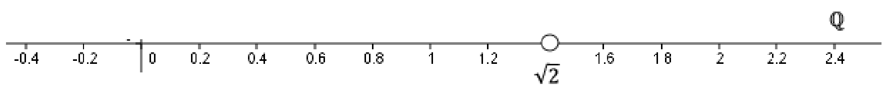

Figure 2.0.12

This is weird, but it's even worse than that!

Recall that between any two rational numbers there is always another. This fact is what led us to represent the rational numbers with a line in the first place.

It's straightforward to show that $\sqrt{3}$, $\sqrt{5}$, etc. are all irrational too. So are π and e, though they aren't as easy to show. It seems that the rational line has a bunch of holes in it. Infinitely many.

And yet, the following theorem is true

Theorem 2.0.13

(a) Between any two distinct real numbers there is a rational number.

(b) Between any two distinct real numbers there is an irrational number.

Both parts of this theorem rely on a judicious use of what is now called the **Archimedean Property** of the Real Number System, which can be formally stated as follows.

Principle 2.0.14 The Archimedean Property. *Given any two positive real numbers, a and b, there is a positive integer, n such that $na > b$.*

Physically this says that we can empty an ocean b with a teaspoon a, provided we are willing to use the teaspoon a large number of times n.

This is such an intuitively straightforward concept that it is easy to accept it without proof. Until the invention of Calculus, and even for some time after that, it was simply assumed. However as the foundational problems posed by the concepts of Calculus were understood and solved we were eventually led to a deeper understanding of the complexities of the real number system and the **Archimedean Property** is no longer taken as an unproved axiom. Rather, it is now understood to be a consequence of other axioms. We will prove this in Section **??**, but for now we will accept it as obviously true just as Archimedes did.

With the invention of Calculus, mathematicians of the seventeenth century began to use objects which didn't satisfy the **Archimedean Property** (in fact, so did Archimedes). As we shall see in the next chapter, when Leibniz wrote the first paper on his version of the Calculus, he followed this practice by explicitly laying out rules for manipulating infinitely small quantities (infinitesimals). These were taken to be actual numbers which are not zero and yet smaller than any real number. The notation he used was $\mathrm{d}x$ (an infinitely small displacement in the x direction), and $\mathrm{d}y$ (an infinitely small displacement in the y direction). These symbols should look familiar to you. They are the same $\mathrm{d}y$ and $\mathrm{d}x$ used to form the derivative symbol $\frac{\mathrm{d}y}{\mathrm{d}x}$ that you learned about in Calculus.

Mathematicians of the seventeenth and eighteenth centuries made amazing scientific and mathematical progress exploiting these infinitesimals, even though they were foundationally suspect. No matter how many times you add the infinitesimal $\mathrm{d}x$ to itself the result will not be bigger than, say 10^{-1000}, which is very bizarre.

When foundational issues came to the forefront, infinitesimals fell somewhat out of favor. You probably didn't use them very much in Calculus. Most of the time you probably used the prime notation, $f'(x)$ introduced by Lagrange in the eighteenth century. Some of the themes in this book are: Why differentials fell out of favor, what were they replaced with and how the modern notations you learned in Calculus evolved over time.

To conclude this aside on the **Archimedean Property**, the idea of infinitesimals was revisited in the twentieth century by the logician Abraham Robinson in [12]. Robinson was able to put the idea of infinitesimals on a solid logical foundation. But in the 18th century, the existence of infinitesimal numbers was shaky to say the very least. However this did not prevent mathematicians from successfully exploiting these infinitely small quantities.

We will come back to this saga in later chapters, but for now we return to Theorem 2.0.13.

Sketch of Proof. We will outline the proof of part (a) of Theorem 2.0.13 and indicate how it can be used to prove part b.

Let α and β be real numbers with $\alpha > \beta$. There are two cases.

- *Case 1:.*

  $\alpha - \beta > 1$. In this case there is at least one integer between α and β. Since integers are rational we are done.

- *Case 2:.*

  $\alpha - \beta \leq 1$. In this case, by the **Archimedean Property** there is a positive integer, say n, such that $n(\alpha - \beta) = n\alpha - n\beta > 1$. Now there will be an integer between $n\alpha$ and $n\beta$. You should now be able to find a rational number between α and β.

■

For part (b), divide α and β by any positive irrational number and apply part a. There are a couple of details to keep in mind. These are considered in the following problem.

Problem 2.0.15

(a) Prove that the product of a nonzero rational number and an irrational number is irrational.

(b) Use the result of part (a) to prove Theorem 2.0.13.

□

As a practical matter, the existence of irrational numbers isn't really very important. In light of Theorem 2.0.13, any irrational number can be approximated arbitrarily closely by a rational number. So if we're designing a bridge and $\sqrt{2}$ is needed we just use 1.414 instead. The error introduced is less than $0.001 = 1/1000$ so it probably doesn't matter.

But from a theoretical point of view this is devastating. When Calculus was invented, the rational numbers were suddenly not up to the task of justifying the concepts and operations we needed to work with.

Newton explicitly founded his version of Calculus on the assumption that we can think of variable quantities as being generated by a continuous motion. If our number system has holes in it such continuous motion is impossible because we have no way to jump over the gaps. So Newton simply postulated that there were no holes. He filled in the hole where $\sqrt{2}$ should be. He simply said, yes there is a number there called $\sqrt{2}$ and he did the same with all of the other holes.

To be sure there is no record of Newton explicitly saying, "Here's how I'm going to fill in the holes in the rational number line." Along with everyone else at the time, he simply assumed there were no holes and moved on. It took about 200 years of puzzling and arguing over the contradictions, anomalies and paradoxes to work out the consequences of that apparently simple assumption. The task may not yet be fully accomplished, but by the 20th century the properties of the real number system ($\mathbb{R}$) as an extension of the rational number system ($\mathbb{Q}$) were well understood. Here are both systems visualized as lines:

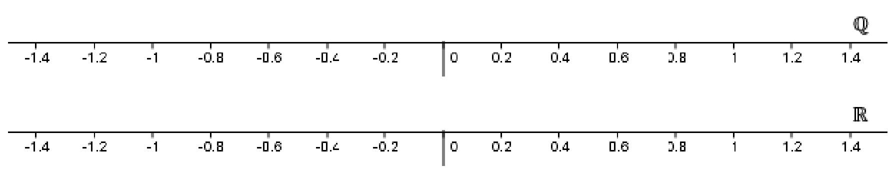

Figure 2.0.16 $\mathbb{R}$ and $\mathbb{Q}$

Impressive, no?

The reason they look alike, except for the labels $\mathbb{R}$ and $\mathbb{Q}$ of course, is that our ability to draw sketches of the objects we're studying utterly fails when we try to sketch $\mathbb{R}$, as different from $\mathbb{Q}$. All of the holes in $\mathbb{Q}$ really are there, but the non-holes are packed together so closely that we can't separate them in a drawing. This inability to sketch the objects we study will be a frequent source of frustration.

Of course, this will not stop us from drawing sketches. When we do, our imaginations will save us because it is possible to imagine $\mathbb{Q}$ as distinct from $\mathbb{R}$. But put away the idea that a sketch is an accurate representation of anything. At best our sketches will only be aids to the imagination.

So, at this point we will simply assume the existence of the real numbers. We will assume also that they have all of the properties that we are used to. This is perfectly acceptable as long as we make our assumptions explicit. However we need to be aware that, so far, the existence and properties of the real numbers is an assumption that has not been logically derived. Any time we make an assumption we need to be prepared to either abandon it completely if we find that it leads to nonsensical results, or to re-examine the assumption in the light of these results to see if we can find another assumption that subsumes the first and explains the (apparently) nonsensical results.

Chapter 3

Calculus in the 17th and 18th Centuries

3.1 Newton and Leibniz Get Started

Leibniz' Calculus Rules

The rules for Calculus were first laid out in the 1684 paper *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales, quantitates moratur, et singulare pro illi calculi genus* (A New Method for Maxima and Minima as Well as Tangents, Which is Impeded Neither by Fractional Nor by Irrational Quantities, and a Remarkable Type of Calculus for This) written by Gottfried Wilhelm Leibniz (1646–1716). Leibniz started with subtraction. That is, if x_1 and x_2 are very close together then their difference, $\Delta x = x_2 - x_1$, is very small. He expanded this idea to say that if x_1 and x_2 are *infinitely* close together (but still distinct) then their difference $\mathrm{d}x$, is infinitesimally small (but not zero).

Figure 3.1.1 Gottfried Wilhelm Leibniz

Calculus Differentialis. This translates, loosely, as the "Calculus of Differences".

This idea is logically very suspect and Leibniz knew it. But he also knew that when he used his *calculus differentialis* he was getting correct answers to some very hard problems. So he persevered.

Leibniz called both Δx and $\mathrm{d}x$ differentials (Latin for difference) because he thought of them as, essentially, the same thing. Over time it has become customary to refer to the infinitesimal $\mathrm{d}x$ as a differential, and to reserve the word difference and the notation Δx for the finite case. This is why Calculus is often called *Differential Calculus*.

In his paper Leibniz gave rules for dealing with these infinitely small differentials. Specifically, given a variable quantity x, $\mathrm{d}x$ represented an infinitesimal change in x. Differentials are related via the slope of the tangent line to a curve. That is, if $y = f(x)$, then $\mathrm{d}y$ and $\mathrm{d}x$ are related by

$$\mathrm{d}y = \text{ (slope of the tangent line) } \cdot \mathrm{d}x.$$

Leibniz then divided by $\mathrm{d}x$ giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \text{ (slope of the tangent line).}$$

The elegant and expressive notation Leibniz invented was so useful that it has been retained through the years despite some profound changes in the underlying concepts. For example, Leibniz and his contemporaries would have viewed the symbol $\frac{\mathrm{d}y}{\mathrm{d}x}$ as an actual quotient of infinitesimals, whereas today we define it via the limit concept first suggested by Newton.

As a result Leibniz' rules governing his differentials are very modern in appearance:

$$\begin{aligned} \mathrm{d}(\text{ constant }) &= 0 \\ \mathrm{d}(z - y + w + x) &= \mathrm{d}z - \mathrm{d}y + \mathrm{d}w + \mathrm{d}x \\ \mathrm{d}(xv) &= x\,\mathrm{d}v + v\,\mathrm{d}x \\ \mathrm{d}\left(\frac{v}{y}\right) &= \frac{y\,\mathrm{d}v - v\,\mathrm{d}y}{yy} \end{aligned}$$

and, when a is an integer:

$$\mathrm{d}(x^a) = ax^{a-1}\,\mathrm{d}x.$$

Leibniz states these rules without proof: ". . . the demonstration of all this will be easy to one who is experienced in such matters . . .". Mathematicians in Leibniz's day would have been expected to understand intuitively that if c is a constant, then

$$\mathrm{d}(c) = c - c = 0.$$

Likewise, $\mathrm{d}(x + y) = \mathrm{d}x + \mathrm{d}y$ is really an extension of

$$\underbrace{(x_2 + y_2) - (x_1 + y_1)}_{\Delta(x+y)} = \underbrace{(x_2 - x_1)}_{\Delta x} + \underbrace{(y_2 - y_1)}_{\Delta y}.$$

Leibniz's Approach to the Product Rule

The explanation of the product rule using differentials is a bit more involved, but Leibniz expected that mathematicans would be fluent enough to derive it. The product $p = xv$ can be thought of as the area of the following rectangle

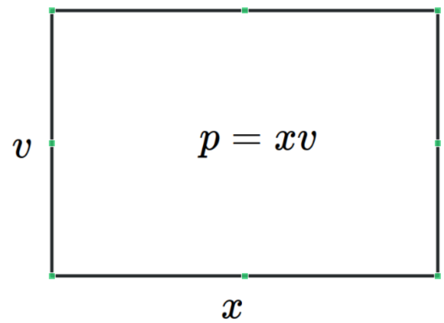

Figure 3.1.2

With this in mind, $\mathrm{d}p = \mathrm{d}(xv)$ can be thought of as the change in area when x is changed by $\mathrm{d}x$ and v is changed by $\mathrm{d}v$. This can be seen as the L shaped region in the following drawing.

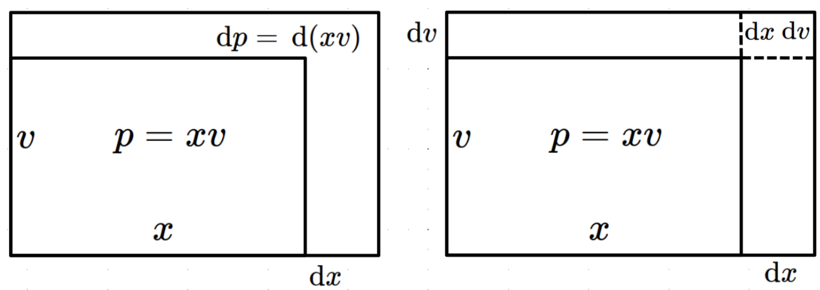

Figure 3.1.3

By dividing the L shaped region into 3 rectangles we obtain

$$\mathrm{d}(xv) = x\,\mathrm{d}v + v\,\mathrm{d}x + \mathrm{d}x\ \mathrm{d}v. \tag{I.1}$$

Even though $\mathrm{d}x$ and $\mathrm{d}v$ are infinitely small, Leibniz reasoned that $\mathrm{d}x\ \mathrm{d}v$ is *even more* infinitely small (quadratically infinitely small?) compared to $x\,\mathrm{d}v$ and $v\,\mathrm{d}x$ and can thus be ignored leaving

$$\mathrm{d}(xv) = x\,\mathrm{d}v + v\,\mathrm{d}x.$$

You should feel some discomfort at the idea of simply tossing the product $\mathrm{d}x\ \mathrm{d}v$ aside because it is "comparatively small." This means you have been well trained, and have thoroughly internalized Newton's dictum [10]: "The smallest errors may not, in mathematical matters, be scorned." It is logically untenable to toss aside an expression just because it is small. Even less so should we be willing to ignore an expression on the grounds that it is "infinitely smaller" than another quantity which is itself "infinitely small."

Newton and Leibniz both knew this as well as we do. But they also knew that their methods worked. They gave verifiably correct answers to problems which had, heretofore, been completely intractable. It is the mark of their genius that both men persevered in spite of the very evident difficulties their methods entailed.

Newton's Approach to the Product Rule

Isaac Newton's (1643–1727) approach to Calculus — his 'Method of Fluxions' — depended fundamentally on motion. He conceived of variables (fluents) as changing (flowing or fluxing) in time. The rate of change of a fluent he called a fluxion. As a theoretical foundation both Leibniz's and Newton's approaches have fallen out of favor, although both are still universally used as a conceptual approach, a "way of thinking," about the ideas of Calculus.

Figure 3.1.4 Isaac Newton

In *Philosophiae naturalis principia mathematica* (this is usually shortened to *Principia*) Newton "proved" the Product Rule as follows: Let x and v be "flowing quantites" and consider the rectangle, R, whose sides are x and v. R is also a flowing quantity and we wish to find its fluxion (derivative) at any time.

First we increment x and v by the half–increments $\frac{\Delta x}{2}$ and $\frac{\Delta v}{2}$ respectively. Then the corresponding half–increment of R is

$$\frac{\Delta R}{2} = \left(x + \frac{\Delta x}{2}\right)\left(v + \frac{\Delta v}{2}\right) = xv + x\frac{\Delta v}{2} + v\frac{\Delta x}{2} + \frac{\Delta x \Delta v}{4}. \qquad \text{(I.2)}$$

Now decrement x and v by the same amounts:

$$-\frac{\Delta R}{2} = \left(x - \frac{\Delta x}{2}\right)\left(v - \frac{\Delta v}{2}\right) = xv - x\frac{\Delta v}{2} - v\frac{\Delta x}{2} + \frac{\Delta x \Delta v}{4}. \qquad \text{(I.3)}$$

Subtracting equation (I.3) from equation (I.2) gives

$$\Delta R = x\Delta v + v\Delta x$$

which is the total change of $R = xv$ over the intervals Δx and Δv and also recognizably the Product Rule.

This argument is no better than Leibniz's as it relies heavily on the number $1/2$ to make it work. If we take any other increments in x and v whose total lengths are Δx and Δv it will simply not work. Try it and see.

In Newton's defense, he wasn't really trying to justify his mathematical methods in the *Principia*. His attention was on physics, not math, so he was really just trying to give a convincing demonstration of his methods. You may decide for yourself how convincing his demonstration is.

Notice that there is no mention of limits of difference quotients or derivatives. In fact, the term derivative was not coined until 1797, by Lagrange as we will see in Section 4.1 . In a sense, these topics were not necessary at the time, as Leibniz and Newton both assumed that the curves they dealt with had tangent lines and, in fact, Leibniz explicitly used the tangent line to relate two differential quantities. This was consistent with the thinking of the time and for the duration of this chapter we will also assume that all quantities are differentiable. As we will see later this assumption leads to difficulties.

Both Newton and Leibniz were satisfied that their Calculus provided answers that agreed with what was known at the time. For example the formulas

$$\mathrm{d}\left(x^2\right) = \mathrm{d}\left(xx\right) = x\,\mathrm{d}x + x\,\mathrm{d}x = 2x\,\mathrm{d}x$$

and

$$\mathrm{d}\left(x^3\right) = \mathrm{d}\left(x^2 x\right) = x^2\,\mathrm{d}x + x\,\mathrm{d}\left(x^2\right) = x^2 + x\left(2x\,\mathrm{d}x\right) = 3x^2\,\mathrm{d}x,$$

were results that had been derived by others using other methods. ways.

Problem 3.1.5 Assume n is a positive integer.

(a) Use Leibniz' product rule $\mathrm{d}\left(xv\right) = x\,\mathrm{d}v + v\,\mathrm{d}x$ to show that

$$\mathrm{d}\left(x^n\right) = nx^{n-1}\,\mathrm{d}x$$

(b) Suppose $y = x^{-1} = \frac{1}{x}$, Use Leibniz' product rule to show

$$dy = -1x^{-2}\,\mathrm{d}x$$

(c) Use the product rule and the result of part (b) to derive the quotient rule

$$d\left(\frac{v}{x}\right) = \frac{x\,\mathrm{d}v - v\,\mathrm{d}x}{x^2}$$

(d) Use the quotient rule to show that

$$d\left(x^{-n}\right) = -nx^{-n-1}\,\mathrm{d}x$$

□

Problem 3.1.6 Suppose $y = x^{\frac{p}{q}}$ with $q \neq 0$, where p and q are integers. Show that

$$\mathrm{d}y = \mathrm{d}\left(x^{\frac{p}{q}}\right) = \frac{p}{q}x^{\frac{p}{q}-1}\,\mathrm{d}x.$$

□

To prove the worth of his Calculus Leibniz also provided applications. As an example he derived Snell's Law of Refraction from his Calculus rules as follows.

Figure 3.1.7 Willebrord Snell (1580–1626)

Given that light travels through air at a speed of v_a and travels through water at a speed of v_w the problem is to find the fastest path from point A to point B.

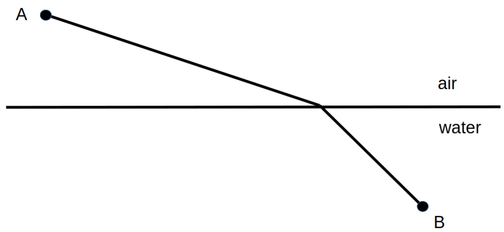

Figure 3.1.8

According to Fermat's Principle of Least Time, this fastest path is the one that light will travel.

Using the fact that Time $= \frac{\text{Distance}}{\text{Velocity}}$ and the labeling in the picture below we can obtain a formula for the time T it takes for light to travel from A to B.

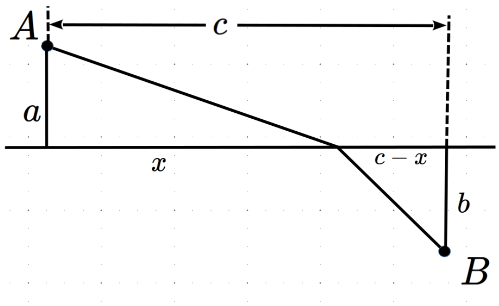

Figure 3.1.9

$$T = \frac{\sqrt{x^2+a^2}}{v_a} + \frac{\sqrt{(c-x)^2+b^2}}{v_w}$$

Using the rules of Leibniz's Calculus, we obtain

$$\begin{aligned} \mathrm{d}T &= \left[\frac{1}{v_a}\frac{1}{2}\left(x^2+a^2\right)^{-\frac{1}{2}}(2x) + \frac{1}{v_w}\frac{1}{2}((c-x)^2+b^2)^{-\frac{1}{2}}(2(c-x)(-1))\right]\mathrm{d}x \\ &= \left[\frac{1}{v_a}\frac{x}{\sqrt{x^2+a^2}} - \frac{1}{v_w}\frac{c-x}{\sqrt{(c-x)^2+b^2}}\right]\mathrm{d}x. \end{aligned}$$

Using the fact that at the minimum value for T, $\mathrm{d}T = 0$, we see that the fastest path from A to B must satisfy

$$\frac{1}{v_a}\frac{x}{\sqrt{x^2+a^2}} = \frac{1}{v_w}\frac{c-x}{\sqrt{(c-x)^2+b^2}}.$$

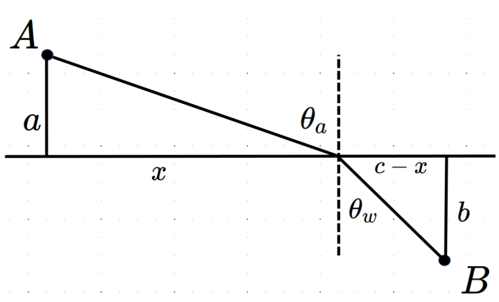

Figure 3.1.10

From Figure 3.1.10 we see that the path that light travels must satisfy $\frac{\sin\theta_a}{v_a} = \frac{\sin\theta_w}{v_w}$ which is Snell's Law.

To compare 18th century and modern techniques we will consider Johann Bernoulli's (1667–1748) solution of the Brachistochrone Problem. In 1696, Bernoulli posed and solved, the Brachistochrone problem: To find the shape of a frictionless wire joining points A and B so that the time it takes for a bead to slide down under the force of gravity is as small as possible.

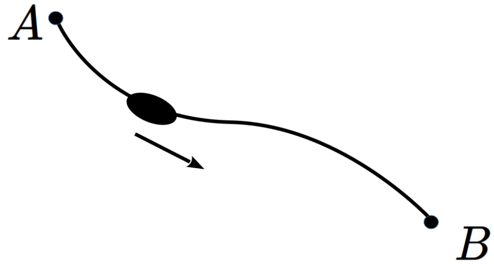

Figure 3.1.11

Bernoulli posed this "path of fastest descent" problem to challenge the mathematicians of Europe and used his solution to demonstrate the power of Leibniz' Calculus as well as his own ingenuity.

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise. [11]

Figure 3.1.12 Johann Bernoulli

In addition to Johann's, solutions were obtained from Newton, Leibniz, Johann's brother Jacob Bernoulli, and the Marquis de l'Hopital [15]. At the time there was an ongoing and very vitriolic controversy raging over whether Newton or Leibniz had been the first to invent Calculus. As an advocate for Leibniz, Bernoulli did not believe Newton would be able to solve the problem using his fluxions. So Bernoulli's this challenge was in part an attempt to embarrass Newton. However Newton solved it easily.

At this point in his life Newton had all but quit science and mathematics and was fully focused on his administrative duties as Master of the Mint. Due in part to rampant counterfeiting, England's money had become severely devalued and the nation was on the verge of economic collapse. The solution was to recall all of the existing coins, melt them down, and strike new ones. As Master of the Mint this job fell to Newton [8]. As you might imagine this was a rather Herculean task. Nevertheless, according to his niece (and housekeeper):

> When the problem in 1696 was sent by Bernoulli–Sir I.N. was in the midst of the hurry of the great recoinage and did not come home till four from the Tower very much tired, but did not sleep till he had solved it, which was by four in the morning.

He is reported to have complained, "I do not love . . . to be . . . teezed by forreigners about Mathematical things" [2].

Newton submitted his solution anonymously, presumably to avoid more controversy. Nevertheless the methods he used were so distinctively Newton's that Bernoulli is said to have exclaimed "*Tanquam ex ungue leonem.*"

Translation: *Tanquam ex ungue leonem.* "I know the lion by his claw."

Newton's solution was clever but it doesn't provide any insights we'll be interested in so we will focus on Bernoulli's ingenious solution which starts, interestingly enough, with Snell's Law of Refraction. He begins by considering the stratified medium in the following figure, where an object travels with velocities $v_1, v_2, v_3, \ldots$ in the various layers.

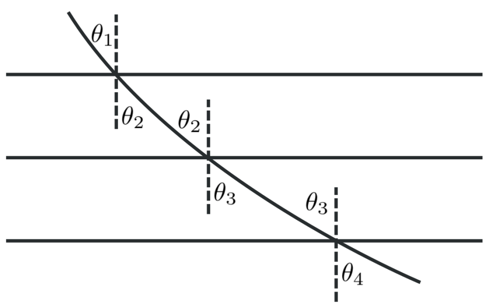

Figure 3.1.13

By repeatedly applying Snell's Law he concluded that the fastest path must satisfy

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} = \frac{\sin\theta_3}{v_3} = \cdots.$$

In other words, the ratio of the sine of the angle that the curve makes with the vertical and the speed remains constant along this fastest path.

If we think of a continuously changing medium as stratified into infinitesimal layers and extend Snell's law to an object whose speed is constantly changing, then along the fastest path, the ratio of the sine of the angle that the curve's tangent makes with the vertical, α, and the speed, v, must remain constant,

$$\frac{\sin\alpha}{v} = c$$

as in Figure 3.1.14 below.

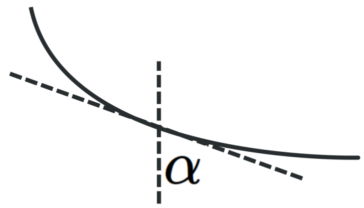

Figure 3.1.14

If we include the horizontal and vertical axes and let P denote the position of the bead at a particular time then we have the following picture.

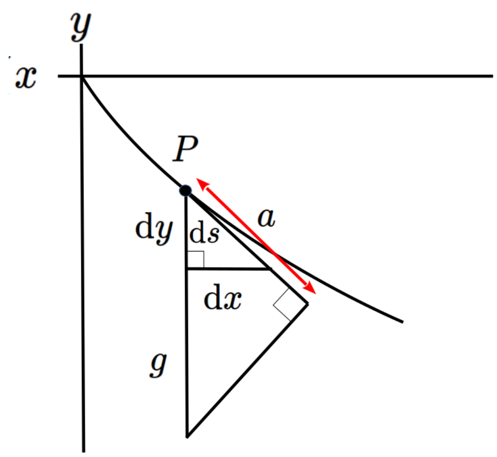

Figure 3.1.15

In Figure 3.1.15, s denotes the length that the bead has traveled down to point P (that is, the arc length of the curve from the origin to that point) and a denotes the tangential component of the acceleration due to gravity g. Since acceleration is the rate of change of velocity with respect to time we see that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a.$$

To get a sense of how physical problems were approached using Leibniz's Calculus we will use the above equation to show that $v = \sqrt{2gy}$.

By similar triangles we have $\frac{a}{g} = \frac{\mathrm{d}y}{\mathrm{d}s}$. As a student of Leibniz, Bernoulli would have regarded $\frac{\mathrm{d}y}{\mathrm{d}s}$ as a fraction so

$$a\,\mathrm{d}s = g\,\mathrm{d}y$$

and since acceleration is the rate of change of velocity we have

$$\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}s = g\,\mathrm{d}y. \tag{I.4}$$

In the seventeenth, eighteenth, and even well into the nineteenth centuries, European mathematicians regarded $\mathrm{d}v$, $\mathrm{d}t$, and $\mathrm{d}s$ as infinitesimally small numbers which nevertheless obey all of the usual rules of algebra. Thus they would simply rearrange equation (I.4), to get

$$\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}v = g\,\mathrm{d}y.$$

Since $\frac{\mathrm{d}s}{\mathrm{d}t}$ is the rate of change of position with respect to time it is, in fact, the velocity of the bead. That is

$$v\,\mathrm{d}v = g\,\mathrm{d}y.$$

Bernoulli would have interpreted this as a statement that two rectangles of height v and g, with respective widths $\mathrm{d}v$ and $\mathrm{d}y$ have equal area. Summing (integrating) all such rectangles we g et:

$$\int v\,\mathrm{d}v = \int g\,\mathrm{d}y$$
$$\frac{v^2}{2} = gy$$

or

$$v = \sqrt{2gy}. \tag{I.5}$$

You are undoubtedly uncomfortable with the cavalier manipulation of infinitesimal quantities you've just witnessed, so we'll pause for a moment now to compare a modern development of equation (I.5) to Bernoulli's. As before we begin with the equation:

$$\frac{a}{g} = \frac{\mathrm{d}y}{\mathrm{d}s}$$
$$a = g\frac{\mathrm{d}y}{\mathrm{d}s}.$$

Moreover, since acceleration is the derivative of velocity this is the same as:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\frac{\mathrm{d}y}{\mathrm{d}s}.$$

Now observe that by the Chain Rule $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$. The physical interpretation of this formula is that velocity will depend on s, how far down the wire the bead has moved, but that the distance traveled will depend on how much time has elapsed. Therefore

$$\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = g\frac{\mathrm{d}y}{\mathrm{d}s}$$

or

$$\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}s} = g\frac{\mathrm{d}y}{\mathrm{d}s}$$

and since $\frac{\mathrm{d}s}{\mathrm{d}t} = v$ we have

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = g\frac{\mathrm{d}y}{\mathrm{d}s}$$

Integrating both sides with respect to s gives:

$$\int v\frac{\mathrm{d}v}{\mathrm{d}s}ds = g\int\frac{\mathrm{d}y}{\mathrm{d}s}ds$$
$$\int vdv = g\int dy$$

and integrating gives

$$\frac{v^2}{2} = gy$$

as before.

In effect, in the modern formulation we have traded the simplicity and elegance of differentials for a comparatively cumbersome repeated use of the

Chain Rule. No doubt you noticed when taking Calculus that in the differential notation of Leibniz, the Chain Rule looks like we are simply "canceling" a factor in the top and bottom of a fraction: $\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$. This is because for 18th century mathematicians, that is exactly what it was, and Leibniz designed his notation to reflect that viewpoint.

To put it another way, 18th century mathematicians wouldn't have recognized a need for what we call the Chain Rule because this operation was a triviality for them. Just reduce the fraction. This begs the question: Why did we abandon such a clear and simple interpretation of our symbols in favor of the comparatively more cumbersome modern interpretation? This is one of the questions we will try to answer in this course.

Returning to the Brachistochrone problem we observe that

$$\frac{\sin\alpha}{v} = c$$

and since $\sin\alpha = \frac{dx}{ds}$ we see that

$$\begin{aligned}
\frac{\frac{dx}{ds}}{\sqrt{2gy}} &= c \\
\frac{dx}{\sqrt{2gy(ds)^2}} &= c \\
\frac{dx}{\sqrt{2gy\left[(dx)^2+(dy)^2\right]}} &= c. \qquad \text{(I.6)}
\end{aligned}$$

Bernoulli was then able to solve this differential equation.

Problem 3.1.16 Show that the equations $x = \frac{\phi - \sin\phi}{4gc^2}$, $y = \frac{1-\cos\phi}{4gc^2}$ satisfy equation equation (I.6). Bernoulli recognized this solution to be an inverted cycloid, the curve traced by a fixed point on a circle as the circle rolls along a horizontal surface. □

This illustrates the state of Calculus in the late 1600s and early 1700s; the foundations of the subject were a bit shaky but there was no denying its power.

3.2 Power Series as Infinite Polynomials

Applied to polynomials, the rules of differential and integral Calculus are straightforward. Indeed, differentiating and integrating polynomials represent some of the easiest tasks in a Calculus course. For example, computing $\int (7 - x + x^2)\,\mathrm{d}x$ is relatively easy compared to computing $\int \sqrt[3]{1+x^3}\,\mathrm{d}x$. Unfortunately, not all functions can be expressed as a polynomial. For example, $f(x) = \sin x$ cannot be since a polynomial has only finitely many roots and the sine function has infinitely many roots, namely $\{n\pi \mid n \in \mathbb{Z}\}$. A standard technique in the 18th century was to write such functions as an "infinite polynomial," or what today we refer to as a **power series**. Unfortunately an "infinite polynomial" is a much more subtle object than a mere polynomial, which by definition is finite. For now we will not concern ourselves with these subtleties. Instead we will follow the example of our forebears and manipulate all "polynomial–like" objects (finite or infinite) as if they are polynomials.

Definition 3.2.1 A **power series centered at a** is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots.$$

◊

Thus a power series centered around zero has the form

$$\sum_{n=0}^{\infty} a_n x^n.$$

In this section we will focus on power series centered around zero. In the next section we will look at power series centered about points other than zero. 0.

A useful comment on notation:. The most advantageous way to represent a power series is using summation notation since there can be no doubt about the pattern in the terms. After all, this notation contains a formula for the general term. However, there are instances where summation notation is not practical. In these cases, it is acceptable to indicate the sum by supplying the first few terms and using ellipses (the three dots). If this is done, then enough terms must be included to make the pattern clear to the reader.

Returning to our definition of a power series, consider the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots.$$

If we multiply this power series by $(1-x)$, we obtain

$$\begin{aligned}(1-x)(1+x+x^2+\cdots) &= (1+x+x^2+\cdots)\\ &\qquad -(x+x^2+x^3+\cdots)\\ &= 1.\end{aligned}$$

Dividing by $1-x$ gives us the power series representation

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \tag{I.7}$$

which is known as the **Geometric Series**.

For each value of x a **power series** reduces to a different ordinary (numerical) **series**. For example, if we substitute $x = \frac{1}{10}$ into the left side of equation (I.7), we obtain the (numerical) **series**

$$\begin{aligned}1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots &= 1 + 0.1 + 0.01 + 0.001 + 0.0001 + \cdots\\ &= 1.1111\cdots.\end{aligned}$$

But substituting into the right side yields

$$\frac{1}{1-\frac{1}{10}} = \frac{10}{9}.$$

If these computations are valid then it must be that $1.111\ldots = \frac{10}{9}$, which seems weird, but you can verify it by entering $\frac{10}{9}$ into any calculator.

If $1.111\ldots = \frac{10}{9}$, then subtracting 1 from both sides and multiplying by 3 gives

$$0.333\ldots = \frac{1}{3}$$

which has probably come up on your calculator more than once. But what seems wierder still is that if we multiply by 3 again we get

$$0.999\ldots = 1,$$

which seems like nonsense. It simply can't be true, can it? What do you think? Is $0.999\ldots = 1$ or that just nonsense? Either way it is clear that the real numbers ($\mathbb{R}$) hide deeper mysteries than the irrationality of $\sqrt{2}$.

There are other issues with these formal manipulations too. Substituting $x = 1$ or $x = 2$ into equation (I.7) yields the questionable results

$$\frac{1}{0} = 1 + 1 + 1 + \cdots \text{ and } \frac{1}{-1} = 1 + 2 + 2^2 + \cdots.$$

When we say "formal manipulations" we mean that we will perform purely algebraic operations on an given expression without concerning ourselves (much) about whether the operations make sense in context. We will formalize them in Chapter **??**

A power series representation of our function seems to work in some cases, but not in others. Obviously we are missing something important here, though it may not be clear exactly what. For now, we will continue to follow the example of our 18th century predecessors. That is, for the rest of this section we will use formal manipulations to obtain and use power series representations of various functions. Keep in mind that this is all highly suspect until we can resolve problems like those we've just seen.

Power series became an important tool in analysis in the 1700s. By representing various functions as power series they could be dealt with as if they were (infinite) polynomials. The following is an example.

Example 3.2.2 Solve the following Initial Value problem: Find $y(x)$ given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, \qquad\qquad y(0) = 1.$$

A few seconds of thought should convince you that the solution of this problem is $y(x) = e^x$. We will ignore this for now in favor of emphasising the technique.

Assuming the solution can be expressed as a power series we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Differentiating gives us

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots.$$

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ we see that

$$a_1 = a_0\,,\ 2a_2 = a_1\,,\ 3a_3 = a_2\,,\ \ldots,\ na_n = a_{n-1}\,,\ldots.$$

This leads to the relationship

$$a_n = \frac{1}{n}a_{n-1} = \frac{1}{n(n-1)}a_{n-2} = \cdots = \frac{1}{n!}a_0.$$

Thus the power series solution of the differential equation is

$$y = \sum_{n=0}^{\infty} \frac{a_0}{n!}x^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

Using the initial condition $y(0) = 1$, we get $1 = a_0(1 + 0 + \frac{1}{2!}0^2 + \cdots) = a_0$. Thus the solution to the initial problem is $y = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. Let's call this function $E(x)$. Then by definition

$$E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots . \tag{I.8}$$

□

Let's examine some properties of this function. The first property is clear from the definition.

Property 1. $E(0) = 1$

Property 2. $E(x+y) = E(x)E(y)$.

To see this we multiply the two series together, so we have

$$\begin{aligned}
E(x)E(y) &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!} y^n\right) \qquad \text{(I.9)}\\
&= \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(\frac{y^0}{0!} + \frac{y^1}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots\right)\\
&= \frac{x^0}{0!}\frac{y^0}{0!} + \frac{x^0}{0!}\frac{y^1}{1!} + \frac{x^1}{1!}\frac{y^0}{0!} + \frac{x^0}{0!}\frac{y^2}{2!} + \frac{x^1}{1!}\frac{y^1}{1!} + \frac{x^2}{2!}\frac{y^0}{0!}\\
&\quad + \frac{x^0}{0!}\frac{y^3}{3!} + \frac{x^1}{1!}\frac{y^2}{2!} + \frac{x^2}{2!}\frac{y^1}{1!} + \frac{x^3}{3!}\frac{y^0}{0!} + \dots\\
&= \frac{x^0}{0!}\frac{y^0}{0!} + \left(\frac{x^0}{0!}\frac{y^1}{1!} + \frac{x^1}{1!}\frac{y^0}{0!}\right)\\
&\quad + \left(\frac{x^0}{0!}\frac{y^2}{2!} + \frac{x^1}{1!}\frac{y^1}{1!} + \frac{x^2}{2!}\frac{y^0}{0!}\right)\\
&\quad\quad + \left(\frac{x^0}{0!}\frac{y^3}{3!} + \frac{x^1}{1!}\frac{y^2}{2!} + \frac{x^2}{2!}\frac{y^1}{1!} + \frac{x^3}{3!}\frac{y^0}{0!}\right) + \dots\\
&= \frac{1}{0!} + \frac{1}{1!}\left(\frac{1!}{0!1!}x^0y^1 + \frac{1!}{1!0!}x^1y^0\right)\\
&\quad + \frac{1}{2!}\left(\frac{2!}{0!2!}x^0y^2 + \frac{2!}{1!1!}x^1y^1 + \frac{2!}{2!0!}x^2y^0\right)\\
&\quad\quad + \frac{1}{3!}\left(\frac{3!}{0!3!}x^0y^3 + \frac{3!}{1!2!}x^1y^2 + \frac{3!}{2!1!}x^2y^1 + \frac{3!}{3!0!}x^3y^0\right) + \dots\\
&= \frac{1}{0!} + \frac{1}{1!}\left(\binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0\right)\\
&\quad + \frac{1}{2!}\left(\binom{2}{0}x^0y^2 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^2y^0\right)\\
&\quad\quad + \frac{1}{3!}\left(\binom{3}{0}x^0y^3 + \binom{3}{1}x^1y^2 + \binom{3}{2}x^2y^1 + \binom{3}{3}x^3y^0\right) + \dots\\
&= \frac{1}{0!} + \frac{1}{1!}(x+y)^1 + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \dots
\end{aligned}$$

so that, finally

$$E(x)E(y) = E(x+y). \tag{I.10}$$

Property 3. If m is a positive integer then $E(mx) = (E(x))^m$. In particular, $E(m) = (E(1))^m$.

Problem 3.2.3 Prove Property 3. □

Property 4. $E(-x) = \frac{1}{E(x)} = (E(x))^{-1}$.

Problem 3.2.4 Prove Property 4. □

Property 5. If n is an integer with $n \neq 0$, then $E(\frac{1}{n}) = \sqrt[n]{E(1)} = (E(1))^{1/n}$.

Problem 3.2.5 Prove Property 5. $\square$

Property 6. If m and n are integers with $n \neq 0$, then $E\left(\frac{m}{n}\right) = (E(1))^{m/n}$.

Problem 3.2.6 Prove Property 6. $\square$

Definition 3.2.7 Let $E(1)$ be denoted by the number e. Using the power series $e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$, we can approximate e to any degree of accuracy. In particular $e \approx 2.71828$. $\Diamond$

In light of Property 6, we see that for any rational number r, $E(r) = e^r$. Not only does this give us the power series representation $e^r = \sum_{n=0}^{\infty} \frac{1}{n!} r^n$ for any rational number r, but it gives us a way to define e^x for irrational values of x as well. That is, we can define

$$e^x = E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

for any real number x.

As an illustration, we now have $e^{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sqrt{2}\right)^n$. The expression $e^{\sqrt{2}}$ is meaningless if we try to interpret it as one irrational number raised to another. What does it mean to raise anything to the $\sqrt{2}$ power? However the power series $\sum_{n=0}^{\infty} \frac{1}{n!}\left(\sqrt{2}\right)^n$ does seem to have meaning and it can be used to extend the exponential function to irrational exponents. In fact, defining the exponential function via this power series answers the question we raised in Chapter 2: What does $4^{\sqrt{2}}$ mean? It means

$$4^{\sqrt{2}} = e^{\sqrt{2}\log 4} = \sum_{n=0}^{\infty} \frac{(\sqrt{2}\log 4)^n}{n!}.$$

This may seem to be the long way around just to define something as simple as exponentiation. But that is a fundamentally misguided attitude. Exponentiation only *seems* simple because we've always thought of it as repeated multiplication (in $\mathbb{Z}$) or root–taking (in $\mathbb{Q}$). When we expand the operation to the real numbers this simply can't be the way we interpret something like $4^{\sqrt{2}}$. How do you take the product of $\sqrt{2}$ copies of 4? The concept is meaningless. What we need is an interpretation of $4^{\sqrt{2}}$ which is consistent with, say $4^{3/2} = \left(\sqrt{4}\right)^3 = 8$. This is exactly what the power series representation of e^x provides.

We also have a means of computing integrals as power series. For example, the famous "bell shaped" curve given by the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is of vital importance in statistics as it must be integrated to calculate probabilities. The power series we developed gives us a method of integrating this function. For example, we have

$$\begin{aligned}
\int_{x=0}^{b} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{x=0}^{b} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-x^2}{2} \right)^n \right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n! 2^n} \int_{x=0}^{b} x^{2n}\,\mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n\, b^{2n+1}}{n! 2^n\,(2n+1)} \right).
\end{aligned}$$

This power series can be used to approximate the integral to any degree of accuracy.

Problem 3.2.8 Write e^{-x^3} as a power series expanded about 0 and use your series to represent

$$\int_{x=0}^{b} e^{-x^3}\,\mathrm{d}x$$

as a power series. □

Problem 3.2.9 Let $a > 0$. Find a power series expansion about 0 for a^x

Hint. $a^x = e^{\ln a^x}$ □

Problem 3.2.10 The ability to express complex functions as **power series** ("infinite polynomials") became a tool of paramount importance for solving differential equations in the 1700s.

(a) Show that if $y = \sum_{n=0}^{\infty} a_n x^n$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y, \tag{I.11}$$

then

$$a_{n+2} = \frac{-1}{(n+2)(n+1)}\,a_n$$

and conclude that

$$\begin{aligned} y = a_0 + a_1 x &- \frac{1}{2!}a_0 x^2 - \frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \frac{1}{5!}a_1 x^5 \\ &- \frac{1}{6!}a_0 x^6 - \frac{1}{7!}a_1 x^7 + \cdots . \end{aligned}$$

(b) Since $y = \sin x$ also satisfies equation (I.11), we see that

$$\begin{aligned} \sin x = a_0 + a_1 x - \frac{1}{2!}a_0 x^2 &- \frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \frac{1}{5!}a_1 x^5 \\ &- \frac{1}{6!}\,a_0 x^6 - \frac{1}{7!}\,a_1 x^7 + \cdots \end{aligned}$$

for some constants a_0 and a_1. Show that in this case $a_0 = 0$ and $a_1 = 1$ and obtain

$$\begin{aligned} \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

□

Problem 3.2.11

(a) Use the power series

$$\begin{aligned} \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \end{aligned}$$

to obtain the power series

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

(b) Let

$$s(x, N) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

and

$$c(x, N) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n)!} x^{2n}$$

and use a computer algebra system to plot these on the interval $-4\pi \leq x \leq 4\pi$, for $N = 1, 2, 5, 10, 15$. Describe what is happening to the graph of the power series as N becomes larger.

□

Problem 3.2.12 Use the power series for $\sin(x)$ to compute

$$\int_{x=0}^{b} \sin\left(x^2\right) \mathrm{d}x$$

as a power series. □

Problem 3.2.13

(a) Use the Geometric series to obtain a power series for

$$f(x) = \frac{1}{1+x},$$

and use your series to show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$.

(b) Use the power series you found in part (a) to obtain a power series for

$$f(x) = \frac{1}{1+x^2},$$

and use your series to show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} = \frac{4}{5}$.

(c) Use the result in part (b) to obtain the power series

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1},$$

and use the series to show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

□

Problem 3.2.14 Compute

$$\int_{x=0}^{1} \frac{1}{1+x^2} \mathrm{d}x$$

as a power series. □

Problem 3.2.15

(a) Use the Geometric series to obtain the power series

$$\begin{aligned}\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}.\end{aligned}$$

Hint. Recall that $\ln(1+x) = \int \frac{1}{1+x}\,\mathrm{d}x$.

(b) Use the result of part (a) to represent the function $\ln\left(1+x^2\right)$ as a power series expanded about 0.

(c) Use the result of part (a) represent the function $\ln(2+x)$ as power series expanded about 0.

Hint. $2+x = 2\left(1+\frac{x}{2}\right)$

□

Problem 3.2.16 Use the Geometric series to find a power series representation for $\frac{2x}{1+x^2}$. Integrate this to obtain a power series representation for $\ln\left(1+x^2\right)$ and compare your answer to part (b) of the previous problem. (This shows that there may be more than one way to obtain a power series representation.)

□

The power series for arctangent was known by James Gregory (1638-1675) and it is sometimes referred to as "Gregory's series." Leibniz independently discovered $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ by examining the area of a circle. Though it gives us a means for approximating π to any desired accuracy, the power series converges too slowly to be of any practical use. For example, if we compute the sum of the first 1000 terms we get

$$4\left(\sum_{n=0}^{1000} (-1)^n \frac{1}{2n+1}\right) \approx 3.142591654$$

which only approximates π to two decimal places.

Figure 3.2.17 James Gregory

Newton knew of these results and the general scheme of using power series to compute areas under curves. He used these results to provide a power series approximation for π as well, which, hopefully, would converge faster. We will use modern terminology to streamline Newton's ideas. First notice that

$\frac{\pi}{4} = \int_{x=0}^{1} \sqrt{1-x^2}\,\mathrm{d}x$ as this integral gives the area of one quarter of the unit circle, $\frac{\pi}{4}$. The trick now is to find a power series that represents $\sqrt{1-x^2}$.

To this end we start with the binomial theorem

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^{N-n} b^n,$$

where

$$\begin{aligned}\binom{N}{n} &= \frac{N!}{n!\,(N-n)!} \\ &= \frac{N\,(N-1)\,(N-2)\cdots(N-n+1\,)}{n!} \\ &= \frac{\prod_{j=0}^{n-1}(N-j)}{n!}.\end{aligned}$$

Unfortunately, we now have a small problem with our notation which will be a source of confusion later if we don't fix it. So we will pause to address this matter. We will come back to the binomial expansion afterward.

This last expression is becoming awkward in much the same way that an expression like

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^k$$

is awkward. Just as this sum is less cumbersome when written as $\sum_{n=0}^{k}\left(\frac{1}{2}\right)^n$ the *product*

$$N\,(N-1)\,(N-2)\cdots(N-n+1\,)$$

is less cumbersome when we write it as $\prod_{j=0}^{n-1}(N-j)$.

A capital pi (Π) is used to denote a product in the same way that a capital sigma (Σ) is used to denote a sum. The most familiar example would be writing

$$n! = \prod_{j=1}^{n} j.$$

Just as it is convenient to define $0! = 1$, we will find it convenient to define $\prod_{j=1}^{0} = 1$. Similarly, the fact that $\binom{N}{0} = 1$ leads to the convention $\prod_{j=0}^{-1}(N-j) = 1$. Strange as this may look, it is convenient and is consistent with the convention $\sum_{j=0}^{-1} s_j = 0$.

Returning to the binomial expansion and recalling our convention

$$\prod_{j=0}^{-1}(N-j) = 1,$$

we can write,

$$(1+x)^N = 1 + \sum_{n=1}^{N}\left(\frac{\prod_{j=0}^{n-1}(N-j)}{n!}\right)x^n = \sum_{n=0}^{N}\left(\frac{\prod_{j=0}^{n-1}(N-j)}{n!}\right)x^n.$$

These two representations probably look the same at first. Take a moment and be sure you see where they differ.

There is an advantage to using this convention (especially when programing a product into a computer), but this is not a deep mathematical insight. It is just a notational convenience and we don't want you to fret over it, so we will use both formulations (at least initially).

Notice that we can extend the above definition of $\binom{N}{n}$ to values $n > N$. In this case, $\prod_{j=0}^{n-1}(N-j)$ will equal 0 as one of the factors in the product will be 0 (the one where $j = N$). This gives us that $\binom{N}{n} = 0$ when $n > N$ and so

$$(1+x)^N = 1 + \sum_{n=1}^{\infty}\left(\frac{\prod_{j=0}^{n-1}(N-j)}{n!}\right)x^n = \sum_{n=0}^{\infty}\left(\frac{\prod_{j=0}^{n-1}(N-j)}{n!}\right)x^n$$

holds true for any nonnegative integer N. Essentially Newton asked if it could be possible that the above equation could hold values of N which are not nonnegative integers. For example, if the equation held true for $N = \frac{1}{2}$, we would obtain

$$(1+x)^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty}\left(\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}\right)x^n = \sum_{n=0}^{\infty}\left(\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}\right)x^n$$

or

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots. \tag{I.12}$$

Notice that since $\frac{1}{2}$ is not an integer the power series no longer terminates. Although Newton did not prove that this power series was correct (nor did we), he tested it by multiplying the power series by itself. When he saw that by squaring the power series he started to obtain $1 + x + 0\,x^2 + 0\,x^3 + \cdots$, he was convinced that the power series was exactly equal to $\sqrt{1+x}$.

Problem 3.2.18 Consider the power series representation

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1 + \sum_{n=1}^{\infty}\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n \\ &= \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n.\end{aligned}$$

Multiply this power series by itself and compute the coefficients for x^0, x^1, x^2, x^3, x^4 in the resulting power series. □

Problem 3.2.19 Let

$$S(x, M) = \sum_{n=0}^{M}\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n.$$

Use a computer algebra system to plot $S(x, M)$ for $M = 5, 10, 15, 95, 100$ and compare these to the graph for $\sqrt{1+x}$. What seems to be happening? For what values of x does the power series appear to converge to $\sqrt{1+x}$? □

Convinced that he had the correct power series, Newton used it to find a power series representation of $\int_{x=0}^{1}\sqrt{1-x^2}\,\mathrm{d}x$.

Problem 3.2.20 Use the power series $(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}\left(\frac{1}{2}-j\right)}{n!}x^n$ to obtain the power series

$$\frac{\pi}{4} = \int_{x=0}^{1}\sqrt{1-x^2}\,\mathrm{d}x$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{\prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{n!} \right) \left(\frac{(-1)^n}{2n+1} \right) \right]$$
$$= 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \cdots .$$

Use a computer algebra system to sum the first 100 terms of this power series and compare the answer to $\frac{\pi}{4}$. □

Again, Newton had a power series which could be verified (somewhat) computationally. This convinced him even further that he had the correct power series.

Problem 3.2.21

(a) Show that

$$\int_{x=0}^{1/2} \sqrt{x - x^2}\, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{\sqrt{2}\, n!\, (2n+3)\, 2^n}$$

and use this to show that

$$\pi = 16 \left(\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{\sqrt{2}\, n!\, (2n+3)\, 2^n} \right).$$

(b) We now have two power series for calculating π: the one from part (a) and the one derived earlier, namely

$$\pi = 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \right).$$

We will explore which one converges to π faster. First define

$$S1(N) = 16 \left(\sum_{n=0}^{N} \frac{(-1)^n \prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{\sqrt{2}\, n!\, (2n+3)\, 2^n} \right)$$

and

$$S2(N) = 4 \left(\sum_{n=0}^{N} \frac{(-1)^n}{2n+1} \right).$$

Use a computer algebra system to compute $S1(N)$and $S2(N)$ for $N = 5, 10, 15, 20$. Which one appears to converge to π faster?

□

In general the power series representation

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \left(\frac{\prod_{j=0}^{n-1} (\alpha - j)}{n!} \right) x^n$$
$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots$$

is called the **binomial series** (or Newton's binomial series). This power series is correct when α is a non-negative integer (after all, that is how we got the series in the first place). We can also see that it is correct when $\alpha = -1$ as we obtain

$$\begin{aligned}(1+x)^{-1} &= \sum_{n=0}^{\infty}\left(\frac{\prod_{j=0}^{n-1}(-1-j)}{n!}\right)x^n \\ &= 1+(-1)x+\frac{-1\,(-1-1)}{2!}x^2+\frac{-1\,(-1-1)\,(-1-2)}{3!}x^3+\cdots \\ &= 1-x+x^2-x^3+\cdots\end{aligned}$$

which can be obtained from the geometric series $\frac{1}{1-x}=1+x+x^2+\cdots$.

In fact, the binomial series is the correct power series representation for all values of the exponent α (though we haven't proved this yet).

Problem 3.2.22

(a) Assuming that the binomial series works for $\alpha=-\frac{1}{2}$, show that

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= \sum_{n=0}^{\infty}\frac{\left(\prod_{j=0}^{n-1}\left(\frac{1}{2}+j\right)\right)}{n!}x^{2n} \\ &= 1+\frac{1}{2}x^2+\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!}x^4+\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{3!}x^6+\ldots.\end{aligned}$$

(b) Integrate the above to obtain the following power series for $\arcsin(x)$.

$$\begin{aligned}\arcsin(x) &= \sum_{n=0}^{\infty}\frac{\left(\prod_{j=0}^{n-1}\left(\frac{1}{2}+j\right)\right)}{n!\,(2n+1)}x^{2n+1} \\ &= x+\frac{\frac{1}{2}}{3}x^3+\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!5}x^5+\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{3!7}x^7+\ldots\end{aligned}$$

(c) Substitute $x=\frac{1}{2}$ into the above power series to obtain a power series representation for $\frac{\pi}{6}$. Add the first four terms of this power series to obtain an approximation for π, and compare with $\pi\approx 3.14159265359$. How close did your approximation come?

□

Problem 3.2.23 Let k be a positive integer. Find the power series, centered at zero, for $f(x)=(1-x)^{-k}$ by

(a) Differentiating the Geometric series $(k-1)$ times.

(b) Applying the binomial series.

(c) Compare the results in parts (a) and (b).

□

Leonhard Euler (1707–1783) was a master at exploiting power series. In 1735, the 28 year-old Euler won acclaim for what is now called the Basel problem: to evaluate the sum

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Other mathematicans had shown that the power series converged, but Euler was the first to find its exact value. The following problem essentially provides Euler's solution.

Figure 3.2.24 Leonhard Euler

Problem 3.2.25 The Basel Problem. Recall that in Problem 3.2.11 we developed a power series representation of the function $\sin x$.

(a) Show that the power series for $\frac{\sin x}{x}$ is given by $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$

(b) Use part (a) to infer that the roots of $1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$ are given by

$$x = \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \ldots$$

(c) Suppose $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial with roots $r_1, r_2, \ldots, r_n$. Show that if $a_0 \neq 0$, then all the roots are non-zero and

$$p(x) = a_0 \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_n}\right).$$

(d) Assuming that the result in part (c) holds for an infinite polynomial (power series), deduce that

$$1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots = \left(1 - \left(\frac{x}{\pi}\right)^2\right)\left(1 - \left(\frac{x}{2\pi}\right)^2\right)\left(1 - \left(\frac{x}{3\pi}\right)^2\right)\cdots$$

(e) Expand this product to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Problem 3.2.26 Euler's Formula.

(a) Use the power series expansion of e^x, $\sin x$, and $\cos x$ to derive **Euler's Formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(b) Use Euler's formula to derive the Addition/Subtraction formulas from Trigonometry:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

(c) Use Euler's formula to show that

$$\sin 2\theta = 2\cos\theta\sin\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

(d) Use Euler's formula to show that

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

(e) Find a formula $\sin(n\theta)$ and $\cos(n\theta)$ for any positive integer n.

□

3.3 Expanding Simple Power Series by Algebraic Methods

We call the power series expansions we'll see in this section "simple" because all that is needed to generate them is prior knowledge of a few series (e.g.,the Geometric Series, the sine and cosine series, the exponential series, the Binomial Series), and a creative use of algebra. In particular Taylor's Theorem is not needed. We assume that you are familiar with the use of Taylor's Theorem from your Calculus course.

As we saw in the last section, it can be particularly fruitful to expand a function as a power series centered at $a = 0$. Unfortunately, this isn't always possible. For example, it is not possible to expand the function $f(x) = \frac{1}{x}$ about zero. (Why not?)

But we are not confined to expanding about zero. Consider that the following is a power series for $f(x) = \frac{1}{x}$ expanded about $a = 1$

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} -1^n(x-1)^n.$$

Of course, there are still questions that need to be resolved. Chief among these is the question, "For which values of x is this series a valid representation of the function we started with?" We will explore this in Section 4.1. For now we will content ourselves with having a representation which seems reasonable.

Problem 3.3.1 Let $a \neq 0$.

(a) Represent $\frac{1}{x}$ as a power series expanded about a. That is, as a power series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$.

Hint. $\frac{1}{x} = \frac{1}{a+x-a} = \frac{1}{a}\left(\frac{1}{1+\frac{x-a}{a}}\right)$

(b) Represent $\ln(x)$ as a power series expanded about a by integrating your solution to part (a).

□

Problem 3.3.2 Let $a > 0$ and use equation (I.12) to represent $\sqrt{x}$ as a power series expanded about a. □

Problem 3.3.3 Let a be a real number. Represent e^x as power series expanded about a. Notice there is no restriction on a. What happens if $a = 0$? □

Problem 3.3.4 Let a be a real number. Represent $x^3 + 2x^2 + 3$ as a power series expanded about a. What happens if $a = 0$? □

Problem 3.3.5 Let a be a real number. Use the power series expansions

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

to obtain the power series representation

$$\begin{aligned} \sin(x) = \sin(a) &+ \cos(a)(x-a) - \frac{1}{2!}\sin(a)(x-a)^2 \\ &- \frac{1}{3!}\cos(a)(x-a)^3 + \frac{1}{4!}\sin(a)(x-a)^4 \\ &+ \frac{1}{5!}\cos(a)(x-a)^5 + \cdots . \end{aligned}$$

This result will come into play in the next section.

Hint. $\sin(x) = \sin(a + (x-a))$ □

Chapter 4

Questions Concerning Power Series

4.1 Taylor's Formula

As we saw in Section 3.3, representing functions as power series was a fruitful strategy for mathematicans in the eighteenth century (as it still is). Differentiating and integrating power series term by term was relatively easy, *seemed* to work, and led to many applications. Furthermore, power series representations for all of the elementary functions could be obtained if one was clever enough. However, cleverness is an unreliable tool. It would be better to have some systematic way to find a power series for a given function that doesn't rely on being sufficiently cleve.

To be sure, there were nagging questions. For example. even if we can find a power series representation of some function, how do we know that the series we've created represents the function we started with? Even worse, is it possible for a function to have more than one power series representation centered at a given value a? This uniqueness issue is addressed by the following theorem.

Theorem 4.1.1 Taylor's Formula. *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$*, then* $a_n = \frac{f^{(n)}(a)}{n!}$*, where* $f^{(n)}(a)$ *represents the* n*th derivative of* f *evaluated at* a*.*

A few comments about Theorem 4.1.1 are in order. Notice that we did *not* start with a function and derive its series representation. Instead we *defined* $f(x)$ to be the series we wrote down. This assumes that the expression $\sum_{n=0}^{\infty} a_n(x-a)^n$ actually has meaning (that it converges). At this point we have every reason to expect that it does, however expectation is not proof so we note that this is an assumption, not an established truth. We've also assumed that we can differentiate an infinite polynomial term-by-term as we would a finite polynomial. As before, we follow in the footsteps of our 18th century forebears in making these assumptions. For now.

Problem 4.1.2 Prove Theorem 4.1.1.

Hint. The "zeroth" derivative (the function itself) at a is given by

$$f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + \cdots = a_0.$$

Differentiate to obtain the other terms. □

From Theorem 4.1.1 we see that if we do start with the function $f(x)$ then no matter how we obtain its power series, the result will always be the same. The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{I.1}$$

is called the ***Taylor series*** *for f expanded about (or centered at) a.* Although this systematic "machine" for obtaining power series for a function seems to have been known to a number of mathematicians in the early 1700s, Brook Taylor (1685–1731) was the first to publish this result in his *Methodus Incrementorum* (1715). The special case when $a = 0$ was included by Colin Maclaurin (1698–1746) in his *Treatise of Fluxions* (1742). Thus when $a = 0$, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ is often called the **Maclaurin Series** for f.

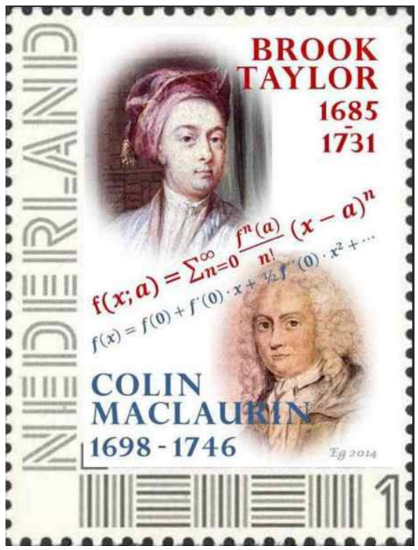

Figure 4.1.3 A postage stamp from the Netherlands honoring Brook Taylor and Colin Maclaurin

Problem 4.1.4 Use Taylor's formula to find the Taylor series of the given function expanded about the given point a.

(a) $f(x) = \ln(1+x)$, $a = 0$

(b) $f(x) = e^x$, $a = -1$

(c) $f(x) = \sin(x)$, $a = \frac{\pi}{2}$

(d) $f(x) = x^3 + x^2 + x + 1$, $a = 0$

(e) $f(x) = x^3 + x^2 + x + 1$, $a = 1$

□

The **prime notation** for the derivative was not used by Taylor, Maclaurin or their contemporaries. It was introduced by Joseph Louis Lagrange in his 1779 work *Thèorie des Fonctions Analytiques*. In that work, Lagrange sought to get rid of Leibniz' infinitesimals and base Calculus on the power series idea. His idea was that by representing every function as a power series, Calculus could be done algebraically by manipulating power series and examining various aspects of the series representation instead of appealing to the controversial notion of infinitesimals. He implicitly assumed that every continuous function could be replaced with its power series representation.

Figure 4.1.5 Joseph-Louis Lagrange

That is, he wanted to think of the Taylor series as a "great big polynomial," because polynomials are easy to work with. It was a very simple, yet exceedingly clever and far-reaching idea. Since $e^x = 1 + x + x^2/2 + \ldots$, for example, why not just define the exponential to be the series and work with the series. After all, the series is just a very long polynomial.

This idea did not come out of nowhere. Particular infinite series, such as the Geometric Series had been known and studied for many years. Later, in the 18th century Leonhard Euler used infinite series to solve many problems, and some of his solutions are still quite breath–taking when you first see them [14].

Taking his cue from the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{I.2}$$

Lagrange observed that the coefficient of $(x-a)^n$ provides the nth derivative of f at a (divided by $n!$). Modifying formula (I.2) to suit his purpose, Lagrange supposed that every differentiable function could be represented as

$$f(x) = \sum_{n=0}^{\infty} g_n(a)(x-a)^n.$$

In that case $g_1(a)$ is the derivative of f at a, $f''(a) = 2g_2(a)$ and generally

$$f^{(n)}(a) = n!g_n(a).$$

Lagrange dubbed his function g_1 the *"fonction dérivée"* from which we get the modern name **derivative**.

Problem 4.1.6 Let $a \neq 0$ be a fixed number. We saw in Problem 3.3.1 that the power series of $\frac{1}{x}$ expanded about a is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}}(x-a)^n = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \ldots$$

(a) Apply Lagrange's idea to show that $f^{(n)}(a) = \frac{(-1)^n n!}{a^{n+1}}$.

(b) Now compute $f^{(n)}(a)$ by directly by differentiating. Do you get the same result?

□