Chapter 15

When the Derivative Doesn't Exist

"I turn away with fright and horror from the lamentable evil of functions which do not have derivatives."

– Charles Hermite (1822 - 1901)

"The brilliant Cerebron, . . . discovered three distinct kinds of dragon: the mythical, the chimerical, and the purely hypothetical. They were all, one might say, nonexistent, but each non-existed in an entirely different way."

– Stanislaw Lem (1921 - 2006)

In Section 8.5 we saw that those points where the derivative doesn't exist are possible optimal points but we didn't pursue this any further then. The time has come for us to re-examine the non-existence of derivatives.

The derivative of a function is by definition a limit, and we know from our work in Chapter 11 that not all limits exist. If we try to compute the value of $f'(2)$ for some function and we find that $\lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{h}$ is meaningless then, by definition, f is not differentiable at $x = 2$. In other words, $f'(2)$ does not exist.

In Section 8.5 we presented evidence that derivatives fail to exist at points where the Principle of Local Linearity does not hold. This is true, but the only tool it provides is to look at a graph. And we have learned not to put our faith in graphs.

Definition #15 says that a function fails to be differentiable at a point precisely when the defining limit fails to exist. This gives us a computational tool we can use to decide the question of differentiability.

Example #86:

For example, when we look at the graph of $f(x) = \frac{x}{x-2}$ (you should do that) it is intuitively clear that its derivative is not defined at $x = 2$.

But graphs provide insight, not rigor, so we'll try to compute the derivative of $f(x) = \frac{x}{x-2}$ at $x = 2$ and see what goes wrong. It is tempting to apply the Quotient Rule but we're asking if the derivative exists, not what form it would take if it did. The existence of the derivative of $f(x)$ at $x = 2$ is equivalent to the existence of the limit:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

Observe that $f(2)$ is meaningless (why?) which means that the limit which defines $f'(2)$:

$$f'(2) \stackrel{?}{=} \lim_{h\to 0} \frac{f(2+h) - \cancel{f(2)}^{\text{meaningless}}}{h}.$$

is also meaningless. So $f'(2)$ is undefined

Problem #522:

(a) Use Definition #15 to show that if[1] $f(x) = \frac{x}{x}$ then $f'(0)$ is undefined.

(b) Use Definition #15 to show that if[2]

$$f(x) = \begin{cases} \frac{x}{x} & x \neq 0 \\ 0 & x = 1 \end{cases}$$

then $f'(0)$ is defined.

End Of Example #86

Example #87: The Absolute Value Function

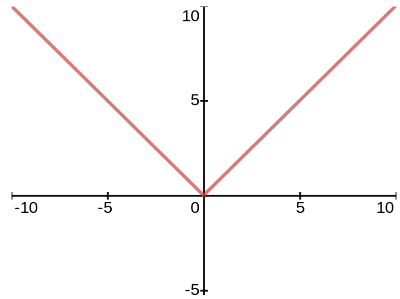
A slightly more abstruse example is the absolute value function.

Definition 20: The Absolute Value

The Absolute Value of x is: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$

[1]It is tempting to say that this is really just $f(x) = 1$ since "anything divided by itself is one" but that is not true. As we saw in Digression 4 division by one is undefined, regardless of the numerator. So the expression $\frac{0}{0}$ is undefined. When discussing indeterminate forms in Section 11.4 we were very careful not to write $\frac{0}{0}$, opting instead for $\frac{(\to 0)}{(\to 0)}$ in order to emphasize that we specifically do not allow the denominator to be equal to zero.

[2]This is a silly way to define this function. We're making a point.



The graph of the Absolute Value function is given at the right. Notice that $f(x) = |x|$ is defined at $x = 0$ since $|0| = 0$, but as we indicated in Section 8.5 the Principle of Local Linearity does not hold at $x = 0$ so we conclude that the Absolute Value function is not differentiable at $x = 0$.

Let's take a look at this using the limit definition. The derivative of $|x|$ at x will be the value of the limit $\lim\limits_{h\to 0} \frac{|x+h|-|x|}{h}$ so the derivative of $|x|$ at $x = 0$ will be:

$$\lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

OK, but what is this limit? Don't jump to conclusions. Think about this carefully for a few minutes. What do you think $\lim\limits_{h\to 0} \frac{|h|}{h}$ is equal to?

Suppose first that h is approaching zero[3] from the *positive*[4] side. Then $|h| = h$ so

$$\lim_{\substack{h\to 0\\ h>0}} \frac{|h|}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{h}{h} = \lim_{\substack{h\to 0\\ h>0}} 1 = 1$$

which says that the right-hand limit will be one. Now suppose h is very small and *negative*. Then $|h| = -h$ so

$$\lim_{\substack{h\to 0\\ h<0}} \frac{|h|}{h} = \lim_{\substack{h\to 0\\ h<0}} \frac{-h}{h} = \lim_{\substack{h\to 0\\ h<0}} -1 = -1$$

which says that the left-hand limit will be negative one.

But since $\frac{|h|}{h}$ can't be close to 1 and close to -1 simultaneously we can not find a value for $\lim\limits_{h\to 0} \frac{|0+h|-|0|}{h}$ which is correct for both positive and negative values of h. In other words this limit does not exist.

Therefore $\left.\frac{\mathrm{d}|x|}{\mathrm{d}x}\right|_{x=0}$ does not exist.

Drill #523: If $\left.\frac{\mathrm{d}(|x|)}{\mathrm{d}x}\right|_{x=0}$ does not exist, then according to Definition 11 $x = 0$ is a possible transition point for the Absolute Value function, so there might be a local extremum of $f(x) = |x|$ at $x = 0$. Is there?

End Of Example #87

The Absolute Value function is not differentiable at $x = 0$ but it is for every other value of x.

Problem #524:
Let $f(x) = |x|$ and use definition 15 to show that:

[3]To say that h "approaches" 0 suggests that h is sliding along the horizontal axis toward the origin. As we will see in Chapter 16 this isn't quite the right way to think of this. But it is adequate to our current purpose so we will continue using it for now.

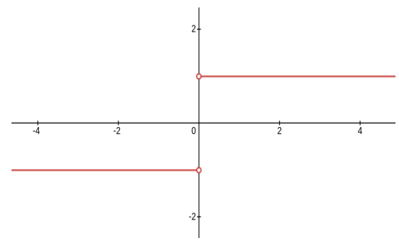
[4]Recall that in Section 11.3 we invented the notations $\lim\limits_{\substack{h\to 0\\ h>0}}$ and $\lim\limits_{\substack{h\to 0\\ h<0}}$ for exactly this situation.

(a) $x > 0$ then $f'(x) = 1$,

(b) $x < 0$ then $f'(x) = -1$.

Example #88: The Heaviside Function

Problem #524 shows that the derivative of the Absolute Value function (shown at the right) is:



$$H(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}.$$

Notice that it is not defined at $x = 0$.

We call $H(x)$ the Heaviside function in honor of Oliver Heaviside. Simple as it is, Heaviside's function is a fundamental tool in signal processing, control theory, and the solution of differential equations.

Problem #525:

(a) Use Definition #15 to show that $H'(x) = 0$ when $x \neq 0$

(b) Use Definition #15 to show that $H'(0)$ does not exist.

(c) Sketch the graph of $H'(x)$.

End Of Example #88

We first introduced one-sided limits in Section 11.3. At the time our concern was to locate vertical asymptotes we were interested in finding vertical asymptotes so we only looked at limits that were either positive or negative infinity. We see now that these limits can take any value and that they are more fundamental than they first appeared to be. Since we didn't provide a formal definition earlier we will do so now.

Definition 21: One-sided Limits

If R is a real number (or ∞) and

$$\lim_{\substack{x \to a \\ x > a}} f(x) = R$$

we say that R is the right-handed limit of $f(x)$ at $x = a$.

If L is a real number (or $-\infty$) and

$$\lim_{\substack{x \to a \\ x < a}} f(x) = L$$

we say that L is the left-handed limit of $f(x)$ at $x = a$.

Because $f(x)$ can't simultaneously approach two different numbers, if the left and right handed limits both exist but do not agree then $\lim\limits_{x\to a} f(x)$ doesn't exist. This fact will be a useful tool for us later so we state it as a theorem. Theorem 38 requires proof. We will revisit this in Chapter 16.

Theorem 38:
$\lim\limits_{x\to a} f(x)$ exists if and only if $\lim\limits_{\substack{x\to a\\ x>a}} f(x) = \lim\limits_{\substack{x\to a\\ x<a}} f(x)$. In that case all three limits are equal.

15.1 One Sided Derivatives

Since limits can be right-handed or left-handed and the derivative is defined as a limit it follows that derivatives can be left or right-handed as well. We will denote the left and right-handed derivatives at $x = a$ with the notation: ${f_\uparrow}'(a)$, and ${f_\downarrow}'(a)$, respectively.

Definition 22: One Sided Derivatives
Given a function, $f(x)$, defined at a point a:

1. f is said to have a right handed derivative at $x = a$ if the limit
$$ {f_\downarrow}'(a) = \lim_{\substack{h\to 0\\ h>0}} \tfrac{f(a+h)-f(a)}{h} $$
exists.
2. f is said to have a left handed derivative at $x = a$ if the limit
$$ {f_\uparrow}'(a) = \lim_{\substack{h\to 0\\ h<0}} \tfrac{f(a+h)-f(a)}{h} $$
exists.

At zero both the right and left-handed derivatives of the Absolute Value function $f(x) = |x|$ exist since
$$ {f_\uparrow}'(0) = \lim_{\substack{h\to 0\\ h<0}} \frac{|0+h|-|0|}{h} = \lim_{\substack{h\to 0\\ h<0}} \frac{-h}{h} = -1 $$
and
$$ {f_\downarrow}'(0) = \lim_{\substack{h\to 0\\ h>0}} \frac{|0+h|-|0|}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{h}{h} = 1, $$
but $f'(0)$ does not exist because $-1 \neq 1$.

Drill #526:

Compute ${H_\uparrow}'(0)$, and ${H_\downarrow}'(0)$ for the Heaviside function. Do your computations agree with the graph of $H(x)$? Explain.

Problem #527:
Use the limit definitions to compute the left and right handed derivative, and the derivative itself, if they exist, of each of the following functions at the $x = -2, -1, 0, 1, 2$. If any of these derivatives don't exist explain why not.

(a) $f(x) = \frac{1}{x}$ **(b)** $f(x) = 3\,|x+1|$ **(c)** $f(x) = x^{2/3}$ **(d)** $f(x) = \sqrt{|x|}$

Problem #528:
Use the limit definitions to compute the left and right handed derivative, of the function

$$f(x) = \begin{cases} 2 & \text{if } x < -1 \\ -2(x+1)^2 + 2 & \text{if } -1 \le x < 0 \\ x^4 - 4x & \text{if } 0 \le x < 2 \\ -14(x-3)^2 + 22 & \text{if } 2 \le x \end{cases}$$

at $x = -1$, $x = 0$ and at $x = 2$. At which of these points is $f(x)$ differentiable?

Problem #529:
Sometimes the function $f(x) = \frac{|x|}{x}$ is also called the Heaviside function. How is $f(x)$ distinct from $H(x)$ as we defined it above? Is $f(x)$ differentiable at $x = 0$?

Problem #530:

(a) **(a)** Let a be a real number and use Theorem #38 to show that $H(x-a)$ is not differentiable at $x = a$.

(b) Sketch $H(x-a)$.

(a) Let a and b be real numbers and use Theorem #38 to show that $f(x) = H(x-a) + H(x-b)$ is not differentiable at $x = a$ or $x = b$.

(b) Sketch $f(x)$.

In Section 8.5 we found that the possible transition points came in three flavors: (1) points where the derivative was zero, (2) points where the derivative was undefined, and, (3) if the function was defined on a closed interval, the endpoints of the interval. We remarked at the time that it seemed odd that the third condition does not involve the derivative while the other two do.

In fact, with our new, deeper understanding of differentiation we can show that there are really only two kinds of possible transition points, and both involve the derivative. The endpoints of are interval are really just points of non-differentiability.

To see this consider any function function, $f(x)$ which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). From our discussion above we see that $f'(b)$ only exists if if ${f_{\uparrow}}'(b)$ and ${f_{\downarrow}}'(b)$ both exist[5]. But the right handed limit at b can't possibly exist because ${f_{\downarrow}}'(b)$ is meaningless. To see this observe that since f is only defined on $[0, b]$ $f(b+h)$ is utterly meaningless when $h > 0$ because $f(b+h)$ asks us to evaluate f at a point outside its domain. Thus

$${f_{\downarrow}}'(b) = \lim_{\substack{h \to 0 \\ h>0}} \frac{\overset{\text{meaningless}}{\cancel{f(b+h)}} - f(b)}{h}$$

does not exist. A similar argument shows that ${f_{\uparrow}}'(a)$ does not exist.

Therefore f' does not exist at the endpoints of a closed interval. As a result the possible transition points of f are *only* those places where the derivative is zero, or undefined.

[5] They also need to be equal, but this is irrelevant right now.