Chapter 16

Formal Limits

". . . in becoming rigorous, mathematical science takes a character so artificial as to strike everyone; it forgets its historical origins; we see how the questions can be answered, we no longer see how and why they are put."

– Henri Poincaré (1854 – 1912)

We began our treatment of limits in Chapter 11 quite informally because it takes time to develop a mindset appropriate to a thorough understanding of limits. As a result of using properties of limits which we have not yet shown to be true we've left a logical hole in the proof of nearly every theorem we've stated. Bishop Berkeley would be most displeased.

It is time at last to fill those holes. In Definition #12 we said that if $x \to \infty$ $f(x)$ gets "closer and closer" to A then $\lim_{x\to\infty} f(x) = A$. It is this phrase "closer and closer" that is the problem.

To illustrate what's wrong with the intuitive approach to limits that we've used so far consider the limit $\lim_{x\to 1} f(x)$, when

$$f(x) = \frac{1}{\pi}\tan^{-1}(10^8(x-1)).$$

In order to get a sense of what this function looks like when we let x get "close" to 1 we've tabulated a few values of $f(x)$ near $x = 1$ in the table at the right.

$f(x) = \frac{1}{\pi}\tan^{-1}(10^8(x-1))$

x	$f(x)$
1.5	4.999
1.4	4.999
1.3	4.999
1.2	4.999
1.1	4.999
1.01	4.999
1.001	4.999
1.0001	4.999

Seems pretty convincing doesn't it? Can we conclude from this table that

$$\lim_{x\to 1}\frac{1}{\pi}\tan^{-1}(10^8(x-1)) = 4.999?$$

No, of course not. This limit is very clearly equal to zero since $f(1) = \frac{1}{\pi}\tan^{-1}(10^8(1-1)) = \frac{1}{\pi}\tan^{-1}(0) = 0$.

Drill #531: Use your favorite computational tool to find a value of x such that $|f(x)| < 10^{-3}$.

The problem with our example is that none of the x values in the first column is close *enough* to 1. Sure, the numbers 1 and 1.0001 are very close together. We don't just want to get close, we want to get *close enough*.

For this particular function[1] we'd have to get much closer to 1 before we start to see $f(x)$ getting close to 0.

And that's the problem. When we decide what "close enough" means for any particular limit the nature of the function we're taking the limit of must be taken into account. This isn't as bad as it sounds, but as always precision is crucial. We need a definition of **limit** that takes the doesn't depend on the nature of the function we're investigating. It also needs to recover the Differentiation Rules in a manner that even Bishop Berkeley would agree is valid.

We will begin with limits "at infinity" because, paradoxically, these are often the easiest to understand. As we proceed through examples the question you want to keep in the back of your mind is, "For this problem how close is close enough?"

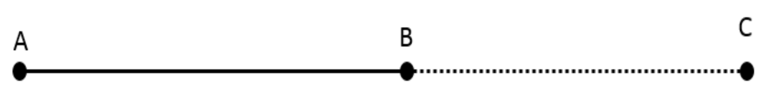
16.1 Limits "at" Infinity

It all comes down to understanding infinity[2]. Or rather, it comes down to realizing that we do not understand infinity at all, so whenever an apparently "infinite quantity" appears we will have to work with it and explain what we are doing in finite terms. This sounds impossible, but it is not. It is merely difficult.

By taking this approach we are well within well established traditions. Infinity had been carefully excluded from serious mathematical consideration from the time of the ancient Greeks until the sixteenth century. The successful exploitation of the infinitely small (differentials) by Galileo and others forced mathematicians to study infinity more closely.

In the modern world we tend to conceive of a straight line in infinite terms; as extending infinitely far in two opposite directions. But when Euclid wrote his geometry text, *The Elements* he very carefully avoided allowing the existence of an "infinite line." For Euclid a straight line was what we would call a line segment: the shortest path between two points.

But this immediately caused problems because for some of his constructions Euclid needed to be able to extend his line segment. For us this is not a problem. Since we allow lines to extend infinitely far in either direction we'd just move to a new point on the line wherever it needs to be.



But when Euclid specified a line segment $\overline{AB}$ he meant that the points A and B were the endpoints of the line segment. Thus when he needed to extend a line to some new point, say C, he couldn't just move to a new point on his (finite) line as we would on our (infinite) lines. He had to extend the line segment $\overline{AB}$ out to the point C, obtaining a new segment, say $\overline{AC}$, which is – and this is the point – still *finite* in length. That is how

[1] It should be clear that we've contrived this function so that we could make this point. It is unlikely that this function is useful for any other purpose.

[2] By "infinity" we mean to indicate both the infinitely large and the infinitely small.

Euclid worked around infinity. He explicitly allowed line segments to be extendable to any finite length. That way he could have a line segment as long as he needed it to be without ever allowing an infinitely long line to exist.

The Infinitely Small

But our problem is not the infinitely long, it is the infinitely short. Differentials are a "convenient fiction" as we've seen but it is very difficult to make precise the notion of a differential. Limits are far more approachable[3].

An example will be helpful so we ask the following question: "If $y = \frac{1}{x}$ and we allow x to grow infinitely large, what happens to y?" The answer is obvious. As x grows larger, y grows smaller.

Of course, we have immediately broken our own stricture against speaking of the infinitely large, so the first thing we have to do is find another way to ask our question. Let's try this: "What happens to y as x grows larger and larger[4]?"

OK. Good enough. Now let's ask this question: As x grows larger and larger will y get close to some fixed number? It seems completely obvious that it does and most people will immediately answer, "Sure, as x grows larger and larger y gets closer to zero."

But the only way we can reach that conclusion is by internally reinterpreting the problem. The question is not clearly formulated. We have to see it as asking more than it does.

Here's what we mean. Suppose x starts at 1 and proceeds to grow "larger and larger" in the following fashion: In the first second it moves halfway to 2, so that $x = 3/2$; in the next second it moves half of the distance remaining to 2, so that $x = 7/4$; in the next second it again moves half of the remaining distance to $x = 15/8$; and so on in the same pattern.

It should be clear that x grows larger each second, so x grows "larger and larger" as required. But it should also be clear that by growing in this manner x will never become larger than 2. So what happens to y? Clearly as x grows "larger and larger" – in this manner – y gets closer to $\frac{1}{2}$, not zero.

Drill #532: Find a formula for x as a function of time (in seconds) where x grows as described above.

Now suppose x moves half of the distance to 3 each second. Or halfway to 10. Or $1,000,000$. In each case x is growing "larger and larger" but y gets closer to $\frac{1}{3}$, $\frac{1}{10}$, or $\frac{1}{1,000,000}$, respectively. So we see that the question, "What happens to y as x grows larger and larger?" is actually very vague because any number at all could be a correct answer, depending on your interpretation of the phrase "larger and larger." In this situation we've described x is said to be **bounded above** because there is an upper bound on how large x can be, despite the fact that x is growing "larger and larger."

But is this how you understood the phrase "x grows larger and larger" when we first asked it? No, of course not. You understood it to mean "Let x grow infinitely large." But because the phrase we chose is ambiguous it is possible for x to growing "larger and larger" but remain below 2. In that case y will not approach zero. It will always be greater than one-half.

[3] Pun intended.

[4] Notice that x can grow as large as we need, as long as it remains finite. This is akin to the way Euclid avoided infinitely long lines.

The fact is that the phrases "larger and larger" and "closer and closer" are just too vague to be useful. We'll have to find a way to convey the meaning we want more precisely.

Let's look first at $x \to \infty$. What we need is a way to say that x increases without any bound at all, so that it can become as large as we'd like for it to be.

But that's the solution! If we ask,"What happens to y as x increases without bound?" then we allow x to become as large as we need, *and* we are avoid infinity.

But let's be careful. Is it clear that we're avoiding infinity? If x increases without bound isn't that just infinity by another name?

No, it isn't. Just as Euclid allowed a line segment $\overline{AB}$ to be extended to any point C *without* extending the line to an infinite length, we can allow x to grow to any finite value *without* allowing it to go all the way to infinity[5].

We can finally ask our question in a meaningful way: If $y = \frac{1}{x}$, what happens to y as x increases without bound? The answer is still clear: y goes to zero.

But of course, that is also too vague.

If y "goes to" zero, doesn't that mean that it eventually gets there? After all, when we "go to" the store, or a friend's house, or Disney World, we actually get there, don't we? But when y "goes to" zero it clearly never arrives. There is no finite value of x for which $y = \frac{1}{x} = 0$. If y never actually gets to zero can we say in any meaningful way that y "goes to" zero?

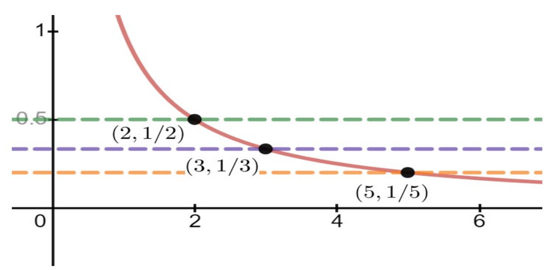
Again, it is not our conception of the problem that is the difficulty. It is the language we're using. We must choose our words carefully.

So how else might we say this? How do we capture in words the idea that y gets "closer and closer[6]" to zero, without ever getting to zero? Let's think this through, being careful to say *exactly* what we mean, no more, no less.

To begin we ask, "Is there a value of x which forces y to be less than, say 1/2?"

Sure. Probably you already see that $x = 2$ is the value that works. But does it? When $x = 2$, y is *exactly equal* to 1/2. But we wanted a value of x that makes y *less than* 1/2, not equal to it.

So we need $y < 1/2$, not $y = 1/2$. Since $y = 1/x$ that means that $1/x < 1/2$. Solving for x (and remembering that if $a < b$ then $\frac{1}{a} > \frac{1}{b}$ we have $x > 2$



So apparently any value of x *strictly greater than* 2 will guarantee that y is less than 1/2. Two is the largest number that *won't* work, because when $x \leq 2$, y is greater than or equal to $\frac{1}{2}$.

Stop and think about that last sentence. Do you see that we've actually discovered more than the original question asked for? Our question was, "Is there a value of x which forces y to be less than 1/2?" But we've actually found *all* of them. We've found that if x is any number greater than 2 then $y = \frac{1}{x} < \frac{1}{2}$, regardless of which number we use.

Can we make $y < 1/3$? Sure. Exactly the same analysis will show that if $x > 3$, then $y = \frac{1}{x} < \frac{1}{3}$, or if $x > 4$, then $y < 1/4$, and so on.

[5]Whatever that means.

[6]Compare this with "larger and larger."

Problem #533:
If $y = \frac{1}{x}$ how large must x be in order to guarantee that

(a) $y < 1/4$ **(b)** $y < 1/10$ **(c)** $y < 1/100$ **(d)** $y < 1/1000000$

It should be clear that we needn't have stopped at one million (10^6). The same argument will show that if we want $y = \frac{1}{x} < 10^{-10}$, we need $x > 10^{10}$. And that if we want $y = \frac{1}{x} < 10^{-1000}$, we need $x > 10^{1000}$.

It should also be clear that there is nothing special about the numbers $3, 4, 10$, or 1000000. We can make y less than any number we choose by an appropriate choice of x. We can make y *arbitrarily close to* zero, even if it never *actually is* zero. And that seems to be what we really intended when we said that y "goes to" zero.

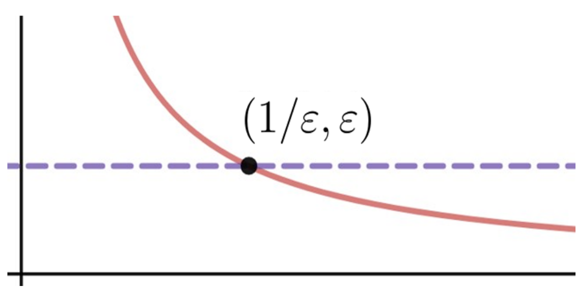
So here, at last, are our question and answer precisely stated:

> **Question:** Given that $y = \frac{1}{x}$, what number does y come arbitrarily close to as x increases without bound?
> **Answer:** As x increases without bound, y comes arbitrarily close to zero.

Convincing Berkeley

Do you think our argument above would convince Bishop Berkeley?

Clearly it won't. If we were to show Berkeley that we can guarantee that $y < 1/2$ by taking $x > 2$, his response would simply be, "So what? How do I know you can make y less than $1/3$." Nor will it be sufficient to show that we can make less than $1/3, 1/4, 1/10000$ or any particular number. Berkeley will simply come back to us with a smaller challenge.



Clearly what we have to do is answer all possible challenges at once. This seems like a lot to ask until we think about it a bit. All we really have to do is suppose that we have some small, positive, *unspecified* number and show that we can find out how large x has to be to make $y = \frac{1}{x}$ less than that number. For the sake of being definite we'll give our number a name. It is traditional to call it ε (epsilon).

Suppose that $\varepsilon > 0$ and we want to figure out how large to make x in order to guarantee that $y = \frac{1}{x} < \varepsilon$. This is essentially the same calculation we did above, but we will repeat it for emphasis. If we want $y < \varepsilon$ we first substitute $\frac{1}{x}$ for y giving $\frac{1}{x} < \varepsilon$. Solving this for x (remember that if $a < b$ then $\frac{1}{a} > \frac{1}{b}$) gives $x > \frac{1}{\varepsilon}$.

Do you see the significance of this? Because we left ε unspecified (other than requiring it to be positive), we've met all possible challenges. If the challenge is to make y less than $10^{-1000000000}$ our response is, "We've already done that. Just take $\varepsilon = 10^{-1000000000}$. Repeating the computation above gives $x > 10^{1000000000}$.

This is the precise meaning of the statement $\lim\limits_{x \to a} f(x) = 0$. We first take ε to be a completely arbitrary positive number. If we can show that no matter the value ε we can always make

$f(x) < \varepsilon$ by taking x larger than some specific number then we say that "the limit as x approaches a of $f(x)$ is zero."

Since ε is a definite (though unspecified) real, positive magnitude and we can respond to the epsilon challenge with "x can be any definite number greater than $1/\varepsilon$," we have avoided any consideration of the infinite, either large or small. To be sure, when we allow ε to be arbitrary, but unspecified we skirt the edge of the infinitely small. But this is the point. If ε is arbitrary then it can be as small as we need for it to be without ever being infinitely small. This is akin to Euclid allowing lines to be extended to any, unspecified, length without allowing them to be infinite in length. This is the idea underlying limits and limit notation.

Be aware that the limit notation and the way we tend to speak about limits can present a lot of problems for the beginner. If we are speaking loosely, among friends, we would read this statement, $\lim\limits_{x\to\infty} \frac{1}{x} = 0$, as follows:

"The limit of one over x as x goes to infinity is equal to zero."

This is an absolutely terrible way to express the idea we are trying to capture. To say that x approaches infinity completely undercuts all of our efforts above because it treats infinity as if it is an actual number that can be approached. This is ridiculous. Infinity is not a number.

But it gets worse! When we express a limit in this way we are stating that the expression on the left is *exactly equal* to zero. This is true despite the fact that $1/x$ itself is never equal to zero. But of course we are not saying that $1/x$ is equal to zero, we are saying that *the limit* of $1/x$ is equal to zero. The "lim" symbol cannot be ignored.

We know that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ really means that x as increases without bound, $1/x$ is getting arbitrarily close to zero but what we *say* is not what the notation *means*. That incongruity can be very confusing at first. In this text we will be very careful not to speak so casually. At least not until we have more experience with limits.

We have only begun, but this is enough for us to offer a first definition of the limit concept. We generalize slightly.

Definition 23: Positive Function With Limit Zero at Infinity

Suppose $f(x) > 0$ for all $x > 0$. Then we say that $\lim\limits_{x\to\infty} f(x) = 0$ if and only if for every $\varepsilon > 0$ we can find a real number B with the property that whenever $x > B$, $f(x) < \varepsilon$.

The parameter B is the lower bound that x has to exceed for $f(x)$ to be less than ε. In our first example we had $B = 1/2$, in our last we had $B = 1/\varepsilon$. Naming the lower bound like this gives us a concrete way to specify how large x has to be. Once B is found we can say that if $x > B$, then $f(x) < \varepsilon$. To a large extent finding B is the whole problem. This is easier to see in an example.

Example #89:

Show that if $f(x) = \frac{1}{x^2}$, then $\lim\limits_{x\to\infty} f(x) = 0$.

As before we need to start by taking $\varepsilon > 0$. Think of epsilon as being handed to you by Bishop Berkeley. You don't get to control it, he does. Moreover all he will tell you about it is that it is a positive number.

Once epsilon is given to you. Your job is to find out how large x has to be in order to guarantee that $f(x) = \frac{1}{x^2} < \varepsilon$. So we work the problem backwards. That is, we start with $f(x) = \frac{1}{x^2} < \varepsilon$ and solve for ε. Thus $x > \frac{1}{\sqrt{\varepsilon}}$.

So if we take $B = \frac{1}{\sqrt{\varepsilon}}$ then it follows that if $x > B = \frac{1}{\sqrt{\varepsilon}}$ then

$$f(x) = \frac{1}{x^2} < \varepsilon.$$

End Of Example #89

Problem #534:

Use Definition 23 to prove that for each of the functions below $\lim\limits_{x\to\infty} f(x) = 0$. That is, assume $\varepsilon > 0$ is given and find a lower bound B such that if $x > B$ then $f(x) < \varepsilon$.

(a) $f(x) = \frac{1}{x^3} = 0$

(b) $f(x) = \frac{1}{x^4} = 0$

(c) $f(x) = \frac{1}{x^5} = 0$

(d) $f(x) = \frac{1}{x+1} = 0$

(e) $f(x) = \frac{1}{2x+1} = 0$

(f) $f(x) = \frac{1}{5x-7} = 0$

Definition 23 works as long as $f(x) > 0$, but without that restriction it fails utterly, as our next example shows:

Example #90:

Suppose $f(x) = -2 - \frac{1}{x}$. What is $\lim\limits_{x\to\infty} f(x)$?

It is intuitively clear that as x increases without bound $\frac{1}{x}$ gets arbitrarily close to zero as before, so clearly $\lim\limits_{x\to\infty} f(x) = -2$. But Definition 23 doesn't capture this. In fact, using Definition 23 we can "prove" that this limit is zero.

To see this let $\varepsilon > 0$ be given. Then B can have *any* positive value since if $x > B$ we have

$$f(x) = -2 - \frac{1}{x} < 0 < \varepsilon,$$

and by Definition 23 we conclude that $\lim\limits_{x\to\infty} f(x) = 0$ since the requirements of our definition have been met.

Of course, this is nonsense. That we are able to "prove" that $f(x)$ goes to zero simply means that Definition 23 doesn't capture everything we need.

We need a better definition of a limit.

End Of Example #90

Based on our experience in Example #90with $f(x) = -2 - \frac{1}{x}$, what would you say needs to be changed?

The problem of course, is that the statement $f(x) < \varepsilon$ doesn't really capture the idea that $f(x)$ is near the number zero, only that it is less than the number ε. For example, -1000 is less than ε but it is nowhere near zero. What we need is a way to measure how far $f(x)$ is from zero, regardless of whether in the positive or the negative direction.

But that is exactly what the absolute value function measures. For example, both 3 and -3 are the same distance, 3, from zero, one in the positive and one in the negative direction, $|3| = |-3| = 3$.

Definition #23 that we want y to be less that ε, when what we really want is for *the distance from* y to zero to be less than ε. We want $|y| < \varepsilon$, not $y < \varepsilon$.

Definition 24: Zero Limit at Infinity

Suppose $f(x)$ is defined for all x. Then we say that $\lim_{x\to\infty} f(x) = 0$ if and only if for every $\varepsilon > 0$ we can find a real number B with the property that whenever $x > B$, $|f(x)| < \varepsilon$.

Example #91:

Suppose $f(x) = -\frac{1}{x}$. We want to prove rigorously that $\lim_{x\to\infty} f(x) = 0$.

Scrapwork

Let $\varepsilon > 0$ be given.

We find the bound B by working the problem backwards. We want to end with $|f(x)| < \varepsilon$, so that's where we start.

$$\begin{aligned} |f(x)| &< \varepsilon \\ |-1/x| &< \varepsilon \\ 1/x &< \varepsilon \\ x &> \frac{1}{\varepsilon}. \end{aligned}$$

Apparently to make $|f(x)| < \varepsilon$ we need $x > \frac{1}{\varepsilon}$, so we take $B = \frac{1}{\varepsilon}$.

End Of Scrapwork

Scrapwork is an essential part of the process of finding a solution but scrapwork is not a part of the proof. Scrapwork is all of the ideas and false starts you make along the way to finding the proof. Because this is a textbook we've only shown you the idea that worked.

Here is the proof.

Proof: Let $\varepsilon > 0$ be given. Take $B > \frac{1}{\varepsilon}$.

If $x > B$ then

$$x > \frac{1}{\varepsilon}.$$

Therefore

$$\frac{1}{x} < \varepsilon$$

and

$$\left|-\frac{1}{x}\right| < \varepsilon$$

and so

$$f(x) < \varepsilon.$$

Therefore $\lim_{x\to\infty} f(x) = 0$. ■

This example displays the format of a limit proof that you need to adhere to. Below is an outline of the format. This is not a course in creative writing. Do not depart from this format.

First: State the challenge, $\varepsilon > 0$.

Second: State the bound, $x > B$.

Third: Show that if $x > B$ then $|f(x)| < \varepsilon$.

Fourth: State your conclusion.

Drill #535:
Identify which statements in the proof in this example correspond to the first, second, third, and fourth parts of the format described above.

End Of Example #91

Problem #536:
Show that $\lim_{x\to\infty} f(x) = 0$.

(a) $f(x) = \frac{1}{x+2}$ **(b)** $f(x) = \frac{1}{x^2}$ **(c)** $f(x) = \frac{1}{x^3}$ **(d)** $f(x) = \frac{1}{x^3+2}$

Definition #24 only tells us when the limit of some function is zero. But as we observed in Example 90 as x increases without bound $f(x) = -2 - \frac{1}{x}$ approaches -2, not 0. We'll need something more general, but all of the important ideas have been introduced. We generalize Definition #24 as follows.

Definition 25: A Limit at $+\infty$

Suppose that L is a real number and that $f(x)$ is defined for all $x > 0$. Then we say that

$$\lim_{x\to\infty} f(x) = L$$

if and only if for every $\varepsilon > 0$ there is a real number B with the property that whenever[7] $x > B$, $|f(x) - L| < \varepsilon$.

[7]Notice that if $L = 0$ this reduces to Definition #24.

Example #92:

Suppose $f(x) = 1 - \frac{1}{x}$. Intuitively, it is clear that $\lim_{x\to\infty} f(x) = 1$ but we need to prove that this is so.

Scrapwork

Suppose $\varepsilon > 0$ is given. We need to find a number B, with the property that if $x > B$ then the following is true:

$$|f(x) - 1| < \varepsilon.$$

So we will work backwards from this inequality.

$$\begin{aligned} |f(x) - 1| &< \varepsilon \\ \left|\left(1 - \tfrac{1}{x}\right) - 1\right| &< \varepsilon \\ \left|-\tfrac{1}{x}\right| &< \varepsilon \\ |x| &> \tfrac{1}{\varepsilon}. \end{aligned}$$

Since we're taking the limit as $x \to \infty$ we are only interested in positive values of x so $|x| = x$. Thus

$$x > \tfrac{1}{\varepsilon}$$

It appears that $|f(x) - 1| < \varepsilon$ as long as $|x|$ is greater than $\frac{1}{\varepsilon}$. Do you see that B can be any number greater than $\frac{1}{\varepsilon}$?

End Of Scrapwork

Proof: Let $\varepsilon > 0$ be given. Take $|x| > B = 1/\varepsilon$. Then

$$\begin{aligned} \left|\tfrac{1}{x}\right| &< \varepsilon \\ \left|\tfrac{-1}{x}\right| &< \varepsilon \\ \left|\left(1 - \tfrac{1}{x}\right) - 1\right| &< \varepsilon \\ |f(x) - 1| &< \varepsilon. \end{aligned}$$

Therefore $\lim_{x\to\infty} f(x) = 1$. ■

End Of Example #92

Example #93:

In Chapter 11.1 we approached the problem of finding a horizontal asymptote of $f(x) = \frac{5x}{x+1}$ in the following highly intuitive manner, using the "$(\to \infty)$" notation.

$$\lim_{x\to\infty} \frac{5x}{x+1} = \lim_{x\to\infty} \frac{5x}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{5}{1+\frac{1}{x}} = \frac{5}{1+(\to\infty)} = \frac{5}{1+0} = 5.$$

So we see that this limit must be equal to five. To prove this rigorously, without referring to infinity, we use Definition 25

Scrapwork

Let $\varepsilon > 0$ be given. As before we work backwards from our goal, $\left|\frac{5x}{x-1} - 5\right| < \varepsilon$.

$$\left|\frac{5x}{x-1} - 5\right| < \varepsilon$$
$$\left|\frac{5x - 5(x+1)}{x+1}\right| < \varepsilon$$
$$\left|\frac{-5}{x+1}\right| < \varepsilon$$
$$\frac{|-5|}{|x+1|} < \varepsilon.$$

As long as $x > -1$ this is the same as

$$\frac{5}{x+1} < \varepsilon. \tag{16.1}$$

so we will stipulate that B (and therefore x) must be at least greater than -1. Solving inequality 16.1 for x we see that

$$x > \frac{5}{\varepsilon} - 1$$

also. So we take to be the greater of -1 and $\frac{5}{\varepsilon} - 1$. We capture this idea with the notation, $B = \max\left(-1, \frac{5}{\varepsilon} - 1\right)$.

End Of Scrapwork

Problem #537:

(a) Show that we only need the condition $B > \frac{5}{\varepsilon} - 1$ by showing that $B > \frac{5}{\varepsilon} - 1$ implies that $B > -1$.

(b) Suppose that $\varepsilon > 0$ and $B = \frac{5}{\varepsilon} - 1$. Prove that if $x > B$ then $\left|\frac{5x}{x+1}\right| < \varepsilon$.

End Of Example #93

Example #94:

We want to prove that $\lim_{x\to\infty} \left(\frac{1}{x^3} + 5\right) = 5$. This time we'll present the formal proof and leave the scrapwork as an exercise.

Proof: Let $\varepsilon > 0$ be given, and take B greater than the larger of zero and $\frac{1}{\sqrt[3]{\varepsilon-5}}$. Then since $x > B$ we have

$$\begin{aligned} x &> \frac{1}{\sqrt[3]{\varepsilon - 5}} \\ \frac{1}{x} &< \sqrt[3]{\varepsilon - 5} \\ \frac{1}{x^3} &< \varepsilon - 5 \\ \frac{1}{x^3} + 5 &< \varepsilon \end{aligned}$$

and since $x > 0$ we see that $\frac{1}{x^3} + 5 = \left|\frac{1}{x^3} + 5\right|$ so

$$\left|\frac{1}{x^3} + 5\right| < \varepsilon$$

and the proof is complete. ■

Problem #538: Do the scrapwork that led to our choice of B in Example 94.

End Of Example #94

Problem #539:
Do the scrapwork, and provide a rigorous proof of each of the limits below.

(a) $\lim_{x\to\infty} \frac{1}{x^{1/3}} = 0$ **(b)** $\lim_{x\to\infty} \left(\frac{2}{x^3} - 1\right) = -1$ **(c)** $\lim_{x\to\infty} \left(\frac{2+x^3}{x^3}\right) = 1$

It should be clear how to define a limit at $-\infty$. All of the same issues of clarity and precision that we encountered before come up here as well. The only difference is that we have to change the sense of our inequalities to reflect that x is decreasing without bound.

Definition 26: A Limit at $-\infty$

Suppose $f(x)$ is defined for all $x < 0$. Then we say that

$$\lim_{x\to-\infty} f(x) = L$$

if and only if for every $\varepsilon > 0$ we can find a real number B with the property that whenever $x < B$, $|f(x) - L| < \varepsilon$.

Problem #540:
Do the scrapwork, and provide a rigorous proof of each of the limits below.

(a) $\lim_{x \to -\infty} \frac{1}{x^{1/3}} = 0$ **(b)** $\lim_{x \to -\infty} \left(\frac{2}{x^3} - 1\right) = -1$ **(c)** $\lim_{x \to -\infty} \left(\frac{2+7x^3}{x^3}\right) = 7$

16.2 Limits at a Real Number

In the previous section we were focused on the relatively simple limits associated with horizontal asymptotes. But our goal is to use the limit in Definition #15 to prove that the Differentiation Rules we've been using are valid. To do that we will the following precise, rigorous definition of a limit as $x \to a$ where a is some real number.

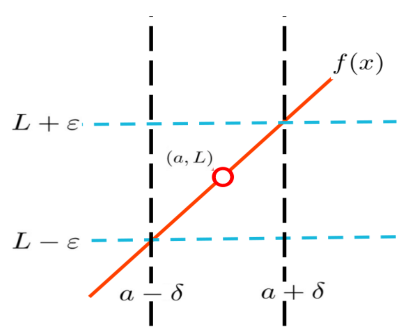
Definition 27: The Limit at a Point
Given a function $f : \mathbb{R} \to \mathbb{R}$, and a real number, a, we say that

$$\lim_{x \to a} f(x) = L$$

for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

We use this definition in a manner similar to the way we used Definition 25: We take $\varepsilon > 0$ as a challenge and find conditions which guarantee that $|f(x) - L| \leq \varepsilon$. Previously we were interested in the behavior of $f(x)$ "close to infinity" so the condition was that $x > B$ (that x was large enough, loosely speaking). Now we are interested in the behavior of $f(x)$ "close to a" so the condition will indicate how close x has to be to a. The $\delta > 0$ in Definition #27 plays the same role that the upper bound B played in Definitions #25 and #26. It locates where x must be in order to guarantee that $|f(x) - L| < \varepsilon$.

The sketch at the right depicts the situation when $\lim_{x \to a} f(x) = L$ visually. The graph of $f(x)$ appears to be a straight line because we are zoomed in very close to the point (a, L) and the Principle of Local Linearity is in play. We have indicated that $f(x)$ is not defined at δ, but it could be. Limits don't care what happens *at* δ, only what happens near δ.



As before ε is the challenge. To show that the limit exists and is equal to L as claimed our task is to find a value of δ such that as long as x is between $a - \delta$ and $a + \delta$ the corresponding $f(x)$ will be between $L - \varepsilon$ and $L + \varepsilon$. Visually, this means that the graph of $f(x)$ will be between the dashed horizontal lines as long as x is between the dotted vertical lines.

Drill #541: Explain why δ must depend on ε.

Definition #27 is the culmination of approximately 200 of effort by some very brilliant people to provide a rigorous foundation for Calculus. Don't expect to absorb this easily. It will take time and effort to fully understand and be able to use it. We will start simply.

Example #95:

Suppose $f(x) = -x^2 + 2$. Show that $\lim_{x \to 0} f(x) = 2$.

First observe that this is obviously a true statement. Our goal is to verify, rigorously, what we already understand, intuitively, to be true.

Scrapwork

Suppose $\varepsilon > 0$ is given. Our goal is to find a $\delta > 0$ such that if[8] $0 < |x - 2| < \delta$ then

$$|f(x) - 2| < \varepsilon.$$

Solving this for x we have

$$\begin{aligned} \left|-x^2 + 2 - 2\right| &< \varepsilon \\ \left|-x^2\right| &< \varepsilon \\ \left|x^2\right| &< \varepsilon. \end{aligned}$$

Recall from Digression #20 that $\sqrt{x^2} = |x|$, so we see that $|x| < \sqrt{\varepsilon}$ or, equivalently

$$-\sqrt{\varepsilon} < |x| < \sqrt{\varepsilon}.$$

It appears that $|f(x) - 2| < \varepsilon$ as long as $|x| < \sqrt{\varepsilon}$, so we take $\delta = \sqrt{\varepsilon}$.

End Of Scrapwork

But this was the scrapwork. The proof consists of showing that the condition we found, $|x| < \delta (= \sqrt{\varepsilon})$, actually works.

Proof: Let $\varepsilon > 0$ be given. Take $\delta = \sqrt{\varepsilon}$.

Assume that $0 < |x - 2| < \delta$. Then

$$\begin{aligned} |x| &< \delta \\ \sqrt{x^2} &< \sqrt{\varepsilon} \\ x^2 &< \varepsilon \\ \left|-x^2\right| &< \varepsilon \\ \left|-x^2 + 2 - 2\right| &< \varepsilon \\ |f(x) - 2| &< \varepsilon. \end{aligned}$$

Therefore $\lim_{x \to 0} f(x) = 2$.

[8] We write $0 < |x - 2| < \delta$ rather than $|x - 2| < \delta$ because "$|x - a| < \delta$" leaves open the possibility that $x - a = 0$ (equivalently, $x = a$). In order to leave no doubt that we are not considering the case, $x = a$, we write $0 < |x - a| < \delta$. This level of precision in our language is what rigor requires.

Incidentally, since $f(0) = -0^2 + 2 = 2$ we have just proved that $f(x)$ is continuous at $x = 0$ as well. See Definition #17.

■

For simple problems like this one the proof consisted of writing the algebraic steps from our scrapwork backwards, as you see. This worked because every algebraic step in the scrapwork was reversible. But don't jump to conclusions. This will not always be the case.

Clearly the scrapwork is the most important part of the solution of this problem. In a very real sense it actually is the solution. We call it scrapwork because it is the part of the work that you don't show anyone else because it is messy and not well organized[9]. The scrapwork is like the scaffolding used to construct a building. The proof is the building.

It is absolutely necessary to have the scaffolding while construction is ongoing but you tear it down and clean everything up before you move in. In the same way your proof should be a cleaned up version of your scrapwork. If this example were a homework problem then your solution would be the part that appears between **Proof:** and the little black[10] square, ■, at the end.

End Of Example #95

Example #96:

Returning to Example #82 from Section 13.1 recall that we had

$$f(x) = \begin{cases} 3x & \text{if } x \neq 2 \\ 10 & \text{if } x = 2. \end{cases}$$

We had shown by an intuitive argument that $\lim_{x \to 2} f(x) = 6$. Our previous proof lacked rigor, especially in the last step. We will provide a fully rigorous proof now.

Proof: Let $\varepsilon > 0$ be given and take $\delta < \frac{\varepsilon}{3}$. Then if $0 < |x - 2| < \delta$ we have

$$\begin{aligned} -\delta < x - 2 < \delta \\ -\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3} \\ -\varepsilon < 3x - 6 < \varepsilon \\ -\varepsilon < f(x) - 6 < \varepsilon \end{aligned}$$

and so

$$|f(x) - 6| < \varepsilon.$$

Therefore $\lim_{x \to 2} f(x) = 6$. ■

Notice that it is again irrelevant that $f(2) = 10$.

[9] We kept it clean and orderly here so you could see the reasoning.

[10] This black square is called a "Halmos" and some variation of it has become the traditional marker at the end of a proof. It is named after Paul Halmos, a preeminent mathematician of the 20th century.

Problem #542:
Use the proof above to recreate the scrapwork that we did before we wrote the proof.

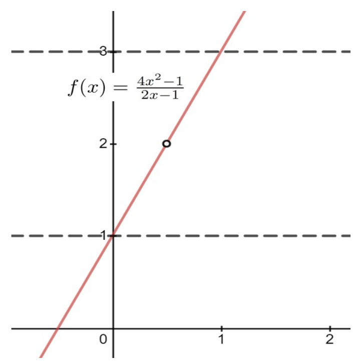
End Of Example #96

Example #97:

We would like to find the value of $\displaystyle\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}$, and prove that the value we find is correct.

Notice that none of our limit definitions tell us how to find the value of a limit, only how to prove that it has a particular value after we've found it. In our examples so far the value of the limits have been fairly clear intuitively so we haven't concerned ourselves with this part of the problem. But before we can prove that a limit has a particular value we obviously need to decide what we believe that value is.

We have several options for doing this. The simplest is guessing, but of course blind guessing is usually unhelpful. If we guess wrong we can waste a lot of time trying to prove that a limit is equal to the wrong value. Nevertheless, guessing is always an option. Can you guess the value of this limit?



Another simple option is to use a calculator and plug the value of the limit point, in this case $x=\frac{1}{2}$, and see what the calculator comes up with. This will work if the function is continuous at the limit point. But $\frac{4x^2-1}{2x-1}$ is not continuous at $x=\frac{1}{2}$ so that won't help with this problem. Try it and see.

We can also sketch the graph of $f(x)$ to see what $f(x)$ is close to near the limit point. The graph of $f(x)=\frac{4x^2-1}{2x-1}$ is given at the right. It is not defined at $x=1/2$ because when $x=1/2$ we get a zero denominator. Nevertheless the limit at $x=1/2$ seems to exist. As you can see as x approaches $1/2$, $f(x)$ appears to approach 2. Based on this graph it seems likely that the value of the limit is 2.

Algebra can help us evaluate this limit too since

$$\frac{4x^2-1}{2x-1}=\frac{(2x+1)(2x-1)}{2x-1}=(2x+1)\cdot\overbrace{\frac{\cancel{2x-1}}{\cancel{2x-1}}}^{=1}=(2x+1)$$

where the cancellation in red is only valid when $x\neq\frac{1}{2}$, our limit point is $x=\frac{1}{2}$ so it is not under consideration.

In fact, once Theorem #16 and Theorem # 18 have been proven (see Section 16.3) the following will be a valid proof:

$$\begin{aligned}\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1} &= \lim_{x\to\frac{1}{2}}\frac{(2x+1)(2x-1)}{2x-1}\\ &= \left[\lim_{x\to\frac{1}{2}}(2x+1)\right]\left[\lim_{x\to\frac{1}{2}}\cancelto{=1}{\frac{2x-1}{2x-1}}\right]\\ &= \lim_{x\to\frac{1}{2}}(2x+1)\\ &= 2.\end{aligned}$$

Problem #543:
Identify where we used Theorem #16 and Theorem # 18 in the development above.

Because we haven't yet proved either Theorem #16 or Theorem # 18 rigorously we can't use them to construct a rigorous proof. Until they are proved they are not known, they are just believed. Belief is not knowledge.

But there is no problem with using them to gather evidence. So far we have strong evidence that this limit is 2. Ordinarily, given this much evidence we would set about constructing the proof. But there is one other evidence gathering technique we can try. This is a L'Hôpital Indeterminate so we can apply L'Hôpital's Rule:

$$\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1} \overset{\mathcal{L}'\mathcal{H}}{=} \lim_{x\to\frac{1}{2}}\frac{8x}{2} = 2. \tag{16.2}$$

If we had proved L'Hôpital's Rule rigorously Equation (16.2) would also be a valid, rigorous proof. But we have not. In fact, we will not be proving L'Hôpital's Rule in this text, so we can never use it for more than evidence gathering[11].

Having gathered our evidence, we now believe that $\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}=2$. Next we need to do the scrapwork for our proof.

Scrapwork

As always we work backwards from our goal. We need to show that $|f(x)-2|<\varepsilon$ so

$$\left|\frac{4x^2-1}{2x-1}-2\right|<\varepsilon.$$

Since we're not interested in the value of anything *at* $x=\frac{1}{2}$ (where the denominator is zero) we can factor and cancel, giving

$$\begin{aligned}\left|\frac{(2x+1)\cancel{(2x-1)}}{\cancel{2x-1}}-2\right| &< \varepsilon\\ |2x+1-2| &< \varepsilon\\ |2x-1| &< \varepsilon.\end{aligned}$$

[11]It is true though. You can trust us.

The Absolute Value function makes this difficult to think about but we can remove the Absolute Value bars by observing that, in general,

$$|x| < B \text{ if and only if } -B < x < B. \tag{16.3}$$

Proceeding, we see that if $|2x-1| < \varepsilon$ then

$$-\varepsilon < 2x - 1 < \varepsilon$$

or

$$\frac{1}{2} - \frac{\varepsilon}{2} < x < \frac{1}{2} + \frac{\varepsilon}{2}.$$

So taking $\delta = \frac{\varepsilon}{2}$ will work for this problem.

End Of Scrapwork

Proof: Let $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{2}$. Then if $0 < \left|x - \frac{1}{2}\right| < \delta$ we have

$$\begin{aligned}
\left|x - \frac{1}{2}\right| &< \frac{\varepsilon}{2} \\
-\frac{\varepsilon}{2} < x - \frac{1}{2} &< \frac{\varepsilon}{2} \\
-\varepsilon < 2x - 1 &< \varepsilon \\
-\varepsilon < 2x + 1 - 2 &< \varepsilon \\
-\varepsilon < \frac{(2x+1)(2x-1)}{2x-1} - 2 &< \varepsilon \\
-\varepsilon < \frac{4x^2-1}{2x+1} - 2 &< \varepsilon \\
-\varepsilon < f(x) - 2 &< \varepsilon,
\end{aligned}$$

and so

$$|f(x) - 2| < \varepsilon.$$

Therefore $\lim_{x \to \frac{1}{2}} \frac{4x^2-1}{2x+1} = 2$. ■

A long list of inequalities like those above can be a little intimidating. Don't let that stop you. Verify each transition from one inequality to the other. If you don't see why a particular transition is valid refer back to the scrapwork.

End Of Example #97

Example #98:

In this example we will show rigorously that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. To do that we need to show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x.$$

Recall that differentiation is a local property so we are thinking of x as a fixed, but unspecified real number. The variable in this example is h.

Scrapwork

For $\varepsilon > 0$ we need to find $\delta > 0$ such that if $|h| < \delta$, then $\left|\frac{f(x+h)-f(x)}{h} - 2x\right| < \varepsilon$. Working backwards from this we have

$$\begin{aligned}
\left|\frac{(a+h)^2 - a^2}{h} - 2a\right| &< \varepsilon \\
\left|\frac{a^2 + 2ah + h^2 - a^2}{h} - 2a\right| &< \varepsilon \\
\left|\frac{2ah + h^2}{h} - 2a\right| &< \varepsilon \\
|2a + h - 2a| &< \varepsilon \\
|h| &< \varepsilon \\
-\varepsilon < h &< \varepsilon,
\end{aligned}$$

so we choose $\delta = \varepsilon$.

End Of Scrapwork

Problem #544:

(a) Use the scrapwork above to prove that if $f(x) = x^2$ then $f'(x) = 2x$.

(b) Use Definition 27 to prove that if $f(x) = x^2$ then

$$f'(x) = \lim_{a \to x} \frac{f(a) - f(x)}{a - x} \tag{16.4}$$

is also equal to $2x$.

(c) If $f(x)$ is an arbitrary function, differentiable at x, show by an appropriate change of variable, that Equation (16.4) is equivalent to the limit in Definition #15.
(**Hint:** In Equation (16.4) identify h Definition 27.)

End Of Example #98

16.3 Limit Laws (Theorems)

If we can prove each of the limit laws in Chapter 13 rigorously (i.e, using the Definition #15) and Definition #27 we will have provided a rigorous foundation for Differential Calculus. We will address these now.

The Limit Laws we need to prove are:

Theorem 18: The Limit of a Constant is the Constant.

Theorem 15: The Limit of a Sum is the Sum of the Limits

Theorem 16: The Limit of a Product is the Product of the Limits

Theorem 22: The Limit of a Composition is the Composition of the Limits, and

Theorem 21: The Squeeze Theorem

We will provide a proof of the "limit at infinity" version of each of these Theorems using Definition #25. We will leave the proof of the "limit at a real number" version using the Definition #27 as an exercise for you. In every case you can model your scrapwork and proof on the ones we provide.

Theorem 39: **[The Limit at Infinity of a Constant Function is the Constant]**
Suppose U and K are real numbers. If $f(x) = K$ for all $x > U$ then $\lim_{x\to\infty} f(x) = K$.

Proof: Let $\varepsilon > 0$ be given. Take $B = 2$. Thus if $x > B$ we see that

$$|f(x) - K| = |K - K| = 0 < \varepsilon.$$

Therefore $\lim_{x\to\infty} f(x) = K$. ■

The formalism of our proof requires that we specify some value for B, so we specified $B = 2$. Any other value for B would work as well in this proof.

Problem #545:
Use the proof of Theorem 39 as a model to construct a proof of Theorem #40 below.

Theorem 40: **[The Limit at Negative Infinity of a Constant Function is the Constant]**
Suppose L and K are real numbers. If $f(x) = K$ for all $x < L$ then $\lim_{x\to-\infty} f(x) = K$.

Problem #546:
Use the proof of Theorem 39 as a model to construct a proof of Theorem #41 below.

Theorem 41: **[The Limit at a Point of a Constant Function is the Constant]**
Suppose K is a real number and a is a point in the domain of $f(x)$. If $f(x) = K$ near a then $\lim_{x\to a} f(x) = K$.
(**Hint:** Recall Definition 16.)

The Limit of a Sum

Theorem 42: **[The Limit of a Sum at Infinity]**
If $\lim_{x\to\infty} f(x) = L_f$ and $\lim_{x\to\infty} g(x) = L_g$ then $\lim_{x\to\infty} (f(x) + g(x)) = L_f + L_g$.

Scrapwork

As always we begin by assuming that $\varepsilon > 0$ has been given.

We want to show that if x is large enough (larger than B) then

$$|(f(x) + g(x)) - (L_f + L_g)| < \varepsilon, \tag{16.5}$$

but the only information we have to work with is the knowledge that

$$\lim_{x\to\infty} f(x) = L_f \text{ and } \lim_{x\to\infty} g(x) = L_g.$$

Observe that the statement $\lim_{x\to\infty} f(x) = L_f$ means that there is a real number we'll call B_f such that if $x > B_f$ then $|f(x) - L_f| < \frac{\varepsilon}{2}$. Similarly[12] there is a B_g such that if $x > B_g$ then $|g(x) - L_g| < \frac{\varepsilon}{2}$.

Rearranging Inequality (16.5) we see that we need to show that

$$|(f(x) - L_f) + (g(x) - L_g)| < \varepsilon \tag{16.6}$$

If we could assert that $|x + y| = |x| + |y|$ we could choose $\delta > 0$ sufficiently small that both

$$|(f(x) - L_f)| < \frac{\varepsilon}{2} \text{ and } |(g(x) - L_g)| < \frac{\varepsilon}{2}$$

so their sum would be less than ε.

Unfortunately it is not always true that $|x + y| = |x| + |y|$.

Fortunately it is always true that $|x + y| \leq |x| + |y|$. This fact is called the **Triangle Inequality**, but it requires proof.

Theorem 43: **[The Triangle Inequality]**
For any real numbers x, and y, $|x + y| \leq |x| + |y|$.

The proof of the Triangle Inequality relies on the fact that if A and B are real numbers then

$$|A| \leq B \text{ if and only if } -B \leq A \leq B. \tag{16.7}$$

(Recall (16.3) in Section 16.2.)

Proof of The Triangle Inequality: Clearly $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding

[12] You will see why we're using $\frac{\varepsilon}{2}$ rather than ε shortly.

these together we have

$$\begin{aligned} -|x| - |y| &\le x + y \le |x| + |y| \\ \underbrace{-(|x| + |y|)}_{-B} &\le \underbrace{x + y}_{A} \le \underbrace{|x| + |y|}_{B}. \end{aligned}$$

so by Equation (16.3) in Section 16.2 we have $|x + y| \le |x| + |y|$. ■

Continuing from Equation (16.6) we see that if $|x|$ is greater than both of B_f and B_g then

$$|(f(x) - L_f) + (g(x) - L_g)| \le |(f(x) - L_f)| + |g(x) - L_g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

End Of Scrapwork

Proof: Let $\varepsilon > 0$ be given.

Since $\lim\limits_{x\to\infty} f(x) = L_f$ there is a bound, B_f such that if $x > B_f$ then $|f(x) - L_f| < \frac{\varepsilon}{2}$.

Since $\lim\limits_{x\to\infty} g(x) = L_g$ there is a bound, B_g such that if $x > B_g$ then $|g(x) - L_g| < \frac{\varepsilon}{2}$.

Take B to be the larger of B_f and B_g. Then if $x > B$ then $x > B_f$ and $x > B_g$. Thus

$$\begin{aligned} |(f(x) + g(x)) - (L_f + L_g)| &= |(f(x) - L_f) + (g(x) - L_g)| \\ &\le |f(x) - L_f| + |g(x) - L_g|. \end{aligned}$$

Since $x > B_g$ and $x > B_f$ we see that

$$\begin{aligned} |f(x) + g(x) - L_f + L_g| &\le |f(x) - L_f| + |g(x) - L_g| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore $\lim\limits_{x\to\infty} (f(x) + g(x)) = L_f + L_g$. ■

Problem #547:
Use the proof of Theorem 42 as a model to construct a proof of Theorem #44 below.

Theorem 44: **[The Limit of a Sum at Negative Infinity]**
If $\lim\limits_{x\to\infty} f(x) = L_f$ and $\lim\limits_{x\to\infty} g(x) = L_g$ then $\lim\limits_{x\to\infty} (f(x) + g(x)) = L_f + L_g$.

Problem #548:
Use the proof of Theorem 42 as a model to construct a proof of Theorem #45 below.

Theorem 45: **[Limit of a Sum at a Point]**
Suppose a is some real number, that $\lim_{x\to a} f(x) = L_f$ and that $\lim_{x\to a} g(x) = L_g$. Then

$$\lim_{x\to a} (f(x) + g(x)) = L_f + L_g.$$

The Squeeze Theorem

Theorem 46: **[The Squeeze Theorem at Infinity]**
If $\alpha(x) \leq f(x) \leq \beta(x)$ on some interval, (c, ∞) and

$$\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = L$$

then $\lim_{x\to\infty} f(x) = L$ also.

Proof: Since $\lim_{x\to\infty} \alpha(x) = L$ there is some a real number B_α, such that if $x > B_\alpha$ then $|\alpha(x) - L| < \varepsilon$. From Equation (16.7) we see that for $x > B_\alpha$:

$$-\varepsilon < \alpha(x) - L < \varepsilon.$$

Similarly there is a real number B_β, such that if $x > B_\beta$ then $|\beta(x) - L| < \varepsilon$ so for $x > B_\beta$:

$$-\varepsilon < \beta(x) - L < \varepsilon.$$

Take B to be greater than the maximum of c, B_α, and B_β. Then if $x > B$

$$-\varepsilon < \alpha(x) - L < f(x) - L < \beta(x) - L < \varepsilon.$$

In particular, $-\varepsilon < f(x) - L < \varepsilon$ so from Equation (16.7) we see that $|f(x) - L| < \varepsilon$.
Therefore $\lim_{x\to\infty} f(x) = L$. ■

Problem #549: Use the proof of Theorem 46 as a model to prove Theorem!47 below.

Theorem 47: **[The Squeeze Theorem at Negative Infinity]**
If $\alpha(x) \leq f(x) \leq \beta(x)$ on some interval, $(c, -\infty)$ and

$$\lim_{x\to-\infty} \alpha(x) = \lim_{x\to-\infty} \beta(x) = L$$

then $\lim_{x\to-\infty} f(x) = L$ also.

Problem #550: Use the proof of Theorem 46 as a model to prove Theorem!48 below.

Theorem 48: **[The Squeeze Theorem, at a Point]**

If $\alpha(x) \leq f(x) \leq \beta(x)$ near c and $\lim\limits_{x\to c} \alpha(x) = \lim\limits_{x\to c} \beta(x) = L$ then $\lim\limits_{x\to c} f(x) = L$ also.

The Limit of a Composition

Theorem 49: **[The Limit of a Composition at Infinity]**

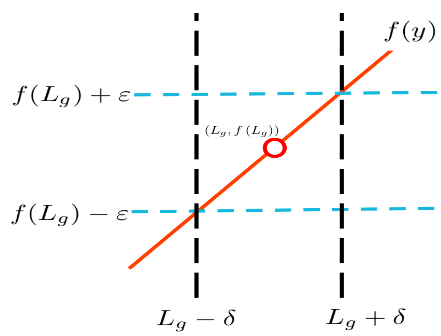
Suppose $\lim\limits_{x\to\infty} g(x) = L_g$ and $f(y)$ is continuous at $y = L_g$. Then

$$\lim_{x\to\infty} f(g(x)) = f(L_g) = f\left(\lim_{x\to\infty} g(x)\right).$$

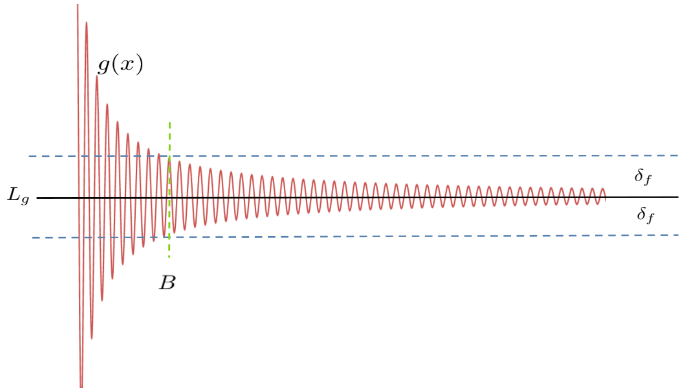
Scrapwork

It is extremely helpful to have a visual guide for this proof so we will rely on diagrams here in the scrapwork. The proof below will not.

First, consider what it means to say that f is continuous at $y = L_g$. From Definition #17 we see that it means that $\lim\limits_{y\to L_g} f(y) = f(L_g)$. But from Definition #27 we also see that this means there is a real number $\delta > 0$ such that if $|y - L_g| < \delta$ then $|f(y) - f(L_g)| < \varepsilon$, as visualized in the sketch at the right.



Next, consider what it means to say that $\lim\limits_{x\to\infty} g(x) = L_g$. It means that for every $\delta_f > 0$, there is a real number B, such that for all $x > B$, $|g(x) - L_G| < \delta_f$, as visualized in the sketch below.



If we take δ_f in this sketch to be the same as δ from the previous sketch we see that if $x > B$ then

$$|g(x) - L_g| < \delta_f = \delta.$$

But since $y = g(x)$ we have

$$|y - L_g| = |g(x) - L_g| < \delta_f = \delta.$$

From the continuity of f at L_g we see that $|y - L_g| = |g(x) - L_g| < \delta_f = \delta$ means that

$$|f(g(x)) - f(L_g)| = |f(y) - f(L_g)| < \varepsilon.$$

End Of Scrapwork

Proof: Let $\varepsilon > 0$.

Since $f(y)$ is continuous at L_g there is a real number $\delta > 0$ such that

$$\text{if } |y - L_g| < \delta \text{ then } |f(y) - f(L_g)| < \varepsilon.$$

Since $\lim_{x\to\infty} g(x) = L_g$ there is a real number B such that for every $\delta_f > 0$

$$\text{if } x > B \text{ then } |g(x) - L_g| < \delta_f.$$

Take B be the upper bound we need to guarantee that $|g(x) - L_g| < \delta$. Then since $y = g(x)$

$$|y - L_g)| = |g(x) - L_g| < \delta,$$

and since $|y - L_g)| \, \delta$ we see, from the continuity of f at L_g that $|f(y) - f(L_g)| < \varepsilon$.

Therefore $\lim_{x\to\infty} f(g(x)) = f(L_g) = f\left(\lim_{x\to\infty} g(x)\right)$.

■

Problem #551:
Use the proof of Theorem 49 as a model to construct a proof of Theorem 50 below.

Theorem 50: **[The Limit of a Composition at Negative Infinity]**
Suppose $\lim\limits_{x\to-\infty} g(x) = L_g$ and $f(y)$ is continuous at $y = L_g$. Then

$$\lim_{x\to-\infty} f(g(x)) = f(L_g) = f\left(\lim_{x\to-\infty} g(x)\right).$$

Problem #552:
Use the proof of Theorem 49 as a model to construct a proof of Theorem 51 below.

Theorem 51: **[The Limit of a Composition at a Point]**
Suppose $g(x)$ is continuous at $x = a$, that $\lim\limits_{x\to a} g(x) = g(a)$, and that $f(x)$ is continuous at $g(a)$. Then $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(g(a))$.

The Limit of a Product

Theorem 52: **[The Limit of a Product at Infinity]**
If $\lim\limits_{x\to\infty} f(x) = L_f$ and $\lim\limits_{x\to\infty} g(x) = L_g$ then $\lim\limits_{x\to\infty} (f(x)\cdot g(x)) = L_f \cdot L_g$.

It is surprisingly difficult to to put together everything we need for a valid, rigorous proof of Theorem #52.

Scrapwork

We begin by assuming that $\varepsilon > 0$ has been given.
We need to make

$$|f(x)\cdot g(x) - L_g \cdot L_f| < \varepsilon.$$

To pull these apart, we will use a trick that is really not intuitive. We can write the left-hand side as

$$|f(x)\cdot g(x) - L_g \cdot L_f| = \left| f(x)\cdot g(x) \underbrace{- f(x)\cdot L_g + f(x)\cdot L_g}_{=0} - L_f \cdot L_g \right|.$$

by "uncanceling" the terms $-f(x)\cdot L_g + f(x)\cdot L_g$ as indicated.

Again, there is no intuition in this step; it is a trick, motivated by our desire to pull things apart and use the Triangle Inequality. We learned from our teachers, as you are now learning it from yours. We (the authors) have called this "uncanceling." Most mathematicians call it "adding zero" since middle terms add to zero, but the name is unimportant. This trick is often used in mathematics and it is hard to tell *a priori* when it will work. Sometimes you just have to try something and see what happens.

We are now able to use the triangle inequality.

$$\begin{aligned}|f(x)\cdot g(x)-L_g\cdot L_f| &= |f(x)\cdot g(x)-f(x)\cdot L_g+f(x)\cdot L_g-L_f\cdot L_g| \\ &\leq |f(x)\cdot g(x)-f(x)\cdot L_g|+|f(x)\cdot L_g-L_f\cdot L_g|\end{aligned}$$

so that

$$|f(x)\cdot g(x)-L_g\cdot L_f| = |f(x)|\,|g(x)-L_g|+|f(x)-L_f|\,|L_g|. \tag{16.8}$$

Next we need to show that for $|x|$ sufficiently large each of the terms in Equation (16.8) is less than $\frac{\varepsilon}{2}$ much as we did in the proof of Theorem #42 limit of a sum before. Once we've done that the proof will be complete since

$$|f(x)\cdot g(x)-L_g\cdot L_f| = |f(x)|\,|g(x)-L_g|+|f(x)-L_f|\,|L_g| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Notice that the two terms in Equation (16.8) are similar. In the first term we have the factor $|f(x)-L_f|$ which we can control by taking x sufficiently large since we know that $\lim\limits_{x\to\infty} f(x)=L_f$. Similarly in the second term we have the factor $|g(x)-L_g|$. It is the other two factors, $|f(x)|$ and $|L_g|$ that will give us difficulty.

We'll show that if x is sufficiently large then the first term is will be less than $\frac{\varepsilon}{2}$ and leave the proof for the second term as an exercise for you.

To guarantee that $|f(x)|\,|g(x)-L_g| < \frac{\varepsilon}{2}$ we need to ensure that $|g(x)-L_g| < \frac{\varepsilon}{2|f(x)|}$. There is an "obvious" strategy to do that which fails for a very subtle reason. We will digress for a moment to examine that difficulty. You may want to skip Digression #25 for now and come back after the proof is see why we made the choices we do.

Digression #25: A very subtle difficulty

Since $\lim\limits_{x\to\infty} g(x)=L_g$ we know from Definition #25 that there is a real number B such that if $x>B$ then $|g(x)-L_g|$ is less than any positive number. Since $\frac{\varepsilon}{2|f(x)|}>0$ it appears that we need only choose B so that $|g(x)-L_g|<\frac{\varepsilon}{2|f(x)|}$.

But we can't. Here's why we can't.

Definition #25 says that if $\lim\limits_{x\to\infty} g(x)=L_g$ then for any *single* real number ε we can find a B such that if $x>B$ then $|g(x)-L_g|<\varepsilon$. But $\frac{\varepsilon}{2|f(x)|}$ is not a single real number. For each distinct value of $x>B$ we'll have a (possibly) different value of $f(x)$.

We have no definition, and no theorem that says we can do this. So we can't.

End Of Digression #25

We need to replace $|f(x)|$ by something which is fixed. The following lemma will allow us to do that.

Lemma 53:

If $\lim\limits_{x\to\infty} f(x)=L_f$ then there are positive real numbers N and β, such that if $x>\beta$, then $|f(x)|<N$.

Drill #553: Draw a convincing diagram of Lemma #53).

Proof: Since $\lim_{x\to\infty} f(x) = L_f$ there is a positive number β such that if $x > \beta$ then $|f(x) - L_f| < 1$. So for $x > \beta$,

$$-1 < f(x) - L_f < 1$$
$$L_f - 1 < f(x) < L_f + 1.$$

Now choose any positive number N with

$$-N < L_f - 1 < f(x) < L_f + 1 < N,$$

and it follows that if $x > \beta$ then $|f(x)| < N$. ■

From Lemma #53) we see that there real numbers N and β, such that for all $x > \beta$

$$|f(x)|\,|g(x) - L_g| < N\,|g(x) - L_g|\,.$$

Since $\lim_{x\to\infty} g(x) = L_g$ there is a real number B such that for all $x > B$

$$|g(x) - L_g| < \frac{\varepsilon}{2N}.$$

Take B_g to be the larger of β and B. Then for all $x > B_g$ we have

$$\begin{aligned} |f(x)|\,|g(x) - L_g| &< N\,|g(x) - L_g| \\ &< \frac{\varepsilon}{2N\,|g(x) - L_g|}\,|g(x) - L_g| \end{aligned}$$

so that

$$|f(x)|\,|g(x) - L_g| < \frac{\varepsilon}{2},$$

which completes our argument for the first term in Equation (16.8).

The argument for the second term in Equation (16.8) has a similar, but less troublesome difficulty as we will see.

To guarantee that

$$|f(x) - L_f|\,|L_g| \le \frac{\varepsilon}{2}$$

we need to ensure that $|f(x) - L_f| < \frac{\varepsilon}{2|L_g|}$. But what if $L_g = 0$? In that case $\frac{\varepsilon}{2|L_g|}$ is meaningless.

Drill #554:
Explain why we can handle the case $L_g = 0$ separately by observing that if $L_g = 0$ then

$$|f(x) - L_f|\,|L_g| = |f(x) - L_f| \cdot 0 = 0 \le \frac{\varepsilon}{2}.$$

To handle this we will make the following small change to Inequality (16.8):

$$|f(x) \cdot g(x) - L_g \cdot L_f| \leq |f(x)|\,|g(x) - L_g| + |f(x) - L_f|\,\left(|L|_g + 1\right) \tag{16.9}$$

This change makes the right side slightly larger, but Inequality (16.9) is still true[13]. Now there is no risk of dividing by zero since $|L_g| + 1 > 0$ regardless of the value of L_g.

Since $\lim\limits_{x\to\infty} f(x) = L_f$ there is a real number B_f such that if $|x| > B_f$ then

$$|f(x) - L_f| < \frac{\varepsilon}{2(|L_g| + 1)}.$$

Thus we see that for all $|x| > B_f$

$$\begin{aligned} |f(x) - L_f|\,|L_g| &< |f(x) - L_f|\,(|L_g| + 1) \\ &< \frac{\varepsilon}{2(|L_g| + 1)}(|L_g| + 1) \end{aligned}$$

and so

$$|f(x) - L_f|\,|L_g| = \frac{\varepsilon}{2}.$$

End Of Scrapwork

Proof of Theorem 52: Let $\varepsilon > 0$ be given.

By Lemma 53 there is a real number N and a real number B_g such that if $x > B_f$

$$|g(x) - L_g| < \frac{\varepsilon}{2N}$$

Since $\lim\limits_{x\to\infty} f(x) = L_f$ there is a real number B_f such that if $x > B_f$

$$|f(x) - L_f| < \frac{\varepsilon}{2\,(|L_g| + 1)}.$$

Let B be the larger of the three numbers B_f and B_g. Then

$$\begin{aligned} |f(x)g(x) - L_gL_f| &= |f(x)g(x) - f(x)L_g + f(x)L_g - L_fL_g| \\ &\leq |f(x)g(x) - f(x)L_g| + |f(x)L_g - L_fL_g| \\ &= |f(x)|\,|g(x) - L_g| + |f(x) - L_f|\,|L_g| \\ &< |f(x)|\,|g(x) - L_g| + |f(x) - L_f|\,|L_g| \\ &< N\,|g(x) - L_g| + |f(x) - L_f|\,(|L_g| + 1) \\ &< N\left(\frac{\varepsilon}{2N}\right) + \frac{\varepsilon}{2\,(|L_g| + 1)}\,(|L_g| + 1) \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

[13] This is a nice property of inequalities that equations don't share. If you change one side of an equation you must make the same change to the other side. But if you make the greater (lesser) side of an inequality even greater (lesser), the inequality remains true.

Thus, by definition, $\lim\limits_{x\to\infty} f(x)g(x) = L_g L_f$. ■

This proof was not for the faint of heart. None of the individual pieces was difficult to follow, but putting them all together in the right order was delicate.

If you found yourself becoming completely absorbed in, and possibly even enjoying the details of this argument you are surely a mathematician by inclination, if not (yet) by training. There is more where that came from. Change your major and come join us. Sometimes we bring cookies.

Problem #555: Use our proof of Theorem 52 as a model to prove Theorem 54 below.

Theorem 54: [**The Limit of a Product at Negative Infinity**]
If $\lim\limits_{x\to-\infty} f(x) = L_f$ and $\lim\limits_{x\to-\infty} g(x) = L_g$ then $\lim\limits_{x\to-\infty} (f(x)\cdot g(x)) = L_f \cdot L_g$.

Problem #556: Use our proof of Theorem 52 as a model to prove Theorem 55 below.

Theorem 55: [**Limit of a Product at a Point**]
Suppose a is some real number, that $\lim\limits_{x\to a} f(x) = L_f$ and $\lim\limits_{x\to a} g(x) = L_g$. Then

$$\lim_{x\to a} (f(x)g(x)) = L_f L_g.$$

Finally, we come to the limit of a quotient. Given how much trouble the limit of a product gave us it is a little scary to think about proving that the limit of a quotient is the quotient of a limit from the definition.

This trepidation is justified. It is very tricky to prove the quotient theorem for the definition. Fortunately we don't have to. We now have enough theorems (tools) available to use them instead of working directly from the definition.

Theorem 56: [**The Limit of a Quotient is the Quotient of the Limits**]
Suppose a is some real number, that $\lim\limits_{x\to a} f(x) = L_f$ and $\lim\limits_{x\to a} g(x) = L_g \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_f}{L_g}.$$

Problem #557:

(a) Prove that if $\lim_{x\to\infty} f(x) = L_f$ and $\lim_{x\to\infty} g(x) = L_g \neq 0$ then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L_f}{L_g}.$$

(**Hint:** Rewrite $\frac{f(x)}{g(x)}$ as $f(x)(g(x))^{-1}$. Which of our known theorems can you apply.)

(b) Prove that if $\lim_{x\to-\infty} f(x) = L_f$ and $\lim_{x\to-\infty} g(x) = L_g \neq 0$ then

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \frac{L_f}{L_g}.$$

(c) Prove that if a is some real number, and $\lim_{x\to a} f(x) = L_f$, and $\lim_{x\to a} g(x) = L_g \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_f}{L_g}.$$

As you can see, rigorous demonstrations, as necessary as they are, can become complicated and tedious. Perhaps this explains why the practice of Calculus predated its theory.

At last, having defined the derivative by Definition 15; having derived all of our differentiation rules via this definition, and having shown – rigorously – that founding Calculus on the theory of limits gives us all of the properties we found useful in Part 1 of this textbook, we have at last placed a solid, logical, and rigorous foundation underneath the Calculus of Newton and Leibniz.

And Bishop Berkeley has nothing left to criticize.