Curvature

Unit tangent vectors

Something remarkable happens when parametrizing curves by arc length: every tangent vector has length 1.

We can do this more generally whenever we know that $\mathbf{f}'(t) \neq 0$.

The unit tangent vector to the curve $\mathbf{f}(t)$ at a point $t = a$ is

$$\mathbf{T}(a) = \frac{\mathbf{f}'(a)}{|\mathbf{f}'(a)|}.$$

Example: for the joey from the "Distance" lecture segment with $\mathbf{f}'(t) = \langle 1, t, \sin(t) \rangle$, the unit tangent vector at time a is

$$\mathbf{T}(a) = \frac{1}{\sqrt{1 + t^2 + \sin^2(t)}} \langle 1, t, \sin(t) \rangle.$$

Practice

Compute the unit tangent to the parabola $y = x^2$ at the point (a, a^2).

Hint: one way to do this is to parametrize the path first, say using $x = t$. Then calculate $\mathbf{f}'(t)/|\mathbf{f}'(t)|$. What if you parametrize (half of) the parabola as $\langle \sqrt{t}, t \rangle$ instead?

The benefits of all of this

Why work with arc length and unit tangents?

- Capture intrinsic geometric information, not artifacts of the choices we made in our description.
- Recovers delicate static information about the shape.

For example: $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$. I.e., $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$.

Indeed,

$$0 = \frac{d}{dt} 1 = \frac{d}{dt} |\mathbf{T}(t)| = \frac{d}{dt} \left(\mathbf{T}(t) \cdot \mathbf{T}(t) \right) = 2\mathbf{T}(t) \cdot \mathbf{T}'(t).$$

Meaning: $\mathbf{T}'(t)$, unlike the acceleration in general, is always changing the tangent to the curve in the "most efficient" way.

Curvature

The curvature of the smooth parametric curve $\mathbf{f}(t)$ is defined to be

$$\kappa(t) = \left|\frac{d\mathbf{T}(t)}{ds}\right|,$$

where s is the arc length function.

Since $s'(t) = |\mathbf{f}'(t)|$, we also have $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{f}'(t)|$.

A lot to digest!

When $\mathbf{f}$ is already paramterized by arc length, this simplifies to

$$\kappa(s) = |\mathbf{f}''(s)|,$$

the acceleration.

In practice, this is *not* how you will compute it (because you won't have paths parametrized by arc length most of the time).

Examples

The Circle of Radius r **and** ***The Parabola Fights Back.***

The equations parametrizing with arc length: $\mathbf{f}(s) = \langle r\cos(s/r), r\sin(s/r)\rangle$.

Thus,

$$\kappa(s) = \left|\left\langle -\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right\rangle\right| = \frac{1}{r}.$$

Makes sense: the curvature of a circle of large radius is small. After all, the path is basically a straight line!

What about for something like the parabola? Try it. You might consider using $\mathbf{f}(t) = \langle t, t^2\rangle$ and $\kappa = |\mathbf{T}'|/|\mathbf{f}'|$, together with your calculation of $\mathbf{T}$ from before. What a mess!

Curvature in practice

Mathematicians have thought about this one pretty hard, and here is what turns out to happen:

The curvature of the smooth path parametrized by $\mathbf{f}(t)$ is

$$\kappa(t) = \frac{|\mathbf{f}'(t) \times \mathbf{f}''(t)|}{|\mathbf{f}'(t)|^3}.$$

Any book or the internet contains a proof!

We can dispatch the parabola $\mathbf{f}(t) = \langle t, t^2 \rangle$:

$$\kappa(t) = \frac{|\langle 1, 2t, 0 \rangle \times \langle 0, 2, 0 \rangle|}{(1 + 4t^2)^{\frac{3}{2}}} = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}.$$

Do one!

Recall the motion of the joey: $\mathbf{f}(t) = \langle t, \frac{1}{2}t^2, 2 - \cos(t) \rangle$.

Calculate the limit of the curvature as $t \to \infty$.

Formulas for curvature:

$$\kappa(t) = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|s'(t)|} = \frac{|\mathbf{f}'(t) \times \mathbf{f}''(t)|}{|\mathbf{f}'(t)|^3}$$