Exercises: Algebraic Structures on Combinatorial Species

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1. Enumeration via Species

(1) (a) * Convince yourself that the species Π is indeed a set species.
 (b) * For any finite set I, consider the collection $P[I] = I \cup \{1, 2\}$. Is P a set species?

(2) Let I be the species of *involutions* (i.e., $\mathrm{I}[J]$ is the set of bijections $\sigma : J \to J$ such that $\sigma^2 = \mathrm{id}$) and P the species of *perfect matchings* (i.e., $\mathrm{P}[J]$ is the set of partitions of J into 2-element sets). Let $\mathrm{E}_{1,2}$ be the species characteristic of singletons and doubletons:

$$\mathrm{E}_{1,2}[J] = \begin{cases} \{*_J\} & \text{if } |J| = 1 \text{ or } 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

 (a) Show that $\mathrm{I} = \mathrm{E} \circ \mathrm{E}_{1,2}$ and deduce that

$$\mathrm{I}(x) = e^{x + \frac{x^2}{2}}.$$

 (b) Show that $\mathrm{I} = \mathrm{E} \cdot \mathrm{P}$. Deduce that

$$\mathrm{P}(x) = e^{\frac{x^2}{2}}$$

 and that the number of perfect matchings on $[2n]$ is $(2n-1)!! := (2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.

(3) Let P and Q be finite set species. Show that

$$(P \cdot Q)(x) = P(x)Q(x).$$

On the right hand side we have the usual product of formal power series, on the left hand side we have the Cauchy product of the two species.

(4) Show that $L \circ E_+ = \Sigma$, and deduce that

$$\Sigma(x) = \frac{1}{2 - e^x}.$$

This is the generating function for the number of set compositions (the ballot numbers).

(5) (a) Let P be a set species. Show that the symmetric group S_n acts on the set $P[n]$ by

$$\sigma \cdot x = P[\sigma](x).$$

 (b) Let $f : P \to Q$ be a morphism of set species. Show that the map $f_{[n]} : P[n] \to Q[n]$ is a morphism of S_n-sets; that is,

$$f_{[n]}(\sigma \cdot x) = \sigma \cdot f_{[n]}(x)$$

 for all $x \in P[n]$ and $\sigma \in S_n$.
 (c) Show that the action of S_n on $L[n]$ has one orbit, while the action on $B[n]$ has as many orbits as partitions of n.
 (d) Deduce that L and B are not isomorphic.

2. Monoids and Hopf monoids

(1) * Convince yourself that concatenation turns the species Σ of set compositions into a monoid, and L into a submonoid.

(2) * Verify that the species Π of set partitions satisfies the compatibility axiom of a bimonoid (with the structure given in the lecture).

(3) (a) Let P and Q be monoid species. Define a monoid structure on the species $\mathrm{P} \cdot \mathrm{Q}$.
(b) Deduce monoid structures on $\mathrm{E} \cdot \mathrm{E}$ and on $\mathrm{E}^{\cdot k}$, and describe them explicitly in terms of subsets and functions.

(4) Show that the antipode of the exponential Hopf monoid E is given by

$$\mathrm{S}_I(*_I) = (-1)^{|I|} *_I .$$

(5) Let H be a bimonoid. Let $\mathrm{End}(\mathbf{H})$ denote the set of all morphisms of vector species $f : \mathbf{H} \to \mathbf{H}$. It is a vector space under pointwise addition and scalar multiplication. The *convolution* $f * g$ of two morphisms f and g is defined by

$$(f * g)_I(x) = \sum_{S \sqcup T = I} f_S(x|_S) \cdot g_T(x/_S).$$

for all finite sets I and all $x \in \mathrm{H}[I]$ (and then extended linearly to $\mathbf{H}[I]$). Let $u : \mathbf{H} \to \mathbf{H}$ be defined by $u(x) = 1$ if $x \in \mathrm{H}[\emptyset]$ and $u(x) = 0$ if $x \in \mathrm{H}[I]$ with $|I| \geq 1$.
(a) Show that convolution is associative and unital, with u being the unit. In this manner $\mathrm{End}(\mathbf{H})$ is an algebra.
(b) Show that H is a Hopf monoid if and only if $\mathrm{id} : \mathbf{H} \to \mathbf{H}$ is invertible in the algebra $\mathrm{End}(\mathbf{H})$, and that in this case S is the convolution inverse of id.
(c) Deduce the uniqueness of the antipode.

(6) Show that a morphism $f : \mathrm{H} \to \mathrm{K}$ of connected bimonoids preserves the antipodes. In other words,

$$f_I(\mathrm{S}_I(x)) = \mathrm{S}_I(f_I(x))$$

for all I and $x \in \mathrm{H}[I]$.