CIMPA Research School:
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Triangulations of polytopes. Glossary (I)

0 Polytopes, affine geometry

- **Affine combination, affine dependence:** An **affine combination** of points $p_1, \ldots, p_n \in \mathbb{R}^d$ is any point of the form

  $$\lambda_1 p_1 + \cdots + \lambda_n p_n$$

  with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\sum \lambda_i = 1$.

  An **affine dependence** is a valid expression of the form

  $$\lambda_1 p_1 + \cdots + \lambda_n p_n = 0$$

  with $\sum \lambda_i = 0$.

  Every affine combination $p = \lambda_1 p_1 + \cdots + \lambda_n p_n$ of n points gives rise to an affine dependence between them and the new one: $\lambda_1 p_1 + \cdots + \lambda_n p_n - p = 0$. Conversely, each point with nonzero coefficient in an affine dependence can be written as an affine combination of the rest.

- **Affine space, affine span:** A subset of $\mathbb{R}^d$ is an affine (sub)-space, or a **flat** if it is closed under affine combinations.

  The affine span of a set $V \subset \mathbb{R}$ is the smallest affine space containing V. Equivalently, it is the set of points that are affine combinations of V. It is denoted by $\mathrm{aff}(V)$.

- **Convex hull:** A **convex combination** is an affine combination with non-negative coefficients. A set is a **convex set** if it is closed under convex combinations.

  The **convex hull** of a set V is the smallest convex set containing V. Equivalently, it is the set of points that are convex combinations of V. It is denoted by $\mathrm{conv}(V)$.

- **Polytope:** A polytope P is the convex hull of a finite set of points V. Its **dimension** is the dimension of its affine span $\mathrm{aff}(P) = \mathrm{aff}(V)$.

  For each linear functional $f : \mathbb{R}^d \to \mathbb{R}$ we call **face** of P induced by f the set

  $$P^f := \{x \in P : f(x) \geq f(y)\ \forall y \in P\}$$

that maximizes f. The empty set is considered a face (of dimension -1) and every face of a polytope is a polytope. The **boundary** of P is the union of all its proper faces, which range from dimensions -1 (the empty set), 0 (called **vertices**), 1 (**edges**), 2, ..., to $d-1$ (**facets**). Its set of vertices will be denoted by $\mathcal{V}(P)$.

- A **polytopal (or polyhedral) complex** is a finite, nonempty collection $\mathcal{S} = \{P_1, \ldots, P_k\}$ of polytopes in $\mathbb{R}^d$ such that every face of each $P_i \in \mathcal{S}$ is in $\mathcal{S}$, and such that $P_i \cap P_j$ is always a common face of both (possibly empty).

  The elements of $\mathcal{S}$ are called **faces of** $\mathcal{S}$, and the maximal elements are called **cells**. Knowing the cells of $\mathcal{S}$ is enough to recover the rest, since all faces are faces of a cell.

  A complex is **pure of dimension** k is all cells are k-dimensional. The i**-skeleton** of $\mathcal{S}$ is the i-dimensional complex consisting of faces of dimension at most i.

  For example, if P is a d-polytope, its faces form a pure d-dimensional complex with a single cell (P itself). Its $(d-1)$-skeleton is the boundary of P (a pure $(d-1)$-complex) and its 1-skeleton is usually called the **graph of** P.

- **Simplex:** A d-dimensional simplex is a d-polytope with exactly $d+1$ vertices. Equivalenty, it is the convex hull of a set of affinely independent points. We will also refer to the set of vertices of a d-simplex as a d-simplex.

  Every face of a simplex is a simplex, and each subset of vertices defines a face. That is, a d-simplex has $\binom{d+1}{i+1}$ faces of each dimension $i = -1, 0, \ldots, d$. (The poset of faces of a simplex is a Boolean poset).

- **Simplicial complex:** A simplicial complex is a polytopal complex whose cells are simplices.

1 Triangulations and subdivisions of point configurations

From now on, V denotes a d-dimensional finite set of n points (we sometimes call it a **point configuration**).

There is no (much) loss of generality in assuming that V is full-dimensional (that is, $V \in \mathbb{R}^d$) because if $V \in \mathbb{R}^D$ and $\dim(V) = d < D$ then we can forget some coordinates and get a d-dimensional configuration V' in $\mathbb{R}^d$ that is affinely isomorphic to V; *all we are going to do is invariant under affine isomorphism.* But sometimes we need to consider a subset W of V as a configuration in itself and, of course, it may be lower dimensional.

- **Faces of a set:** Let W be a subset of V. We say W is a **face of** V if there is a face F of the polytope $P = \mathrm{conv}(V)$ for which $W = V \cap F$. Note that W may include points that are not vertices of F.

- **Subdivision:** A **(polyhedral) subdivision of** V is a finite collection $\mathcal{S} = \{S_1, \ldots, S_m\}$ of subsets of V, called **cells**, such that:

(DP) For each $i \in \{1, \ldots, m\}$, $P_i := \mathrm{conv}(S_i)$ is d-dimensional (a d-polytope);

(UP) P is the union of $P_1, \ldots, P_m$; and

(IP) If $i \neq j$ then $F := S_i \cap S_j$ is a common (possibly empty) proper face of S_i and S_j, and $P_i \cap P_j = \mathrm{conv}(F)$.

For example, $\{1245, 1345\}$ and $\{124, 134\}$ are subdivisions of the configuration in Figure 1, and we consider them different subdivisions. Observe that the second one does not use point 5, but that is ok. But $\{1245, 134\}$ is not a subdivision. The triangles $\mathrm{conv}\{1245\}$ and $\mathrm{conv}\{134\}$ intersect in a common face (the edge from 1 to 4) but the *sets* $\{1245, 134\}$ do not: their intersection is $\{14\}$, which is not a face of $\{1245\}$.

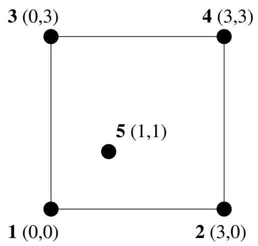

Figure 1: 5 points in the plane

The collection of polytopes $P_1, \ldots, P_m$, together with their faces, is a pure polytopal complex.

When V is the vertex set of a polytope P we say that $\mathcal{S}$ is a **subdivision of** P with no extra vertices.

- **Triangulation:** A subdivision of V is a triangulation if every cell is affinely independent (a simplex). Since every subset of a simplex is a face of it, for triangulations property (IP) reduces to

(IP) If $i \neq j$ then $\mathrm{conv}(S_i) \cap \mathrm{conv}(S_j) = \mathrm{conv}(S_i \cap S_j)$.

For example, $\{124, 134\}$ is a triangulation but $\{1245, 1345\}$ is not a triangulation, in the above point configuration.

- **Refinement of a subdivision:** Suppose $\mathcal{S} = \{S_1, \ldots, S_l\}$ and $\mathcal{T} = \{T_1, \ldots, T_m\}$ are two subdivisions of V. We say $\mathcal{T}$ is a refinement of $\mathcal{S}$ if for each j, $1 \leq j \leq m$, there exists i, $1 \leq i \leq l$, such that $T_j \subseteq S_i$. We write $\mathcal{T} \preceq \mathcal{S}$ if $\mathcal{T}$ refines $\mathcal{S}$ because refinement of subdivisions is a partial order (poset). The unique maximal element in the poset is the **trivial subdivision** (the subdivision $\{V\}$ with a single cell, the set V). The minimal elements are the **triangulations**.

  For example, in the above configuration we have the following chain from a triangulation to the trivial subdivision:

$$\{124, 134\} \preceq \{1245, 1345\} \preceq \{12345\}$$

Triangulations of polytopes. Glossary (II)

2 Oriented matroids

- **Affine dependences and affine evaluations:** Let $V = \{p_1, \dots, p_n\}$ be a set of n points with $\dim(V) = d$.

  We regard each **affine dependence** $\sum \lambda_i p_i = 0$ with $\sum \lambda_i = 0$ as a vector $\lambda \in \mathbb{R}^n$. Similarly, we are interested in **affine evaluations**: for each affine function $f : \mathbb{R}^d \to \mathbb{R}$, the affine evaluation produced by f is the vector $(f(p_1), \dots, f(p_n)) \in \mathbb{R}^n$. It is easy to show that:

  Lemma 1 *Affine dependences and affine evaluations of V are complementary linear subspaces of $\mathbb{R}^n$, of dimensions $n - d - 1$ and $d + 1$, respectively.*

  In fact, since every affine functional is of the form $f(x_1, \dots, x_d) = a_0 + a_1 x_1 + \cdots + a_d x_d$, affine evaluations are generated by the evaluations corresponding to a constant functional and to the coordinate functionals. The fact the V is full-dimensional implies these $d + 1$ evaluations are independent vectors in $\mathbb{R}^n$.

  A more compact way of looking at dependences and evaluations is to consider the $(d + 1) \times n$ matrix

  $$A = \begin{pmatrix} p_1 & \dots & p_n \\ 1 & \dots & 1 \end{pmatrix}.$$

  In these conditions:

  Lemma 2 *Affine dependences are the kernel of A and affine evaluations are the rowspace of A (the linear subspace generated by rows, which is the same as the image of A^t).*

- **Signature** of a vector. For a vector $\lambda \in \mathbb{R}^n$ the signature of λ is the vector $(s_1, \dots, s_n) \in \{-1, 0, +1\}^n$ of signs of λ. (That is, $s_i = \lambda_i / |\lambda_i|$, with $0/0$ taken as 0). The signature of a vector will normally be represented as a pair (S^+, S^-) where

  $$S^+ := \{i : \lambda_i > 0\} \qquad \text{and} \qquad S^- := \{i : \lambda_i < 0\}.$$

  That is, there is a bijection between possible signatures of vectors in $\mathbb{R}^n$ and pairs (S^+, S^-) of disjoint subsets of $\{1, \dots, n\}$.

- **(Oriented) circuits, Radon partitions:** A circuit is a set $V = \{v_1, \ldots, v_k\}$ of affinely dependent points such that every proper subset is affinely independent. This implies, in particular, that $k = \dim(V)+2$ and that there is a unique (up to rescaling) affine dependence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of V. Since λ has no zero entries, the signature (C^+, C^-) of λ is a partition of V into two parts. (Here we are identifying each point c_i with its label i). This partition is called the **oriented circuit** or **Radon partition** of V and it is the unique partition of V into two disjoint subsets such that $\mathrm{conv}(C^+) \cap \mathrm{conv}(C^-) \neq \emptyset$.

  For a general V, we call **circuits of** V its minimal affinely dependent subsets and **oriented circuits of** V the corresponding signatures. Each circuit produces two opposite oriented circuits. Put differently: *oriented circuits of* V *are the dependence signatures with minimal support.*

  As an example, the following is the full list of circuits of the configuration in Figure 1. For brevity, sets are written as strings of labels. That is, 123 denotes the set $\{1, 2, 3\}$ or the set $\{p_1, p_2, p_3\}$.

$$\begin{array}{cccc} (14,5), & (123,5), & (14,23), & (45,23), \\ (5,14), & (5,123), & (23,14), & (23,45). \end{array}$$

- **(Oriented) cocircuits:** It makes sense to do the same for evaluations as we did above for dependences. We call **cocircuits** of V the minimal supports of affine evaluations in V, and call *oriented cocircuits* their signatures. Each cocircuit produces two opposite oriented cocircuits. For example, the list of cocircuits of our running example is:

$$\begin{array}{ccccccc} (\emptyset,125), & (\emptyset,135), & (\emptyset,245), & (\emptyset,345), & (1,24), & (1,34), & (2,3), \\ (125,\emptyset), & (135,\emptyset), & (245,\emptyset), & (345,\emptyset), & (24,1), & (34,1), & (3,2). \end{array}$$

- **Oriented matroid of a point configuration** V**:** The oriented matroid of a point configuration V is its set of oriented circuits, or its set of oriented cocircuits (both carry the same information about V). Observe that from the oriented matroid of V we can recover several things:

  - The facets of V (as a set) are the complements of the positive cocircuits. That is, F is a facet if and only if $(V \setminus F, \emptyset)$ is a cocircuit.
  - The faces of V are all possible intersections of facets.
  - If W is a subset of V, the circuits of S are the circuits of V contained in W and the cocircuits of W are a subset of the restrictions to W of the cocircuits of V (explanation: every evaluation

on V, restricted to W, is an evaluation on W. But some of the evaluations that have minimal support in V may not have minimal support in W; those have to be discarded from the list of cocircuits).

 – In particular, we can compute faces of all subsets of V.
 – Affnely independent subsets of V are those that do not contain any circuit. In particular, from the oriented matroid we can compute the rank of V and the rank of every subset of V.

With some extra effort one can prove the following:

Theorem 3 *Let S be a collection of subsets of a point set V. Then, knowing the oriented matroid of V (and forgetting the points) is enough information to check whether S is a subdivision of V, and whether it is a triangulation.*

- **Oriented matroid of a *vector* configuration** V:

  Oriented matroids can be defined (and are usually defined) for **vector configurations**. If $V = \{v_1, \dots, v_n\}$ is a vector configuration (a finite subset of vectors of $\mathbb{R}^n$) then we define its dependences, evaluations, circuits and cocircuits as above, simply changing the word *affine* to *linear* everywhere. Of course, one can say that a point configuration and a vector configuration are the same thing, a finite subset of $\mathbb{R}^d$. What makes a difference is whether we are doing *affine geometry* or *linear algebra* in $\mathbb{R}^d$.

  Observe that the oriented matroid of the *point configuration* $\{p_1, \dots, p_n\}$ is the same as the oriented matroid of the *vector configuration* $\left\{ \binom{p_1}{1}, \dots, \binom{p_n}{1} \right\}$.

- **Gale transform, dual oriented matroid**: Let V and W be two vector configurations with n elements in $\mathbb{R}^k$ and $\mathbb{R}^l$, with $k + l = n$, and represent them as a $k \times n$ A and an $l \times n$ matrix B, respectively (putting the vectors as columns). We say that W is a **Gale transform** of V if the row spaces of A and B are orthogonal complements to one another.

  If this happens, then the linear dependences of V are the same as the linear evaluations of W, and vice-versa, so the circuits of V are the cocircuits of W and vice-versa. We say that the oriented matroids of V and W are **dual** to one another.

  For every V there is some Gale transform W. Simply write V as a matrix A, compute a basis of the orthogonal complement of the row space of A, and use that as the rows for W. One trick to quickly compute a Gale transform is: without loss of generality, assume that the first k vectors of V are independent (if this is not the case, reorder them) and change coordinates to make these vectors be the standard basis of $\mathbb{R}^k$. That is, your matrix A is of the form $(I|A')$ where $I$is the identity matrix and A' is some $k \times (n-k)$ matrix. Then, take

$$B = (-A'^t|I)$$

and let W consist of the columns of B.

Let us compute, as an example, the dual oriented matroid of the point configuration of Figure 1. The matrix A is

$$A = \begin{pmatrix} 0 & 3 & 0 & 3 & 1 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

One Gale transofrm is as follows (check that row spaces of A and B are orthogonal), which is drawn as a set of vectors in Figure **??**.

$$B = \begin{pmatrix} -1 & -1 & -1 & 0 & 3 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

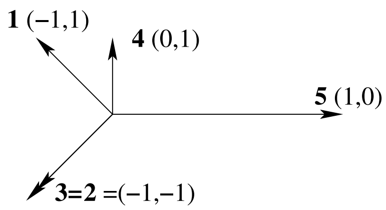

Figure 2: 5 vectors in the plane

Observe that vectors 2 and 3 coincide. This is in agreement with the fact that $(2,3)$ was a cocircuit in V (corresponding to the evaluation $(0,3,-3,0,0)$ of the functional $x-y$), so $(2,3)$ must now be a cocircuit and $(0,3,-3,0,0)$ a linear dependence. These two "equal vectors" must be considered different elements in the oriented matroid, and in the vector configuration, distinguished by their labels.