An introduction to
Subword Complexes of Coxeter Groups

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האוניברסיטה העברית בירושלים
The Hebrew University of Jerusalem

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Hoy es un día muy especial

Fête nationale du Québec
(Saint-Jean-Baptiste)

Luis, Nantel, Emerson, Cesar, JP

Acknowledgements

These slides are inspired from many different talks from the last few years.

Many thanks to Nantel Bergeron, Cesar Ceballos, Vincent Pilaud, and Christian Stump for their help in writing, commenting and improving the current slides.

A nice encryption method

Book cypher: Encode a secret message using letters or words in a popular text

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Example (Buena Vista Social Club – El cuarto de Tula)

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Ay mama, ¿qué pasó? ¡Ay, mamá! ¿qué pasó? ¡Ay!

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Cyphers the message:

El bar "La Casa" arde
Un loro con penas llenó su capa
Sin amo, ay qué soy?

In this talk

Today: Decypher mathematical structures!

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The cypher uses

- Coxeter groups and
- the Bruhat order

In this talk

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The cypher uses

- Coxeter groups and
- the Bruhat order

to hide nice structures:

- cyclic polytopes, associahedra, multi-associahedra
- cluster complexes
- tropical planes
- Schubert polynomials
- ...

Plan of the talk

1. Basics on Coxeter groups
2. Subword Complexes
3. Subword Complexes approach to multi-triangulations

Preliminaries – Coxeter groups

Symmetric group $\mathbb{S}_{n+1}$:
The group of permutations of $\{1, \ldots, n+1\}$

generators:	$\{s_1, \ldots, s_n\}$, $s_i = (i\ i+1)$
length of $w \in \mathbb{S}_{n+1}$:	smallest r such that $w = s_{i_1} \ldots s_{i_r}$
longest element $w_\circ$:	the permutation $[n+1,\ \ldots\ ,1]$
reduced expression of w:	expression for w of smallest length

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Finite Coxeter groups:

Groups obtained by a presentation with generators and relations:

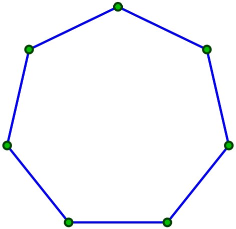
$$W = \langle S | e = s^2 = (st)^{m_{s,t}}; \quad \forall s, t \in S \rangle$$

and $m_{s,t} \in \{2, 3, \dots, \} \cup \{\infty\}$

Generators: S
Coxeter system: (W, S)

Preliminaries – Reflection groups

A reflection fixes an hyperplane and flips a complementary vector



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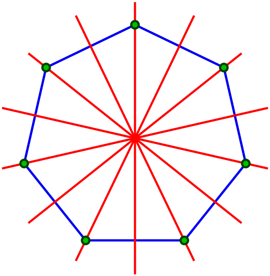

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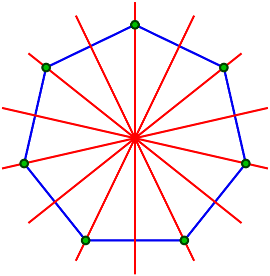

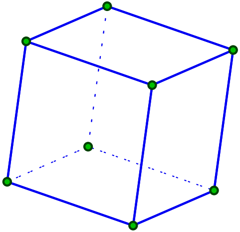

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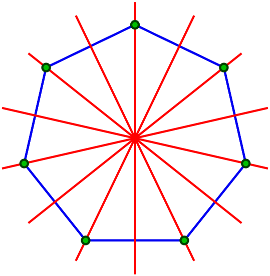

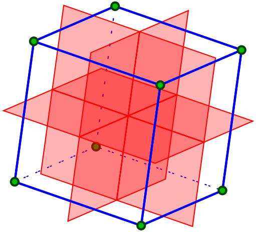

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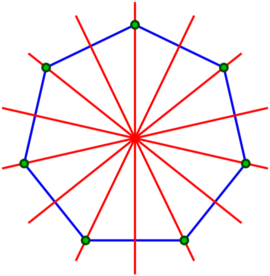
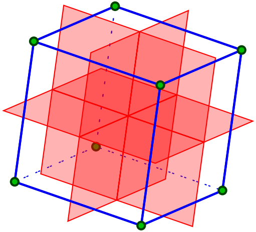

Theorem (Coxeter, 1934)

Finite reflection groups of Euclidean spaces are exactly finite Coxeter groups

Classification: $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$

Preliminaries – Bruhat order

Bruhat order $(W, \prec)$:

$$w_1 \prec w_2 \iff \begin{array}{l} w_1 \text{ can be expressed as a subword} \\ \text{of a reduced expression of } w_2 \end{array}$$

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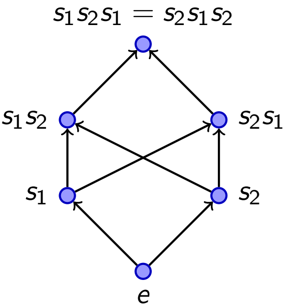

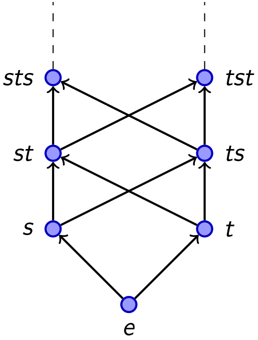

$\mathbb{S}_3$: $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$ $\quad I_2(\infty) : \langle s, t \mid s^2 = t^2 = e \rangle$

2. Subword Complexes

Subword Complexes of Coxeter groups

Analogous to book ciphers!

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(W, S) finite Coxeter system

$Q = (q_1, \ldots, q_m)$ a word in S

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Definition (Knutson–Miller, 2004)

The subword complex $\Delta(Q, \pi)$ is the simplicial complex whose

faces $\longleftrightarrow$ subwords P of Q such that $Q \setminus P$ contains a reduced expression of π

Knutson–Miller. *Gröbner geometry of Schubert polynomials*. Ann. Math., 161(3), '05

Knutson–Miller. *Subword complexes in Coxeter groups*. Adv. Math., 184(1), '04

Example

In type A_2:

$W = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

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$$Q = \begin{pmatrix} s_1, s_2, s_1, s_2, s_1 \\ q_1, q_2, q_3, q_4, q_5 \end{pmatrix} \text{ and } \pi = [3\ 2\ 1]$$

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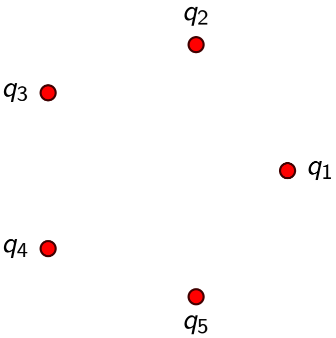

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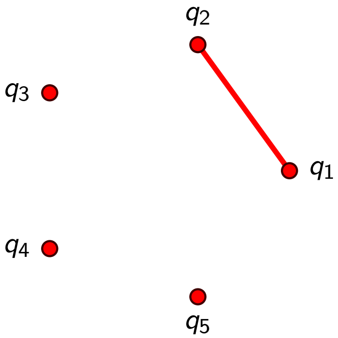

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$Q = \begin{matrix} (& s_1, & & , & & , s_2, s_1 &) \\ & & , q_2, q_3, & & , & & \end{matrix}$ and $\pi = [3\ 2\ 1] = s_1 s_2 s_1$

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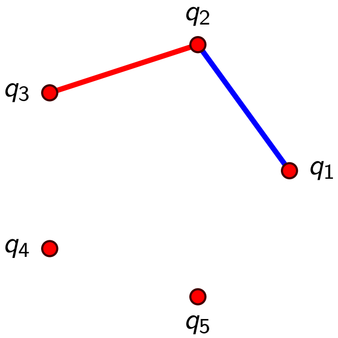

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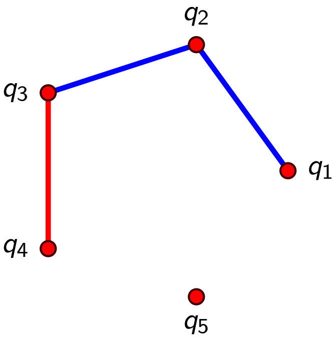

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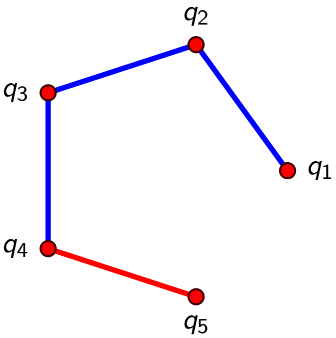

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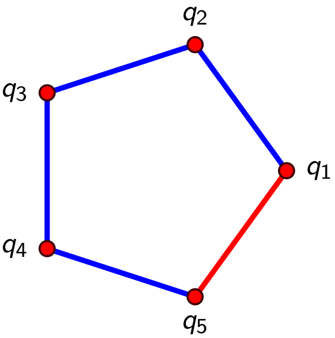

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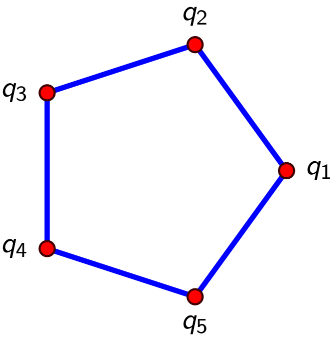

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In type A_3:

$W = \mathbb{S}_4$, $S = \{s_1, s_2, s_3\} = \{(1\ 2), (2\ 3), (3\ 4)\}$

$$Q = \begin{pmatrix} s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_1 \\ q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9 \end{pmatrix} \text{ and } \pi = [4\ 3\ 2\ 1].$$

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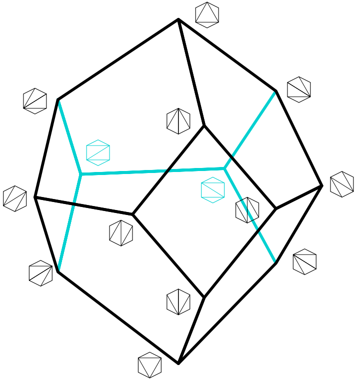
$\Delta(Q, \pi)$ is isomorphic to the dual of the associahedron



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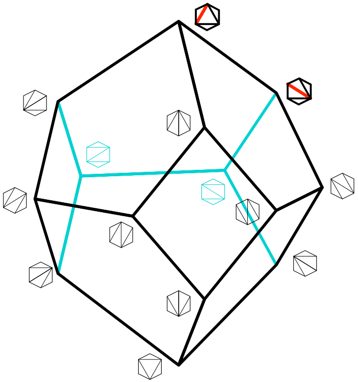


Subword Complexes - Generalized associahedra

Theorem (Ceballos–L.–Stump, 2014)

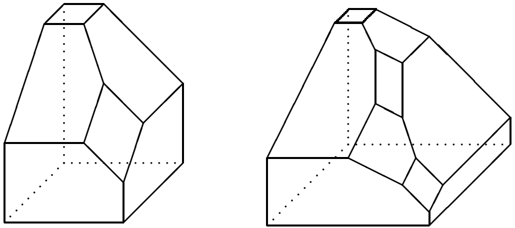
For any finite Coxeter group, if $Q_c = cw_\circ(c)$ and $\pi = w_\circ$ then the subword complex

$$\begin{aligned}\Delta(Q_c, \pi) &\cong \textit{dual generalized associahedron}\\ &\cong \textit{c-cluster complex}\end{aligned}$$

where $c = (s_{i_1}, \ldots, s_{i_n})$ is a Coxeter element and $w_\circ(c)$ is the first lexicographically subword of c^∞ which is a reduced expression of $w_\circ$.

Subword Complexes - Generalized associahedra

Generalized associahedra are remarkable polytopes which encode combinatorics of mutation graphs in cluster algebras of finite type.



Fomin–Zelevinsky. *Cluster algebras I: Foundations*. J. Amer. Math. Soc., 15(2), '02
Fomin–Zelevinsky. *Cluster algebras II: Finite type classification*. Invent. Math. 154, '03
Fomin–Zelevinsky. *Y-systems and generalized associahedra*. Ann. of Math. 158, '03

Zoology of subword complexes

Theorem (Knutson–Miller, 2004)

Subword complexes are $vertex\text{-}decomposable$ spheres or balls.

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They are found in various areas:

Discrete geometry: simplices, even-dim. cyclic polytopes, pseudotriangulation polytope of planar point sets, dual of brick polytopes (Ceballos, L., Pilaud, Pocchiola, Santos, Stump)

Algebra: Finite cluster algebras (c-cluster complexes, denominator vectors), Hopf algebras, Cambrian lattices (Bergeron, Ceballos, L., Lange, Pilaud, Stump)

Alg. geometry: Schubert varieties, Brick varieties, Total positivity, Schubert patches, tropical planes (Armstrong, Brodsky, Escobar, Hersh, Knutson, Meszaros, Miller)

Some recent results

Bergeron–Ceballos (2015): Introduced an Hopf algebra of subword complexes

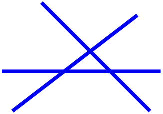
Escobar–Meszaros (2015): Describe geometric realizations of certain contractible subword complexes related to Toric matrix Schubert varieties

Brodsky–Ceballos–L. (2015): Describe a combinatorial model for (combinatorial type) of tropical planes in $\mathbb{TP}^5$.

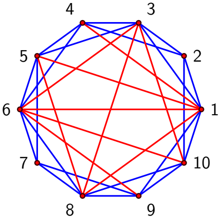
3. Subword Complexes approach to multi-triangulations

Multi-associahedra

k-triangulations: maximal sets of diagonals where no $k+1$ diagonals mutually cross



no 3-crossings



2-triangulation

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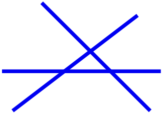

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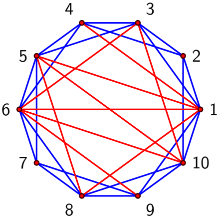

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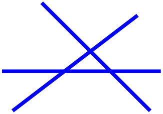

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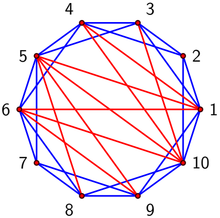

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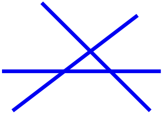

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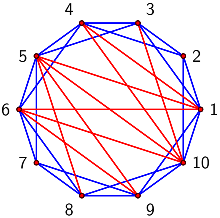

2-triangulation

Multi-associahedron: simplicial complex $\Delta_{m,k}$ whose facets correspond to k-triangulations of a convex m-gon (Jonsson '03-'05)

Some properties

Topology: pure vertex-decomposable simplicial sphere
(Capoyleas–Pach '92, Dress–Koolen–Moulton '02,
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Facets: Bijection with k-fans of Dyck paths and plane partitions of height k (Jonsson '05, Stump–Serrano '12)

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Subword approach: Every multi-associahedron can be obtained as a well chosen subword complex of type A (Pilaud–Pocchiola, Serrano–Stump, Stump)

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Generalization: Generalized multi-triangulations for finite Coxeter groups (Ceballos–L.–Stump)

Subword Complexes - Multi-associahedra

Subword complexes provide simple proofs of non trivial facts.

- V. Capoyleas and J. Pach, A Turán-type theorem on chords of a convex polygon, J. Combinatorial Theory, Ser. B 56 (1992)
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Holy Grail Question

Question (Knutson–Miller, 2004)

Is every spherical subword complex the boundary of a convex polytope?

Conjecture (Jonsson, 2005)

Every multi-associahedron is the boundary of a convex polytope.

Conjecture (Soll–Welker, 2009)

Every type B multi-associahedron is the boundary of a convex polytope.

The Cenacle – Jerusalem

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Multi-associahedra: polytopality

Polytopality of spheres is a very difficult problem

$d = 3 \rightarrow$ Steinitz's theorem (1922)

$d \geq 4 \rightarrow$ "Steinitz's problem"

Multi-associahedra: polytopality

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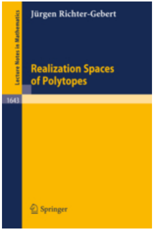
$d = 3 \to$ Steinitz's theorem (1922)

$d \geq 4 \to$ "Steinitz's problem"

Universality theorem: Realization spaces of polytopes can take arbitrary (semi-algebraic) shapes and thus can exhibit all kinds of pathologies.

The realizability problem for 4-polytopes is NP-hard.



Status of the problem

Polytopal constructions:

- $k = 1$: dual of a classical associahedron
- $m = 2k + 1$: single vertex
- $m = 2k + 2$: simplex
- $m = 2k + 3$: cyclic polytope (Pilaud–Santos, Ceballos–L.–Stump)
- $\Delta_{8,2}$: 6-dimensional polytope (Bokowski–Pilaud, Ceballos, Bergeron–Ceballos–L.)

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Fan realizations:

$m = 2k + 4$, and sub. compl. $(A_n\ (n \leq 3), B_2)$ (Bergeron–Ceballos–L. 2015)

$k = 2$ and $m \leq 13$ (Manneville 2016^{+})

Still many structural questions

- What is the f-vector of the multi-associahedron?

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- What is the diameter of multi-associahedron?

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- What is the f-vector of the multi-associahedron?
- What is the diameter of multi-associahedron?
- Are multi-associahedron hamiltonian?

Hasta pronto todos!!

Muchisimas gracias a todos!!

Nos vemos en 2 años!!