Lecture 3 Practice Problems

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(1) Recall that $H(z) := \sum_{n\geq 0} h_n z^n$ and that ps denotes stable principal specialization.
   (a) Show that
$$\mathrm{ps}(H(z)) = \sum_{n\geq 0} \frac{z^n}{(1-q)\dots(1-q^n)}.$$
   (b) Show that by taking principal stable specialization of
$$\frac{(1-t)H(z)}{H(zt)-tH(z)}$$
   and then replacing z with $z(1-q)$, one gets
$$\frac{(1-t)\exp_q(z)}{\exp_q(tz)-t\exp_q(z)}.$$

(2) Verify

$$(*) \qquad \sum_{i\in \mathrm{DEX}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$$

   for each of the following permutations.
   (a) $\sigma = 41637852$
   (b) $\sigma = 54321$
   (c) all of $\mathfrak{S}_3$
   (d) all of $\mathfrak{S}_{(4)}$.

(3) Prove (*) for all $\sigma \in \mathfrak{S}_n$.

(4) Why does h-positivity impliy Schur-positivity and p-positivity?

(5) (a) Show that if a homogenous symmetric function f of degree n is Schur-positive and
$$\mathrm{ps}(f) = \frac{g(q)}{(1-q)\dots(1-q^n)}$$
   then $g(q)$ is a polynomial with positive coefficients.
   (b) Explain why Schur-unimodality of $\sum_{j\geq 0} Q_{\lambda,j} t^j$ implies q-unimodality of $A_\lambda(q,t)$.

(6) Draw all ornaments of type $\lambda = (4)$, weight $x_2 x_3 x_6^2$, with two red letters.

(7) For $\sigma = 32675814$ and sequence $s = (9,9,8,7,7,3,3,1)$ give the corresponding ornament under the bijection given in the lecture.

(8) Let $\Gamma_{n,i}$ be the coefficient of $t^i(1+t)^{n-1-2i}$ in the expansion of $\sum_{j=0}^{n-1} Q_{n,j} t^j$. Show
$$\mathrm{ps}(\Gamma_{n,i}) = \frac{\sum_\sigma q^{\mathrm{maj}(\sigma^{-1})}}{(1-q)(1-q^2)\dots(1-q^n)}$$
where the sum in the numerator ranges over all permutations with no double descent, no final descent and with i descents.

(9) Give a combinatorial proof of the fact that $Q_{\lambda,j}$ is a symmetric function for all λ and j. Suggestion: Use the ornament characterization.