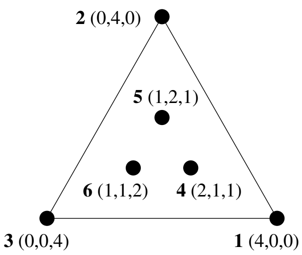
CIMPA Research School:
Algebraic, Enumerative and Geometric Combinatorics - ECCO 2016

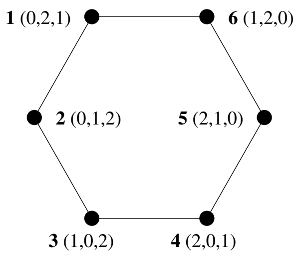
Triangulations of polytopes. Problem sheet.

1 Triangulations and subdivisions

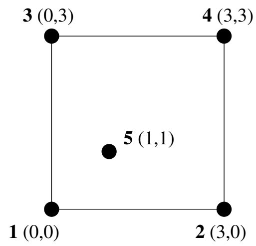
1. *Draw all triangulations of one of the following three point sets. The boldface number is the label for the point, so that you can refer to the triangle given by points labelled 1, 2 and 3 simply as "123". The first two configurations in a plane in $\mathbb{R}^3$. If that disturbs you, forget the third coordinate; this gives an affine projection to $\mathbb{R}^2$, which preserves the set of triangulations:



The m. o. a. e.[1]



A hexagon



5 points in the plane

2. (a) *Construct and all the subdivisions of a triangular prism (the product of an equilateral triangle and a segment perpendicular to it). Clue: modulo symmetries there are only three subdivisions: triangulations (all equivalent under symmetry), the trivial subdivision, and another class. Draw also the poset of subdivisions.
   (b) Describe all the triangulations of $\Delta_k \times \Delta_1$, the product of a k-simplex and a segment. Clue: there is a natural bijection between triangulations of $\Delta_k \times \Delta_1$ and permutations of $\{1, \ldots, k+1\}$, where the $k+1$ symbols $\{1, \ldots, k+1\}$ correspond to the vertices of Δ_k.

3. *Let V consist of 4 points in $\mathbb{R}^2$, not all in a line (that is, they affinely span $\mathbb{R}^2$). Show that there are three subdivisions of V, no matter what the points are: the trivial subdivision and two triangulations.

4. *Do the same for 5 points in $\mathbb{R}^3$, not all in a plane.

5. Do the same for 5 points in $\mathbb{R}^d$, not all in a hyperplane. For this:
   (a) Observe that there is (modulo multiplication by a constant) a unique affine dependence among the points. Let V^+ and V^- be the sets of points with positive and negative coefficient in it.

[1] "Mother Of All Examples"

(b) Show that every full-dimensional simplex in V is of the form $V \setminus \{v_i\}$ where v_i is in $V^+ \cup V^-$.

(c) Show that two such simplices $V \setminus \{v_i\}$ and $V \setminus \{v_j\}$ intersect properly (that is, they satisfy (IP)) if and only if v_i and v_j lie in the same subset V^+ or V^-.

(d) Show that $\{V \setminus \{v_i\} : v_i \in V^+\}$ and $\{V \setminus \{v_i\} : v_i \in V^-\}$ are two triangulations of V, and conclude from the above that they are the only non-trivial subdivisions of it.

6. (a) *Consider the vertex set of a 1×2 grid. That is, let $V = \{(i,j) : i \in \{0,1\}, j \in \{0,1,2\}\}$. Construct all the triangulations of V. How many of them use all the points?

   (b) Consider the vertex set of a $1 \times k$ grid. That is, let $V = \{(i,j) : i \in \{0,1\}, j \in \{0,1,\ldots,k\}\}$. Show that V has $\binom{2k}{k}$ triangulations that use all points.

7. Show that the 3-dimensional regular cube has six symmetry classes of triangulations, one with five tetrahedra and five with six tetrahedra. (Clue: first show that every triangulation uses either one of the big, regular, tetrahedra inscribed the cube, or one of the four long diagonals between opposite vertices. In the first case this completely determines the triangulation; in the second case look at the possible configurations of tetrahedra around that diagonal).

8. For each $n \geq 3$, let T_n be the set of all triangulations of a convex n-gon. The goal is to show that $|T_n|$ equals $\frac{1}{n-1}\binom{2n-4}{n-2}$. For this:

   (a) Show that every triangulation of the n-gon has $2n-3$ edges (the n edges of the n-gon plus $n-3$ internal diagonals). Deduce that the average degree of vertices in a triangulation is $4 - \frac{6}{n}$.

   (b) Look at a particular vertex, say vertex 1. Explain why the average degree of *that vertex* among *all triangulations* is also $4 - \frac{6}{n}$.

   (c) Consider the map $f : T_{n+1} \to T_n$ that "contracts" the edge $\{1, n+1\}$, converting each triangulation of the $(n+1)$-gon into a triangulation of the n-gon.[2] Show that for each $\mathcal{T} \in T_n$, the number of triangulations on $f^{-1}(\mathcal{T})$ equals $\deg_{\mathcal{T}}(1)$ (the degree of vertex 1 in $\mathcal{T}$).

   (d) Conclude form (b) and (c) that the *average* size of $f^{-1}(\mathcal{T})$ is exactly $\frac{4n-6}{n}$.

   (e) Deduce that

$$|T_{n+1}| = \frac{4n-6}{n}|T_n|,$$

   and from this prove the formula for $|T_n|$ by induction.

[2] We consider the vertices of the n-gon to be labeled from 1 to n in cyclic order.

Remark: the numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are called *Catalan numbers*. The exercise shows that $|T_n|$ equals the $(n-2)$-nd Catalan number C_{n-2}.

9. *Show bijections between the following two sets (clue: the bijection becomes easier to visualize if you draw the n-gon with a long horizontal edge $1n$ on top and a chain of $n-1$ small edges below it. Think of half an arepa; the edge $1n$ corresponds to the cut you made):

   (a) Triangulations of a convex $(n+2)$-gon. For example, for $n = 3$ there are $C_3 = \frac{1}{4}\binom{6}{3} = 5$ of them.

   (b) Ways of putting n pairs of parentheses in the $(n+1)$-term multiplication $a_1 \cdots a_{n+1}$, in order to break it into 2-term multiplications. For example, for $n = 3$ there are five ways: $(((a_1a_2)a_3)a_4)$, $((a_1(a_2a_3))a_4)$, $(a_1((a_2a_3)a_4))$, $(a_1(a_2(a_3a_4)))$, $((a_1a_2)(a_3a_4))$.

10. Show a bijection between the above two sets and Dyck paths of length $2n$: monotone paths in the integer grid going from $(0,0)$ to (n,n) and staying always above the diagonal $x = y$. For example, for $n = 3$ there are five paths: $NNNEEE$, $NNEENE$, $NENNEE$, $NENENE$, $NNENEE$. Here N and E denote a "north" and "east" step, resp.