Tropical theta functions

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1. What are tropical theta functions?
2. Voronoi tilings
3. Applications
4. Tropicalizing classical theta functions

History

- Tropical abelian varieties and tropical theta functions were introduced by Mikhalkin and Zarkov in 2007 in a combinatorial setting. They work out the tropical Jacobian of a metric graph and studied the tropical Riemann theta function on it.

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- Some ideas were already present before in different contexts, namely graph theory and degenerations of abelian varieties.
- In 2013 Baker and Rabinoff showed that the 'tropicalization' of the Jacobian of a smooth projective connected curve is the tropical Jacobian of the associated 'tropical' curve.

Novelties

In this work, the main results are:

- Compatibility between the algebraic and the tropical setting: i.e. the tropicalization of a (non-archimedean) theta function is a tropical theta function.

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- Application to faithful troipicalization of abelian varieties.

To obtain this result, we have used the combinatorics of the tropical theta functions.

What are tropical theta functions?

Roughly speaking...

Tropical theta functions are functions defined on a tropical abelian variety (real torus) or, equivalently, are functions defined on a real vector space which are quasi-periodic respect to a full rank lattice.

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But....

- What do we mean by a tropical abelian variety?
- Which kind of functions do we obtain?
- What is the relation with classical theta functions?

Tropical abelian varieties

We fix the following data *

- Let $M \simeq \mathbb{Z}^n$, $M' \simeq \mathbb{Z}^n$ and $[\cdot,\cdot] : M' \times M \to \mathbb{R}$ be a non-degenerate pairing.
- Set $N_\mathbb{R} := \mathrm{Hom}(M, \mathbb{R})$, $N'_\mathbb{R} := \mathrm{Hom}(M', \mathbb{R})$ and $\lambda : M' \to M$ be a group morphism. It induces an integral affine morphism $\varphi : N_\mathbb{R} \to N'_\mathbb{R}$.
- Let $\langle \cdot,\cdot \rangle : M \times N_\mathbb{R} \to \mathbb{R}$ be the canonical pairing.
- Assume that $[\cdot, \lambda(\cdot)] : M' \times M' \to \mathbb{R}$ is positive definite.

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Remark

Note that M (and M') is a full rank lattice in $N'_{\mathbb{R}}$ (in $N_{\mathbb{R}}$) via the pairing $[\cdot,\cdot]$, as it is non-degenerate. That is, we have $m \in M \mapsto [\cdot, m] \in N'_{\mathbb{R}}$ and $m' \in M' \mapsto [m', \cdot] \in N_{\mathbb{R}}$.
Note that $\varphi(M') \subset M'$.

Tropical abelian varieties

With this data, we are ready to define tropical abelian varieties.

Definition

Given the data *, a tropical abelian variety is given by the real torus $\Sigma = N_{\mathbb{R}}/M'$. The dual tropical abelian variety is given by $\Sigma' = N'_{\mathbb{R}}/M$ and the induced morphism $\overline{\varphi} : \Sigma \to \Sigma'$ (or λ) is called a polarization. If φ is an isomorphism, the polarization is called principal.

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Remark

The image of N in Σ (N' in Σ') gives the integral structure in the (dual) tropical abelian variety.

Example: Tropical elliptic curve

If $N_{\mathbb{R}} \simeq \mathbb{R}$ and $M' \simeq \gamma\mathbb{Z}$ with $\gamma \in \mathbb{R}_{>0}$. A tropical elliptic curve Σ is isomorphic to a circle $\mathbb{R}/\gamma\mathbb{Z}$.

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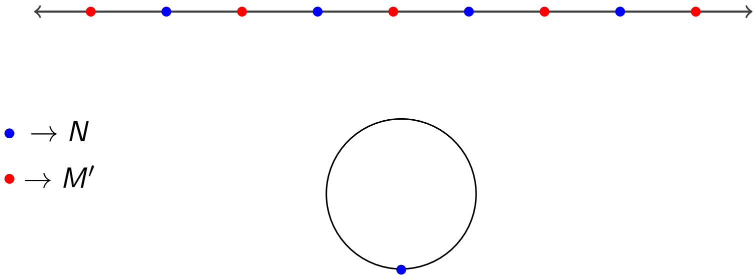

Figure: Tropical elliptic curve.

Tropical theta functions

Definition

Let $\Sigma = N_{\mathbb{R}}/M'$ be a tropical abelian variety with polarization λ and let $c : M' \to \mathbb{R}$ be a function satisfying

$$c(u'_1 + u'_2) - c(u'_1) - c(u'_2) = [u'_1, \lambda(u'_2)],$$

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$$c(u_1' + u_2') - c(u_1') - c(u_2') = [u_1', \lambda(u_2')],$$

for all $u_1', u_2' \in M'$. A *tropical theta function* respect to (λ, c) is a piece-wise integral affine function $\theta_{\text{trop}} : N_{\mathbb{R}} \to \mathbb{R}$ such that

$$\theta_{\text{trop}}(w) = \theta_{\text{trop}}(w + u') + c(u') + \langle \lambda(u'), w \rangle,$$

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A tropical theta function descends and give rise to a well defined function $\overline{\theta_{\text{trop}}} : \Sigma \to \mathbb{R}$ on the tropical abelian variety.

Tropical Riemann theta function

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Tropical Riemann theta function

If we identify M' with M via λ, we have

$$\theta_{\mathrm{trop}}(w) = \min_{u \in M} \left\{ \frac{1}{2}[u, u] + \langle u, w \rangle \right\}.$$

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$$\theta_{\mathrm{trop}}(w) = \min_{u \in M} \left\{ \frac{1}{2}[u,u] + \langle u, w \rangle \right\}.$$

and the transformation rule is given by

$$\theta_{\mathrm{trop}}(w) = \theta_{\mathrm{trop}}(w+u) + \frac{1}{2}[u,u] + \langle u, w \rangle,$$

for all $w \in N_{\mathbb{R}}$ and $u \in M$.

Voronoi polytopes

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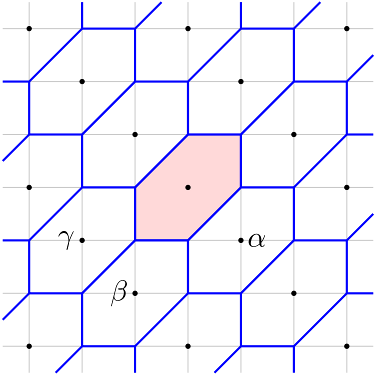
The following properties hold.

- $\mathrm{Vor}_u = \mathrm{Vor}_0 + u$.
- Vor_0 is centrally symmetric.
- $\{\mathrm{Vor}_u\}_{u \in M}$ gives a polyhedral decomposition of $N_{\mathbb{R}}$.
- Vor_0 is a fundamental domain w.r.t. M.

A Voronoi decomposition of the plane

For example, we have the following Voronoi decomposition of the plane w.r.t. $\begin{pmatrix} [e_1, e_1] & [e_1, e_2] \\ [e_2, e_1] & [e_2, e_2] \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$



Corner locus of tropical theta functions

From now on, we will restrict to principally polarized tropical abelian varieties. The Riemann tropical theta function can be written as

$$\begin{aligned} \theta_{\mathrm{trop}}(w) &= \min_{u \in M} \left\{ \tfrac{1}{2}[u,u] - [u,w] \right\} \\ &= \tfrac{1}{2} \min_{u \in M} \left\{ [u-w, u-w] \right\} - \tfrac{1}{2}[w,w], \end{aligned}$$

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It follows that the corner locus of θ_{trop} is given by the boundary of the Voronoi polytopes induced by $[\cdot,\cdot]$.

Tropical theta function for elliptic curves

For a tropical elliptic curve, we have

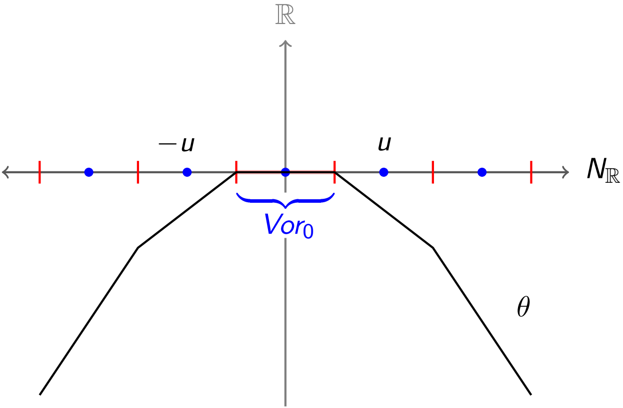

Figure: The graph of θ_{trop}.

Tropical Jacobians

Let Γ be a connected finite metric graph. Consider $H^1(\Gamma, \mathbb{Z})$ and $H^1(\Gamma, \mathbb{R}) = H^1(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Define the edge length pairing on $C_1(\Gamma, \mathbb{Z})$ by

$$[e, e'] := \begin{cases} l(e) & \text{if } e = e' \\ 0 & \text{otherwise} \end{cases}$$

for $e, e' \in E$ the edges of Γ.

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Definition

The Jacobian of the metric graph Γ is the principally polarized tropical abelian variety $\Sigma := H^1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z})$ with lattice $H_1(\Gamma, \mathbb{Z})$ and the pairing $[\cdot, \cdot]$ defined above. The integral structure is given by $H^1(\Gamma, \mathbb{Z})$.

Voronoi decomposition

Fact

We have that Vor_0 is a zonotope generated by the segments $[-e/2, e/2]$, for $e \in H^1(\Gamma, \mathbb{R})$, with e a non-bridge edge.

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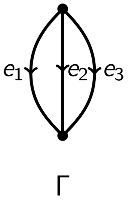
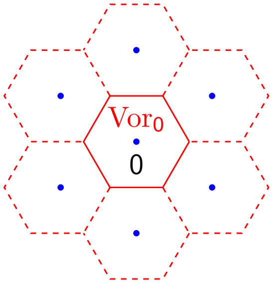
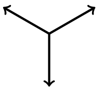
Example in dimension 2.







Connection with tropical geometry

As we have said, one of the main results is to used tropical theta functions in order to construct faithful tropicalizations.

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$$\boxed{\text{Algebraic Varieties}} \xrightarrow{\text{tropicalization}} \boxed{\text{Polyhedral complexes}}$$

Tropicalization

A field K is is called *non-archimedean* if it is endowed with a non-archimedean norm, i.e. $|\cdot| : K \to \mathbb{R}$ such that for all $x, y \in K$

$$|xy| = |x||y|, \quad |x+y| \leq \max\{|x|, |y|\},$$

and $|x| = 0$ iff $x = 0$. We will assume that K is complete and algebraically closed.

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Tropicalization

Let $X \subset (K^\times)^n$ be a closed subvariety. Then the set

$$\mathrm{trop}(X) := \overline{\{(-\log|x_1|, \ldots, -\log|x_n|) : (x_1, \ldots, x_n) \in X\}} \subset \mathbb{R}^n$$

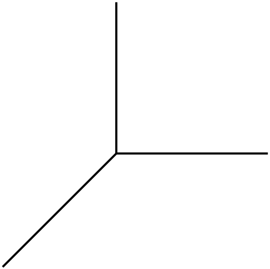
is a polyhedral complex.

Tropical line

Consider $X \subset (K^{\times})^2$ given by

$$x + y + 1 = 0.$$

The tropical variety is



Tropicalization and analytic geometry

To a variety X defined over a non-archimedean field K one can associate an “analytic variety” X^{an}. There exists a piece-wise affine space S, called the skeleton, such that X^{an} is a strong deformation retract of X^{an}.

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Analytic Varieties $\xrightarrow{\text{tropicalization}}$ Polyhedral complexes

Faithful tropicalization

The tropicalization is called faithful, if the skeleton S of X^{an} can be represented in the tropical variety, i.e. $\mathrm{trop}|_S$ is a homeomorphism onto its image and is unimodular, with $\mathrm{trop} : X^{\mathrm{an}} \to \mathbb{R}^n$.

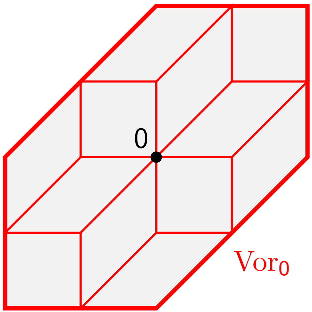
Faithful tropicalization of tropical Jacobians

Consider the case of dimension 2. We use tropical theta functions to construct the map $\psi = (\psi_1, \psi_2, \psi_3) : N_{\mathbb{R}} \to \mathbb{R}$, with

$$\begin{aligned}
\psi_\alpha(w) &= \theta_{\mathrm{trop}}\left(w + \tfrac{1}{2}\alpha\right) + \theta_{\mathrm{trop}}\left(w - \tfrac{1}{2}\alpha\right) - 2\,\theta_{\mathrm{trop}}(w) \\
\psi_\beta(w) &= \theta_{\mathrm{trop}}\left(w + \tfrac{1}{2}\beta\right) + \theta_{\mathrm{trop}}\left(w - \tfrac{1}{2}\beta\right) - 2\,\theta_{\mathrm{trop}}(w) \\
\psi_\gamma(w) &= \theta_{\mathrm{trop}}\left(w + \tfrac{1}{2}\gamma\right) + \theta_{\mathrm{trop}}\left(w - \tfrac{1}{2}\gamma\right) - 2\,\theta_{\mathrm{trop}}(w).
\end{aligned}$$

With α, β, γ the relevant vectors respect to the Voronoi decomposition of $N_{\mathbb{R}}$.

The domain of linearity of ψ gives a subdivision of Vor_0.



This map descends to map $\overline{\psi} : \Sigma \to \mathbb{R}^3$ which is unimodular but is not injective.

To obtain injectivity, we consider a new map

$$\psi' : \Sigma \to \mathbb{R}^3 \times \mathbb{R}^3.$$

given by $\psi'(w) := (\psi(w), \psi(w + \epsilon\alpha))$, for $\alpha \in (0, 1/2)$.

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Theorem

The map ψ' is faithful, i.e. unimodular e injective.

Non-archimedean theta functions

Let A be a principally polarized abelian variety over K which is totally degenerate, i.e. $A \simeq (K^\times)^n/\Lambda$, with Λ a full rank lattice.

A theta function on A is given by a function $\theta : (K^\times)^n \to K$ defined as

$$\theta = \sum_{m \in M} q(m,m)^{1/2} \chi^m,$$

satisfying $\theta(x) = c(\lambda)\chi^{\sigma(\lambda)}(x)\theta(\lambda x)$, with $\sigma : \Lambda \to M$ a group isomorphism, $c : \Lambda \to K^\times$ a morphism and q is a symmetric bilinear form.

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Tropicalizing and restricting to the points in the skeleton of A we obtain

$$\theta_{\mathrm{trop}}(w) = \min_{m \in M} \{\frac{1}{2}[m,m] + \langle m, w \rangle\},$$

with $[m,m] = -\log|q(m,m)|$.

Faithful tropicalization

Due to the previous slide, there exists an algebraic map

$$\Psi : A \dashrightarrow (K^3) \times (K^3)$$

that tropicalizes to ψ'.

$$\Psi_\alpha(w) = \theta\big(w\gamma^{1/2}\big)\theta\big(w\gamma^{-1/2}\big) / \theta(w)^2 \ldots\ldots$$

This implies faithful tropicalization for abelian surfaces.

Remark

It remains open how to construct faithful maps on general tropical abelian varieties.

Thank you!!
Gracias!!