Lecture 2 Practice Problems

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(1) Write out $s_\lambda(x_1, x_2, x_3, 0, 0, \dots)$ for each partition λ of 4 and 5. That is, only include the terms of s_λ that contain x_1, x_2, x_3. This is called a Schur polynomial.

(2) Expand $h_{3,2,1}$ and $e_{3,2,1}$ in the Schur basis.

(3) Expand $h_{3,2}$ in the p-basis.

(4) Recall $\omega : QSym_n \to QSym_n$ is defined on Gessel's fundamental basis by

$$\omega(F_S) = F_{[n-1]\setminus S}.$$

Use this definition to show that $\omega(s_\lambda) = s_{\lambda'}$.

(5) Recall for $f(x) \in \mathbb{R}[X]$,

$$\mathrm{ps}(f(x_1, x_2, \dots)) := f(1, q, q^2, \dots).$$

Prove that

$$\mathrm{ps}(F_S) = \frac{q^{\sum S}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Suggestion: First try proving this for $S = \emptyset$ and $S = [n-1]$.

(6) Verify the q-hook length formula

$$\sum_{T \in SYT_\lambda} q^{\mathrm{maj}(T)} = q^{b(\lambda)} \frac{[n]_q!}{\prod_{x\in\lambda} [h_x]_q},$$

for $\lambda = (3, 2)$.

(7) Prove:

$$h_k h_{n-k} = \sum_{\sigma \in \mathfrak{S}_{n,k}} F_{DES(\sigma)},$$

where $\mathfrak{S}_{n,k}$ is the set of words of length n over alphabet $\{1, 2\}$ with k 1's.

(8) Prove: For all $\lambda \vdash n$,

$$s_\lambda = \sum_{T \in SYT_\lambda} F_{DES(T)}.$$