EXERCISES - FRIDAY - ECCO 2016

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The exercises marked * are recommended to review the understanding of the material.

1. RSK algorithm*

1. Apply the RSK algorithm to the matrix A

$$\begin{matrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{matrix}$$

and get two semi standard young tableaux of the same shape.
2. Compute the length of the first row of the tableaux from the matrix A.

3. Tranform the two tableaux into a pair of Gelfand-Tsetlin triangles.

Recall that a plane partition $\Pi = (\pi_{i,j})$ is an array of non negative integers such that $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$. Its weight $|\Pi|$ is $\sum_{i,j} \pi_{i,j}$.

4. Glue the two triangles in order to get a plane partition.

5. Is it possible to compute the weight of the plane partition from the matrix A? If the matrix has n rows and m columns, how many rows and columns does the plane partition have?

2. Plane partitions

Given a plane partition $\Pi = (\pi_{i,j})$ its trace $\mathrm{tr}(\Pi)$ is $\sum_i \pi_{i,i}$. Let $\mathcal{P}_n$ be the set of plane partitions with at most n rows and n columns. That is $\Pi = (\pi_{i,j})$ with $1 \leq i, j \leq n$.
Exercise 1*. Draw all the plane partitions of 4 with at most 2 rows and 2 columns. Compute their trace.
Exercise 2.

We want to compute the generating function :

$$P_n(q,t) = \sum_{\Pi \in \mathcal{P}_n} q^{|\Pi|} t^{\mathrm{tr}(\Pi)}.$$

1.$*$ Use the decomposition of plane partitions into two Gelfand-Tsetlin triangles with the same first row to show that :

$$P_n(q,t) = \sum_{\lambda} q^{-|\lambda|} t^{|\lambda|} s_\lambda(q, \ldots, q^n) s_\lambda(q, \ldots, q^n).$$

2. Show that

$$x^{|\lambda|} s_\lambda(q, \ldots, q^n) = s_\lambda(xq, \ldots, xq^n).$$

3. Use the Cacuchy identity

$$\sum_{\lambda} s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_n) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}$$

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to compute $P_n(q,t)$.

2.1. **Reverse plane partitions.** **Exercise 1** A reverse plane partition of shape λ is a filling of the cells of λ by nonnegative integers. The entries are increasing in rows and columns. Its weight is the sum of the entries.

For example

$$\begin{array}{ccc} 0 & 0 & 7 \\ 1 & 4 & 7 \\ 1 & 6 & \end{array}$$

is a reverse plane partition of shape $(3,3,2)$.

1* Let $\lambda=(3,3,2)$. Show that there exists a weight preserving bijection between reverse plane partitions of shape λ and sequences of partitions $\Lambda=(\lambda^{(0)},\ldots,\lambda^{(6)})$ such that $\lambda^{(0)},\ldots,\lambda^{(6)}=\emptyset$ et

$$\lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \succ \lambda^{(3)} \prec \lambda^{(4)} \succ \lambda^{(5)} \succ \lambda^{(6)}.$$

Recall that $\mu \succ \lambda$ if and only of $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots$.

For example $\Lambda=(\emptyset,(1),(6,1),(4,0),(7,0),(7),\emptyset)$ is such a sequence.

2.*. Deduce that the generating function of such reverse plane partitions of shape $\lambda=(3,3,2)$ can be written as :

$$\langle\emptyset|\,\Gamma_-(q^{-1})\Gamma_-(q^{-2})\Gamma_+(q^3)\Gamma_-(q^{-4})\Gamma_+(q^5)\Gamma_+(q^6)\,|\emptyset\rangle\,.$$

3.* Show that this series is equal to :

$$\frac{1}{(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)^2(1-q^5)}.$$

4. Compute the generating function of reverse plane partition of shape λ for any partition λ.

Exercise 3 1. Define a polynomial

$$g_\lambda(x_0,x_1,\ldots,x_m)=\sum_{\Pi} x^{\Pi},$$

where the sum is over reverse plane partitions Π with entries $\leq m$ of shape λ. Compute $g_{21}(x_0,x_1,x_2)$. Is g_λ symmetric? Is g_λ quasisymmetric?

2. Define a polynomial

$$G_\lambda(x_0,x_1,\ldots,x_m)=\sum_{\Pi} x_0^{c_0(\Pi)}x_1^{c_1(\Pi)}\cdots\,,$$

where the sum is over reverse plane partitions Π of shape λ and $c_i(\Pi)$ is the number of columns of Π with i. Compute $g_{21}(x_0,x_1,x_2)$. Is it symmetric?