Enumerating lattice 3-polytopes

Mónica Blanco, Francisco Santos

Departamento de Matemáticas Estadística y Computación
Universidad de Cantabria, SPAIN

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5 Encuentro Colombiano de Combinatoria,
Universidad Nacional Sede Medellín

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- **Lattice polytope** $P :=$
  convex hull of a finite set of points in $\mathbb{Z}^d$
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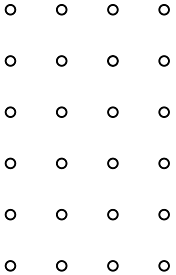

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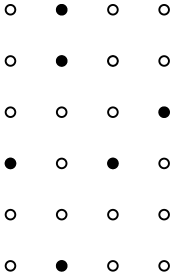

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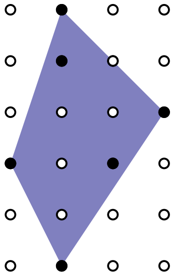

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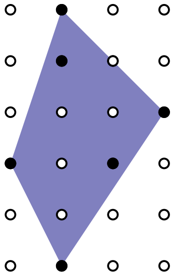

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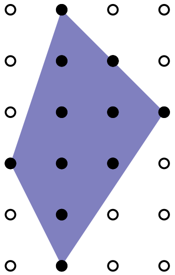

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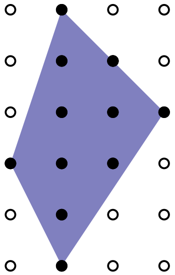

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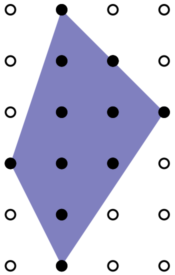

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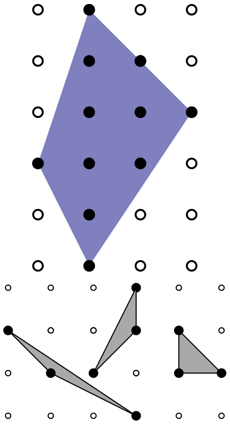

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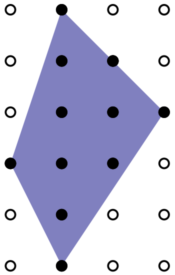

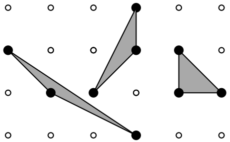

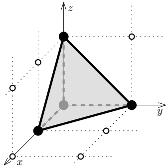

Unimodular equivalence

A **unimodular transformation** is a linear integer map $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice. That is,

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for $A \in \mathbb{Z}^{d \times d}$, $\det(A) = \pm 1$ and $b \in \mathbb{Z}^d$.
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Remark

Size and volume are invariant modulo unimodular equivalence.

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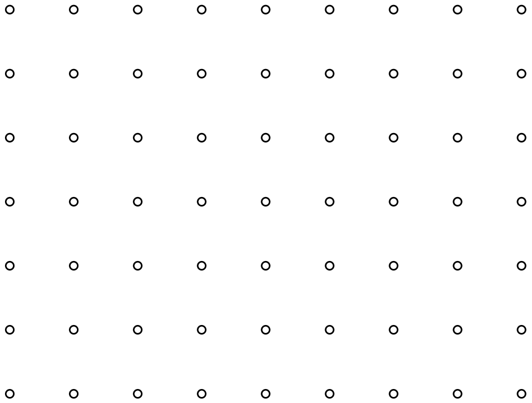

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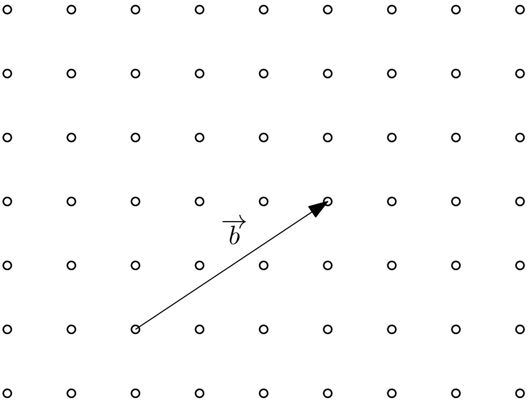

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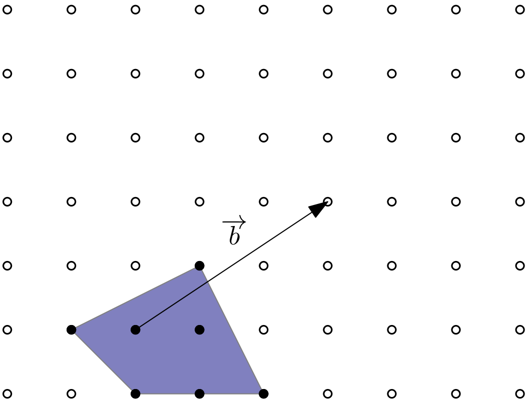

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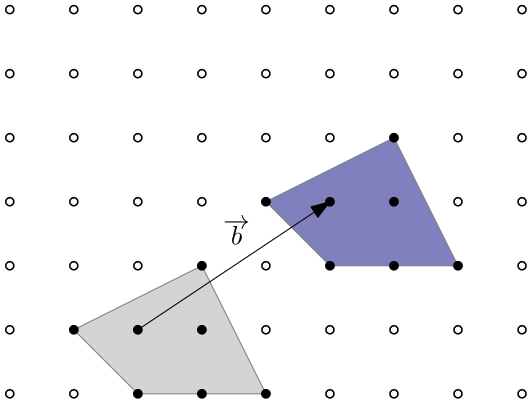

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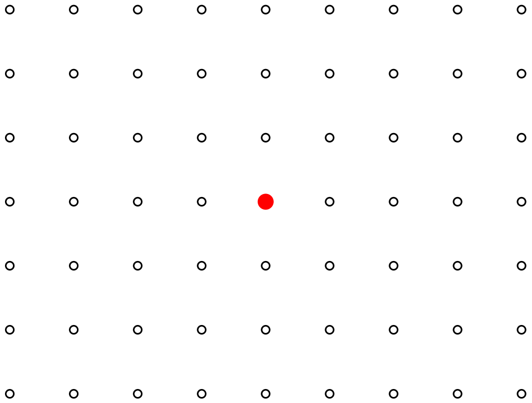

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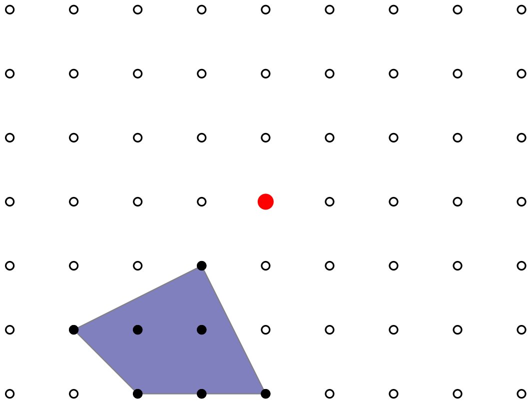

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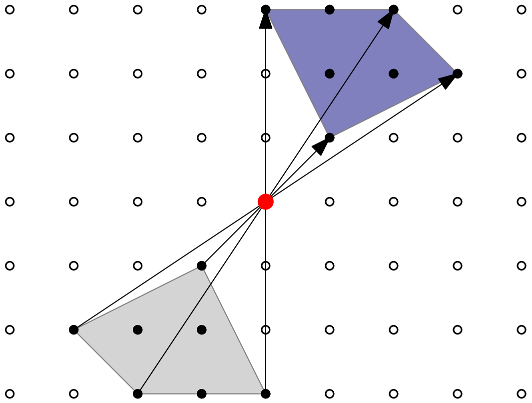

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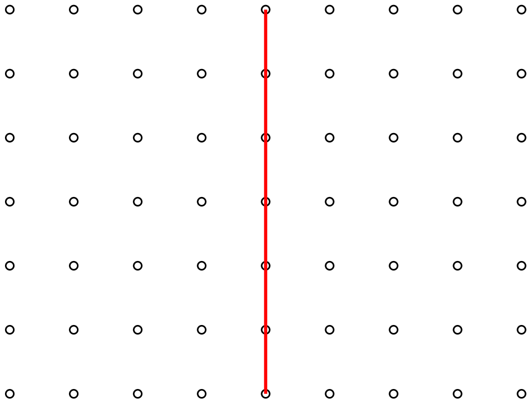

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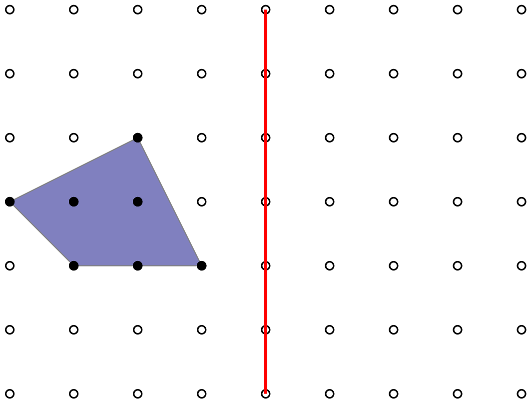

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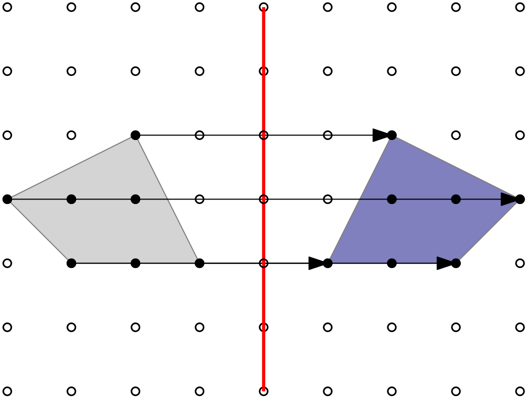

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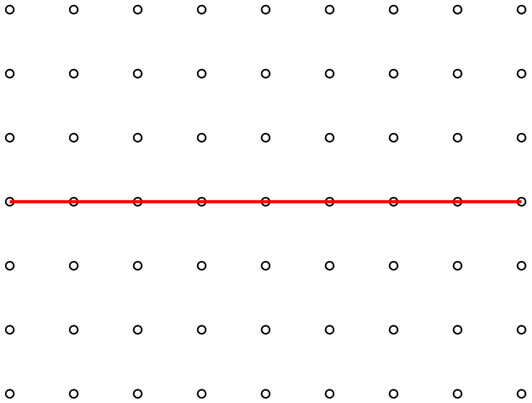

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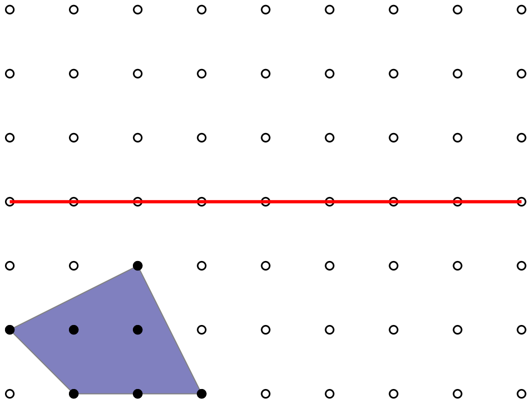

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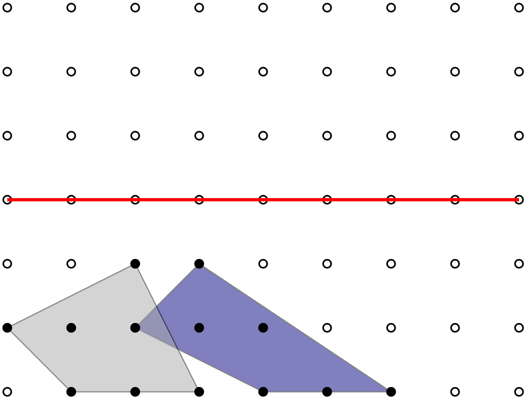

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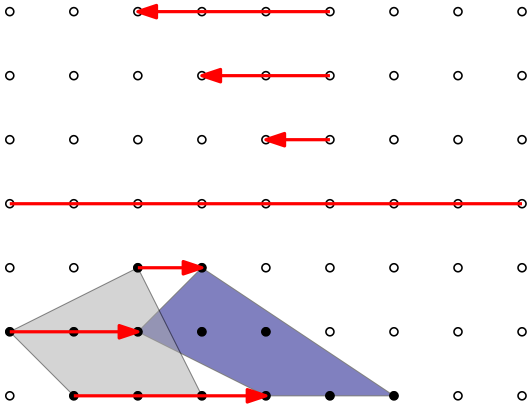

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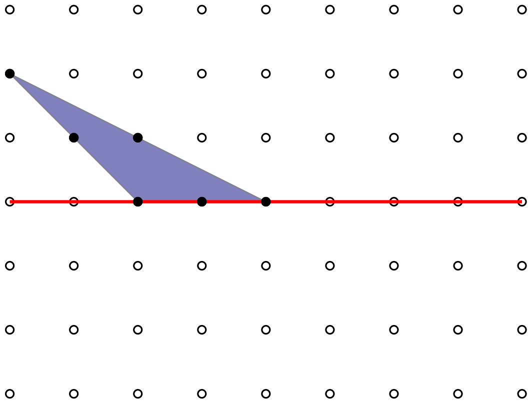

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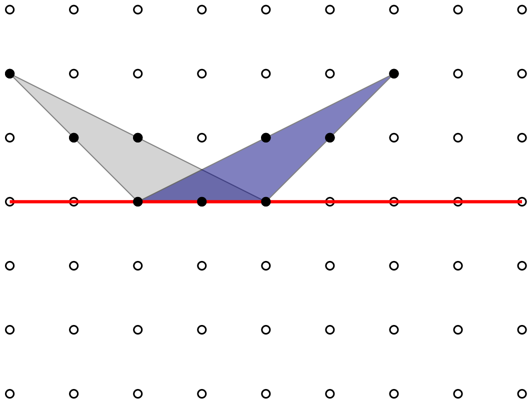

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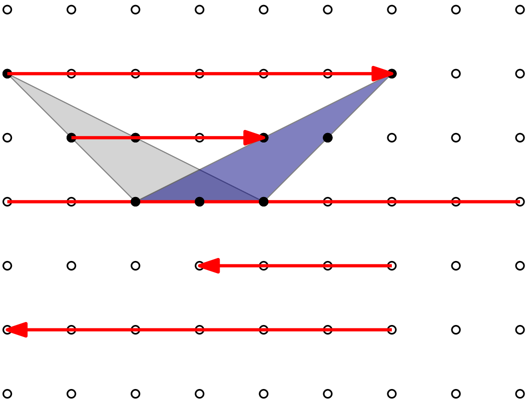

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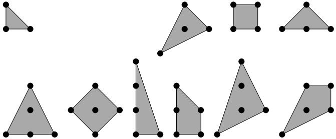

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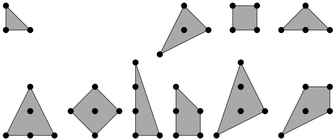

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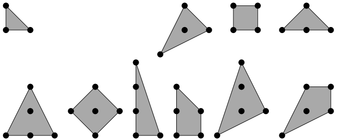

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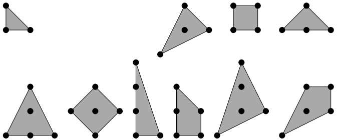

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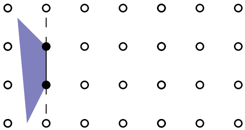

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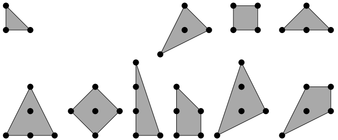

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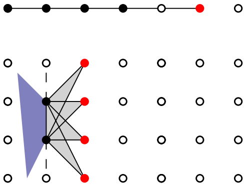

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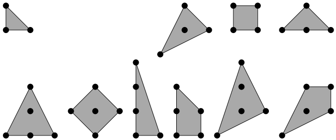

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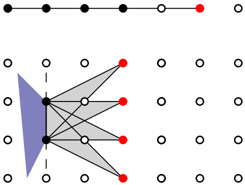

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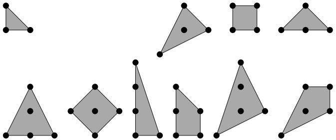

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$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

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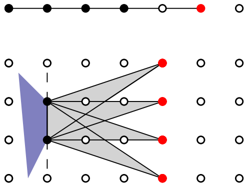

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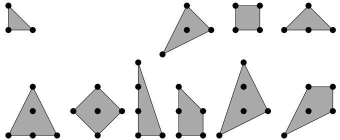

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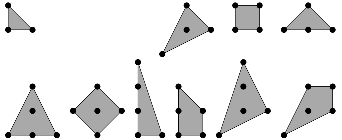

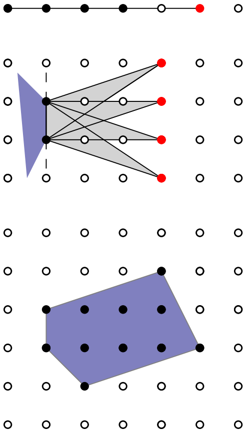

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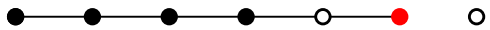

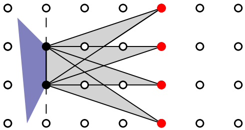

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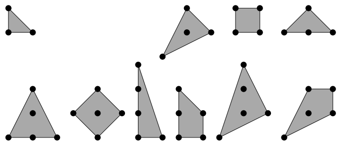

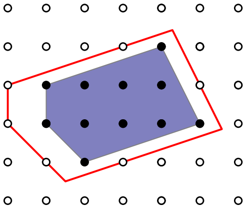

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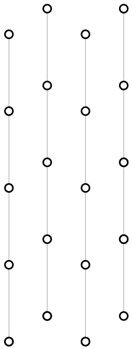

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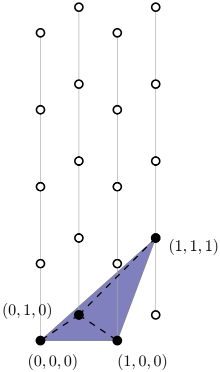

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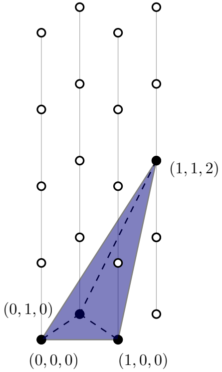

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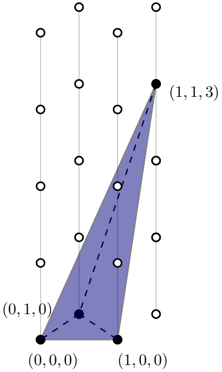

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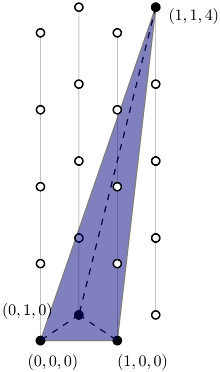

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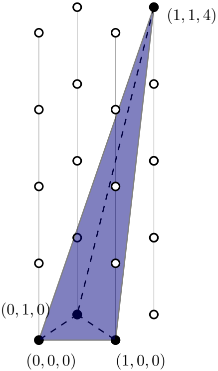

Elements in $\mathcal{P}_3(4)$ are called **empty tetrahedra**: tetrahedra in which the only lattice points are the four vertices. Their classification is classical (White 1964):

$$\mathcal{P}_3(4) = \{ T(p,q) \mid p,q \in \mathbb{Z},\ 0 < p \leq q,\ \gcd(p,q) = 1 \},$$

where $T(p,q) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$.

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Size	4	5	6
width 1	∞	∞	∞
width 2	–	9	74
width 3	–	–	2

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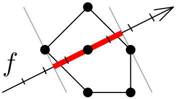

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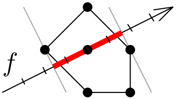

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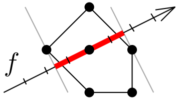

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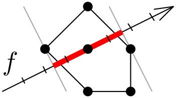

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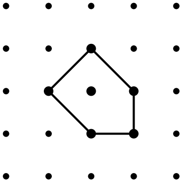

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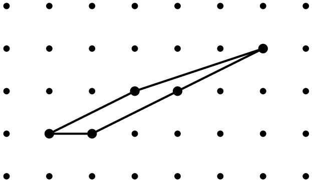

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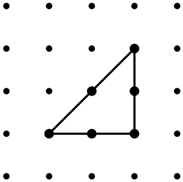

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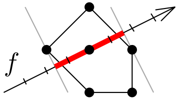

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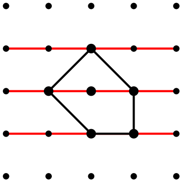

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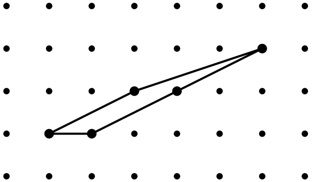

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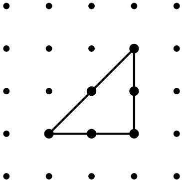

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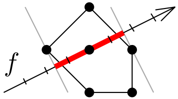

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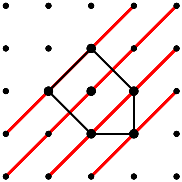

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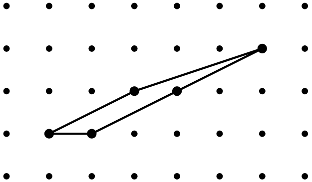

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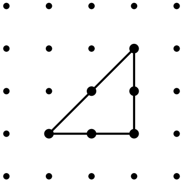

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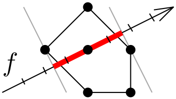

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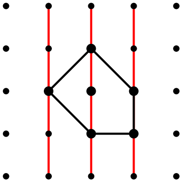

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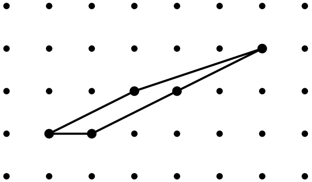

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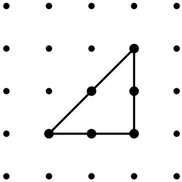

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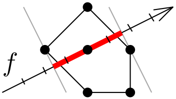

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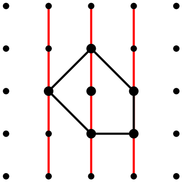

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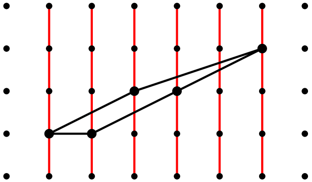

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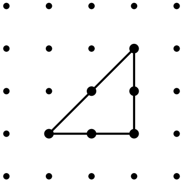

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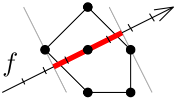

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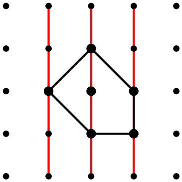

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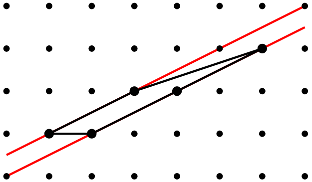

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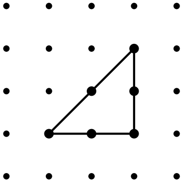

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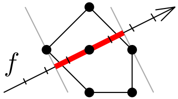

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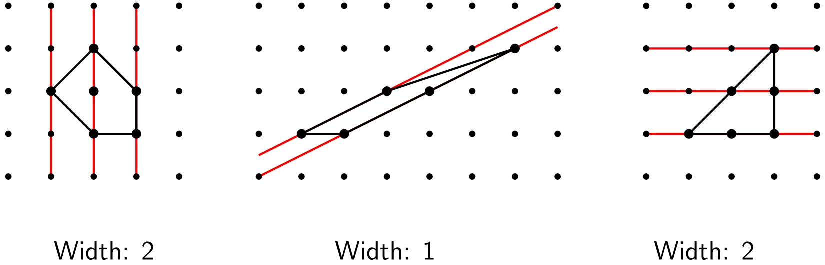

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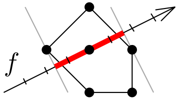

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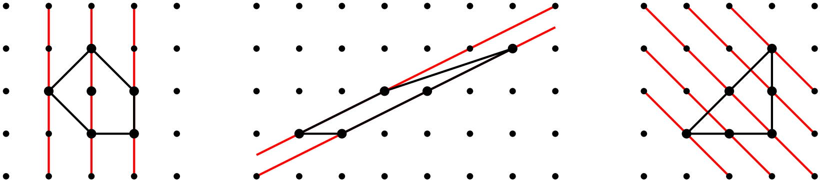

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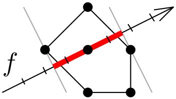
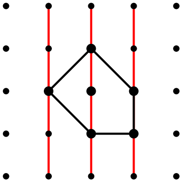
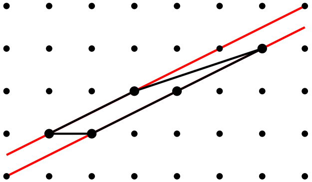


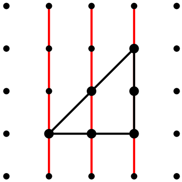
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1



Width: 2

Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

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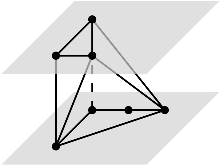

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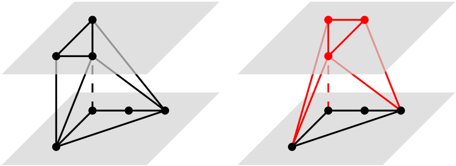

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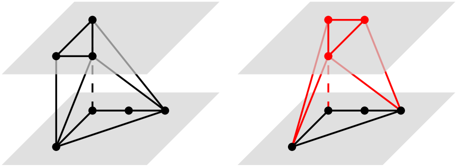

- If the width is > 1:

Lemma (B.-Santos, 2014+)

For each $n \geq 4$, there are **finitely** *many lattice 3-polytopes of width greater than one and size n. That is,*

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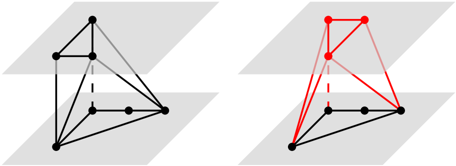

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WE CAN ENUMERATE the complete list $\mathcal{P}_3^*(n)$ of lattice 3-polytopes of size n AND WIDTH > 1, for each n

Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

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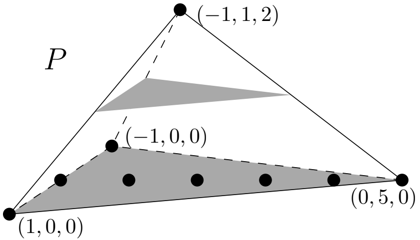

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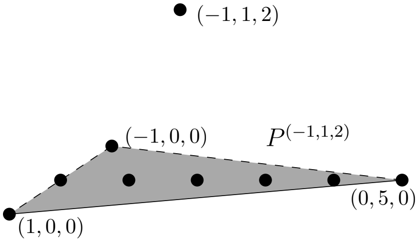

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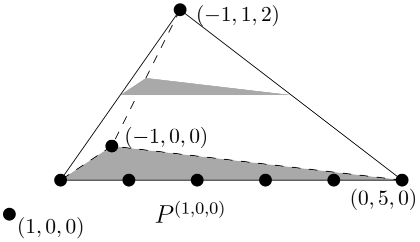

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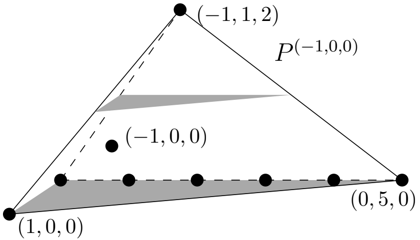

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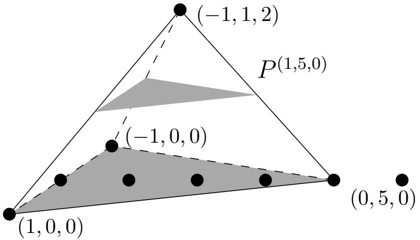

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Quasiminimal vs. Merged

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Let $P \in \mathcal{P}_d^*(n)$.

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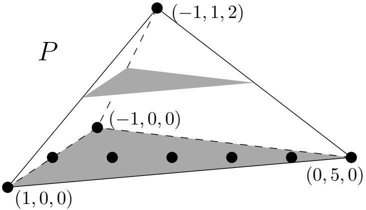

$$\left.\begin{array}{l} P^{(-1,1,2)} \text{ is 2-dimensional} \\ P^{(1,0,0)} \text{ has width one} \\ P^{(-1,0,0)} \text{ has width one} \end{array}\right\} \begin{array}{l} \textit{essential} \\ \text{vertices} \end{array}$$

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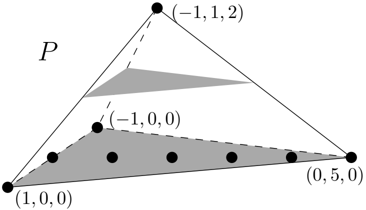

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We will denote by $\mathcal{Q}_d(n)$ the set of quasiminimal d-polytopes of size n.

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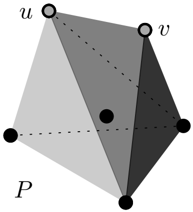

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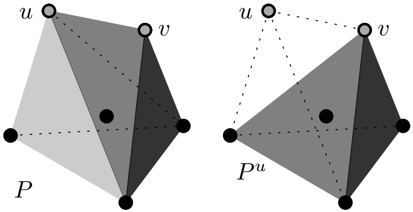

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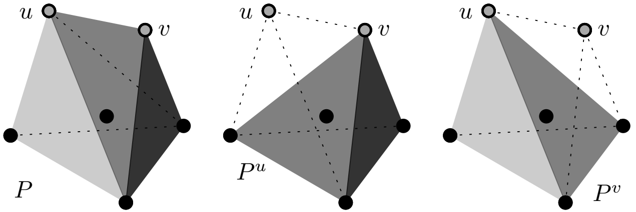

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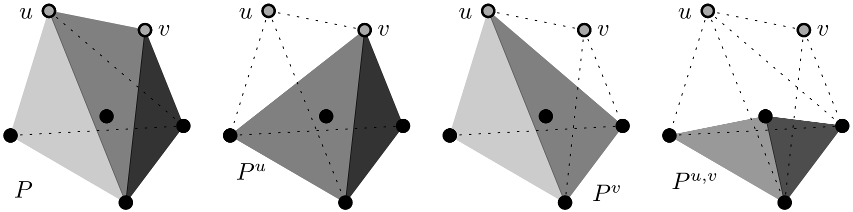

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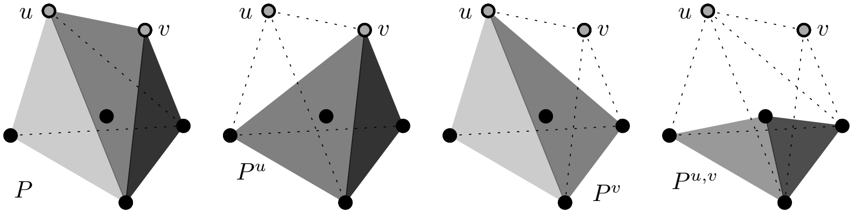

We will denote by $\mathcal{M}_d(n)$ the set of merged d-polytopes of size n.

Exceptions

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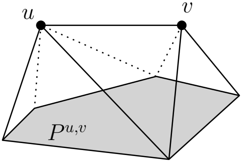

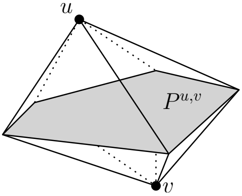

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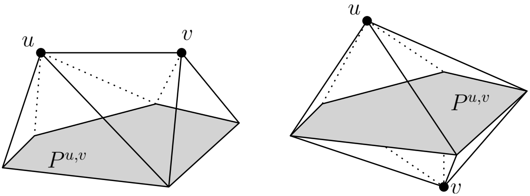


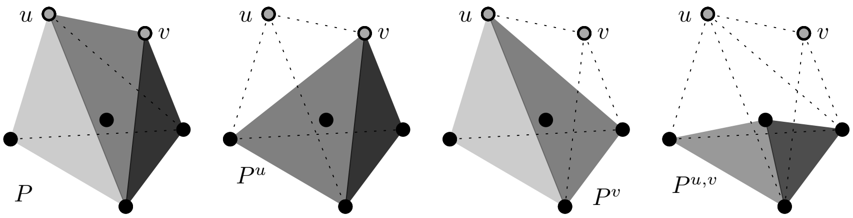
Theorem (Blanco and Santos)

There is a single lattice 3*-dimensional exception, and it is of size* $n = 6$*:*

$$|\mathcal{P}_3^*(6) \setminus (\mathcal{Q}_3(6) \cup \mathcal{M}_3(6))| = 1, \qquad \mathcal{P}_3^*(n) = \mathcal{Q}_3(n) \cup \mathcal{M}_3(n), \text{ for all } n \geq 7.$$

Notice that $\mathcal{P}_3^*(4) = \emptyset$ and, hence, $\mathcal{P}_3^*(5) = \mathcal{Q}_3(5)$.

Classifying merged polytopes: size $n \to n-1$



Size $n-1 \to n$: `Merging` algorithm

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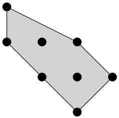
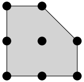


P_2



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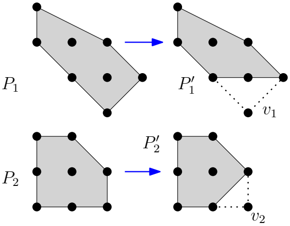

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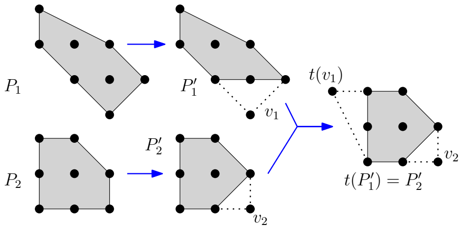
2. Check if P_1' and P_2' are d-dimensional and equivalent. If they are, let $t : \mathbb{Z}^d \to \mathbb{Z}^d$ be an equivalence sending P_1' to P_2' (there are finitely many possibilities for t; do step 3 for each).



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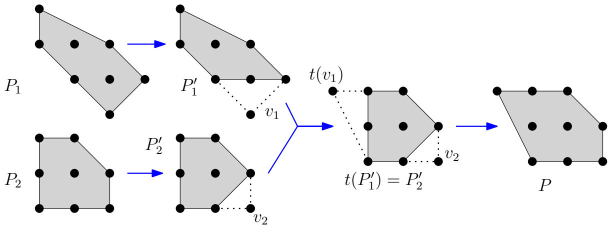
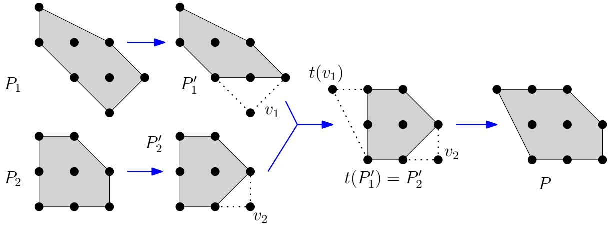
3. If the size of $P := \mathrm{conv}(t(P_1) \cup P_2)$ equals n, add P to L'.



Size $n-1 \to n$: Merging algorithm

INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width > 1.
OUTPUT: the list $L' = \texttt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.
For each $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:

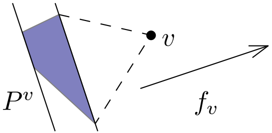
1. Let $P_1' = \operatorname{conv}(\mathbb{Z}^d \cap P_1 \setminus \{v_1\})$ and $P_2' = \operatorname{conv}(\mathbb{Z}^d \cap P_2 \setminus \{v_2\})$.
2. Check if P_1' and P_2' are d-dimensional and equivalent. If they are, let $t : \mathbb{Z}^d \to \mathbb{Z}^d$ be an equivalence sending P_1' to P_2' (there are finitely many possibilities for t; do step 3 for each).
3. If the size of $P := \operatorname{conv}(t(P_1) \cup P_2)$ equals n, add P to L'.



Dimension 3: By definition, and since $\mathcal{P}_3^*(n-1)$ is a finite list:
$\mathcal{M}_3(n) = \texttt{Merging}(\mathcal{P}_3^*(n-1))$, for all n.

Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.



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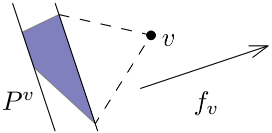

Definition (Boxed vs. spiked)

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *most of their lattice points lie in the vertices of a d-parallelepiped*.

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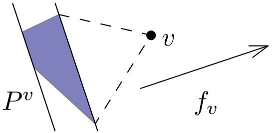

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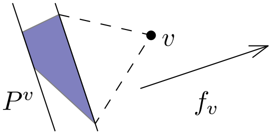

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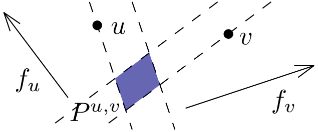

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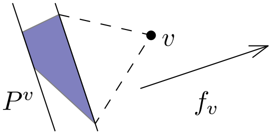

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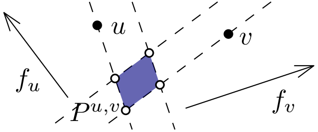

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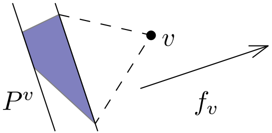

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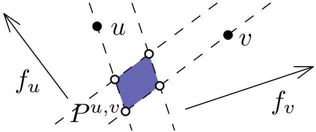

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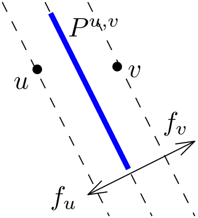

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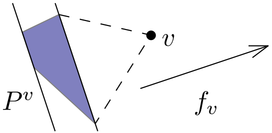

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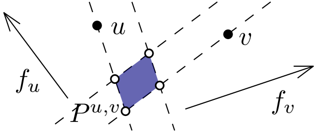

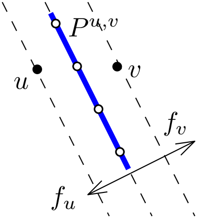

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In dimension three this implies that there are finitely many.

We have enumerated those of dimension 3 with computer help. Let the list of them, for each size $n \in \{7, \ldots, 11\}$, be denoted $\texttt{Boxed}_3(n)$.

Spiked polytopes

Spiked d-polytopes can, however, have arbitrarily large size. Still, it turns out that the projection of spiked d-polytopes in the direction of *the* lattice segment has very specific properties:

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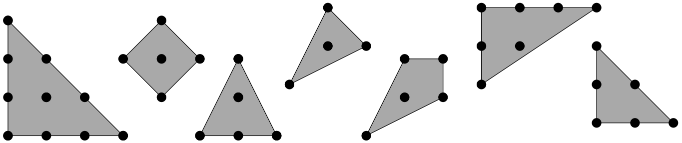
Theorem (Blanco and Santos)

Every spiked 3-polytope of size $n \geq 7$ projects to one of the following 2-polytopes in such a way that all the vertices in the projection have a unique element in the preimage.



This allows us to explicit list spiked 3-polytopes for each given size $n \geq 7$. We denote this list by $\texttt{Spiked}_3(n)$.

Quasiminimal polytopes

Putting these things together, we present the full classification of quasiminimal 3-polytopes:

Theorem (Blanco and Santos)

For $7 \leq n \leq 11$, $\mathcal{Q}_3(n) = \mathtt{Boxed}_3(n) \cup \mathtt{Spiked}_3(n)$, *and it has* 50, 42, 44, 46 *and* 49 *elements, respectively.*
For $n > 11$, $\mathcal{Q}_3(n) = \mathtt{Spiked}_3(n)$ *and it has* $4n + 7$ *elements if* $n \equiv 0 \pmod 3$, *and* $4n + 5$ *otherwise.*

SUMMARY & RESULTS

- $\mathcal{P}_3^*(5)$ and $\mathcal{P}_3^*(6)$ (classified by B-Santos).
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Size	4	5	6	7	8	9	10	11
width 2	0	9	74	477	2524	10862	40885	137803
width 3	0	0	2	19	151	836	4148	18635
width 4	0	0	0	0	0	0	2	26
quasiminimal	0	9	35	50	42	44	46	49
merged	0	0	40	446	2633	11654	44989	156415
exceptions	0	0	1	0	0	0	0	0
total	0	9	76	496	2675	11698	45035	156464

Thank you for your patience!!