Symmetric Functions and Eulerian Polynomials

Lecture 4: Chromatic Quasisymmetric functions

Michelle Lynn Wachs Gajzenberg de Galloway
University of Miami

Cuarto Encuentro Colombiano de Combinatoria 2016

Steps of the proof of $\sum Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)}$

1. Alternative characterization of $Q_{\lambda,j}$ involving ornaments.

$\Longrightarrow$ 2. Bijection from ornaments to banners is used to give another characterization of $Q_{n,j}$

3. Banner characterization is used to obtain a recurrence relation, which yields the generating function formula.

Step 1 - Ornaments

Recall cycle-type Eulerian quasisymmetric function. For $\lambda \vdash n$ and $j \geq 0$,

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_\lambda \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}.$$

Ornament characterization

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} x^R.$$

(Last time I said $wt(R)$ instead of x^R.)

Step 2 - Banners

A banner is a word over alphabet

$$\{1, 1, 2, 2, \ldots\}$$

such that

- blue letter is followed by letter greater than or equal in value or is last
- red letter is followed by a letter less than or equal in value

Example: 22757547

Let $\mathcal{B}_{n,j}$ be the set of banners of length n with j red letters.

Step 2 - Banners

A banner is a word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

such that

- blue letter is followed by letter greater than or equal in value or is last
- red letter is followed by a letter less than or equal in value

Example: 22757547

Let $\mathcal{B}_{n,j}$ be the set of banners of length n with j red letters.

We construct a bijection

$$\cup_{\lambda \vdash n} \mathcal{R}_{\lambda,j} \to \mathcal{B}_{n,j}.$$

$$(2, 2)\,(7, 5)\,(7, 5, 4, 7) \mapsto 22 \cdot 75 \cdot 7547 \mapsto 22757547$$

Order the bicolored alphabet $1 < 1 < 2 < 2 < \dots$ and choose the lexicographically largest way to write the circular word. Then order the words in lexicographically increasing order. (Here a prefix of a word is considered to be lex. greater than the word.)

Step 3 -Recurrence Relation

The bijection allows us to conclude that

$$Q_{n,j} = \sum_{B \in \mathcal{B}_{n,j}} x^B.$$

We use this characterization of $Q_{n,j}$ to obtain a recurrence relation that implies

$$\sum_{n \geq 0} \left(\sum_{j=0}^{n-1} Q_{n,j} t^j \right) z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}$$

An observation of Stanley

By P-partition reciprocity

$$Q_{n,j} = \sum_{B \in \mathcal{B}_{n,j}} x^B = \omega\Big(\sum_{\substack{w \in W_n \\ \mathrm{des}(w) = j}} x^w \Big),$$

where

$$W_n = \{w \in \mathbb{P}^n : w_i \neq w_{i+1} \ \ \forall i \in [n-1]\}.$$

Example: $3153141 \in W_6$.

An observation of Stanley

By P-partition reciprocity

$$Q_{n,j} = \sum_{B \in \mathcal{B}_{n,j}} x^B = \omega\Big(\sum_{\substack{w \in W_n \\ \operatorname{des}(w) = j}} x^w \Big),$$

where

$$W_n = \{w \in \mathbb{P}^n : w_i \neq w_{i+1} \;\; \forall i \in [n-1]\}.$$

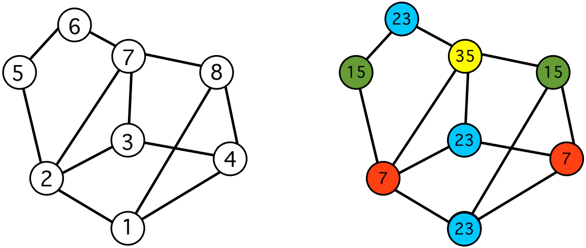
Example: $3153141 \in W_6$.

A word in W_n can be viewed as a proper coloring of the path graph $1 - 2 - 3 - \cdots - n$ with colors from $\mathbb{P}$. Hence

$$\sum_{w \in W_n} x^w$$

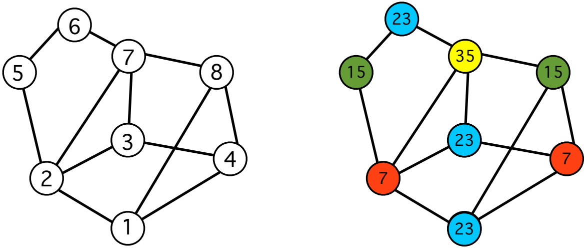
is Stanley's chromatic symmetric function for the path graph.

Chromatic symmetric functions



Let $C(G)$ be set of proper colorings of graph $G = ([n], E)$, where a proper coloring is a map $c : [n] \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E$.

Chromatic symmetric functions

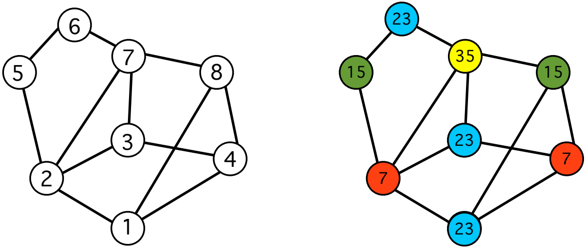


Let $C(G)$ be set of proper colorings of graph $G = ([n], E)$, where a proper coloring is a map $c : [n] \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E$.

Chromatic symmetric function (Stanley, 1995)

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

Chromatic symmetric functions



Let $C(G)$ be set of proper colorings of graph $G = ([n], E)$, where a proper coloring is a map $c : [n] \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E$.

Chromatic symmetric function (Stanley, 1995)

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \dots x_{c(n)}$$

$$X_G(\underbrace{1, 1, \dots, 1}_{m}, 0, 0, \dots) = \chi_G(m)$$

Chromatic symmetric functions

Let Π_G be the bond lattice of G.
Whitney (1932):

$$\chi_G(m) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) m^{|\pi|}$$

Chromatic symmetric functions

Let Π_G be the bond lattice of G.
Whitney (1932):

$$\chi_G(m) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) m^{|\pi|}$$

Stanley (1995): Let p_λ denote the power-sum symmetric function associated with $\lambda \vdash n$. Then

$$X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}(\mathbf{x})$$

Consequently

$$\omega X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} |\mu(\hat{0}, \pi)| p_{\mathrm{type}(\pi)}(\mathbf{x}).$$

Hence $\omega X_G(\mathbf{x})$ is p-positive.

Chromatic symmetric functions

Let Π_G be the bond lattice of G.
Whitney (1932):

$$\chi_G(m) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) m^{|\pi|}$$

Stanley (1995): Let p_λ denote the power-sum symmetric function associated with $\lambda \vdash n$. Then

$$X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}(\mathbf{x})$$

Consequently

$$\omega X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} |\mu(\hat{0}, \pi)| p_{\mathrm{type}(\pi)}(\mathbf{x}).$$

Hence $\omega X_G(\mathbf{x})$ is p-positive.

What about positivity for other bases for the symmetric functions?

Chromatic symmetric functions

Let Π_G be the bond lattice of G.
Whitney (1932):

$$\chi_G(m) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) m^{|\pi|}$$

Stanley (1995): Let p_λ denote the power-sum symmetric function associated with $\lambda \vdash n$. Then

$$X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}(\mathbf{x})$$

Consequently

$$\omega X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} |\mu(\hat{0}, \pi)| p_{\mathrm{type}(\pi)}(\mathbf{x}).$$

Hence $\omega X_G(\mathbf{x})$ is p-positive.

What about positivity for other bases for the symmetric functions?
Recall h-positivity is stronger that p-positivity.

Chromatic symmetric functions

Let Π_G be the bond lattice of G.
Whitney (1932):

$$\chi_G(m) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) m^{|\pi|}$$

Stanley (1995): Let p_λ denote the power-sum symmetric function associated with $\lambda \vdash n$. Then

$$X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}(\mathbf{x})$$

Consequently

$$\omega X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} |\mu(\hat{0}, \pi)| p_{\mathrm{type}(\pi)}(\mathbf{x}).$$

Hence $\omega X_G(\mathbf{x})$ is p-positive.

What about positivity for other bases for the symmetric functions?
Recall h-positivity is stronger that p-positivity. Is $\omega X_G(\mathbf{x})$ h-positive?
That would mean $X_G(\mathbf{x})$ is e-positive

e-positivity

$$X_{K_{3,1}} = 4e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}$$

e-positivity

$$X_{K_{3,1}} = 4e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}$$

- The incomparability graph $\mathrm{inc}(P)$ of a finite poset P on $[n]$ is the graph whose edges are pairs of incomparable elements of P.
- A poset P is said to be $(a+b)$-free if P contains no induced subposet isomorphic to the disjoint union of an a element chain and a b element chain.

e-positivity

$$X_{K_{3,1}} = 4e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}$$

- The incomparability graph $\mathrm{inc}(P)$ of a finite poset P on $[n]$ is the graph whose edges are pairs of incomparable elements of P.
- A poset P is said to be $(a + b)$-free if P contains no induced subposet isomorphic to the disjoint union of an a element chain and a b element chain.

Conjecture (Stanley-Stembridge (1993))

If P is $(3 + 1)$-free then $X_{\mathrm{inc}(P)}$ is e-positive.

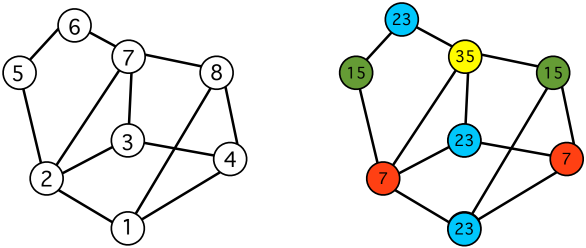
Stanley-Stembridge e-positivity conjecture

Conjecture (Stanley-Stembridge (1993))

If P is $(3+1)$-free then $X_{\mathrm{inc}(P)}$ is e-positive.

- Gasharov (1994): expansion in the Schur basis $\{s_\lambda\}$
- Chow (1996): expansion in the fundamental quasisymmetric function basis $\{F_\mu\}$
- Guay-Paquet (2013): If true for unit interval orders (posets that are both (3+1)-free and (2+2)-free) then true in general i.e. for posets that are (3+1)-free.

Quasisymmetric refinement



Chromatic quasisymmetric function (Shareshian and MW)

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\mathrm{des}(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

where

$$\mathrm{des}(c) := |\{\{i,j\} \in E(G) : i < j \text{ and } c(i) > c(j)\}|.$$

If $G = 1 - 2 - \cdots - n$, then

$$X_G(\mathbf{x}, t) = \sum_{w \in W_n} t^{\mathrm{des}(w)} x^w = \omega \sum_{j=0}^{n-1} Q_{n,j} t^j$$

Quasisymmetric refinement

$G =$ 

$$X_G(\mathbf{x}, t) = e_3 + (e_3 + e_{2,1})t + e_3 t^2$$

$G =$ 

$$X_G(\mathbf{x}, t) = (e_3 + F_{\{1\}}) + 2e_3 t + (e_3 + F_{\{2\}})t^2$$

Chromatic quasisymmetric functions that are symmetric

A natural unit interval order is a unit interval order with a certain natural canonical labeling.

Example: The poset $P_{n,r}$ on $[n]$ with order relation given by $i <_P j$ if $j - i \geq r$. Let

$$G_{n,r} := \mathrm{inc}(P_{n,r}) = ([n], \{\{i,j\} : 0 < j - i < r\})$$

When $r = 2$, $G_{n,r}$ is the path

$$1 - 2 - \cdots - n$$

and

$$X_{G_{n,r}} = \sum_{w \in W_n} t^{\mathrm{des}(w)} x^w.$$

Chromatic quasisymmetric functions that are symmetric

Theorem (Shareshian and MW)

If G is the incomparability graph of a natural unit interval order then $X_G(\mathbf{x}, t)$ is symmetric in $\mathbf{x}$ and palindromic.

$$
\begin{aligned}
X_{G_{3,2}} &= e_3 + (e_3 + e_{2,1})t + e_3 t^2 \\
X_{G_{4,2}} &= e_4 + (e_4 + e_{3,1} + e_{2,2})t + (e_4 + e_{3,1} + e_{2,2})t^2 + e_4 t^3
\end{aligned}
$$

Chromatic quasisymmetric functions that are symmetric

Theorem (Shareshian and MW)

If G is the incomparability graph of a natural unit interval order then $X_G(\mathbf{x}, t)$ is symmetric in $\mathbf{x}$ and palindromic.

$$
\begin{aligned}
X_{G_{3,2}} &= e_3 + (e_3 + e_{2,1})t + e_3 t^2 \\
X_{G_{4,2}} &= e_4 + (e_4 + e_{3,1} + e_{2,2})t + (e_4 + e_{3,1} + e_{2,2})t^2 + e_4 t^3
\end{aligned}
$$

Conjecture (Shareshian and MW - refinement of Stan-Stem)

If G is the incomparability graph of a natural unit interval order then $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal.

Verified for $G_{n,r}$ when $r \leq 2$ and $r \geq n - 2$.

Schur-positivity

Definition (Gasharov (1994))

Let P be a poset of size n and λ be a partition of n. A P-tableau of shape λ is a filling of a Young diagram of shape λ (in English notation) with elements of P such that

- each element of P appears exactly once,
- if $y \in P$ appears immediately to the right of $x \in P$ then $y >_P x$,
- if $y \in P$ appears immediately below $x \in P$ then $y \not<_P x$.

Recall $P_{n,r}$ is the poset on $[n]$ with order relation given by $i <_P j$ if $j - i \geq r$. Let $P = P_{9,3}$.

good

2	6	9
1	4	8
3	7	
5		

bad

2	4	9
5	6	8
1	7	
3		

Schur-positivity

Let $\mathcal{T}_P$ be the set of all P-tableaux.

For $T \in \mathcal{T}_P$ and graph G on $[n]$, define a *G-inversion* of T to be an edge $\{i, j\}$ of G such that $i < j$ and i appears lower than j in T. Let $\mathrm{inv}_G(T)$ be the number of G-inversions of T.

For

$$G_{9,3} := ([9], \{\{i, j\} : 0 < j - i < 3\})$$

and

$T =$

2	6	9
1	4	8
3	7	
5		

$$\mathrm{inv}_G(T) = |\{12, 34, 46, 56, 57, 78, 79, 89\}| = 8.$$

Schur-positivity

Theorem (Shareshian and MW, t=1 Gasharov)

Let G be the incomparability graph of a natural unit interval order P. Then

$$X_G(\mathbf{x}, t) = \sum_{T \in \mathcal{T}_P} t^{\mathrm{inv}_G(T)} s_{\mathrm{shape}(T)}.$$

Consequently $X_G(\mathbf{x}, t)$ is Schur-positive.

Schur-positivity

Theorem (Shareshian and MW, t=1 Gasharov)

Let G be the incomparability graph of a natural unit interval order P. Then

$$X_G(\mathbf{x}, t) = \sum_{T \in \mathcal{T}_P} t^{\mathrm{inv}_G(T)} s_{\mathrm{shape}(T)}.$$

Consequently $X_G(\mathbf{x}, t)$ is Schur-positive.

- Gasharov's proof uses the Jacobi-Trudi identitly and a sign-reversing involution. We had to modify the sign-reversing involution so that it is also inv_G preserving.

Schur-positivity

Theorem (Shareshian and MW, t=1 Gasharov)

Let G be the incomparability graph of a natural unit interval order P. Then

$$X_G(\mathbf{x}, t) = \sum_{T \in \mathcal{T}_P} t^{\mathrm{inv}_G(T)} s_{\mathrm{shape}(T)}.$$

Consequently $X_G(\mathbf{x}, t)$ is Schur-positive.

- Gasharov's proof uses the Jacobi-Trudi identitly and a sign-reversing involution. We had to modify the sign-reversing involution so that it is also inv_G preserving.

Schur-positivity of a symmetric function f means there is a representation of the symmetric group whose Frobenius characteristic is f.

A representation

Conjecture (Shareshian and MW)

Let G be the incomparability graph of a natural unit interval order on $[n]$. Let $\mathcal{H}_G$ be the Hessenberg variety associated with G and let $\mathrm{ch}H^{2j}(\mathcal{H}_G)$ be the Frobenius characteristic of Tymoczko's representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_G)$. Then

$$\omega X_G(\mathbf{x}, t) = \sum_{j\geq 0} \mathrm{ch}H^{2j}(\mathcal{H}_G)t^j.$$

Recently proved by Brosnan and Chow and by Guay-Paquet.

A representation

Conjecture (Shareshian and MW)

Let G be the incomparability graph of a natural unit interval order on $[n]$. Let $\mathcal{H}_G$ be the Hessenberg variety associated with G and let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ be the Frobenius characteristic of Tymoczko's representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_G)$. Then

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

Recently proved by Brosnan and Chow and by Guay-Paquet.

- Our expansion in Schur function basis now gives the multiplicity of irreducibles in Tymoczko's representation.
- Coeffiecients of p-expansion (conjectured by us, proved by Athanasiadis) now gives character of Tymoczko's representation.
- By the hard Lefshetz theorem this implies Schur-unimodality of $X_G(\mathbf{x}, t)$.
- To prove our refinement of the Stanley-Stembridge e-positivity conjecture we would also need to show that Tymoczko's representation on $H^{2j}(\mathcal{H}_G)$ is a permutation representation for which each point stabilizer is a Young subgroup.

An algebraic interpretation

Clearman, Hyatt, Shelton and Skandera have given an algebraic interpretation of the chromatic quasisymmetric functions of incomparability graphs of natural unit interval orders in terms of characters of type A Hecke algebras evaluated at Kazhdan-Lusztig basis elements.

Specialization and permutation statistics

Theorem (Shareshian and MW, t=1 Chow)

$$\omega X_{G_{n,r}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{<r}(\sigma)} F_{\mathrm{DES}_{\geq r}(\sigma)}.$$

Corollary

For all $r \in [n]$,

$$\mathrm{ps}(\omega X_{G_{n,r}}(\mathbf{x}, t)) = \frac{\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)} t^{\mathrm{inv}_{<r}(\sigma)}}{(1-q)\dots(1-q^n)},$$

where

$$\begin{aligned} \mathrm{inv}_{<r}(\sigma) &:= |\{(i,j) : 1 \leq i < j \leq n, \quad 0 < \sigma(i) - \sigma(j) < r\}| \\ \mathrm{maj}_{\geq r}(\sigma) &:= \sum_{i : \sigma(i) - \sigma(i+1) \geq r} i \end{aligned}$$

Generalized q-Eulerian polynomials

Note that the (<2)-inversions of σ are the descents of σ^{-1}.

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{<2}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)}$$

Recall

$$\sum_{j=0}^{n-1} Q_{n,j}t^j = \omega X_{G_{n,2}}(\mathbf{x},t)$$

By taking principal stable specialization, we have

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}t^{\mathrm{exc}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})}t^{\mathrm{des}(\sigma)}$$

Problem: Find a bijective proof. For $q=1$, there is a classical bijection.

Generalized q-Eulerian polynomials

Now define the generalized q-Eulerian polynomials

$$A_n^{(r)}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)} t^{\mathrm{inv}_{<r}(\sigma)}$$

Theorem (consequence of Schur-unimodality of $\omega X_{G_{n,r}}(\mathbf{x},t)$)

$A_n^{(r)}(q,t)$ is palindromic and q-unimodal for all $r \in [n]$.

$$A_4^{(2)}(q,t) = 1+(3+2q+3q^2+2q^3+q^4)t+(3+2q+3q^2+2q^3+q^4)t^2+t^3$$

$$A_5^{(2)}(q,t) = 1+(4+3q+5q^2+\dots)t+(6+6q+11q^2+\dots)t^2+\dots$$

Open Problem: Find an elementary combinatorial proof of q-unimodality. This one is algebro-geometric; it involves representations of the symmetric group on cohomology of Hessenberg varieties and the hard Lefshetz theorem.

p-positivity of $\omega X_G(\mathbf{x}, t)$

Let P be a poset on $[n]$. We say that a permutation $\sigma \in \mathfrak{S}_n$ has a

- P-descent at j if $\sigma(j) >_P \sigma(j+1)$
- left-to-right P-maximum at j if $\sigma(j) >_P \sigma(i)$ for all $i \in [j-1]$

Example: Let $P = P_{6,3}$ and $\sigma = 215634$.

- P-descent at: 4
- left-to-right P-maximum at: $1, 3$

p-positivity of $\omega X_G(\mathbf{x}, t)$

Let P be a poset on $[n]$. We say that a permutation $\sigma \in \mathfrak{S}_n$ has a

- P-descent at j if $\sigma(j) >_P \sigma(j+1)$
- left-to-right P-maximum at j if $\sigma(j) >_P \sigma(i)$ for all $i \in [j-1]$

Example: Let $P = P_{6,3}$ and $\sigma = 215634$.

- P-descent at: 4
- left-to-right P-maximum at: $1, 3$

For $\lambda \vdash n$, let $\mathcal{N}_{P,\lambda}$ be the set of permutations σ in $\mathfrak{S}_n$ such that when σ is split into contiguous segments of lengths $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}$, each segment has neither a P-descent nor a nontrivial left-to-right P-maximum.

Example: Same as above: Let $\lambda = (2, 2, 2)$. Then σ splits into

$$21 \cdot 56 \cdot 34$$

None of the segments have P-descents nor nontrivial left-to-right P-maximums. So $\sigma \in \mathcal{N}_{P,\lambda}$.

p-positivity of $\omega X_G(\mathbf{x}, t)$

For a graph $G = ([n], E)$ and $\sigma \in \mathfrak{S}_n$, a G-inversion of σ is an inversion $(\sigma(i), \sigma(j))$ of σ such that $\{\sigma(i), \sigma(j)\} \in E$. Let $\mathrm{inv}_G(\sigma)$ be the number of G-inversions of σ.

Example: Let $G = G_{6,3}$ and $\sigma = 215634$. Then the G-inversions of σ are $(2, 1), (5, 3), (5, 4), (6, 4)$. No $(6, 3)$ since $\{6, 3\} \notin E$. Hence $\mathrm{inv}_G(\sigma) = 4$.

p-positivity of $\omega X_G(\mathbf{x}, t)$

For a graph $G = ([n], E)$ and $\sigma \in \mathfrak{S}_n$, a G-inversion of σ is an inversion $(\sigma(i), \sigma(j))$ of σ such that $\{\sigma(i), \sigma(j)\} \in E$. Let $\mathrm{inv}_G(\sigma)$ be the number of G-inversions of σ.

Example: Let $G = G_{6,3}$ and $\sigma = 215634$. Then the G-inversions of σ are $(2,1), (5,3), (5,4), (6,4)$. No $(6,3)$ since $\{6,3\} \notin E$. Hence $\mathrm{inv}_G(\sigma) = 4$.

Theorem (Athanasiadis (2014), conjectured by Shareshian and MW)

Let $G = \mathrm{inc}(P)$ where P is a natural unit interval order. Then

$$\omega X_G(\mathbf{x}, t) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda \sum_{\sigma \in \mathcal{N}_{P,\lambda}} t^{\mathrm{inv}_G(\sigma)}$$

Consequently $\omega X_G(\mathbf{x}, t)$ is p-positive.

p-unimodality is still open.

Consequence of p-positivity

$$A_4^{(2)}(q,t) = 1+(3+2q+3q^2+2q^3+q^4)t+(3+2q+3q^2+2q^3+q^4)t^2+t^3$$

$$A_4^{(2)}(1,t) \quad = \quad 1+11t+11t^2+t^3$$

Consequence of p-positivity

$$A_4^{(2)}(q,t) = 1+(3+2q+3q^2+2q^3+q^4)t+(3+2q+3q^2+2q^3+q^4)t^2+t^3$$

$$\begin{aligned} A_4^{(2)}(1,t) &= 1+11t+11t^2+t^3 \\ A_4^{(2)}(-1,t) &= 1+3t+3t^2+t^3 \end{aligned}$$

Consequence of p-positivity

$$A_4^{(2)}(q,t) = 1+(3+2q+3q^2+2q^3+q^4)t+(3+2q+3q^2+2q^3+q^4)t^2+t^3$$

$$\begin{aligned} A_4^{(2)}(1,t) &= 1+11t+11t^2+t^3 \\ A_4^{(2)}(-1,t) &= 1+3t+3t^2+t^3 \\ A_4^{(2)}(i,t) &= 1+t+t^2+t^3 \end{aligned}$$

Consequence of p-positivity

$$A_4^{(2)}(q,t) = 1+(3+2q+3q^2+2q^3+q^4)t+(3+2q+3q^2+2q^3+q^4)t^2+t^3$$

$$\begin{aligned} A_4^{(2)}(1,t) &= 1+11t+11t^2+t^3 \\ A_4^{(2)}(-1,t) &= 1+3t+3t^2+t^3 \\ A_4^{(2)}(i,t) &= 1+t+t^2+t^3 \end{aligned}$$

Theorem (consequence of p-expansion formula for $\omega X_{G_{n,r}}(\mathbf{x},t)$)

Let $dm = n$ and let ξ_d be a primitive dth root of unity. Then

$$A_n^{(r)}(\xi_d,t) = \sum_{\sigma \in \mathcal{N}_{P_{n,r},d^m}} t^{\mathrm{inv}_{<r}(\sigma)}$$

Consequently $A_n^{(r)}(\xi_d,t) \in \mathbb{N}[t]$.

Unimodality of $A_n^{(r)}(\xi_d,t)$ is still open and is a consequence of p-unimodality conjecture. Proved for $d = n$ and for $r = 1, 2, n-2, n-1, n$.