

Symmetric Functions and Eulerian Polynomials

Lecture 1: Permutation Statistics and Eulerian polynomials

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Lecture 1: Permutation Statistics and Eulerian Polynomials

Lecture 2: Symmetric and Quasisymmetric Functions

Lecture 3: Eulerian quasisymmetric functions

Lecture 4: Chromatic quasisymmetric functions

Binomial Coefficients and Eulerian numbers

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

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- coeff's of polynomial

$$(t+1)^n = \sum_{j=0}^n \binom{n}{j} t^j$$

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- coeff's of Eulerian polynomial

$$A_n(t) = \sum_{j=0}^{n-1} \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle t^j$$

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- rows add to 2^n

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- rows add to $n!$

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- rows add to 2^n
- subsets of $\{1, 2, \dots, n\}$ of size j

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- rows add to $n!$
- permutations in \mathfrak{S}_n with j descents

- Rows are palindromic and unimodal.

Eulerian polynomials - Euler's definition

$$\sum_{i \geq 1} t^i = \frac{t}{1-t}$$

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Leonhard Euler
(1707-1783)

Euler's triangle

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Euler's definition

$$\sum_{i \geq 1} i^n t^i = \frac{t A_n(t)}{(1-t)^{n+1}}$$

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Euler's definition

$$\sum_{i \geq 1} i^n t^i = \frac{t A_n(t)}{(1-t)^{n+1}}$$

Euler's exponential generating function formula

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z} - t}$$

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Permutations

Given a set A , a **permutation** on A is a bijection on A .

The group of permutations on A under composition is called the **symmetric group** on A and is denoted by \mathfrak{S}_A .

Let $[n]$ denote the set $\{1, 2, \dots, n\}$ and let $\mathfrak{S}_n := \mathfrak{S}_{[n]}$.

Two line notation:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{bmatrix} \in \mathfrak{S}_5$$

One line notation:

$$\sigma = [4\ 5\ 2\ 1\ 3] \in \mathfrak{S}_5$$

Cycle notation:

$$\sigma = (1, 4)(2, 5, 3) \in \mathfrak{S}_5$$

Partitions and Compositions

A **partition** of $n \in \mathbb{P}$ is a weakly decreasing sequence of positive integers whose sum is n .

A **composition** of $n \in \mathbb{P}$ is a sequence of positive integers whose sum is n .

Partitions of 4

(4)

(3,1)

(2,2)

(2,1,1)

(1,1,1)

Compositions of 4

(4)

(3,1), (1,3)

(2,2)

(2,1,1), (1,2,1), (1,1,2)

(1,1,1,1)

If λ is a partition of n , we say $\lambda \vdash n$ and $|\lambda| = n$.

If μ is a composition of n , we say $\mu \models n$ and $|\mu| = n$.

$l(\lambda)$ denotes the length of a partition (or composition) λ .

Associate a partition of n with a permutation $\sigma \in \mathfrak{S}_n$, by writing σ in cycle form and letting $\lambda(\sigma)$ be the sequence of cycle sizes listed in weakly decreasing order. The partition $\lambda(\sigma)$ is the **cycle type** of σ .

Example: $\lambda((1, 4), (2, 7, 5)(3, 6)) = (3, 2, 2)$

Eulerian polynomials - combinatorial interpretation

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\text{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \quad \text{DES}(\sigma) = \{1, 3, 4\}$$

Define $\text{des}(\sigma) := |\text{DES}(\sigma)|$. So

$$\text{des}(32541) = 3$$

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Excedance set: $\text{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i\}$

$$\sigma = 32541 \quad \text{EXC}(\sigma) = \{1, 3\}$$

Define $\text{exc}(\sigma) := |\text{EXC}(\sigma)|$. So

$$\text{exc}(32541) = 2$$

Eulerian polynomials - combinatorial interpretation

\mathfrak{S}_3	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$\sum_{\sigma \in \mathfrak{S}_3} t^{\text{des}(\sigma)} = 1 + 4t + t^2$$

$$\sum_{\sigma \in \mathfrak{S}_3} t^{\text{exc}(\sigma)} = 1 + 4t + t^2$$

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Eulerian polynomial

$$A_n(t) = \sum_{j=0}^{n-1} \left\langle n \atop j \right\rangle t^j = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

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MacMahon (1905) showed equidistribution of des and exc.

Carlitz and Riordin (1955) showed equals $A_n(t)$.

The characterizations of $A_n(t)$

Combinatorial

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

Recurrence relation

$$\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle = (n-j) \left\langle \begin{matrix} n-1 \\ j-1 \end{matrix} \right\rangle + (j+1) \left\langle \begin{matrix} n-1 \\ j \end{matrix} \right\rangle$$

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Palindromicity and Unimodality

A polynomial $f(t) = \sum_{i=0}^n a_i t^i \in \mathbb{R}[t]$ is

- **palindromic** (with center of symmetry $\frac{n}{2}$) if $a_i = a_{n-i}$ for all i
- **unimodal** if for some c

$$a_0 \leq a_1 \leq \cdots \leq a_c \geq \cdots \geq a_{n-1} \geq a_n$$

- **positive** if $a_i \geq 0$ for all i

Example: $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$

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Sum and Product Lemma

Let $A(t)$ and $B(t)$ be **P**ositive, **U**nimodal, **P**alindromic with respective centers of symmetry c_A and c_B . Then

- $A(t)B(t)$ is **PUP** with center of symmetry $c_A + c_B$.
- If $c_A = c_B$ then $A(t) + B(t)$ is **PUP** with center of symmetry c_A .

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Example 1: $[5]_t[2]_t + [6]_t + [4]_t[3]_t$, where $[k]_t := 1 + t + \cdots + t^{k-1}$.

Example 2: Palindromicity and unimodality of rows of Pascal's triangle are consequences since the polynomial $(1+t)^n$ is product of PUP's.

Palindromicity and Unimodality

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Theorem: The Eulerian polynomials are palindromic and unimodal.

Proof: From Euler's exponential generating formula, we can derive

$$A_n(t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1, \dots, k_m \geq 2} \binom{n}{k_1 - 1, k_2, \dots, k_m} t^{m-1} \prod_{i=1}^m [k_i - 1]_t$$

where

$$[k]_t := 1 + t + \dots + t^{k-1}.$$

Center of symmetry of $t^{m-1} \prod_{i=1}^m [k_i - 1]_t$ is

$$m - 1 + \sum_{i=1}^m \frac{k_i - 2}{2} = \frac{n - 1}{2}. \quad \square$$

γ -positivity

$f(t) \in \mathbb{R}[t]$ is palindromic $\iff \exists \gamma_k \in \mathbb{R}$ such that

$$f(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k}.$$

If $\gamma_k \geq 0$ for all k then $f(t)$ said to be γ -positive.

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γ -positive \implies palindromic and unimodal

Example: $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$.

$$\begin{aligned} 1t^0(1+t)^4 &= 1 + 4t + 6t^2 + 4t^3 + t^4 \\ 22t^1(1+t)^2 &= 22t + 44t^2 + 22t^3 \\ 16t^2(1+t)^0 &= 16t^2 \end{aligned}$$

So

$$A_5(t) = 1t^0(1+t)^4 + 22t^1(1+t)^2 + 16t^2(1+t)^0.$$

Thus $A_5(t)$ is γ -positive.

γ -positivity

Foata & Schutzenberger (1970):

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n : \sigma \text{ has no double descents, no final descent \& } \text{des}(\sigma) = k\}|$.

3.2.14 has a double descent. 124.3 has a final descent.

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\mathfrak{S}_3	des
123	0
1 32	
213	1
2 31	
312	1
3 21	

$$\begin{aligned} A_3(t) &= 1t^0(1+t)^2 + 2t^1(1+t)^0 \\ &= 1 + 4t + t^2. \end{aligned}$$

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Another property stronger than unimodality is log-concavity; real-rootedness is still stronger. $A_n(t)$ has only real roots.

Mahonian permutation statistics and q -analog

Let $\sigma \in \mathfrak{S}_n$.

Inversion Number:

$$\text{inv}(\sigma) := |\{(i, j) : 1 \leq i < j \leq n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\text{inv}(3142) = 3$$

Major Index:

$$\text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i$$

$$\text{maj}(3142) = \text{maj}(3.\textcolor{red}{1}4.\textcolor{red}{2}) = 1 + 3 = 4$$



Major Percy Alexander MacMahon
(1854 - 1929)

Mahonian Permutation Statistics - q-analogs

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Theorem (MacMahon 1905)

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$

Other examples of q -analogs

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$$\sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\text{inv}(\sigma)} = \left[\begin{matrix} n \\ k \end{matrix} \right]_q$$

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Proof of first equality: Use Foata bijection.

Proof of second equality: Show

$$\sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\text{inv}(\sigma)} [k]_q! [n-k]_q! = [n]_q!$$

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Proof of second equality: Show

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Use the map $\phi : \mathfrak{S}_{n,k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k} \rightarrow \mathfrak{S}_n$ that takes (σ, α, β) to the word obtained from σ by replacing the subword of 1's by α and the subword of 2's by $\tilde{\beta}$, where $\tilde{\beta}$ is obtained from β by replacing each letter i by $i + k$. Check that

- ϕ is a bijection
- $\text{inv}(\tilde{\beta}) = \text{inv}(\beta)$
- $\text{inv}(\phi(\sigma, \alpha, \beta)) = \text{inv}(\sigma) + \text{inv}(\alpha) + \text{inv}(\tilde{\beta})$

Example: $n = 5, k = 2, \sigma = 12212, \alpha = 21, \beta = 231$. Then $\tilde{\beta} = 453$ and

$$\phi(\sigma, \alpha, \beta) = 24513$$

Other examples of q -analogs

Fact: The number of derangements (i.e. permutations with no fixed points) in \mathfrak{S}_n is given by

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q -analog: Let \mathcal{D}_n be the set of derangements in \mathfrak{S}_n . Then

$$\sum_{\sigma \in \mathcal{D}_n} q^{\text{maj}(\sigma)} = [n]_q! \sum_{k=0}^n q^{\binom{k}{2}} \frac{(-1)^k}{[k]_q!}.$$

Due independently to Gessel and MW (1989).

Doesn't work for inv .

q -analogs of Eulerian polynomials

$$A_n^{\text{inv},\text{des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{maj},\text{des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{des}(\sigma)}$$

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Theorem (~~Carlitz 1954~~ MacMahon 1916)

$$\sum_{i \geq 1} [i]_q^n t^i = \frac{t A_n^{\text{maj},\text{des}}(q, t)}{\prod_{i=0}^n (1 - tq^i)}$$

q-analogs of Euler's exp. generating function formula

Theorem (Stanley 1976)

$$\sum_{n \geq 0} A_n^{\text{inv, des}}(q, t) \frac{z^n}{[n]_q!} = \frac{1 - t}{\text{Exp}_q(z(t - 1)) - t}$$

where

$$\text{Exp}_q(z) := \sum_{n \geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

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Theorem (Shareshian & MW 2006)

$$\sum_{n \geq 0} A_n^{\text{maj, exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1 - tq) \exp_q(z)}{\exp_q(z tq) - tq \exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}$$

q -Eulerian polynomials and q -Eulerian numbers

Theorem (Shareshian & MW 2006)

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, tq^{-1}) \frac{z^n}{[n]_q!} = \frac{(1-t) \exp_q(z)}{\exp_q(z) - t \exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}$$

Proof uses symmetric function theory, which we will talk about next time.

From now on

$$A_n(q, t) := A_n^{\text{maj,exc}}(q, tq^{-1}) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)}$$

and

$$\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle_q := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} q^{\text{maj}(\sigma) - \text{exc}(\sigma)}$$

Palindromicity and unimodality of the q -Eulerian numbers

$n \setminus j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2 + q + q^2$	1		
4	1	$3 + 2q + 3q^2 + 2q^3 + q^4$	$3 + 2q + 3q^2 + 2q^3 + q^4$	1	
5	1	$4 + 3q + 5q^2 + \dots$	$6 + 6q + 11q^2 + \dots$	$4 + 3q + 5q^2 + \dots$	1

Theorem (Shareshian and MW)

The q -Eulerian polynomial $A_n(q, t) = \sum_{t=0}^{n-1} \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle_q t^j$ is

- **palindromic** in the sense that $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle_q = \left\langle \begin{matrix} n \\ n-1-j \end{matrix} \right\rangle_q$ for $0 \leq j \leq \frac{n-1}{2}$
- **q -unimodal** in the sense that $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle_q - \left\langle \begin{matrix} n \\ j-1 \end{matrix} \right\rangle_q \in \mathbb{N}[q]$ for $1 \leq j \leq \frac{n-1}{2}$

Palindromicity and unimodality of the q -Eulerian numbers

Theorem (Shareshian and MW)

The q -Eulerian polynomial $A_n(q, t) = \sum_{t=0}^{n-1} \left\langle n \atop j \right\rangle_q t^j$ is palindromic and q -unimodal.

Proof: We use our q -analog of Euler's exponential generating function formula to prove

$$A_n(q, t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1, \dots, k_m \geq 2} \left[\begin{matrix} n \\ k_1 - 1, k_2, \dots, k_m \end{matrix} \right]_q t^{m-1} \prod_{i=1}^m [k_i - 1]_t,$$

where

$$\left[\begin{matrix} n \\ k_1, \dots, k_m \end{matrix} \right]_q := \frac{[n]_q!}{[k_1]_q! \cdots [k_m]_q!}$$

Then apply the Sum & Product Lemma.

q - γ -positivity of q -Eulerian polynomials

Recall: Foata & Schutzenberger (1970):

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n : \sigma \text{ has no double descents, no final descent \& } \text{des}(\sigma) = k\}|$.

Theorem (Shareshian and Wachs (2010))

Let $\Gamma_{n,k}$ be the set of permutations in \mathfrak{S}_n with no double descents, no final descent and $\text{des}(\sigma) = k$. Let

$$\gamma_{n,k}(q) := \sum_{\sigma \in \Gamma_{n,k}} q^{\text{inv}(\sigma)}.$$

Then

$$A_n(q, t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k},$$

Proof uses our q -analog of Euler's exponential generating function and a symmetric function identity of Gessel.

Cycle-type Eulerian polynomials

For $\lambda \vdash n$, let \mathfrak{S}_λ be the set of permutations of cycle type λ . Define the **cycle-type Eulerian polynomial** as follows

$$A_\lambda(t) := \sum_{\sigma \in \mathfrak{S}_\lambda} t^{\text{exc}(\sigma)}$$

For $\lambda \vdash n$ and $i \in \mathbb{P}$, let $m_i(\lambda)$ be the number of occurrences of i in λ .

Brenti (1993): $A_\lambda(t)$ is palindromic and unimodal with center of symmetry $c = \frac{n - m_1(\lambda)}{2}$.

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Now define the **cycle-type q -Eulerian polynomial**

$$A_\lambda(q, t) := \sum_{\sigma \in \mathfrak{S}_\lambda} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)}.$$

Henderson and MW (2010): $A_\lambda(q, t)$ is palindromic and q -unimodal with center of symmetry $c = \frac{n - m_1(\lambda)}{2}$.

Proof uses symmetric function theory and representation theory.

Derangements

Corollary. Let \mathcal{D}_n be the set of derangements in \mathfrak{S}_n and let

$$D_n(q, t) := \sum_{\sigma \in \mathcal{D}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)}.$$

Then $D_n(q, t)$ is palindromic and q -unimodal with center of symmetry $\frac{n}{2}$.

Also a consequence of

Shareshian and MW (2010):

$$\sum_{n \geq 0} D_n(q, t) z^n = \frac{1 - t}{\exp_q(tz) - t \exp_q(z)}$$

and of

Shareshian and MW (2010): Let $\Gamma_{n,k}$ be the set of permutations in \mathfrak{S}_n with no double descents, no final descent, **no initial descent**, and $\text{des}(\sigma) = k$. Let

$$\gamma_{n,k}(q) := \sum_{\sigma \in \Gamma_{n,k}} q^{\text{inv}(\sigma)}.$$

Then

$$D_n(q, t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^{k+1} (1+t)^{n-2k-2},$$

Log-concavity

A sequence (a_0, a_1, \dots, a_n) is **log-concave** if $a_j^2 > a_{j-1}a_{j+1}$ for all j .
We will say a polynomial $\sum_{j=0}^n a_j t^j$ is log concave if its sequence of coefficients (a_0, a_1, \dots, a_n) is log-concave.

Example. $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$.

Theorem. For all n , $A_n(t)$ is log-concave.

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Conjecture (Shareshian and MW)

- For all n , $A_n(q, t)$ is q -log-concave.
- For all n and $\lambda \vdash n$, $A_\lambda(q, t)$ is q -log-concave.

We checked this up to $n = 8$.

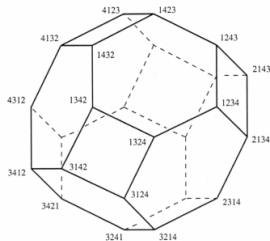
Geometric interpretation of Eulerian polynomials

The **h -polynomial** of a d -dimensional convex polytope \mathcal{P} is defined by

$$h_{\mathcal{P}}(t) := \sum_{j=0}^d f_{d-1-j}(t-1)^j$$

where f_i is the number of faces of \mathcal{P} of dimension i .

The **permutohedron** \mathcal{P}_n is the convex hull of points in \mathbb{R}^n of the form $(\sigma(1), \dots, \sigma(n))$, where $\sigma \in \mathfrak{S}_n$. This is an $(n-1)$ -dimensional polytope embedded in \mathbb{R}^n .



For each convex polytope \mathcal{P} , there is another convex polytope \mathcal{P}^* called the **polar dual**. The number of i -dimensional faces of \mathcal{P}^* equals the number of $(d-i)$ -dimensional faces of \mathcal{P} for each i .

Theorem: $A_n(t) = h_{\mathcal{P}_n^*}(t)$.

Geometric interpretation of Eulerian polynomials

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Dehn-Sommerville equations: The h -polynomial of every simplicial convex polytope is palindromic.

Stanley (1980): The h -polynomial of every simplicial convex polytope is unimodal.

This is part of the celebrated g -theorem of Billera, Lee, and Stanley.

Gal's conjecture (2005): The h -polynomial of a flag simplicial convex polytope \mathcal{P} is γ -positive.

Fact: \mathcal{P}_n^* is simplicial and flag.

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We will see a geometric interpretation of the q -Eulerian polynomials.