Markov Processes on branching graphs of classical Lie groups

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Plan of the talk

1. Setup: Markov jump processes, BC branching graph and z-measures
2. Construction of the Markov dynamics
3. Invariance of the z-measures
4. Further results

Markov processes

A general homogeneous Markov process is a stochastic process $X(t)$ not depending on past history. It's determined by:

- E: countable set
- $\nu : E \to [0, 1]$: initial probability distribution
- $P(t) = [P(t; i, j)]_{i,j \in E}$, $t \geq 0$: transition matrices

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$$\nu(i) = Prob(X(0) = i)$$

- $P(t) = [P(t; i,j)]_{i,j \in E}$, $t \geq 0$: transition matrices

$$P(t; i,j) = Prob(X(t+s) = j | X(s) = i)$$

Markov jump processes

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Kolmogorov's backward and forward equations:

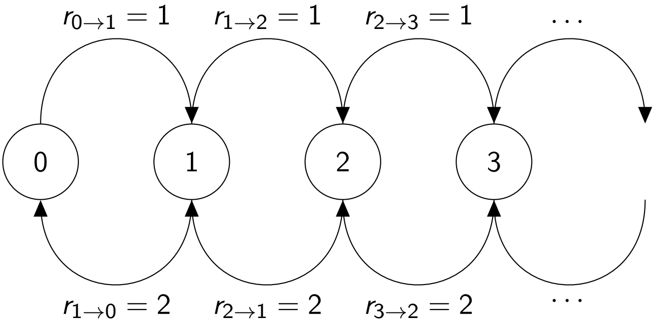
$$\begin{aligned} P'(t) &= RP(t) \\ P'(t) &= P(t)R \end{aligned}$$

Initial condition:

$$P(0) = Id$$

Special case: birth and death processes

$$E = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$$



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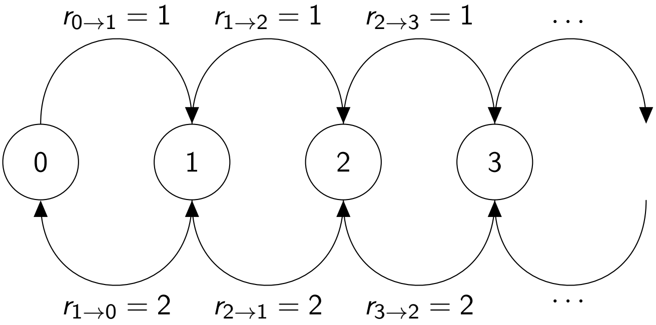

Invariant measure: $\mu_0 = 1/2, \mu_1 = 1/4, \mu_2 = 1/8, \ldots$

BC branching graph (the levels)

For $N \geq 1$, $\mathbb{GT}_N^+ =$ set of partitions of length exactly N. Zeroes are not omitted! e.g., $(3, 2, 0, 0) \neq (3, 2)$. For $N = 1$: $\mathbb{GT}_1^+ = \mathbb{Z}_+$.

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For later:

$$V = \sqcup_{N \geq 1} \mathbb{GT}_N^+$$

is the set of vertices of the BC branching graph.

z-measures (at level N)

Let

$$M_N(\lambda|z,z',a,b) = const \cdot \prod_{1\le i<j\le N} ((\lambda_i+N-i+\epsilon)^2-(\lambda_j+N-j+\epsilon)^2)^2$$

$$\times \prod_{i=1}^{N} W_{z,z',a,b|N}(\lambda_i+N-i),$$

where

$$W_{z,z',a,b|N}(x) = (x+\epsilon)\frac{\Gamma(x+2\epsilon)\Gamma(x+a+1)}{\Gamma(x+b+1)\Gamma(x+1)}$$

$$\times \frac{1}{\Gamma(z-x+N)\Gamma(z'-x+N)\Gamma(z+x+N+2\epsilon)\Gamma(z'+x+N+2\epsilon)}$$

$$\left\{ \epsilon = \frac{a+b+1}{2} \right\}$$

Construction of the Markov dynamics

The z-measures are examples of determinantal point processes.

Question:
Does there exists a Markov process that has the z-measures as unique invariant measures?

Markov dynamics on $\mathbb{GT}_1^+ = \mathbb{Z}_+$

$$
\begin{aligned}
r_{x\to x+1} &= \frac{(x+a+b+1)(x+a+1)(x-z)(x-z')}{(x+\epsilon)(2x+2\epsilon+1)}, \ x \geq 0 \\
r_{x\to x-1} &= \frac{x(x+b)(x+z+a+b+1)(x+z'+a+b+1)}{(x+\epsilon)(2x+2\epsilon-1)}, \ x \geq 1
\end{aligned}
$$

$$
\left\{ \epsilon = \frac{a+b+1}{2} \right\}
$$

Proposition (C.)

The rates $r_{x\to x\pm 1}$ determine a unique regular birth and death process on $\mathbb{GT}_1^+ = \mathbb{Z}_+$.

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How to find $r_{x \to x+1}, r_{x \to x-1}$?

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How to find $r_{x\to x+1}, r_{x\to x-1}$?

Treat $W_{z,z',a,b|1}(x)$ as a weight function on the quadratic lattice $\{\epsilon^2, (\epsilon+1)^2, (\epsilon+2)^2, \ldots\}$.
It yields *orthogonal polynomials* $\mathfrak{p}_k((x+\epsilon)^2)$ whose second order difference eqn. is:

$$r_{x\to x+1}\mathfrak{p}_n((x+1+\epsilon)^2) - (r_{x\to x+1} + r_{x\to x-1})\mathfrak{p}_n((x+\epsilon)^2)$$
$$+ r_{x\to x-1}\mathfrak{p}_n((x-1+\epsilon)^2) = \gamma_n \mathfrak{p}_n((x+\epsilon)^2).$$

Markov dynamics on $\mathbb{GT}_N^+$

$$r^{(N)}_{\lambda\to\nu} = \frac{\prod_{i<j}((\nu_i+N-i+\epsilon)^2-(\nu_j+N-j+\epsilon)^2)}{\prod_{i<j}((\lambda_i+N-i+\epsilon)^2-(\lambda_j+N-j+\epsilon)^2)}$$
$$\left(r_{l_1\to n_1}\mathbf{1}_{\{l_i=n_i, i\neq 1\}}+\ldots+r_{l_N\to n_N}\mathbf{1}_{\{l_i=n_i, i\neq N\}}\right)-c_N\mathbf{1}_{\lambda=\nu},$$

where

$$\begin{aligned}\epsilon &= \frac{a+b+1}{2}\\ c_N &= \frac{N(N-1)(N-2)}{3}-\frac{N(N-1)}{2}(z+z'+b+2N-2)\end{aligned}$$

Markov dynamics on $\mathbb{GT}_N^+$: equivalent definition

$r^{(N)}_{\lambda\to\mu} = 0$ unless λ, μ differ in at most one position; if so:

$$r^{(N)}_{\lambda\to\nu} = \begin{cases} r_{l_i\to l_i+1}\cdot\prod_{j\neq i}\frac{\widehat{(l_i+1)}-\widehat{l_j}}{\widehat{l_i}-\widehat{l_j}} & \text{if } \nu_i = \lambda_i + 1 \\ r_{l_i\to l_i-1}\cdot\prod_{j\neq i}\frac{\widehat{(l_i-1)}-\widehat{l_j}}{\widehat{l_i}-\widehat{l_j}} & \text{if } \nu_i = \lambda_i - 1. \end{cases}$$

$$\begin{aligned} l_i &= \lambda_i + N - i \\ \widehat{l_i} &= (l_i + \frac{a+b+1}{2})^2. \end{aligned}$$

Proposition (C.)

The rates $r^{(N)}_{\lambda\to\mu}$ determine a unique regular Markov jump process on $\mathbb{GT}_N^+$.

Invariance of the z-measures

Invariance means: $\mu P(t) = \mu$, $t \geq 0$

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$$\sum_{\lambda_1 \geq \ldots \geq \lambda_N} \left(\prod_{i=1}^{N} W(l_i) \right) \prod_{i<j} ((l_i + \epsilon)^2 - (l_j + \epsilon)^2) \times$$
$$\times \left(r_{l_1 \to n_1} \mathbf{1}_{\{l_i = n_i, i \neq 1\}} + \ldots + r_{l_N \to n_N} \mathbf{1}_{\{l_i = n_i, i \neq N\}} - c_N \mathbf{1}_{\{l = n\}} \right) = 0$$

Invariance of the z-measures: orthogonal polynomials

Recall $\mathfrak{p}_k((x+\epsilon)^2)$ orthogonal w.r.t. $W(x)$ and

$$\mathcal{D}\mathfrak{p}_k((x+\epsilon)^2) = \sum_n r_{x\to y}\mathfrak{p}_k((y+\epsilon)^2) = \gamma_n \mathfrak{p}_n((x+\epsilon)^2),$$

$$\begin{Bmatrix} \gamma_n = n(n+1-z-z'-b-2N) \\ R = [r_{x\to y}]_{x,y\in\mathbb{Z}_+}, r_{x\to x} = -r_{x\to x+1} - r_{x\to x-1} \end{Bmatrix}$$

Invariance of the z-measures: proof

Key easy fact: $\prod_{i<j}((n_i+\epsilon)^2-(n_j+\epsilon)^2) = \det_{i,j}[\mathfrak{p}_j((n_i+\epsilon)^2)]$

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$$\prod_{i=1}^{N} W(n_i)(\mathcal{D}^{[1]}+\ldots+\mathcal{D}^{[N]})\det_{i,j}[\mathfrak{p}_j((n_i+\epsilon)^2)]$$

$$= \prod_{i=1}^{N} W(n_i)(\gamma_1+\ldots+\gamma_N)\det_{i,j}[\mathfrak{p}_j((n_i+\epsilon)^2)]$$

Invariance of the z-measures: proof

Right side is

$$\sum_{\lambda_1 \geq \ldots \geq \lambda_N} \left(\prod_{i=1}^{N} W(l_i) \right) (c_N \mathbf{1}_{\{l=n\}}) \det[\mathfrak{p}_j((l_i + \epsilon)^2)] = 0$$

Left side is

$$\sum_{\lambda_1 \geq \ldots \geq \lambda_N} \prod_i W(n_i)(r_{n_1 \to l_1} + \ldots + r_{n_N \to l_N}) \det[\mathfrak{p}_j((l_i + \epsilon)^2)] = 0$$

Use $W(x) r_{x \to y} = W(y) r_{y \to x}$

Motivation: BC branching graph

The compact Lie groups $G(N) = Sp(2N), SO(2N+1)$ have (polynomial) irreducible representations parametrized by partitions $\lambda \in \mathbb{GT}_N^+$.

There are natural embeddings $G(N-1) \hookrightarrow G(N)$.

If V_λ is an irreducible representation of $G(N)$, then $V_\lambda|_{G_{N-1}}$ is a representation of $G(N-1)$, but not irreducible:

$$V_\lambda|_{G(N-1)} = \oplus_\mu m(\lambda, \mu) V_\mu.$$

Motivation: BC branching graph

BC branching graph is a graded branching graph.
$\lambda \in \mathbb{GT}^+_{N+1}$ and $\mu \in \mathbb{GT}^+_N$ are joined by an edge if V_μ appears in $V_\lambda|_{G(N-1)}$ with nonzero multiplicity $m(\lambda, \mu) \neq 0$.

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BC branching graph gives the basic info about representation theory of $Sp(2\infty), SO(2\infty + 1)$.
Should be compared to Young's lattice which is the analogue for the infinite symmetric group $S(\infty)$.

Motivation: coherence of the z-measures

z-measures are special:

$$M_N = M_{N+1}\Lambda_N^{N+1}$$

Λ_N^{N+1}: $\mathbb{GT}_{N+1}^+ \times \mathbb{GT}_N^+$ matrix of *cotransition probabilities*.
E.g. for $(a, b) = (1/2, \pm 1/2)$:

$$\Lambda_N^{N+1}(\lambda, \mu) = \begin{cases} \frac{m(\lambda,\mu)Dim(V_\mu)}{Dim(V_\lambda)} & \text{if } \mu - \lambda \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

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Consequence:
$(M_N(\cdot|z, z', a, b))_{N\geq 1}$ determines a probability measure $M_{z,z',a,b}$ on an infinite-dimensional space Ω, the <u>spectral z-measure</u>.

Coherence of the Markov dynamics $X_N^{z,z',a,b}$

Dynamics are special:

$$P_{N+1}(t)\Lambda_N^{N+1} = \Lambda_N^{N+1} P_N(t), t \geq 0$$

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Consequence ("Method of intertwiners" by Borodin-Olshanski):
$\{P_N(t)\}_{N\geq 1}$ determines a Markov process on the infinite-dimensional space Ω preserving the spectral z-measures.

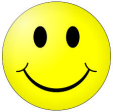
Conclusion

- Constructed transition matrices $(P_N(t))_{t \geq 0}$ on $\mathbb{GT}_N^+$ that are coherent.
- The transition matrices give rise to Markov processes on $\mathbb{GT}_N^+$ that preserve projections of the z-measures.
- The coherence allow us to define Markov processes on an infinite-dimensional space Ω that preserve the spectral z-measures.
- Result is probabilistically a Markov dynamics on a system with infinitely many particles on $\mathbb{R}_{>0} \backslash \{1\}$.

Questions

Any questions?

Thanks



Thank you for listening!