Chapter 4

Counting with operators: an example

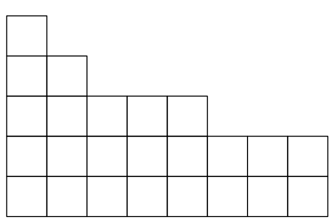
In this chapter, we will introduce the so-called *vertex operators*, that are very powerful tools that have been developed only in recent years by combinatorialists, theoretical physicists, and mathematicians. Here we will only survey the most combinatorial part of the theory (we will not talk about the deep algebraic structure hidden behind these operators - symmetric functions, representations of the symmetric group, Fock spaces to name only the closest to combinatorics). We will use the vertex operators as "tools" to prove, in a very easy and computationless way one of the most famous and beautiful results of combinatorics: MacMahon's formula for plane partitions. The price to pay for this very elegant proof is that everything will be a little bit more abstract than usual.

4.1 Integer partitions, plane partitions and MacMahon's formula

4.1.1 Integer partitions

Let $n \geq 0$ be an integer. An *integer partition* of n, or *partition* of n, is a sequence $\lambda_1 \geq \lambda_2 \geq \dots \lambda_\ell > 0$ of non-increasing positive integers summing to n. We write $\lambda \vdash n$ for "λ is a partition of n". We also say that n is the *size* of λ. The integers $\lambda_1, \dots, \lambda_\ell$ are called the *parts* of λ and the number of parts is denoted by $l(\lambda) := \ell$. We often see a partition as infinite sequence $\lambda_1 \geq \lambda_2 \geq \dots$ by completing it with an infinite number of zeros.

A partition is often represented by its *Ferrer's diagram*. An example is better than any definition, so here is the Ferrer's diagram of the partition $\lambda = [8, 8, 5, 2, 1]$ of 24:



$\lambda = (8, 8, 5, 2, 1)$

We let $\mathcal{P}$ be the set of all integer partitions and for $n \geq 0$ we let $\mathcal{P}_n$ be the subset made by partitions of n. For $n \geq 0$, we let $p(n) = \#\mathcal{P}_n$ be the number of partitions of the integer n.

By convention, the empty parition $\emptyset$ is a partition of 0, so that $p(0) = 1$. The first values of $p(n)$ can be found by listing all the partitions of small sizes:

$p(0) = 1$	$0 =$ empty sum
$p(1) = 1$	$1 = 1$
$p(2) = 2$	$2 = 2$
	$2 = 1 + 1$
$p(3) = 3$	$3 = 3$
	$3 = 2 + 1$
	$3 = 1 + 1 + 1$
$p(4) = 5$	$4 = 4$
	$4 = 3 + 1$
	$4 = 2 + 2$
	$4 = 2 + 1 + 1$
	$4 = 1 + 1 + 1 + 1$
$p(5) = 7$	(check it!)

Here are more values of the sequence $(p(n))_{n\geq 0}$:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, \ldots$$

For partitions, it turns out that the numbers don't have a nice closed expression, but the generating function does. We have the easy theorem:

Theorem 36 (Euler)**.** *The generating function of partitions is given by:*

$$P(q) := \sum_{n\geq 0} p(n) q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}. \tag{4.1}$$

Proof. The set of partitions having all parts $\leq k$ is specified as a combinatorial class by:

$$\mathrm{SEQ}(\Box) \times \mathrm{SEQ}(\Box\Box) \times \mathrm{SEQ}(\Box\Box\Box) \times \cdots \times \mathrm{SEQ}(\underbrace{\Box\Box \ldots \Box}_{k \text{ boxes}}),$$

so its generating function is $\frac{1}{1-q} \times \frac{1}{1-q^2} \times \cdots \times \frac{1}{1-q^k}$. Now let k tend to infinity (we are considering formal power series here, so it is clear that taking the limit makes sense – since for any $l \geq 0$ the coefficient of q^l in the last expression is stationary k large enough). $\Box$

4.1.2 Plane partitions and MacMahon's formula

A *plane partition* is a finite array of integers, weakly decreasing along rows and columns. When we write a plane partition we do not write the entries equal to zero, so we obtain a finite array looking like that:

$$\begin{array}{ccccc} 7 & 7 & 3 & 3 & 1 \\ 5 & 5 & 3 & 2 & \\ 4 & 3 & 3 & & \\ 2 & 1 & & & \end{array}$$

The *size* of a plane partition is the sum of its entries. In our example, the size is thus 49. We let $\mathcal{R}_n$ be the set of plane partitions of size n, and we let $\mathcal{R} = \cup_{n\geq 0}\mathcal{R}_n$ be the set of all plane partitions. We let $r(n) = \#\mathcal{R}_n$ be the number of plane partitions of size n. Note that there is a unique plane partition of size 0 (the array full of zeros), so $r(0) = 1$.

Plane partitions are much more complicated objects than integer partitions. However, their generating function has an amazingly simple form:

Theorem 37 (MacMahon – main formula of this chapter). *The generating function of plane partitions is given by the explicit formula:*

$$R(q) := \sum_{n\geq 0} r(n)q^n = \prod_{i=1}^{\infty}\left(\frac{1}{1-q^i}\right)^i. \tag{4.2}$$

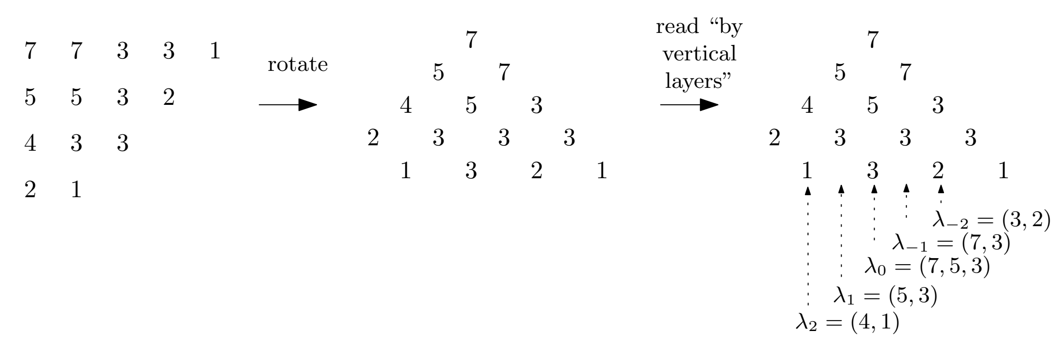
Remark 10. This formula looks very similar to Euler's formula for the generating function of integer partitions. However it is *much* more complicated to prove!

Remark 11. Using this formula one can easily determine the first numbers of plane partitions:

$$R(q) = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + 86\,q^7 + 160\,q^8 + 282\,q^9 + 500\,q^{10} + \dots$$

Remark 12. It is natural to guess that similar formulas will exist for variants of partitions defined similarly in higher dimensions (3D-partitions, 4D-partitions, etc.). However this intuition is false: no such simple formula seems to exist in dimensions others than 1 and 2.

In order to prove MacMahon's formula, we will first rephrase the result in terms of sequences of *integer* partitions. Take the plane partition of the previous example, and rotate it by 45 degrees. Then, in "each vertical layer", you read an integer partition. This defines a bi-infinite sequence $(\dots,\lambda_{-1},\lambda_0,\lambda_1,\dots)$ of integer partitions as in the following figure:



Since the original array of numbers was a plane partition, this sequence of partitions is not arbitrary. It satisfies constraints that are better expressed using the notion of *interlacing.*

Definition 38. Let λ and μ be two partitions. We write $\lambda \succ \mu$, and we say "λ and μ are interlaced", if for all $i \geq 1$ one has $\lambda_i \geq \mu_i \geq \lambda_{i+1}$, i.e.:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \dots$$

Looking at the previous picture, it is clear that $\lambda_0 \succ \lambda_{-1} \succ \lambda_{-2}$ and that $\lambda_0 \succ \lambda_1 \succ \lambda_2$. More generally, it is clear that these constraints express nothing but the fact that the original array of numbers (before rotation of 45 degrees) was non-decreasing along rows and columns. This viewpoint will actually be easier to work with:

Definition 39 (Plane partition – alternate definition). A *plane partition* is a bi-infinite sequence of partitions $(\dots, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \dots)$, such that $\lambda_i = \lambda_{-i} = \emptyset$ for i large enough, and satisfying the interlacing conditions:

$$\dots \prec \lambda_{-3} \prec \lambda_{-2} \prec \lambda_{-1} \prec \lambda_0 \succ \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \dots \tag{4.3}$$

4.2 A vector space, and two operators

In this section we will start doing the "more abstract" things promised in the introduction. We will do several dangerous things (e.g. take infinite linear combinations in vector spaces, take infinite sums of operators, etc.). All of them can be properly justified in an appropriate algebraic setting, but this would only make the exposition more complicated, so we will avoid this. Just be convinced that everything we will do "makes sense" (for example, be convinced that each time we apply an operator to a vector there are no infinite coefficients that appear).

We let V be the vector space made by formal linear combinations of partitions over $\mathbb{Q}$ (or $\mathbb{C}$ if you prefer). For example:

$$3.5 \cdot \emptyset - \frac{3}{7} \cdot (7,6,4) + \frac{11}{2} \cdot (8,8,1,1,1)$$

is an element of V. Sometimes we will also consider formal linear combinations with coefficients being formal power series in some variable (usually t, or t' or q). In this case we will sometimes consider *infinite* linear combinations. Typically we will consider elements of the form

$$\sum_{n \geq 0} q^n v_n$$

where for all $n \geq 0$, v_n is an element of V. The important thing will be that for any partition λ and for any power q^k of q, the coefficient of $q^k \lambda$ is well defined and finite. I don't give any more details here, and I don't specify properly over which field we are working (as you will see everything will be pretty obvious). Let's just say somehow inprecisely that we denote by Λ the vector space of all "infinite linear combinations of partitions with coefficients which are formal series in the parameters q, t, t'".

We equip the vector space Λ with the scalar product defined by:

$$(\lambda|\mu) = \mathbf{1}_{\lambda=\mu}.$$

In particular, for any $v \in \Lambda$ and $\lambda \in \mathcal{P}$, $(\lambda|v)$ is "the coefficient of λ in v".

We now define the two *vertex operators*[1] $\Gamma_+(t)$ and $\Gamma_-(t')$. They respectively interlace "upwards" and "downwards" the partitions on which they operate:

Definition 40 (Vertex operators). The operator $\Gamma_+(t)$ and $\Gamma_-(t')$ are the linear operators on Λ defined by:

$$\Gamma_+(t)\lambda = \sum_{\substack{\mu \in \mathcal{P} \\ \mu \succ \lambda}} t^{|\mu|-|\lambda|} \cdot \mu.$$

$$\Gamma_-(t')\lambda = \sum_{\substack{\mu \in \mathcal{P} \\ \mu \prec \lambda}} t'^{|\lambda|-|\mu|} \cdot \mu.$$

[1]The terminology "vertex operators" comes from theoretical physics, and I never met anyone that could explain to me why it is called like that.

In concrete terms, $\Gamma_+(t)$ has the effect of "interlacing upwards" a partition in all possible ways, and the exponent of t remembers the extra-size added by the interlacing. $\Gamma_-(t')$ has a similar effect, but interlaces "downwards".

Using Definition 39, the generating function of plane partitions can be easily rewritten in terms of the vertex operators:

Proposition 41. *Let $R_{\leq k}(q)$ be the generating function of plane partitions that fit in a rectangle of size $k \times k$. Equivalently, let $R_{\leq k}(q)$ be the generating function of sequences:*

$$\emptyset = \lambda^{(k)} \prec \lambda^{(k-1)} \prec \cdots \prec \lambda^{(1)} \prec \lambda^{(0)} \succ \lambda^{(-1)} \succ \cdots \succ \lambda^{(k-1)} \succ \lambda^{(-k)} = \emptyset.$$

Then $R_{\leq k}$ is given by the scalar product:

$$R_{\leq k}(q) = \Big(\emptyset \,\Big|\, \underbrace{\Gamma_-(q^{-1})\Gamma_-(q^{-2})\ldots\Gamma_-(q^{-k})}_{k \text{ operators}}\underbrace{\Gamma_+(q^{k+1})\ldots\Gamma_+(q^{2k-1})\Gamma_+(q^{2k})}_{k \text{ operators}}\cdot\emptyset\Big). \tag{4.4}$$

Proof. The idea is that each sequence of the form

$$\lambda^{(k)} \prec \lambda^{(k-1)} \prec \cdots \prec \lambda^{(1)} \prec \lambda^{(0)} \succ \lambda^{(-1)} \succ \cdots \succ \lambda^{(k-1)} \succ \lambda^{(-k)} = \emptyset$$

can be constructed, from right to left, by starting with $\lambda^{(-k)} = \emptyset$, interlacing k times upwards, and interlacing k times downwards. Algebraically, by definition of the operators Γ_+, Γ_-, we have:

$$\begin{aligned} &\Gamma_-(q^{-1})\Gamma_-(q^{-2})\ldots\Gamma_-(q^{-k})\Gamma_+(q^{k+1})\ldots\Gamma_+(q^{2k-1})\Gamma_+(q^{2k})\cdot\emptyset \\ = &\sum_{\lambda^{(k)}\prec\cdots\prec\lambda^{(1)}\prec\lambda^{(0)}\succ\lambda^{(-1)}\succ\cdots\succ\lambda^{(-k)}=\emptyset} q^{\sum_{i=1}^{k}(i-1-k)(|\lambda_{i-1}|-|\lambda_i|)+\sum_{i=1-k}^{0}(k+1-i)(|\lambda_i|-|\lambda_{i-1}|)}\cdot\lambda^{(k)}. \end{aligned}$$

Taking the scalar product of this element with $\emptyset$ keeps only the sequences such that $\lambda^{(k)} = \emptyset$, which are exactly the sequences we want to count in the series $R_{\leq k}(q)$. So the only thing left to check is that the exponent of q in the last expression is indeed the total size of the sequence of partitions.

For $(1-k) \leq i \leq k$, write $u_i = |\lambda^{(i)}| - |\lambda^{(i-1)}|$ the difference of size between the two consecutive partitions. Note that $u_i \geq 0$ if $i \leq 0$ and $u_i \leq 0$ else. We have, for $-k \leq p \leq k$:

$$|\lambda^{(p)}| = \sum_{i=1-k}^{p} u_i,$$

so that the total size of the sequence is:

$$\sum_{p=-k}^{k} |\lambda^{(p)}| = \sum_{i=1-k}^{k} (k+1-i)u_i.$$

This shows that the exponent of q in the last formula indeed counts the total size. $\square$

4.3 The commutation relation

We now are going to prove MacMahon's formula by evaluating (4.4). How? First, notice that, if we had taken the operators Γ_+ and Γ_- in the other direction, the product would have been easily evaluated. Indeed, for any $t_1, t_2, \dots, t_{2k}$ one has:

$$\left(\emptyset \,\middle|\, \Gamma_+(t_1)\dots\Gamma_+(t_2)\Gamma_+(t_k)\Gamma_-(t_{k+1})\Gamma_-(t_{k+2})\dots\Gamma_-(t_{2k})\cdot\emptyset\right) = 1. \tag{4.5}$$

Indeed, since the only partition μ such that $\emptyset \succ \mu$ is $\mu = \emptyset$, we have: $\Gamma_-(t_{k+1})\Gamma_-(t_{k+2})\dots\Gamma_-(t_{2k})\emptyset = \emptyset$. For the same reason, it is clear that the coefficient of $\emptyset$ in $\Gamma_+(t_1)\Gamma_+(t_2)\dots\Gamma_+(t_k)\emptyset$ is equal to 1.

Now, how do we go from (4.4) to something of the form (4.5)? The answer is clear: we commute operators! The only thing we will need is:

Proposition 42 (Commutation relation — a.k.a. the magic of combinatorics!). *The operators Γ_- and Γ_+ satisfy the following "quasi-commutation" relation:*

$$\Gamma_-(t')\Gamma_+(t) = \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t'). \tag{4.6}$$

Remark 13. The relation says that, for any partition α, one has:

$$\Gamma_-(t')\Gamma_+(t)\alpha = \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t')\alpha,$$

that is, that for any partitions α and β one has:

$$\left(\beta \,\middle|\, \Gamma_-(t')\Gamma_+(t)\alpha\right) = \left(\beta \,\middle|\, \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t')\alpha\right).$$

By definition of the operators Γ_-, Γ_+, this is equivalent to saying that for any partitions α and β one has:

$$\sum_{\substack{\lambda \text{ such that} \\ \beta\prec\lambda\succ\alpha}} t'^{|\lambda|-|\beta|}t^{|\lambda|-|\alpha|} = \frac{1}{1-tt'}\cdot \sum_{\substack{\mu \text{ such that} \\ \beta\succ\mu\prec\alpha}} t^{|\beta|-|\mu|}t'^{|\alpha|-|\mu|}. \tag{4.7}$$

To prove the commutation relation, we are going to prove this last formula. More, we will prove it in a *bijective* way.

Bijective proof of (4.7)*, hence of Proposition 42.* Let $\alpha \succ \lambda \prec \beta$ and $k \geq 0$. Define $(t_i)_{i\geq 0}$ by $t_0 = k$ and $t_i = \min(\alpha_i, \beta_i) - \lambda_i$ for $i \geq 1$.

Then set $\mu_i = \max(\alpha_i, \beta_i) + t_{i-1}$. It is easy to check that μ is a partition such that $\alpha \prec \mu \succ \lambda$, and that $|\mu| = |\alpha| + |\beta| - |\lambda| + k$. Indeed one has:

- We have $\mu_i \geq \alpha_i$ since $\mu_i - \alpha_i = (\max(\alpha_i, \beta_i) - \alpha_i) + (\min(\alpha_{i-1}, \beta_{i-1}) - \lambda_{i-1})$, and both terms are ≥ 0 (for the second term, this is because $\lambda \prec \alpha$ and $\lambda \prec \beta$, so $\lambda_{i-1} \leq \alpha_{i-1}$ and idem for β).
- We have $\mu_{i+1} \leq \alpha_i$ since $\mu_{i+1} - \alpha_i = (\max(\alpha_{i+1}, \beta_{i+1}) - \lambda_i) + (\min(\alpha_i, \beta_i) - \alpha_i)$, and both terms are ≤ 0 (for the first term, this is because $\lambda \prec \alpha$ and $\lambda \prec \beta$, so $\lambda_i \geq \alpha_{i+1}$ and idem for β).

Moreover we have that $|\mu| = \sum_{i\geq 1}\mu_i = k + \sum_{i\geq 1}(\min(\alpha_i, \beta_i) + \max(\alpha_i, \beta_i)) = k + |\alpha| + |\beta|$.

We leave the reciprocal bijection as an exercise.

□

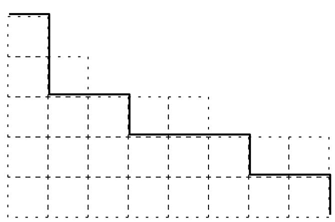
4.3.1 Another bijective proof of (4.7), with many pictures (from 2012)

Here is the proof that I gave last year. It is essentially the same, but with pictures, but ten times longer.

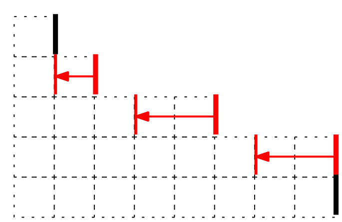
Bijective proof of (4.7), *hence of Proposition 42.* This is the only point where we really have to understand what we are doing combinatorially. More precisely, we have to study in details what the *interlacing* of partitions is. Recall that $\alpha \succ \mu$ means that for every $i \geq 1$ one has:

$$\alpha_{i+1} \leq \mu_i \leq \alpha_i.$$

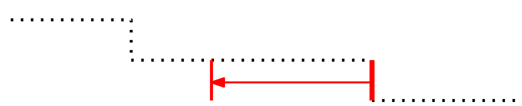
Graphically[2] on the Ferrers diagram of the partition, μ is obtained from α by "moving vertical bars to the left".



$$\alpha = (8, 8, 5, 2, 1)$$
$$\mu = (8, 6, 3, 1, 1)$$



More precisely, all the vertical bars of α can move (simultaneously) towards the left, but each bar cannot go further that the initial position of the bar immediately to its left.

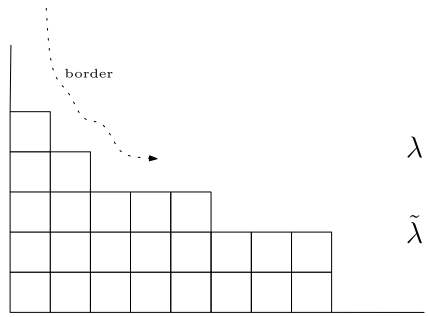


In order to represent the phenomenon more clearly, we introduce the *fermionic representation* of a partition. Given a partition $\lambda_1 \geq \lambda_2 \geq \ldots$, that we see as an infinite sequence by completing it with infinitely many parts equal to 0, define the sequence:

$$\tilde{\lambda}_i = \lambda_i + 1 - i \text{ for } i \geq 1.$$

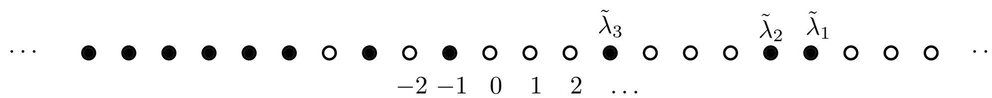
Note that, since (λ_i) is non-decreasing, $(\tilde{\lambda}_i)$ is strictly decreasing. Now consider an infinite line of "positions", indexed by $\mathbb{Z}$, and place a *particle* at position $\tilde{\lambda}_i$ for all $i \geq 1$. One obtains the fermionic representation of λ:

[2] Everything from now on should be much clearer if you attended the lecture... sorry if you didn't. But remember, you can always contact me if you want more explanations.

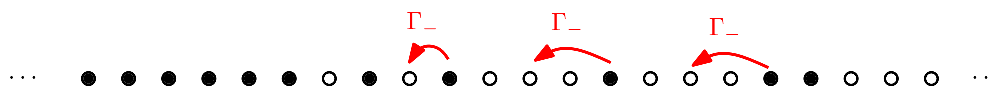


$$\lambda = (8, 8, 5, 2, 1, 0, 0, 0, \ldots)$$

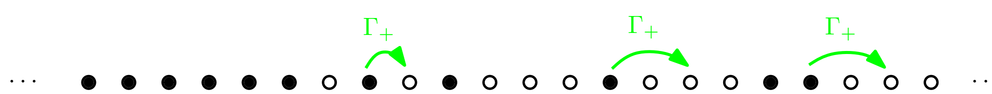
$$\tilde{\lambda} = (8, 7, 3, -1, -3, -5, -6, -7, \ldots)$$



Equivalently, the fermionic representation is the word obtained by reading the *border* of the partition from left to right, and writing a letter $\bullet$ for each vertical step, and a letter $\circ$ for each horizontal step. The effect of the operator $\Gamma_-(t')$ on fermions is to make *some* particles jump to the left:



The only constraint is that a particle cannot jump further than the original position (before jump) of the particle on its left. Note that $\Gamma_-(t)$ is the same, but makes particle jump to their *right*.



Note finally that in both cases the exponent of t (or t') counts the total displacement of particules.

We now proceed with the proof. Keeping Remark 13 in mind, and in the sake of proving (4.7), we *fix* two partitions α and β. We are now going to describe, on the fermionic representation:

(a) the set of partitions μ such that $\beta \succ \mu \prec \alpha$,

(b) the set of partitions λ such that $\beta \prec \lambda \succ \alpha$,

and we will see that there is a simple connection between them.

Description of the set (a)

Let α, β, μ be three partitions such that $\beta \succ \mu \prec \alpha$. Then for every $i \geq 2$:

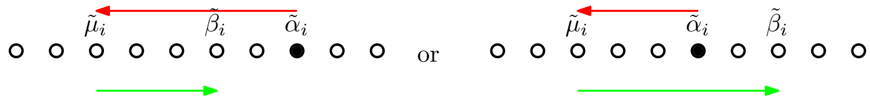
$$\max(\alpha_{i+1}, \beta_{i+1}) \leq \mu_i \leq \min(\alpha_i, \beta_i),$$

so that:

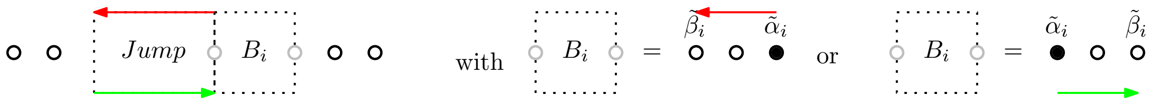
$$\max(\tilde{\alpha}_{i+1}, \tilde{\beta}_{i+1}) < \tilde{\mu}_i \leq \min(\tilde{\alpha}_i, \tilde{\beta}_i),$$

That is, for every $i \geq 2$, the interval $[\![\tilde{\mu}_i, \tilde{\mu}_{i-1}[\![$ contains the two numbers (the two "particles") $\tilde{\alpha}_i$ and $\tilde{\beta}_i$. However, we may have $\tilde{\alpha}_i \geq \tilde{\beta}_i$ or $\tilde{\beta}_i \geq \tilde{\alpha}_i$ (both cases are possible).

We now look at the fermionic representations. When passing from α to μ, then to β, the particle in position $\tilde{\alpha}_i$ jumps to the left to $\tilde{\mu}_i$, then to the right to $\tilde{\beta}_i$. We represent the left jump with a red arrow, and the right jump with a green arrow, so the displacement of the particle looks like one of these two pictures:



with the convention that the up arrow is taken first. This situation can be summarized by:



So in short, a triple $\beta \succ \mu \prec \alpha$ can be represented as a sequence of Jumps, Blocks, and empty positions as follows:

Description of the set (b)

Let α, λ, μ be three partitions such that $\beta \prec \lambda \succ \alpha$. Then for every $i \geq 2$:

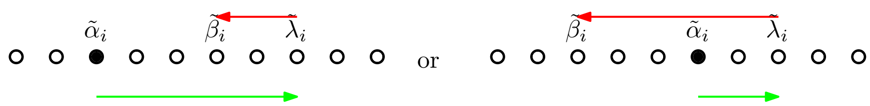
$$\max(\alpha_i, \beta_i) \leq \lambda_i \leq \min(\alpha_{i-1}, \beta_{i-1}),$$

so that:

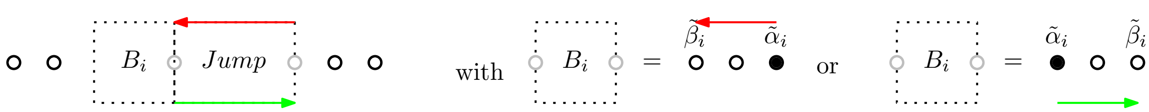
$$\max(\tilde{\alpha}_i, \tilde{\beta}_i) \leq \tilde{\lambda}_i < \min(\tilde{\alpha}_{i-1}, \tilde{\beta}_{i-1}),$$

That is, for every $i \geq 1$, the interval $]\!]\tilde{\lambda}_{i+1}, \tilde{\lambda}_i]\!]$ contains the two numbers (the two “particles”) $\tilde{\alpha}_i$ and $\tilde{\beta}_i$.

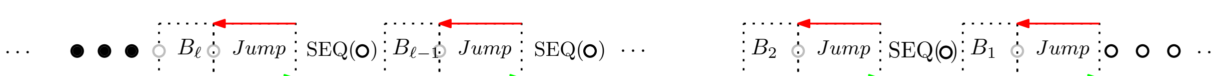
We now look at the fermionic representations. When passing from α to λ, then to β, the particle in position $\tilde{\alpha}_i$ jumps to the right to $\tilde{\lambda}_i$, then to the left to $\tilde{\beta}_i$. As before we represent the left jump with a red arrow, and the right jump with a green arrow, so the displacement of the particle looks like one of these two pictures:



where in this case we use the convention that the bottom arrow is taken first. As before we summarize the situation by:

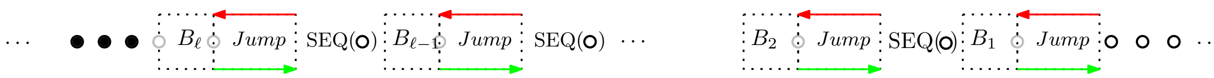


So in short, a triple $\beta \prec \lambda \succ \alpha$ can be represented as a sequence of Jumps, Blocks, and empty positions as follows:



Conclusion of the proof of the commutation relation. Clearly, to go from a sequence of the form

to a sequence of the form



one must do two things:

- For $\ell - 1 \geq i \geq 1$, between the block B_{i+1} and B_i, permute the Jump and the sequence of empty positions.
- add an extra Jump at the very right.

The extra-jump at the very right can have any length $k \in [\![0, \infty)$, so the generating function contribution of this extra jump is:

$$\sum_{k=0}^{\infty} (tt')^k = \frac{1}{1 - tt'}.$$

The construction is clearly reversible (just remove the rightmost Jump, and between the blocks permute again the Jumps and the sequences of empty positions). Moreover, except from the rightmost Jump, the total red displacement and the total green displacement do not change in the construction. So we have proved:

$$\sum_{\substack{\lambda \text{ such that} \\ \beta \prec \lambda \succ \alpha}} t'^{|\lambda| - |\beta|} t^{|\lambda| - |\alpha|} = \frac{1}{1 - tt'} \cdot \sum_{\substack{\mu \text{ such that} \\ \beta \succ \mu \prec \alpha}} t^{|\beta| - |\mu|} t'^{|\alpha| - |\mu|}.$$

that is, we have proved (4.7). $\square$

4.3.2 End of the proof of MacMahon's formula

Recall Equation (4.4):

$$R_{\leq k}(q) = \Big(\emptyset \,\Big|\, \underbrace{\Gamma_-(q^{-1})\Gamma_-(q^{-2}) \dots \Gamma_-(q^{-k})}_{k \text{ operators}} \underbrace{\Gamma_+(q^{k+1}) \dots \Gamma_+(q^{2k-1})\Gamma_+(q^{2k})}_{k \text{ operators}} \cdot \emptyset\Big).$$

In this formula, take the operator $\Gamma_+(q^{k+1})$ and send it to the left. To do that, you must "make it commute" with the operators $\Gamma_-(q^{-k}), \Gamma_-(q^{1-k}), \dots, \Gamma_-(q^{-1})$. For each of them, you must apply the commutation relation (Proposition 42, leading in total to a factor:

$$\frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \dots \cdot \frac{1}{1-q^k} = \prod_{i=1}^{k} \frac{1}{1-q^i}.$$

Now, do the same with the operator $\Gamma_+(q^{k+1})$. Again, you have to "make it commute" with the operators $\Gamma_-(q^{-k}), \Gamma_-(q^{1-k}), \ldots, \Gamma_-(q^{-1})$. Apply the commutation relation (Proposition 42, this leads to a factor:

$$\frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdot \cdots \cdot \frac{1}{1-q^{k+1}} = \prod_{i=1}^{k} \frac{1}{1-q^{i+1}}.$$

Similarly, for all $1 \leq j \leq k$ passing the $\Gamma_+(q^{k+j})$ operator to the left, the multiplicative factor comming from the commutation relations is:

$$\prod_{i=1}^{k} \frac{1}{1-q^{i+j-1}}.$$

When all the Γ_+ operators have been passed to the left, we can apply (4.5). The remaining scalar product is 1. Therefore we have proved:

Proposition 43. *The generating function of plane partitions fitting in a rectangle of size $k \times k$ is given by:*

$$R_{\leq k}(q) = \prod_{i=1}^{k} \prod_{i=1}^{k} \frac{1}{1-q^{i+j-1}}.$$

Now, let k tend to infinity, so that $R_{\leq k}(q)$ tends (coefficientwise) to the generating function $R(q)$ of all plane partitions. We obtain tMacMahon's formula:

$$R(q) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-q^{i+j-1}} = \prod_{i=1}^{\infty} \left(\frac{1}{1-q^i} \right)^i.$$