Lecture 1 Practice Problems

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(1) Give the cycle type of each of the following permutations.
  (a) [397468152]
  (b) [234561]
  (c) [1234]

(2) For $\lambda \vdash n$, let $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$, where m_i is the number of occurrences of i in λ. Show that the number of permutations of cycle type λ is $\frac{n!}{z_\lambda}$.

(3) In this problem we show that the various characterizations of the Eulerian polynomial $A_n(t)$ are consistent with each other and with Euler's original definition:

$$\sum_{i \geq 1} i^n t^i = \frac{t\, A_n(t)}{(1-t)^{n+1}}.$$

  (a) Using Euler's original definition of $A_n(t)$, prove his exponential generating function formula

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z} - t}.$$

  (b) Using Euler's original definition of $A_n(t)$, prove that the coefficients of $A_n(t)$ satisfy the recurrence

$$\left\langle {n \atop j} \right\rangle = (n-j) \left\langle {n-1 \atop j-1} \right\rangle + (j+1) \left\langle {n-1 \atop j} \right\rangle$$

  (c) Using the combinatorial characterization of $A_n(t)$ involving des, prove that the coefficients of $A_n(t)$ satisfy the above recurrence.
  (d) Using the combinatorial characterization of $A_n(t)$ involving exc, prove that the coefficients of $A_n(t)$ satisfy the above recurrence.

(4) Give a bijective proof that des and exc are equidistributed on $\mathfrak{S}_n$.

(5) Let C_n be the set of permutations in $\mathfrak{S}_n$ of cycle type (n). Find a formula for

$$\sum_{\sigma \in C_n} t^{\mathrm{exc}(\sigma)}.$$

(6) A binary tree is a rooted tree in which each node has at most two children and each child is either a left child or a right child. An increasing binary tree is a binary tree with n nodes labeled with $1, 2, \ldots, n$ so that every node has a label that is bigger than its parent. Let $\mathcal{T}_n$ be the set of increasing binary trees with n nodes. For $T \in \mathcal{T}_n$, let $l(T)$ be the number of left children in T. Prove

(a)
$$\sum_{T\in\mathcal{T}_n} t^{l(T)} = A_n(t).$$
(b)
$$A_n(t) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}\, t^k(1+t)^{n-1-2k},$$
where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n : \sigma$ has no double descents, no final descent, and $\mathrm{des}(\sigma) = k\}|$.

(7) Show that γ-positivity implies palindromicity and unimodality, but the converse is false.

(8) Prove:
(a)
$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!.$$
(b) Let $\mathfrak{S}_{k_1,k_2,\ldots,k_m}$ be the set of permutations of the multiset $\{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$. In other words $\mathfrak{S}_{k_1,k_2,\ldots,k_m}$ is the set of words with k_i i's for each i. Prove that
$$\sum_{\sigma\in\mathfrak{S}_{k_1,k_2,\ldots,k_m}} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_q.$$

(9) Let
$$A_n(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$
and let Alt_n be the set of permutations σ in $\mathfrak{S}_n$ such that
$$\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \sigma(5) > \cdots < \sigma(n).$$
Prove that
$$A_n(q,-1) = \begin{cases} (-1)^{\frac{n-1}{2}} \sum_{\sigma\in\mathrm{Alt}_n} q^{\mathrm{inv}(\sigma)} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

(10) Prove the closed form formula
$$A_n(q,t) = \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{k_1,\ldots,k_m\geq 2} \begin{bmatrix} n \\ k_1-1, k_2, \ldots, k_m \end{bmatrix}_q t^{m-1} \prod_{i=1}^{m} [k_i-1]_t$$