Some families of reduced Kronecker coefficients, plane partitions and quasipolynomials

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ECCO 2016 June 20, 2016

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Symmetric functions

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- Description in terms of *integral hives* and *honeycombs*, by T. Tao and A. Knutson (1999).

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Let X and Y be two alphabets.

- Sum: $X + Y = x_1 + y_1 + x_2 + y_2 + \dots$
- Product: $XY = x_1 y_1 + x_1 y_2 + \dots + x_2 y_1 + x_2 y_2 + \dots$

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The Littlewood-Richardson coefficients can be obtained as

$$s_\nu[X+Y] = \sum_{\lambda,\mu} c^\nu_{\lambda\mu} s_\lambda[X] \cdot s_\mu[Y]$$

where we sum over all the partitions such that $|\lambda| + |\mu| = |\nu|$.

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The Kronecker coefficients define a *new* product on *Sym*

$$\begin{array}{ccc} Sym \otimes Sym & \longrightarrow & Sym \\ (s_\lambda, s_\mu) & \longrightarrow & s_\lambda * s_\mu \end{array}$$

where

$$s_\lambda * s_\mu := \sum_\nu g_{\lambda\mu}^{\nu} s_\nu$$

Contents

Reduced Kronecker coefficients: $\overline{g}_{\lambda\mu}^{\nu}$

$$\begin{aligned} s_{(10,2)} * s_{(10,2)} &= s_{(12)} + s_{(11,1)} + 2 \cdot s_{(10,2)} + s_{(10,1,1)} + \\ &+ s_{(9,3)} + 2 \cdot s_{(9,2,1)} + s_{(9,1,1,1)} + \\ &+ s_{(8,4)} + s_{(8,3,1)} + s_{(8,2,2)} \end{aligned}$$

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The **reduced Kronecker coefficients** $\overline{g}_{\lambda\mu}^{\nu}$ are the stable values of the sequence of Kronecker coefficients, after we disregard the first part.

Relation with Kronecker coefficients

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- Kronecker coefficients can be recovered from reduced Kronecker coefficients.

$$g_{\lambda\mu}^{\nu} = \sum_{i=1}^{\ell(\lambda)\ell(\mu)} (-1)^{i+1} \overline{g}_{\overline{\lambda}\overline{\mu}}^{\nu^{\dagger i}}$$

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Proposition (E. Briand - R. Orellana - M. Rosas, 2011)

$$\overline{g}_{\lambda\mu}^{\nu} = g_{\lambda[n]\ \mu[n]}^{\nu[n]}$$

with $\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \dots)$ *and n sufficiently large.*

Families

- **Family 1** $\overline{g}^{(k)}_{(k^a),(k^a)}$
- **Family 2** $\overline{g}^{(k)}_{((k+i)^a),(k^a)}$
- **Family 3** $\overline{g}^{(k)}_{(k^{a+1}),(k+i,k^a)}$

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Family 1

Theorem (L. C. - M. Rosas, 2015)

Consider the reduced Kronecker coefficients $\left\{\overline{g}^{(k)}_{(k^a),(k^a)}\right\}_{k\geq 0}$.

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$$\mathcal{F}_a = \frac{1}{(1-x)(1-x^2)^2\cdots(1-x^a)^2(1-x^{a+1})}$$

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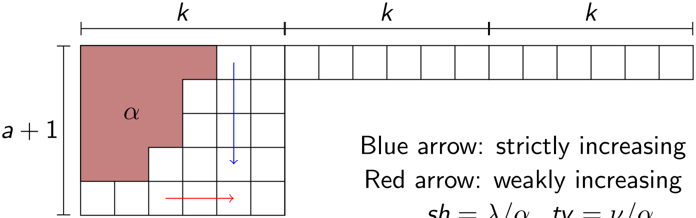
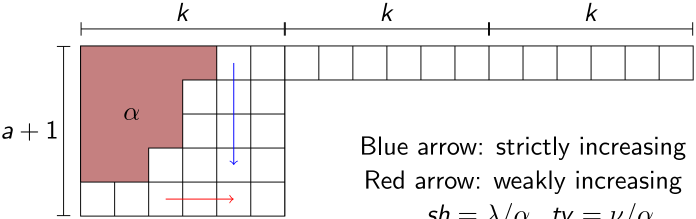


Blue arrow: strictly increasing

Red arrow: weakly increasing

$sh = \lambda/\alpha,\ ty = \nu/\alpha$

Kronecker tableau



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- α–condition:
  - If $\alpha_1 = \alpha_2$, there is no condition.
  - If $\alpha_1 > \alpha_2$, then

$$\begin{aligned}(\#1)_{2nd\ Row} &= \alpha_1 - \alpha_2\\ (\#2)_{1st\ Row} &= \alpha_1 - \alpha_2\end{aligned}$$

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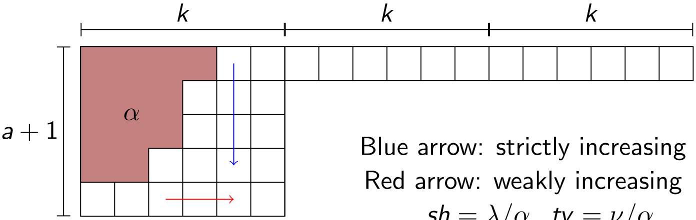

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- The reverse reading word is an α– lattice permutation.

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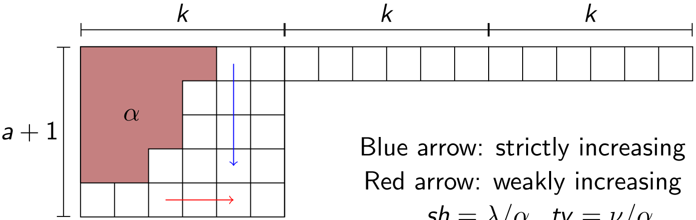

Blue arrow: strictly increasing
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$sh = \lambda/\alpha, \ ty = \nu/\alpha$

Example for $a = 4, \ k = 7, \ \alpha = (3, 2, 1, 1)$,
$sh = (21, 7, 7, 7, 7)/\alpha, \ ty = (18, 5, 6, 6, 7)$

			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	3	5
		1	2	2	2	2														
	1	3	3	3	3	3														
	4	4	4	4	4	4														
1	5	5	5	5	5	5														

Sketch of the proof for case $a = b$

Using a result of R. Orellana and C. Ballantine,

$$\overline{g}^{(k)}_{(k^a)(k^a)} = \# \left\{ \begin{array}{c} \text{Kronecker tableau with} \\ sh = (3k, k^a)/\alpha \\ ty = (3k, k^a)/\alpha \\ \alpha \vdash k \end{array} \right\}$$

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Looking at the generating function $\mathcal{F}_a$,

$$\mathcal{F}_a = \frac{1}{(1-x)(1-x^2)^2 \cdots (1-x^a)^2(1-x^{a+1})}$$

$$\begin{array}{c} \text{Coefficient of} \\ x^k \text{ in } \mathcal{F}_a \end{array} = \# \left\{ \begin{array}{c} \text{colored sequences of } k \\ \text{with parts in} \\ \{1, 2, \overline{2}, \ldots, a, \overline{a}, a+1\} \end{array} \right\}$$

Sketch of the proof for case $a = b$

There exists a bijection

$$\left\{ \begin{array}{c} \text{colored sequences of } k \\ \text{with parts in} \\ \{1, 2, \overline{2}, \ldots, a, \overline{a}, a+1\} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Kronecker tableau} \\ sh = (3k, k^a)/\alpha \\ ty = (3k, k^a)/\alpha \\ \alpha \vdash k \end{array} \right\}$$

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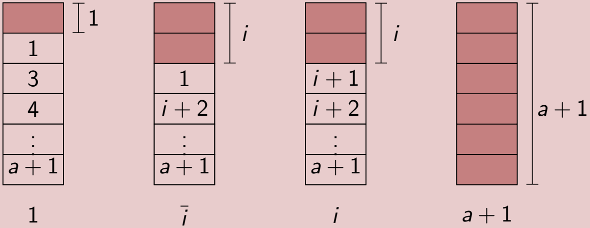
It is based on the following identification



for $i \in \{2, 3, \dots, a-1, a\}$

EXAMPLE: $a = 4$ AND $\beta = (1, \overline{2}, \overline{4})$

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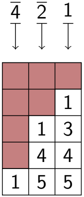
STEP 1. Column identification



STEP 2. Identify α:
Looking at the picture, we have that $\alpha = (3, 2, 1, 1)$, which is a partition of 7.

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STEP 1. Column identification

$\overline{4}$ ↓	$\overline{2}$ ↓	1 ↓
		1
	1	3
	4	4
1	5	5

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STEP 3. Complete rest of the tableau

			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	3	5
		1	2	2	2	2														
	1	3	3	3	3	3														
	4	4	4	4	4	4														
1	5	5	5	5	5	5														

Family 2: $\overline{g}^{(k)}_{((k+i)^a),(k^a)}$

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Table: Case $a = 2$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
i= 0	1	1	3	4	7	9	14	17	24	29	38	45	57	66
i= 1	0	0	0	1	1	3	4	7	9	14	17	24	29	38
i= 2	0	0	0	0	0	0	1	1	3	4	7	9	14	17
i= 3	0	0	0	0	0	0	0	0	0	1	1	3	4	7

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i= 2	0	0	0	0	0	0	1	1	3	4	7	9	14	17
i= 3	0	0	0	0	0	0	0	0	0	1	1	3	4	7

Theorem (L. C., 2015)

After some initial zeros, the generating function for the reduced Kronecker coefficients $\overline{g}^{(k)}_{((k+i)^a)(k^a)}$ *is exactly* $\mathcal{F}_a$.

Family 3: $\overline{g}^{(k)}_{((k+i),k^{a-1}),(k^a)}$

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Table: Case $a = 2$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
i= 0	**1**	1	3	4	7	9	14	17	24	29	38	45	57
i= 1	0	**1**	**2**	4	7	11	16	23	31	41	53	67	83
i= 2	0	0	**1**	**2**	**5**	8	14	20	30	40	55	70	91
i= 3	0	0	0	**1**	**2**	**5**	**9**	15	23	34	47	64	84
i= 4	0	0	0	0	**1**	**2**	**5**	**9**	**16**	24	37	51	71
i= 5	0	0	0	0	0	**1**	**2**	**5**	**9**	**16**	**25**	38	54
i= 6	0	0	0	0	0	0	**1**	**2**	**5**	**9**	**16**	**25**	**39**

Family 3: $\overline{g}^{(k)}_{((k+i),k^{a-1}),(k^a)}$

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i= 3	0	0	0	**1**	**2**	**5**	**9**	15	23	34	47	64	84

Theorem (L. C., 2015)

The stable value of the j^{th} diagonal corresponds to the reduced Kronecker coefficients $\overline{g}^{(k)}_{(k^a),(2k-j,k^{a-1})}$, when $k \geq 2j$ and their generating function is

$$\mathcal{G}_a = \frac{1}{(1-x)^2(1-x^2)^3\dots(1-x^{a-1})^3(1-x^a)^2(1-x^{a+1})}$$

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Plane partitions

A **plane partition** is a two-dimensional array of non-negative integers $n_{i,j}$ (with positive integer indices i and j) that is non-increasing in both indices and for which only finitely many of the $n_{i,j}$ are nonzero.

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2	2		
2	1		

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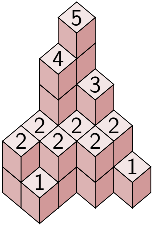

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Theorem (MacMahon, 1915)

Let $r = \min(a, l)$ and $s = \max(a, l)$. Then, the generating function for the plane partitions fitting inside an $l \times a$ rectangle is

$$\prod_{j=r}^{s} \left(\frac{1}{1-x^j}\right)^r \cdot \prod_{i=1}^{r-1} \left(\frac{1}{1-x^i}\right)^i \left(\frac{1}{1-x^{s+i}}\right)^{r-i}$$

Relation with Families 1 and 3

Theorem (L. C. - M. Rosas, 2015)

The reduced Kronecker coefficient $\overline{g}^{(k)}_{(k^a),(k^a)}$ counts the number of plane partitions of k fitting inside a $2 \times a$ rectangle.

Relation with Families 1 and 3

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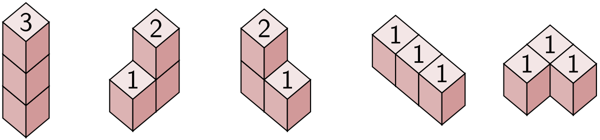
The reduced Kronecker coefficient for $a = 4$ and $k = 3$ is $\overline{g}^{(3)}_{(3,3,3,3),(3,3,3,3)} = 5$.



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Theorem (L. C. , 2015)

For the stable values of the j^{th} diagonal in Family 3,

$$\overline{\overline{g}}_a(j) = \sum_{l=0}^{j} \# \left\{ \begin{array}{c} \textit{plane partitions} \\ \textit{of } l \\ \textit{in } 3 \times (a-1) \end{array} \right\} \# \left\{ \begin{array}{c} \textit{plane partitions} \\ \textit{of } j-l \\ \textit{in } 2 \times 1 \end{array} \right\}$$

Contents

Theorem (L. C. - M. Rosas, 2015)

Let ℓ be the least common multiple of $1, 2, \ldots, a, a+1$. The coefficients $\overline{g}_{(k^a),(k^a)}^{(k)}$ are described by a quasipolynomial of degree $2a-1$ and period dividing ℓ. In fact, we have checked that the period is exactly ℓ for $a \leq 10$.

For $a = 2$, the coefficients are given by the quasipolynomial of degree 3 and period 6:

$$\begin{cases} 1/72k^3 + 1/6k^2 + \ 2/3k + \ 1 & \text{if } k \equiv 0 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 5/18 & \text{if } k \equiv 1 \mod 6 \\ 1/72k^3 + 1/6k^2 + \ 2/3k + 8/9 & \text{if } k \equiv 2 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 1/2 & \text{if } k \equiv 3 \mod 6 \\ 1/72k^3 + 1/6k^2 + \ 2/3k + 7/9 & \text{if } k \equiv 4 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 7/18 & \text{if } k \equiv 5 \mod 6 \end{cases}$$

Description for Family 3

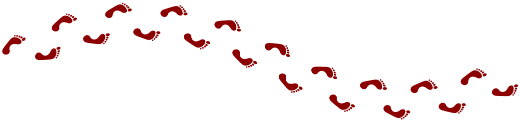
Theorem (L. C., 2015)

Let ℓ be the least common multiple of $1, 2, \ldots, a, a+1$. The coefficients $\overline{\overline{g}}_a(j)$ are described by a quasipolynomial of degree $3a-2$ and period dividing ℓ. In fact, we have checked that the period is exactly ℓ for $a \leq 7$.

For $a = 2$, the coefficients are given by the quasipolynomial of degree 4 and period 6:

$$\begin{cases} 1/288j^4 + 1/16j^3 + 7/18j^2 + j + 1 & j \equiv 0 \mod 6 \\ 1/288j^4 + 1/16j^3 + 7/18j^2 + 15/16j + 175/288 & j \equiv 1 \mod 6 \\ 1/288j^4 + 1/16j^3 + 7/18j^2 + j + 8/9 & j \equiv 2 \mod 6 \\ 1/288j^4 + 1/16j^3 + 7/18j^2 + 15/16j + 23/32 & j \equiv 3 \mod 6 \\ 1/288j^4 + 1/16j^3 + 7/18j^2 + j + 8/9 & j \equiv 4 \mod 6 \\ 1/288j^4 + 1/16j^3 + 7/18j^2 + 15/16j + 175/288 & j \equiv 5 \mod 6 \end{cases}$$

Thank you very much!



¡Muchas gracias!