ALGEBRAIC STRUCTURES ON COMBINATORIAL SPECIES

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Abstract. These are notes for a course in ECCO 2016. We study algebraic structures in Joyal's category of species, with particular emphasis on Hopf monoids (objects analogous to groups or Hopf algebras). Combinatorial structures which compose and decompose give rise to Hopf monoids. We study several examples of this nature, focusing on the combinatorial analysis of the antipode map. We also touch upon objects analogous to rings; these involve two kinds of compositions playing the role of the addition and the multiplication in a ring. We emphasize the central role played by the braid arrangement and the Tits product of faces.

Contents

1. Enumeration via species

Combinatorics organizes cardinalities into generating functions. After Joyal, one may similarly organize sets into species. In this context, the word species is to be understood as *combinatorial type*. We illustrate the usefulness of this approach by means of one or two very simple enumerative applications. Many more are given in [6, 10].

1.1. **Set species.** A *set species* P consists of the following data.

- For each finite set I, a set $\mathrm{P}[I]$.
- For each bijection $\sigma : I \to J$, a map
$$\mathrm{P}[\sigma] : \mathrm{P}[I] \to \mathrm{P}[J].$$
These should be such that

$$\mathrm{P}[\sigma \circ \tau] = \mathrm{P}[\sigma] \circ \mathrm{P}[\tau] \quad \text{and} \quad \mathrm{P}[\mathrm{id}] = \mathrm{id}. \tag{1}$$

It follows that each map $\mathrm{P}[\sigma]$ is invertible, with inverse $\mathrm{P}[\sigma^{-1}]$.

Sometimes we refer to an element $x \in \mathrm{P}[I]$ as a *structure* (of species P) on the set I, and to a map $\mathrm{P}[\sigma]$ as a *relabeling*.

A *morphism* $f : \mathrm{P} \to \mathrm{Q}$ between set species P and Q is a collection of maps

$$f_I : \mathrm{P}[I] \to \mathrm{Q}[I]$$

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which satisfy the following *naturality* axiom: for each bijection $\sigma : I \to J$,

$$f_J \circ \mathrm{P}[\sigma] = \mathrm{Q}[\sigma] \circ f_I. \tag{2}$$

Example 1. The species L is defined as follows. For any finite set I, $\mathrm{L}[I]$ is the set of all linear orders on I. If ℓ is a linear order on I and $\sigma : I \to J$ is a bijection, then $\mathrm{L}[\sigma](\ell)$ is the linear order on J for which $j_1 < j_2$ if $\sigma^{-1}(j_1) < \sigma^{-1}(j_2)$ in ℓ. In other words, if we regard ℓ as a list consisting of the elements of I, then $\mathrm{L}[\sigma](\ell)$ is the list obtained by replacing each $i \in I$ for $\sigma(i) \in J$.

For instance,

$$\mathrm{L}[a, b, c] = \{abc,\ bac,\ acb,\ bca,\ cab,\ cba\},$$

and if $\sigma : \{a, b, c\} \to \{x, y, z\}$ is

$$\begin{tikzcd} a \arrow[d] & b \arrow[d] & c \arrow[d] \\ y & z & x \end{tikzcd},$$

then $\mathrm{L}[\sigma] : \mathrm{L}[a, b, c] \to \mathrm{L}[x, y, z]$ is

$$\begin{tikzcd} abc \arrow[d] & bac \arrow[d] & acb \arrow[d] & bca \arrow[d] & cab \arrow[d] & cba \arrow[d] \\ yzx & zyx & yxz & zxy & xyz & xzy \end{tikzcd}.$$

When defining species, we will often omit the specification of the relabeling maps.

Example 2. A *partition* of I is a collection X of disjoint nonempty subsets of I whose union is I. The subsets in X are the *blocks* of the partition. For example,

$$X = \{gol,\ de,\ \mathcal{J}ames\}$$

is a partition of the set $I = \{a, e, e, d, g, \mathcal{J}, l, m, o, s\}$ into 3 blocks. We write $X \vdash I$ to indicate that X is a partition of I.

Let $\Pi[I]$ be the set of all partitions of I. With the obvious definition for the relabeling maps, we obtain the species Π of set partitions.

A *composition* of I is an ordered tuple of disjoint nonempty subsets of I whose union is I. We write $F \vDash I$ to indicate that F is a composition of I. Set compositions are also called ordered set partitions or preferential arrangements.

Let Σ denote the species of compositions. There is a surjective morphism

$$\pi : \Sigma \to \Pi \tag{3}$$

which forgets the order among the blocks of a composition. The compositions

$$F = (\mathcal{U}ruguay,\ 1,\ Colombia,\ 0) \quad \text{and} \quad G = (Colombia,\ 1,\ \mathcal{U}ruguay,\ 0)$$

are distinct but map to the same partition. Here I has 17 elements. The naturality axiom (2) holds: we may first relabel the elements and then forget the order among the blocks, or vice versa, with the same result.

There is an injective morphism

$$\mathrm{L} \to \Sigma$$

which regards a linear order as a composition into singletons. (We say L is a subspecies of Σ.)

Example 3. The *exponential* species E is such that

$$\mathrm{E}[I] = \{*_I\}.$$

In other words, for each finite set I, $\mathrm{E}[I]$ consists of a single element (denoted $*_I$ for convenience). Given any set species P, there is a unique morphism

$$\mathrm{P} \to \mathrm{E}$$

(since for each I there is a unique map $\mathrm{P}[I] \to \mathrm{E}[I]$). In other words, E is terminal in the category of set species.

Exercise 4. Let $\mathsf{set}^\times$ be the category whose objects are finite sets and whose morphisms are bijections between such sets. (It is a groupoid.) Let Set be the category whose objects are arbitrary sets and whose morphisms are arbitrary functions.

(a) Observe that a set species is precisely a functor $\mathsf{set}^\times \to \mathsf{Set}$, and a morphism of species is precisely a natural transformation between such functors.
(b) What changes if replace Set for $\mathsf{Set}^\times$, the category whose objects are arbitrary sets but whose morphisms are bijective functions?

For definitions regarding category theory, see [11].

1.2. **Generating functions.** Let P be a *finite* set species. This means that each set $\mathrm{P}[I]$ is finite. The *exponential generating function* of P is

$$\mathrm{P}(x) = \sum_{n\geq 0} \#\mathrm{P}[n]\frac{x^n}{n!}.$$

For example,

$$\mathrm{E}(x) = e^x \quad \text{and} \quad \mathrm{L}(x) = \frac{1}{1-x}.$$

1.3. **Cauchy product.** Given two set species P and Q, their *Cauchy product* is the species $\mathrm{P}\cdot\mathrm{Q}$ defined by

$$(\mathrm{P}\cdot\mathrm{Q})[I] := \coprod_{S\sqcup T=I} \mathrm{P}[S]\times\mathrm{Q}[T], \tag{4}$$

for all finite sets I. This is the disjoint union of several cartesian products, one for each ordered decomposition $S\sqcup T = I$ of the set I. This notation indicates that $S\cup T = I$ and $S\cap T=\emptyset$. There is one decomposition for each subset S of I, since T is then the complement of S. In particular, note that there is one term for $S\sqcup T$ and another for $T\sqcup S$.

Explicitly, a structure of species $\mathrm{P}\cdot\mathrm{Q}$ on a set I consists of a decomposition $S\sqcup T=I$, a structure of species P on S and a structure of species Q on T.

The Cauchy product is associative and we may consider iterated products

$$\mathrm{P}_1\cdot\cdots\cdot\mathrm{P}_k.$$

A structure of this species on a set I consists of an ordered decomposition $S_1\sqcup\cdots\sqcup S_k = I$ and a structure of species P_i on S_i, for each i.

The *unit* species 1 is defined by

$$1[I] = \begin{cases} \{*\} & \text{(a singleton) if } I=\emptyset,\\ \emptyset & \text{otherwise.}\end{cases} \tag{5}$$

It is concentrated on the empty set. It is the unit for the Cauchy product: for any set species P, there are canonical isomorphisms

$$1\cdot\mathrm{P} = \mathrm{P} = \mathrm{P}\cdot 1.$$

Exercise 5. The generating function of the Cauchy product is the product of the generating functions of the factors: let P and Q be finite set species, then

$$(\mathrm{P}\cdot\mathrm{Q})(x) = \mathrm{P}(x)\mathrm{Q}(x). \tag{6}$$

On the right hand side we have the usual product of formal power series (which corresponds to the Cauchy product of the coefficient sequences).

Example 6. A structure of species $\mathrm{E}\cdot\mathrm{E}$ on I is simply a subset S of I.

Fix a nonnegative integer k. A structure of species $\mathrm{E}^{\cdot k}$ on I is simply a function $f : I\to[k]$. Indeed, we may identify such a function with the decomposition $f^{-1}(1)\sqcup\cdots\sqcup f^{-1}(k) = I$.

Employing (6) we deduce that

$$(\mathrm{E}\cdot\mathrm{E})(x) = e^{2x} \quad\text{and}\quad (\mathrm{E}^{\cdot k})(x) = e^{kx}.$$

Expanding these power series we recover the facts that the number of subsets of $[n]$ is 2^n and the number of functions $[n] \to [k]$ is k^n.

Example 7. Let B be the species of bijections and D that of derangements. A B-structure on I is simply a bijection $\sigma : I \to I$, while a D-structure is a bijection with no fixed points (a derangement). We have

$$\mathrm{B} = \mathrm{E} \cdot \mathrm{D}.$$

Indeed, a bijection is uniquely determined by the subset of fixed points and the induced derangement on the complement.

Since $\#\mathrm{B}[n] = n!$, we have $\mathrm{B}(x) = \frac{1}{1-x}$. Employing (6) we deduce that

$$\mathrm{D}(x) = \frac{e^{-x}}{1-x}$$

and then that the number of derangements of $[n]$ is

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Warning. While $\mathrm{B}[n] = n! = \mathrm{L}[n]$, the species B (bijections) and L (linear orders) are not isomorphic. The following exercise arrives at this conclusion.

Exercise 8.

(a) Let P be a set species. Show that the symmetric group S_n acts on the set $\mathrm{P}[n]$ by

$$\sigma \cdot x = \mathrm{P}[\sigma](x).$$

(b) Let $f : \mathrm{P} \to \mathrm{Q}$ be a morphism of set species. Show that the map $f_{[n]} : \mathrm{P}[n] \to \mathrm{Q}[n]$ is a morphism of S_n-sets; that is,

$$f_{[n]}(\sigma \cdot x) = \sigma \cdot f_{[n]}(x)$$

for all $x \in \mathrm{P}[n]$ and $\sigma \in \mathrm{S}_n$.

(c) Show that the action of S_n on $\mathrm{L}[n]$ has one orbit, while the action on $\mathrm{B}[n]$ has as many orbits as partitions of n.

(d) Deduce that L and B are not isomorphic.

1.4. **Substitution.** Let Q be a set species with $\mathrm{Q}[\emptyset] = \emptyset$ and P an arbitrary set species. Their *substitution* is the species $\mathrm{P} \circ \mathrm{Q}$ defined by

$$(\mathrm{P} \circ \mathrm{Q})[I] = \coprod_{X \vdash I} \Big(\mathrm{P}[X] \times \prod_{S \in X} \mathrm{Q}[S] \Big). \tag{7}$$

for all finite sets I. This is the disjoint union of several cartesian products, one for each partition of I.

Explicitly, a structure of species $\mathrm{P} \circ \mathrm{Q}$ on a set I consists of a partition X of I, a structure of species P on the set X (whose elements are the blocks) and a structure of species Q on each of the blocks of X (which are subsets of I).

Substitution is associative and unital. The unit is the species X defined by

$$\mathrm{X}[I] = \begin{cases} \{*\} & \text{(a singleton) if } I \text{ is a singleton,} \\ \emptyset & \text{otherwise.} \end{cases}$$

It is concentrated on singletons. (Do not confuse the species X with a partition X).

Exercise 9. Verify that for any set species P, $\mathrm{P} \circ \mathrm{X} = \mathrm{P} = \mathrm{X} \circ \mathrm{P}$ (the latter if $\mathrm{P}[\emptyset] = \emptyset$).

Exercise 10. The generating function of the substitution is the substitution of the generating functions of the factors: let P and Q be as above and suppose in addition they are finite, then

$$(\mathrm{P} \circ \mathrm{Q})(x) = \mathrm{P}\big(\mathrm{Q}(x)\big). \tag{8}$$

On the right hand side we have the usual substitution of formal power series (which is defined when $\mathrm{Q}(0) = 0$).

In particular,

$$(\mathrm{E} \circ \mathrm{Q})(x) = e^{\mathrm{Q}(x)}. \tag{9}$$

This is referred to as the *exponential formula* in [17, Section 5.1]. A structure of species $\mathrm{E} \circ \mathrm{Q}$ on I consists of a partition of I and a Q-structure on each block.

Given a species Q, its *positive part* is the species Q_+ defined by

$$\mathrm{Q}_+[I] = \begin{cases} \emptyset & \text{if } I = \emptyset, \\ \mathrm{Q}[I] & \text{otherwise.} \end{cases}$$

Compositions of the form $\mathrm{P} \circ \mathrm{Q}_+$ are always defined, since $\mathrm{Q}_+[\emptyset] = \emptyset$.

Remark. The definition of substitution can be extended to the case in which the species Q is arbitrary. See [3, Section B.4].

Example 11. Consider the species $\mathrm{E} \circ \mathrm{E}_+$. A structure on I is simply a partition of I. Thus,

$$\mathrm{E} \circ \mathrm{E}_+ = \Pi.$$

Employing (9) we deduce that

$$\Pi(x) = e^{e^x - 1}.$$

This is the generating function for the number of set partitions (the Bell numbers).

Exercise 12.

(a) Show that $\mathrm{L} \circ \mathrm{E}_+ = \Sigma$.

(b) Deduce that

$$\Sigma(x) = \frac{1}{2 - e^x}.$$

This is the generating function for the number of set compositions (the ballot numbers).

Exercise 13. A *rooted tree* (with node set I) consists of an acyclic graph (with vertex set I) and the choice of a vertex (the root of the tree). The same structure can be defined recursively: a rooted tree with node set I consists of the choice of a node $r \in I$ (the root), a partition X of $I \setminus \{x\}$, and a rooted tree with node set S for each $S \in X$. Let A be the species of rooted trees.

(a) Show that $\mathrm{A} = \mathrm{X} \cdot (\mathrm{E} \circ \mathrm{A})$. (Note in particular that $\mathrm{A}[\emptyset] = \emptyset$).

(b) Deduce that $\mathrm{A}(x) = xe^{\mathrm{A}(x)}$.

Exercise 14. In a rooted tree with root r, the children of a node s are those nodes adjacent to s that are not between s and r. A *planar rooted tree* is a rooted tree in which the children of each node are given a linear order. Let $\vec{\mathrm{A}}$ denote the species of planar rooted trees.

(a) Show that $\vec{\mathrm{A}} = \mathrm{X} \cdot (\mathrm{L} \circ \vec{\mathrm{A}})$.

(b) Deduce that

$$\vec{\mathrm{A}}(x) = \frac{x}{1 - \vec{\mathrm{A}}(x)} \quad \text{and} \quad \vec{\mathrm{A}}(x) = \frac{1 - \sqrt{1 - 4x}}{2}.$$

Thus the number of planar rooted trees on n nodes is $n!C_{n-1}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Exercise 15. Let I be the species of *involutions* and P the species of *perfect matchings*. Let $\mathrm{E}_{1,2}$ be the species characteristic of singletons and doubletons:

$$\mathrm{E}_{1,2}[I] = \begin{cases} \{*_I\} & \text{if } |I| = 1 \text{ or } 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

(a) Show that $\mathrm{I} = \mathrm{E} \circ \mathrm{E}_{1,2} = \mathrm{E} \cdot \mathrm{P}$.

(b) Deduce that

$$\mathrm{I}(x) = e^{x + \frac{x}{2}} \quad \text{and} \quad \mathrm{P}(x) = e^{\frac{x}{2}}.$$

(c) Deduce that the number of perfect matchings on $[n]$ is $(2n-1)!!$.

2. Monoid species

Sets may carry algebraic structures, such as that of a group. Species may carry analogous structures. This aspect of the theory goes beyond enumeration. In this section we discuss monoids

2.1. **Monoids.** Let us review the basic notion of monoid. A monoid is a set M endowed with an operation that is associative and possesses a unit element $1 \in M$. This notion can be captured in terms of maps and diagrams. The operation is a map

$$\mu : M \times M \to M, \quad \mu(x, y) = xy,$$

and associativity is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\mu\times\mathrm{id}"'] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd}$$

Indeed, starting from an element $(x, y, z) \in M^{\times 3}$ and going around the top results in $x(yz)$, while going around the bottom results in $(xy)z$. Specifying an element $1 \in M$ is the same as providing a map

$$\iota : \{*\} \to M.$$

The domain is a singleton set (fixed once and for all) and the map is $\iota(*) = 1$. Unitality is equivalent to the commutativity of the diagrams below.

$$\begin{tikzcd} M & M \times M \arrow[l, "\mu"'] \\ & \{*\} \times M \arrow[ul, equal] \arrow[u, "\iota\times\mathrm{id}"'] \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[r, "\mu"] & M \\ M \times \{*\} \arrow[u, "\mathrm{id}\times\iota"] \arrow[ur, equal] & \end{tikzcd}$$

2.2. **Monoid species.** What should a monoid species be? If in Section 2.1 we replace cartesian products for Cauchy products, and the singleton $\{*\}$ for the unit species 1, we arrive at the definition. A *monoid species* is a species M endowed with morphisms of species

$$\mu : \mathrm{M} \cdot \mathrm{M} \to \mathrm{M} \quad \text{and} \quad \iota : 1 \to \mathrm{M}$$

such that the following diagrams commute.

$$\begin{tikzcd} \mathrm{M}\cdot\mathrm{M}\cdot\mathrm{M} \arrow[r, "\mathrm{id}\cdot\mu"] \arrow[d, "\mu\cdot\mathrm{id}"'] & \mathrm{M}\cdot\mathrm{M} \arrow[d, "\mu"] \\ \mathrm{M}\cdot\mathrm{M} \arrow[r, "\mu"'] & \mathrm{M} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{M} & \mathrm{M}\cdot\mathrm{M} \arrow[l, "\mu"'] \\ & 1\cdot\mathrm{M} \arrow[ul, equal] \arrow[u, "\iota\cdot\mathrm{id}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{M}\cdot\mathrm{M} \arrow[r, "\mu"] & \mathrm{M} \\ \mathrm{M}\cdot 1 \arrow[u, "\mathrm{id}\cdot\iota"] \arrow[ur, equal] & \end{tikzcd}$$

As for ordinary monoids, these conditions may be made explicit. First of all, a morphism $\mathrm{M} \cdot \mathrm{M} \to \mathrm{M}$ consists of a collection of maps $(\mathrm{M} \cdot \mathrm{M})[I] \to \mathrm{M}[I]$, one for each finite set I. Then note that, in view of (4), this data is equivalent to a collection of maps

$$\mu_{S,T} : \mathrm{M}[S] \times \mathrm{M}[T] \to \mathrm{M}[I],$$

one for each finite set I and each decomposition $S \sqcup T = I$. We may now write

$$\mu_{S,T}(x, y) = x \cdot y \in \mathrm{M}[I]$$

whenever $x \in \mathrm{M}[S]$ and $y \in \mathrm{M}[T]$. In view of (5), the morphism $1 \to \mathrm{M}$ boils down to a single map

$$\iota_\emptyset : \{*\} \to \mathrm{M}[\emptyset],$$

and this in turn to the selection of an element $1 = \iota_\emptyset(*) \in \mathrm{M}[\emptyset]$. The axioms may now be formulated in familiar terms:

$$x(yz) = (xy)z \quad \text{and} \quad 1 \cdot x = x = x \cdot 1. \tag{10}$$

The former should hold for all I, all decompositions $I = R \sqcup S \sqcup T$, and all elements $x \in \mathrm{M}[R]$, $y \in \mathrm{M}[S]$, $z \in \mathrm{M}[T]$. The latter for all I and $x \in \mathrm{M}[I]$.

There is one bit more. We need to account for axiom (2) stipulating the naturality of the morphism μ. (As it turns out, this is automatic for ι.) The condition is that for any bijection $\sigma : I \to J$, $x \in \mathrm{M}[S]$, and $y \in \mathrm{M}[T]$,

$$\mathrm{M}[\sigma]\big(\mu_{S,T}(x,y)\big) = \mu_{\sigma(S),\sigma(T)}\big(\mathrm{M}[\sigma](x), \mathrm{M}[\sigma](y)\big). \tag{11}$$

In other words, relabeling may be performed before or after multiplying, with the same result. This completes the definition of monoid species.

A morphism of monoids $f : \mathrm{M} \to \mathrm{N}$ is a morphism of species such that

$$f_\emptyset(1) = 1 \quad \text{and} \quad f_I(x \cdot y) = f_S(x) \cdot f_T(y)$$

for all $x \in \mathrm{M}[S]$, $y \in \mathrm{M}[T]$, and all decompositions $I = S \sqcup T$.

Example 16. The species L carries a monoid structure. Let I be a finite set and $S \sqcup T = I$ a decomposition. Given linear orders ℓ_1 on S and ℓ_2 on T, we let $\ell_1 \cdot \ell_2$ be their concatenation. This is a linear order on I. In more detail, if $\ell_1 = s_1 \ldots s_i$ and $\ell_2 = t_1 \ldots t_j$, then

$$\ell_1 \cdot \ell_2 := s_1 \ldots s_i \, t_1 \ldots t_j.$$

The unit is the unique linear order on the empty set. Axioms (10) and (11) are clearly satisfied.

Note that $\ell_1 \cdot \ell_2$ and $\ell_2 \cdot \ell_1$ are different orders: the monoid L is not commutative. We say that a monoid species M is commutative if

$$\mu_{S,T}(x,y) = \mu_{T,S}(y,x)$$

for all $I = S \sqcup T$, $x \in \mathrm{M}[S]$, and $y \in \mathrm{M}[T]$.

Example 17. An even simpler example of monoid is afforded by the species E. We simply define

$$\mu_{S,T} : \mathrm{E}[S] \times \mathrm{E}[T] \to \mathrm{E}[I], \quad \mu_{S,T}(*_S, *_T) = *_I.$$

This is the only possible definition, and the axioms hold trivially. The monoid E is commutative.

Example 18. The species B of bijections is a monoid. Given bijections $g \in \mathrm{B}[S]$ and $h \in \mathrm{B}[T]$, we let $f = g \cdot h \in \mathrm{B}[I]$ be the unique function $I \to I$ such that $f|_S = g$ and $f|_T = h$. Then f is a bijection. Again the unit is the unique element in $\mathrm{B}[\emptyset]$.

The species Π of partitions is a monoid. Given partitions X of S and Y of T, their union is a partition $X \cdot Y$ of I. (A block of $X \cdot Y$ is either a block of X or a block of Y.) Similarly, concatenation turns the species Σ of compositions into a monoid, and L into a submonoid. The map $\pi : \Sigma \to \Pi$ from (3) is a morphism of monoids.

The monoids B and Π are commutative, while the monoid Σ is not.

Exercise 19.

(a) Let P and Q be monoid species. Define a monoid structure on the species $\mathrm{P} \cdot \mathrm{Q}$.
(b) Deduce monoid structures on $\mathrm{E} \cdot \mathrm{E}$ and on $\mathrm{E}^{\cdot k}$, and describe them explicitly in terms of subsets and functions.

Remark. The notion of monoid makes sense in any monoidal category. The category of sets under cartesian product is monoidal and a monoid therein is an ordinary monoid. A monoid species is a monoid in the monoidal category of species under Cauchy product. Substitution endows the category of positive species with another monoidal structure. A monoid therein is an *operad*.

Exercise 20. Show that the species E_+ and L_+ carry an operad structure. These are the commutative and associative operad, respectively.

2.3. **Free monoids.** The free monoid on a set X is X^*, the set of all finite sequences of elements of X. The product is concatenation of sequences and the empty sequence is the unit element. (One commonly refers to X as an *alphabet*, to its elements as *letters*, and to the elements of X^* as *words*.) There is an analogous construction for species.

Let Q be a species with $\mathrm{Q}[\emptyset] = \emptyset$. Consider the species $\mathrm{L} \circ \mathrm{Q}$: a structure on I consists of a pair (F, x) where $F = (S_1, \ldots, S_k)$ is a compsition of I and $x = (x_1, \ldots, x_k)$ is a sequence of Q-structures $x_i \in \mathrm{Q}[S_i]$. This follows from (7), noting that a linear order on a partition turns it into a composition.

Once this is understood, one may easily turn $\mathrm{L} \circ \mathrm{Q}$ into a monoid. Given I and a decomposition $I = S \sqcup T$, take two $\mathrm{L} \circ \mathrm{Q}$-structures (F, x) and (G, y), the first on S and the other on T. We thus have compositions $F = (S_1, \ldots, S_k) \vDash S$, $G = (T_1, \ldots, T_h) \vDash T$, and sequences $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_h)$ of Q-structures, x_i on S_i and y_j on T_j for each i and j. Concatenating we obtain a composition

$$F \cdot G = (S_1, \ldots, S_k, T_1, \ldots, T_h) \vDash I$$

and a sequence

$$x \cdot y = (x_1, \ldots, x_k, y_1, \ldots, y_h)$$

consisting of Q-structures on each of the parts of $F \cdot G$. This defines the product

$$(F, x) \cdot (G, y) = (F \cdot G, x \cdot y).$$

There is only one $\mathrm{L} \circ \mathrm{Q}$ structure on the empty set and this is the unit. The monoid axioms are easily satisfied.

The *free monoid on* Q is the species $\mathrm{L} \circ \mathrm{Q}$ endowed with this monoid structure. The terminology is justified by the following result. First note that we may view Q as a subspecies of $\mathrm{L} \circ \mathrm{Q}$: a Q-structure x on I gives rise to the $\mathrm{L} \circ \mathrm{Q}$-structure $((I), x)$ on I, where (I) is the unique composition of I with one block.

Proposition 21. *Let* Q *be as above and* M *a monoid species. Given a morphism of species* $f : \mathrm{Q} \to \mathrm{M}$*, there is a unique morphism of monoid species* $\hat{f} : \mathrm{L} \circ \mathrm{Q} \to \mathrm{M}$ *such that* $\hat{f}_I((I), x) = f_I(x)$ *for all* I *and* $x \in \mathrm{Q}[I]$.

Proof. Given $(F, x) \in (\mathrm{L} \circ \mathrm{Q})[I]$ as above, define $\hat{f}(F, x) = f_{S_1}(x_1) \cdot \ldots \cdot f_{S_k}(x_k)$. □

Example 22. The monoid Σ is the free monoid on E_+. Indeed, the equality $\Sigma = \mathrm{L} \circ \mathrm{E}_+$ expresses that a composition is an ordered partition, and the product of Σ is concatenation, as in the free monoid.

Exercise 23. Let Q be as above.

(a) Describe a structure of species $\mathrm{E} \circ \mathrm{Q}$.
(b) Turn the species $\mathrm{E} \circ \mathrm{Q}$ into a monoid.
(c) State and prove the analog of Proposition 21, showing that $\mathrm{E} \circ \mathrm{Q}$ is the *free commutative monoid on* Q.

Example 24. The monoid Π is free on E_+. The monoid B is free on the species of *cycles*.

The monoid of natural numbers (under addition) is free on one generator. What is its analog for monoid species? There are two valid anwers. The monoid L, which is the free monoid on the species X, and the monoid E, which is the free commutative monoid on the species X. The monoid of natural numbers happens to be commutative, so it is both the free monoid and the free commutative monoid on one generator. But in the world of species, these two notions diverge.

3. Hopf monoids and the antipode

We turn to bimonoids and Hopf monoids in species. The latter play the role of groups. The role of inversion is played by the antipode. To accommodate for this notion, however, we must broaden our considerations and allow formal linear combinations of combinatorial structures. For a more thorough discussion, see [3, Part II].

3.1. **Groups.** Let us go back to ordinary monoids. The passage to groups requires existence of inverses:

$$xx^{-1} = 1 = x^{-1}x. \tag{12}$$

Can we make sense of an analogous notion for species?

An immediate obstacle arises: in order to multiply in a monoid species M, we must have $x \in \mathrm{M}[S]$ and $x^{-1} \in \mathrm{M}[T]$ for some $I = S \sqcup T$. But then the product is in $\mathrm{M}[I]$, while $1 \in \mathrm{M}[\emptyset]$.

We adopt a different perspective. We first consider *bimonoid* species. These objects carry not only a monoid structure, but also a compatible *comonoid* structure. We skip an explicit consideration of comonoids and move directly to bimonoids. Then we proceed to Hopf monoids. These objects may be seen as analog to groups in the world of species, or perhaps more properly, to Hopf algebras, which are themselves analogous to groups.

3.2. **Bimonoid species.** A *bimonoid species* consists of the following data.

- A set species H.
- For each finite set I and each decomposition $I = S \sqcup T$, maps

$$\mathrm{H}[S] \times \mathrm{H}[T] \xrightarrow{\mu_{S,T}} \mathrm{H}[I] \quad \text{and} \quad \mathrm{H}[I] \xrightarrow{\Delta_{S,T}} \mathrm{H}[S] \times \mathrm{H}[T].$$

These are subject to the axioms below. They include the monoid axioms from Section 2.2, which we list again for convenience. Before stating them, we discuss some terminology and notation. The collection of maps μ (resp. Δ) are called the *product* (resp. the *coproduct*) of the bimonoid H. We often employ the same terminology for an individual map $\mu_{S,T}$ (resp. $\Delta_{S,T}$) in the collection. Fix a decomposition $I = S \sqcup T$. Given $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$, let

$$x \cdot y \in \mathrm{H}[I]$$

denote the image of (x, y) under the map $\mu_{S,T} \colon \mathrm{H}[S] \times \mathrm{H}[T] \to \mathrm{H}[I]$. We call it the *product* of the structures x and y. Also, given $z \in \mathrm{H}[I]$, let $z|_S$ and $z/_S$ denote the components of the pair $\Delta_{S,T}(z) \in \mathrm{H}[S] \times \mathrm{H}[T]$. Thus,

$$\Delta_{S,T}(z) = (z|_S, z/_S).$$

Note that $z|_S \in \mathrm{H}[S]$ and $z/_S \in \mathrm{H}[T]$. We refer to the former as the *restriction of z to S* (a structure on S) and to the latter as the *contraction of S from z* (a structure on T). There is a distinguished element $1 \in \mathrm{H}[\emptyset]$ called the *unit* of H. The axioms are as follows.

Naturality. For each decomposition $I = S \sqcup T$ and each bijection $\sigma : I \to J$, we must have

$$\mathrm{H}[\sigma](x \cdot y) = \mathrm{H}[\sigma|_S](x) \cdot \mathrm{H}[\sigma|_T](y) \text{ for all } x \in \mathrm{H}[S],\ y \in \mathrm{H}[T];$$
$$\mathrm{H}[\sigma](z)|_S = \mathrm{H}[\sigma|_S](z|_S) \quad \text{and} \quad \mathrm{H}[\sigma](z)/_S = \mathrm{H}[\sigma|_T](z/_S) \text{ for all } z \in \mathrm{H}[I].$$

Unitality. For each I and $x \in \mathrm{H}[I]$, we must have

$$x \cdot 1 = x = 1 \cdot x \quad \text{and} \quad x|_I = x = x/_\emptyset.$$

Associativity. For each decomposition $I = R \sqcup S \sqcup T$, we must have

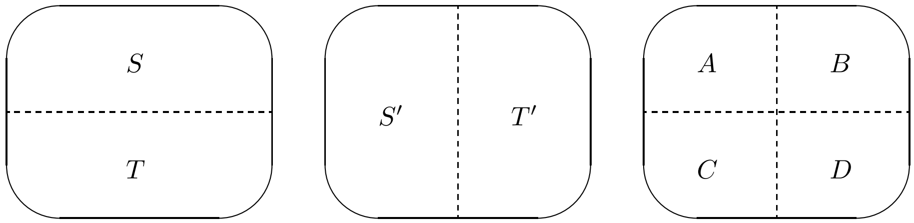
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ for all } x \in \mathrm{H}[R],\ y \in \mathrm{H}[S],\ z \in \mathrm{H}[T];$$
$$(z|_{R \sqcup S})|_R = z|_R, \quad (z|_{R \sqcup S})/_R = (z/_R)|_S, \quad \text{and} \quad z/_{R \sqcup S} = (z/_R)/_S \text{ for all } z \in \mathrm{H}[I].$$

Compatibility. Fix decompositions $S \sqcup T = I = S' \sqcup T'$, and consider the resulting pairwise intersections:

$$A := S \cap S',\ B := A \cap T',\ C := T \cap S',\ D := T \cap T',$$

as illustrated below.

(13)



In this situation, for any $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$, we must have

$$(x \cdot y)|_{S'} = x|_A \cdot y|_C \quad \text{and} \quad (x \cdot y)/_{S'} = x/_A \cdot y/_C.$$

This completes the list of axioms and the definition of bimonoid species.

A *morphism* $f : \mathrm{H} \to \mathrm{K}$ between bimonoids H and K is a morphism of species which preserves products, restrictions and contractions. Thus, we have

$$f_J\big(\mathrm{H}[\sigma](x)\big) = \mathrm{K}[\sigma]\big(f_I(x)\big) \text{ for all } x \in \mathrm{H}[I] \text{ and all bijections } \sigma : I \to J,$$
$$f_\emptyset(1) = 1,$$
$$f_I(x \cdot y) = f_S(x) \cdot f_T(y) \text{ for all } x \in \mathrm{H}[S],\ y \in \mathrm{H}[T], \text{ and all decompositions } I = S \sqcup T,$$
$$f_S(z|_S) = f_I(z)|_S \text{ and } f_T(z/_S) = f_I(z)/_S \text{ for all } z \in \mathrm{H}[I] \text{ and all decompositions } I = S \sqcup T.$$

A bimonoid H is *commutative* if for any $I = S \sqcup T$, $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$ we have

$$x \cdot y = y \cdot x.$$

It is *cocommutative* if for any $I = S \sqcup T$ and $z \in \mathrm{H}[I]$ we have

$$z/_S = z|_T.$$

Example 25. The species L of linear orders is a bimonoid. We defined its monoid structure in Example 16: the product is concatenation. To define the coproduct, given a linear order ℓ on I, we let $\ell|_S$ be the list consisting of the elements of S written in the order in which they appear in ℓ. In addition, we set $\ell/_S = \ell|_T$. In other words,

$$\Delta_{S,T} : \mathrm{L}[I] \to \mathrm{L}[S] \times \mathrm{L}[T], \quad \Delta_{S,T}(\ell) = (\ell|_S, \ell|_T).$$

Consider the situation (13). Given linear orders ℓ_1 on S and ℓ_2 on T, the required commutativity boils down to the fact that the concatenation of $\ell_1|_A$ and $\ell_2|_C$ agrees with the restriction to S' of $\ell_1 \cdot \ell_2$. Thus the compatibility axiom is satisfied. The verification of the remaining axioms is similar. Thus L is a bimonoid. Note also that L is cocommutative (though not commutative).

Example 26. The exponential species E is a bimonoid. The structure maps are

$$\begin{array}{ccc} \mathrm{E}[S] \times \mathrm{E}[T] \xrightarrow{\mu_{S,T}} \mathrm{E}[I] & \qquad & \mathrm{E}[I] \xrightarrow{\Delta_{S,T}} \mathrm{E}[S] \times \mathrm{E}[T] \\ (*_S, *_T) \longmapsto *_I & & *_I \longmapsto (*_S, *_T). \end{array}$$

The axioms are trivially satisfied. The bimonoid E is both commutative and cocommutative.

Example 27. The species Π of set partitions is a bimonoid. The structure maps are

$$\begin{array}{ccc} \Pi[S] \times \Pi[T] \xrightarrow{\mu_{S,T}} \Pi[I] & \qquad & \Pi[I] \xrightarrow{\Delta_{S,T}} \Pi[S] \times \Pi[T] \\ (X_1, X_2) \longmapsto X_1 \cdot X_2 & & X \longmapsto (X|_S, X|_T). \end{array}$$

The union $X_1 \cdot X_2$ of partitions is as in Example 18. The restriction of $X \vdash I$ to S is the partition $X|_S$ whose blocks are the nonempty intersections between S and each of the blocks of X. The Hopf monoid Π is commutative and cocommutative.

The species Σ of compositions is a bimonoid with product and coproduct defined by

$$\begin{aligned} \Sigma[S] \times \Sigma[T] &\xrightarrow{\mu_{S,T}} \Sigma[I] & \qquad \Sigma[I] &\xrightarrow{\Delta_{S,T}} \Sigma[S] \times \Sigma[T] \\ (F_1, F_2) &\longmapsto F_1 \cdot F_2 & F &\longmapsto (F|_S, F|_T). \end{aligned} \tag{14}$$

The concatenation $F_1 \cdot F_2$ is as in Example 18. The restriction of $F \vDash I$ to S is the composition $F|_S$ whose blocks are the nonempty intersections between S and each of the blocks of F, ordered as in F. The bimonoid Σ is cocommutative but not commutative.

The map $\pi : \Sigma \to \Pi$ from (3) is a morphism of bimonoids. L is a subbimonoid of Σ.

All of the preceding are in fact examples of Hopf monoids. We turn to this notion next. Additional examples of Hopf monoids are given in Section 5. Many more appear in [3, Chapters 11–13] and [1, 2, 4].

3.3. **Linearization.** We fix a field $\Bbbk$. All vector spaces are over $\Bbbk$.

Given a set species P and a finite set I, we let $\mathbf{P}[I]$ denote the vector space with basis the set $\mathrm{P}[I]$. In other words, $\mathbf{P}[I]$ consists of all formal linear combinations of elements of P. Given a bijection $\sigma : I \to J$, we let $\mathbf{P}[\sigma] : \mathbf{P}[I] \to \mathbf{P}[J]$ denote the unique linear transformation extending $\mathrm{P}[\sigma]$.

As P is set species, $\mathbf{P}$ is a *vector species*: a functor $\mathsf{set}^{\times} \to \mathsf{Vec}$. The vector species $\mathbf{P}$ is the linearization of the set species P. Not every vector species is the linearization of a set species. In this notes we only deal with set species and their linearizations. For details about vector species and Hopf monoids therein, see [3, Chapter 8] or [4].

3.4. **Hopf monoids and the antipode.** Let H be a bimonoid. We say that H is a Hopf monoid if for each finite set I and each $x \in \mathrm{H}[I]$, there exists a formal linear combination

$$\mathrm{s}_I(x) \in \mathbf{H}[I]$$

with the following properties.

Naturality. For each bijection $\sigma : I \to J$,

$$\mathbf{H}[\sigma](\mathrm{s}_I(x)) = \mathrm{s}_J(\mathrm{H}[\sigma](x)).$$

Inversion. For each decomposition $I = S \sqcup T$ and $x \in \mathrm{H}[I]$,

$$\sum_{S \sqcup T = I} \mathrm{s}_S(x|_S) \cdot x/_S = \sum_{S \sqcup T = I} x|_S \cdot \mathrm{s}_T(x/_S) = \begin{cases} 1 & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

This axiom is the replacement of the familiar (12). It follows from it that the set $\mathrm{H}[\emptyset]$ is an ordinary group and $\mathrm{s}_\emptyset(x) = x^{-1}$ for all $x \in \mathrm{H}[\emptyset]$. In addition, if two maps s and s′ satisfy these conditions, they are equal. Thus, when such a map s exists, it is uniquely determined. It is called the *antipode* of the Hopf monoid H.

Exercise 28. Let H be a bimonoid. Let $\mathrm{End}(\mathbf{H})$ denote the set of all morphisms of vector species $f : \mathbf{H} \to \mathbf{H}$. It is a vector space under pointwise addition and scalar multiplication. The *convolution* $f * g$ of two morphisms f and g is defined by

$$(f * g)_I(x) = \sum_{S \sqcup T = I} f_S(x|_S) \cdot g_T(x/_S). \tag{15}$$

for all finite sets I and all $x \in \mathrm{H}[I]$ (and then extended linearly to $\mathbf{H}[I]$). Let $u : \mathbf{H} \to \mathbf{H}$ be the linear extension of $\iota\epsilon : \mathrm{H} \to \mathrm{H}$.

- (a) Show that convolution is associative and unital, with u being the unit. In this manner $\mathrm{End}(\mathbf{H})$ is an algebra.
- (b) Show that H is a Hopf monoid if and only id : $\mathbf{H} \to \mathbf{H}$ is invertible in the algebra $\mathrm{End}(\mathbf{H})$, and that in this case s is the convolution inverse of id.
- (c) Deduce the uniqueness of the antipode.

All bimonoids H we have considered are such that $\mathrm{H}[\emptyset] = \{1\}$; that is, they possess only one structure on the empty set. Such bimonoids are called *connected*. In this case, the inversion axiom is trivially satisfied when $I = \emptyset$ ($\mathrm{H}[\emptyset]$ is the trivial group) and the axiom states that

$$\sum_{S \sqcup T = I} \mathrm{s}_S(x|_S) \cdot x/_S = 0 = \sum_{S \sqcup T = I} x|_S \cdot \mathrm{s}_T(x/_S) \tag{16}$$

for all nonempty sets I and $x \in \mathrm{H}[I]$. The following result shows that any such bimonoid is a Hopf monoid. This applies in particular to E, L, Π, Σ, and other examples to be treated later.

Proposition 29. *Let* H *be a connected bimonoid. Then the antipode exists (so* H *is a Hopf monoid). Moreover, the antipode is uniquely determined by any of the following two recursions.*

$$\mathrm{s}_I(x) = -\sum_{\emptyset \subseteq S \subset I} \mathrm{s}_S(x|_S) \cdot (x/_S) \qquad \text{and} \qquad \mathrm{s}_I(x) = -\sum_{\emptyset \subset S \subseteq I} x|_S \cdot \mathrm{s}_T(x/_S), \tag{17}$$

where I is nonempty and $x \in \mathrm{H}[I]$.

Proof. When the antipode exists, the recursive formulas (17) follow from (16). Conversely, either of them defines a map s that satisfies one of the equalities in (16). Each of these says that s is a one-sided inverse of id in the algebra $\mathrm{End}(\mathbf{H})$, and hence imply that it is a two-sided inverse, so the remaining equality in (16) follows. □

Inversion reverses products. The same is true for antipodes.

Proposition 30. *Let* H *be a connected bimonoid. Then its antipode reverses products. In other words, for any $I = S \sqcup T$, $x \in \mathrm{H}[S]$, and $y \in \mathrm{H}[T]$, we have*

$$\mathrm{s}_I(x \cdot y) = \mathrm{s}_T(y) \cdot \mathrm{s}_S(x). \tag{18}$$

Proof. We argue by induction on the size of I. We employ the first recursion in (17) and then make use of the compatibility axiom as in (13). We obtain

$$\mathrm{s}_I(x \cdot y) = -\sum_{\emptyset \subseteq S' \subset I} \mathrm{s}_{S'}\big((x \cdot y)|_{S'}\big) \cdot (x \cdot y)/_{S'} = -\sum_{\emptyset \subseteq S' \subset I} \mathrm{s}_{S'}(x|_A \cdot y|_C) \cdot x/_A \cdot y/_C.$$

The induction hypothesis allows to derive that

$$\mathrm{s}_I(x \cdot y) = -\sum_{\substack{\emptyset \subseteq A \subseteq S \\ \emptyset \subseteq C \subseteq T \\ A \sqcup C \neq I}} \mathrm{s}_C(y|_C) \cdot \mathrm{s}_A(x|_A) \cdot x/_A \cdot y/_C.$$

We split the last sum according to whether $A = S$, $C = T$, or both A is a proper subset of S and C is a proper subset of T, and employ the unit axiom which tells us that

$$x|_S = x, \quad x/_S = 1, \quad y|_T = y, \quad \text{and} \quad y/_T = 1.$$

We obtain

$$\mathrm{s}_I(x \cdot y) = -\sum_{\emptyset \subseteq C \subset T} \mathrm{s}_C(y|_C) \cdot \mathrm{s}_S(x) \cdot y/_C - \sum_{\emptyset \subseteq A \subset S} \mathrm{s}_T(y) \cdot \mathrm{s}_A(x|_A) \cdot x/_A - \sum_{\substack{\emptyset \subseteq A \subset S \\ \emptyset \subseteq C \subset T}} \mathrm{s}_C(y|_C) \cdot \mathrm{s}_A(x|_A) \cdot x/_A \cdot y/_C.$$

Employing (17) for $\mathrm{s}_S(x)$ on the last two terms we deduce

$$\mathrm{s}_I(x \cdot y) = -\sum_{\emptyset \subseteq C \subset T} \mathrm{s}_C(y|_C) \cdot \mathrm{s}_S(x) \cdot y/_C + \mathrm{s}_T(y) \cdot \mathrm{s}_S(x) + \sum_{\emptyset \subseteq C \subset T} \mathrm{s}_C(y|_C) \cdot \mathrm{s}_S(x) \cdot y/_C = \mathrm{s}_T(y) \cdot \mathrm{s}_S(x),$$

as needed. □

Exercise 31. Show that the antipode reverses coproducts. In other words, for any $z \in \mathbf{H}[I]$,

$$\Delta_{S,T}\big(\mathrm{s}_I(z)\big) = \sum \mathrm{s}_S(z/_T) \otimes \mathrm{s}_T(z|_T). \tag{19}$$

COMPLETE

The next result involves iterated products and coproducts. Let H be a bimonoid. The following is a consequence of the associativity axiom. For any decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ with $k \geq 2$, there is a unique map

$$\mathrm{H}[S_1] \times \cdots \times \mathrm{H}[S_k] \xrightarrow{\mu_{S_1,\ldots,S_k}} \mathrm{H}[I] \tag{20}$$

obtained by iterating the maps μ in any meaningful way. For instance, given $I = R \sqcup S \sqcup S \sqcup T$, the map

$$\mathrm{H}[R] \times \mathrm{H}[S] \times \mathrm{H}[T] \xrightarrow{\mu_{R,S,T}} \mathrm{H}[I]$$

is obtained as either

$$\mathrm{H}[R] \times \mathrm{H}[S] \times \mathrm{H}[T] \xrightarrow{\mathrm{id}\times\mu_{S,T}} \mathrm{H}[R] \times \mathrm{H}[S \sqcup T] \xrightarrow{\mu_{R,S\sqcup T}} \mathrm{H}[I]$$

or

$$\mathrm{H}[R] \times \mathrm{H}[S] \times \mathrm{H}[T] \xrightarrow{\mu_{R,S}\times\mathrm{id}} \mathrm{H}[R \sqcup S] \times \mathrm{H}[T] \xrightarrow{\mu_{R\sqcup S,T}} \mathrm{H}[I].$$

We write

$$\mu_{S_1,\ldots,S_k}(x_1, \ldots, x_k) = x_1 \cdot \ldots \cdot x_k \in \mathrm{H}[I]$$

whenever $x_i \in \mathrm{H}[S_i]$ for $i = 1, \ldots, k$.

Similarly, there is a unique map

$$\mathrm{H}[I] \xrightarrow{\Delta_{S_1,\ldots,S_k}} \mathrm{H}[S_1] \times \cdots \times \mathrm{H}[S_k] \tag{21}$$

obtained by iterating the maps Δ. If $z \in \mathrm{H}[I]$ and

$$\Delta_{S_1,\ldots,S_k}(z) = (z_1, \ldots, z_k),$$

we refer to $z_i \in \mathrm{H}[S_i]$ as the *i-th minor* of z corresponding to the decomposition $I = S_1 \sqcup \cdots \sqcup S_k$. It is obtained from z by combining restrictions and contractions in any meaningful way.

For $k = 1$, we define

$$\mu_I = \Delta_I = \mathrm{id} : \mathrm{H}[I] \to \mathrm{H}[I]. \tag{22}$$

Proposition 32. *Let* H *be a connected bimonoid. The antipode is explicitly given by the following formula.*

$$\mathrm{s}_I = \sum_{\substack{S_1\sqcup\cdots\sqcup S_k=I \\ S_i\neq\emptyset \;\; k\geq 1}} (-1)^k \mu_{S_1,\ldots,S_k} \Delta_{S_1,\ldots,S_k}. \tag{23}$$

The sum is over all ordered decompositions of I into nonempty *subsets S_i.*

Proof. This follows from either of the recursions (17), by induction on the size of I. $\square$

We refer to (17) as the *Milnor-Moore* recursions, and to (23) as *Takeuchi's* formula, since analogous results for Hopf algebras are due to those authors.

Exercise 33. Show that a morphism $f : \mathrm{H} \to \mathrm{K}$ of connected bimonoids preserves the antipodes. In other words,

$$f_I(\mathrm{s}_I(x)) = \mathrm{s}_I(f_I(x))$$

for all I and $x \in \mathrm{H}[I]$.

Remark. It is true, more generally, that if H and K are Hopf monoids (not necessarily connected) and $f : \mathrm{H} \to \mathrm{K}$ is a morphism of bimonoids, then f preserves the antipodes. The result of Proposition 30 also holds for arbitrary Hopf monoids. They may be deduced by means of ideas similar to those in Exercise 28.

3.5. **The antipode problem.** Cancellations frequently take place in Takeuchi's formula (23). Understanding these cancellations is often a challenging combinatorial problem. The antipode problem asks for an explicit, cancellation-free, formula for the antipode of a given Hopf monoid H. The problem is to determine the structure constants of s_I on the basis $\mathrm{H}[I]$ of the vector space $\mathbf{H}[I]$, for each finite set I. We may also be interested in other linear bases of $\mathbf{H}[I]$, and the corresponding structure constants.

Example 34. For the exponential Hopf monoid E, the antipode is given by

$$\mathrm{s}_I(*_I) = (-1)^{|I|} *_I .$$

In this case, axiom (16) boils down to the basic identity

$$\sum_{S\subseteq I} (-1)^S = 0$$

for $I \neq \emptyset$, or $(1-1)^n = 0$ for $n > 0$. It may also be derived from Takeuchi's formula (23) by using

$$\sum_{\substack{S_1\sqcup\cdots\sqcup S_k=I\\ S_i\neq\emptyset\ k\geq 1}} (-1)^k = (-1)^{|I|},$$

or equivalently

$$\sum_{\substack{s_1+\cdots+s_k=n\\ s_i\geq 1\ k\geq 1}} \binom{n}{s_1,\ldots,s_k}(-1)^k = (-1)^n.$$

This is a well-known identity. It provides the reduced Euler characteristic of the Coxeter complex of type A_{n-1}, which is a sphere of dimension $n-2$. (Given that the antipode formula may be settled directly, the precedin identity may be derived from Takeuchi's formula.)

Example 35. The antipode of the Hopf monoid L of linear orders is given by

$$\mathrm{s}_I(l) = (-1)^{|I|}\, \bar{l},$$

where $\bar{l}$ is the linear order on I obtained by reversing the linear order l. As for E, this can be verified in various ways. Let us discuss how it may be derived from Takeuchi's formula. This yields

$$\mathrm{s}_I(l) = \sum_{\substack{S_1\sqcup\cdots\sqcup S_k=I\\ S_i\neq\emptyset\ k\geq 1}} (-1)^k\, l|_{S_1}\cdots l|_{S_k}.$$

The right-hand side involves concatenations of restrictions of the linear order l. To reconcile this with the earlier formula, one has to show that for any pair of linear orders l and l' on I, we have

$$\sum_{\substack{(S_1,\ldots,S_k)\\ k\geq 1}} (-1)^k = \begin{cases} (-1)^{|I|} & \text{if } l' = \bar{l}\\ 0 & \text{otherwise,}\end{cases}$$

where the sum if over all ordered decompositions $I = S_1 \sqcup \cdots \sqcup S_k$ into nonempty subsets for which $l|_{S_1}\cdots l|_{S_k} = l'$. Such decompositions are partially ordered by refinement and in this manner they form a Boolean poset. They are decompositions into intervals of l', and on each interval the orders l and l' must agree. The thinnest one is always the decomposition into singleton intervals. If $l' \neq \bar{l}$, then there are longer intervals of l' on which l and l' agree. The result then follows by inclusion-exclusion.

Exercise 36. Derive the antipode formulas for E and L from Proposition 30. Recall that these monoids are generated by singleton structures.

The following exercise involves set partitions and compositions. The length $l(X)$ of a partition X is the number of blocks. When a partition Y refines a partition X, we write $X \leq Y$. This means each block of Y is contained in a block of X, or equivalently, each block of X is a union of blocks of Y. In this case,

$$(Y/X)! = \prod_{B\in X} (n_B)!,$$

where $n_B = l(Y|_B)$ is the number of blocks of Y that refine the block B of X. The length of a composition is the number of blocks. A composition G refines a composition F if each block of F is obtained by merging a number of contiguous blocks of G. In this case, we write $F \leq G$. The opposite of a composition $F = (S_1, \dots, S_k)$ is the composition $\overline{F} = (S_k, \dots, S_1)$.

Exercise 37.

(a) Show that the antipode of the Hopf monoid Σ of set compositons is given by

$$\mathrm{s}_I(F) = \sum_{G:\, \overline{F} \leq G} (-1)^{l(G)}\, G.$$

(Apply Proposition 30. Recall that the monoid $\Sigma = \mathrm{L} \circ \mathrm{E}_+$ and treat the one-block case via (23).)

(b) Deduce the antipode formula for L applying Exercise 33 to the inclusion $\mathrm{L} \to \Sigma$.

(c) Show that the antipode of the Hopf monoid Π of set partitions is given by

$$\mathrm{s}_I(X) = \sum_{Y:\, X \leq Y} (-1)^{l(Y)}\, (Y/X)!\, Y,$$

in two ways: by applying Proposition 30 as above, or by applying Exercise 33 and employing the morphism π from (3).

4. Enumeration via Hopf monoids

We discuss the notion of character on a connected Hopf monoid and the associated polynomial invariant. The main result is Proposition 43, which relates the values of the invariant on an integer and on its opposite by means of the antipode of the Hopf monoid. The results of this section are taken from [1].

Intuitively, the Hopf monoid consists of a class of combinatorial structures, the character selects a particular subclass (or property of the structures), and the invariant enumerates decompositions of the given structure in terms of minors in the subclass. This point of view is illustrated in Section 5, where we return to examples.

The notion of character is most conveniently formulated in the context of vector species, and we work directly in this context. We work over a field $\Bbbk$ of characteristic 0.

4.1. Characters.

Let $\mathbf{H}$ be a connected Hopf monoid. The iterated product and coproduct of $\mathbf{H}$ are linear maps as follows.

$$\mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \xrightarrow{\mu_{S_1,\dots,S_k}} \mathbf{H}[I] \quad \text{and} \quad \mathbf{H}[I] \xrightarrow{\Delta_{S_1,\dots,S_k}} \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k]$$

To connect with the examples of earlier sections, we may assume that the vector species $\mathbf{H}$ is the linearization of a set species H. In this case, the above maps are the unique linear extensions of the maps (20) and (21).

A *character* on $\mathbf{H}$ is a morphism of vector species $\zeta : \mathbf{H} \to \mathbf{E}$. Explicitly, ζ consists of a collection of linear maps

$$\zeta_I : \mathbf{H}[I] \to \Bbbk,$$

one for each finite set I, subject to the following axioms. When $\mathbf{H}$ is the linearization of a set species H, ζ_I is determined by its values on the basis $\mathrm{H}[I]$ of $\mathbf{H}[I]$.

Naturality. For each bijection $\sigma : I \to J$ and $x \in \mathbf{H}[I]$,

$$\zeta_J\big(\mathbf{H}[\sigma](x)\big) = \zeta_I(x).$$

Multiplicativity. For each decomposition $I = S \sqcup T$, $x \in \mathbf{H}[S]$ and $y \in \mathbf{H}[T]$,

$$\zeta_I(x \cdot y) = \zeta_S(x)\zeta_T(y).$$

Unitality. The map $\zeta_\emptyset : \mathbf{H}[\emptyset] \to \Bbbk$ sends $1 \in \Bbbk = \mathbf{H}[\emptyset]$ to $1 \in \Bbbk$:

$$\zeta_\emptyset(1) = 1.$$

The *trivial character* $\epsilon : \mathbf{H} \to \mathbf{E}$ is defined by

$$\epsilon_I(x) = \begin{cases} 1 & \text{if } I = \emptyset \text{ and } x = 1, \\ 0 & \text{if } I \neq \emptyset. \end{cases}$$

Exercise 38. Let $\mathbb{X}(\mathbf{H})$ denote the set of all characters on a connected bimonoid $\mathbf{H}$. The convolution of two characters ζ and ξ is defined as in (15) by

$$(\zeta * \xi)_I(x) = \sum_{S \sqcup T = I} \zeta_S(x|_S)\xi_T(x/_S). \tag{24}$$

The product on the right takes place in $\Bbbk$. As in Exercise 28, convolution is associative. The unit is the trivial character ϵ.

(a) Show that $\zeta * \xi$ and $\zeta \circ \mathrm{s}$ are characters.
(b) Show that $\zeta \circ \mathrm{s}$ is the convolution inverse of ζ.
(c) Deduce that $\mathbb{X}(\mathrm{H})$ is a group under convolution.

Remark. Convolution entered in Exercises 28 and 38. More generally, the convolution product is defined on the set $\mathrm{Hom}(\mathbf{C}, \mathbf{M})$ of morphisms of species between a comonoid $\mathbf{C}$ and a monoid $\mathbf{M}$.

4.2. **Invariants.** Fix a connected Hopf monoid $\mathbf{H}$ and a character ζ on $\mathbf{H}$.

Define, for each nonempty finite set I, each element $x \in \mathbf{H}[I]$, and each natural number $n \in \mathbb{N}$,

$$\chi_I(x)(n) := \sum_{I = S_1 \sqcup \cdots \sqcup S_n} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(x). \tag{25}$$

The sum is over all decompositions of I into n (disjoint) subsets. The subsets are allowed to be empty. Thus, there are precisely $n^{|I|}$ terms in the sum. The sum takes place in the field $\Bbbk$; the identification

$$\Bbbk \otimes \cdots \otimes \Bbbk \cong \Bbbk$$

is implicit. For fixed I and x, the function $\chi_I(x)$ is defined on $\mathbb{N}$ and takes values on $\Bbbk$.

We have

$$\chi_I(x)(0) = 0 \tag{26}$$

and

$$\chi_I(x)(1) = \zeta_I(x). \tag{27}$$

The former holds since a decomposition of a nonempty set has at least 1 part; the latter in view of (22).

Proposition 39. *Let $\mathbf{H}$, ζ and χ be as above. Fix a finite set I and an element $x \in \mathbf{H}[I]$. Then*

$$\chi_I(x)(n) = \sum_{k=0}^{|I|} \chi_I^{(k)}(x) \binom{n}{k}$$

where, for each $k = 0, \ldots, |I|$,

$$\chi_I^{(k)}(x) = \sum_{(T_1,\ldots,T_k) \vDash I} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x) \in \Bbbk.$$

Therefore, $\chi_I(x)$ is a polynomial function of n of degree at most $|I|$.

The sum in $\chi_I^{(k)}(x)$ is over compositions of I: the subsets T_i are disjoint and nonempty.

Proof. Given a decomposition $I = S_1 \sqcup \cdots \sqcup S_n$, let $(T_1, \ldots, T_k)$ be the composition of I obtained by removing the empty S_i's and keeping the remaining ones in order. In view of unitality of Δ and ζ, we have

$$(\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(x) = (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x).$$

On the other hand, the number of decompositions $I = S_1 \sqcup \cdots \sqcup S_n$ which give rise to one same composition $(T_1, \ldots, T_k)$ is $\binom{n}{k}$. It follows that

$$\chi_I(x)(n) \;= \sum_{k=0}^{n} \sum_{(T_1,\ldots T_k)\models I} \binom{n}{k} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x).$$

In this sum, k is bounded both by n and by the cardinality of I. Therefore,

$$\chi_I(x)(n) \;= \sum_{k=0}^{|I|} \Bigg(\sum_{(T_1,\ldots T_k)\models I} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x) \Bigg) \binom{n}{k} = \sum_{k=0}^{|I|} \chi_I^{(k)}(x) \binom{n}{k}.$$

For each k, the coefficient $\chi_I^{(k)}(x)$ does not depend on n. Since each $\binom{n}{k}$ is a polynomial function of n of degree k, the result follows. $\square$

Proposition 40. *Let* $\mathbf{H}$*,* ζ *and* χ *be as above. Let* $\sigma : I \to J$ *be a bijection,* $x \in \mathbf{H}[I]$ *and* $y := \mathbf{H}[\sigma](x) \in \mathbf{H}[J]$*. Then*

$$\chi_I(x) = \chi_J(y).$$

Proof. This follows from the naturality of Δ and ζ. $\square$

Let $\Bbbk[t]$ denote the polynomial algebra in the formal variable t. Since $\Bbbk$ is of characteristic 0, we may identify polynomials and polynomial functions. Proposition 39 states that to each structure $x \in \mathbf{H}[I]$ there is associated a polynomial $\chi_I(x) \in \Bbbk[t]$, whose values on nonnegative integers n are given by (25).

Proposition 40 states that two isomorphic structures have the same associated polynomial. Thus, the function $\chi_I(x)$ is a *polynomial invariant* of the structure x (canonically associated to the Hopf monoid $\mathbf{H}$ and the character ζ). The following result gives some algebraic properties of the invariant.

Proposition 41. *Let* $\mathbf{H}$*,* ζ *and* χ *be as above. Let* I *be a finite set.*

(i) χ_I *is a linear map from* $\mathbf{H}[I]$ *to* $\Bbbk[t]$.

(ii) *Let* $I = S \sqcup T$ *be a decomposition. For any* $x \in \mathbf{H}[S]$ *and* $y \in \mathbf{H}[T]$*,*

$$\chi_I(x \cdot y) = \chi_S(x)\chi_T(y)$$

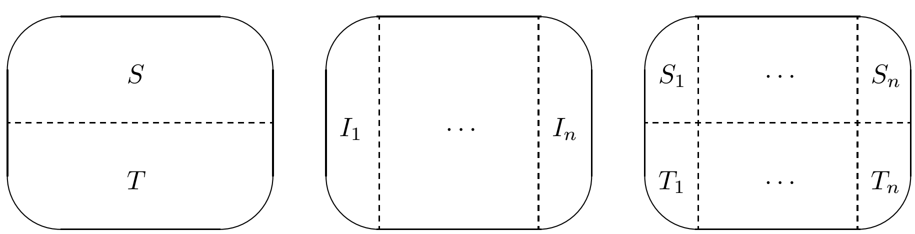
(product of polynomials).

(iii) *For any* $x \in \mathbf{H}[I]$ *and formal variables* s *and* t*,*

$$\chi_I(x)(s+t) = \sum_{I=S\sqcup T} \chi_S(x|_S)(s)\chi_T(x/_S)(t).$$

Proof. Property (i) follows from the linearity of Δ and ζ.

Property (ii) follows from the compatibility between μ and Δ and the multiplicativity of ζ. We provide the details. First, decompositions $I = I_1 \sqcup \cdots \sqcup I_n$ into n parts are in bijection with pairs of decompositions $S = S_1 \sqcup \cdots \sqcup S_n$ and $T = T_1 \sqcup \cdots \sqcup T_n$, where $S_i = I_i \cap S$ and $T_i = I_i \cap T$.



The compatibility between μ and Δ and the associativity of the latter imply that if we write

$$\Delta_{S_1,\ldots,S_n}(x) = \sum x_1 \otimes \cdots \otimes x_n \quad \text{and} \quad \Delta_{T_1,\ldots,T_n}(y) = \sum y_1 \otimes \cdots \otimes y_n,$$

then

$$\Delta_{I_1,\ldots,I_n}(x \cdot y) = \sum (x_1 \cdot y_1) \otimes \cdots \otimes (x_n \cdot y_n).$$

This, together with the multiplicativity of ζ, yield

$$\chi_I(x\cdot y)(n) = \sum_{I=I_1\sqcup\cdots\sqcup I_n} (\zeta_{I_1}\otimes\cdots\otimes\zeta_{I_n})\circ\Delta_{I_1,\dots,I_n}(x\cdot y)$$

$$= \sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n\\ T=T_1\sqcup\cdots\sqcup T_m}} \sum \zeta_{I_1}(x_1\cdot y_1)\cdots\zeta_{I_n}(x_n\cdot y_n) = \sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n\\ T=T_1\sqcup\cdots\sqcup T_m}} \sum \zeta_{S_1}(x_1)\zeta_{T_1}(y_1)\cdots\zeta_{S_n}(x_n)\zeta_{T_n}(y_n)$$

$$= \Big(\sum_{S=S_1\sqcup\cdots\sqcup S_n} (\zeta_{S_1}\otimes\cdots\otimes\zeta_{S_n})\circ\Delta_{S_1,\dots,S_n}(x)\Big)\Big(\sum_{T=T_1\sqcup\cdots\sqcup T_n} (\zeta_{T_1}\otimes\cdots\otimes\zeta_{T_n})\circ\Delta_{T_1,\dots,T_n}(y)\Big)$$

$$= \chi_S(x)(n)\,\chi_T(y)(n).$$

Thus, $\chi_I(x\cdot y) = \chi_S(x)\chi_T(y)$ as needed.

For property (iii), note that decompositions of I into $n+m$ parts are in bijection with tuples

$$(S, S_1, \dots, S_n, T, T_1, \dots, T_m)$$

where $I = S\sqcup T$, $S = S_1\sqcup\cdots\sqcup S_n$, and $T = T_1\sqcup\cdots\sqcup T_m$. In addition, associativity of Δ implies that

$$\Delta_{S_1,\dots,S_n,T_1,\dots,T_m} = \big(\Delta_{S_1,\dots,S_n}\otimes\Delta_{T_1,\dots,T_m}\big)\circ\Delta_{S,T}.$$

Therefore,

$$\chi_I(x)(n+m) = \sum_{I=S_1\sqcup\cdots\sqcup S_n\sqcup T_1\sqcup\cdots\sqcup T_m} (\zeta_{S_1}\otimes\cdots\otimes\zeta_{S_n}\otimes\zeta_{T_1}\otimes\cdots\otimes\zeta_{T_m})\circ\Delta_{S_1,\dots,S_n,T_1,\dots,T_m}(x)$$

$$= \sum_{I=S\sqcup T}\sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n\\ T=T_1\sqcup\cdots\sqcup T_m}} (\zeta_{S_1}\otimes\cdots\otimes\zeta_{S_n}\otimes\zeta_{T_1}\otimes\cdots\otimes\zeta_{T_m})\circ\big(\Delta_{S_1,\dots,S_n}\otimes\Delta_{T_1,\dots,T_m}\big)\circ\Delta_{S,T}(x)$$

$$= \sum_{I=S\sqcup T}\sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n\\ T=T_1\sqcup\cdots\sqcup T_m}} \Big((\zeta_{S_1}\otimes\cdots\otimes\zeta_{S_n})\circ\Delta_{S_1,\dots,S_n}(x|_S)\Big)\Big((\zeta_{T_1}\otimes\cdots\otimes\zeta_{T_m})\circ\Delta_{T_1,\dots,T_m}(x/_S)\Big)$$

$$= \sum_{I=S\sqcup T} \chi_S(x|_S)(n)\,\chi_T(x/_S)(m).$$

The above yields the desired result when $s=n$ and $t=m$ are nonnegative integers. This suffices since both sides of the equation are polynomial functions of (n,m), in view of Proposition 39. □

The following result states that if two characters are related by a morphism of Hopf monoids, then the same relation holds for the corresponding polynomial invariants.

Proposition 42. *Let* $\mathbf{H}$ *and* $\mathbf{K}$ *be two Hopf monoids. Suppose* $\zeta^{\mathbf{H}}$ *is a character on* $\mathbf{H}$*,* $\zeta^{\mathbf{K}}$ *is a character on* $\mathbf{K}$*, and* $f:\mathbf{H}\to\mathbf{K}$ *is a morphism of Hopf monoids such that*

$$\zeta^{\mathbf{K}}_I\big(f_I(x)\big) = \zeta^{\mathbf{H}}_I(x)$$

for every I *and* $x\in\mathbf{H}[I]$*. Let* $\chi^{\mathbf{H}}$ *and* $\chi^{\mathbf{K}}$ *be the polynomial invariants corresponding to* $\zeta^{\mathbf{H}}$ *and* $\zeta^{\mathbf{K}}$*, respectively. Then*

$$\chi^{\mathbf{K}}_I\big(f_I(x)\big) = \chi^{\mathbf{H}}_I(x)$$

for every I *and* $x\in\mathbf{H}[I]$*.*

Proof. Since f preserves coproducts, we have $\Delta_{S,T}\big(f_I(x)\big) = (f_S\otimes f_T)\big(\Delta_{S,T}(x)\big)$ and a similar fact for iterated coproducts. This and the hypothesis give the result. □

Remark. Most of the results in this section hold under weaker hypotheses (different ones for each result). For instance, Proposition 39 holds for any collection of linear maps $\zeta_I:\mathbf{H}[I]\to\Bbbk$ which is unital (with the same proof). In n and m are nonnegative integers, statement (iii) in Proposition 41 holds for any collection of linear maps $\zeta_I:\mathbf{H}[I]\to\Bbbk$. Proposition 42 holds for any morphism of comonoids which preserves the characters.

4.3. **Reciprocity.** For a character ζ on a Hopf monoid $\mathbf{H}$, the construction of Section 4.2 produces a polynomial invariant χ whose values at nonnegative integers are understood (in terms of the character and the Hopf monoid structure). In particular, as already stated (27),

$$\chi_I(x)(1) = \zeta_I(x)$$

for any structure $x \in \mathbf{H}[I]$. What about the values on negative integers? A connection with the antipode occurs at this point.

Proposition 43. *Let $\mathbf{H}$, ζ and χ be as in Section 4.2. Let s be the antipode of $\mathbf{H}$. Then*

$$\chi_I(x)(-1) = \zeta_I\big(\mathrm{s}_I(x)\big). \tag{28}$$

More generally, for every scalar n,

$$\chi_I(x)(-n) = \chi_I\big(\mathrm{s}_I(x)\big)(n). \tag{29}$$

Proof. Since $\binom{-1}{k} = (-1)^k$, Proposition 39 implies

$$\chi_I(x)(-1) \ = \sum_{k=0}^{|I|}\left(\sum_{(T_1,\dots T_k)\vDash I} (\zeta_{T_1}\otimes\cdots\otimes\zeta_{T_k})\circ\Delta_{T_1,\dots,T_k}(x)\right)(-1)^k.$$

Using multiplicativity of ζ and Takeuchi's formula (23), this may be rewritten as

$$\begin{aligned}
\chi_I(x)(-1) \ &= \sum_{k=0}^{|I|}\left(\sum_{(T_1,\dots T_k)\vDash I} \zeta_I\circ(\mu_{T_1}\otimes\cdots\otimes\mu_{T_k})\circ\Delta_{T_1,\dots,T_k}(x)\right)(-1)^k \\
&= \zeta_I\left(\sum_{k\geq 0}(-1)^k \sum_{(T_1,\dots T_k)\vDash I} \mu_{T_1,\dots,T_k}\circ\Delta_{T_1,\dots,T_k}(x)\right) \\
&= \zeta_I\big(\mathrm{s}_I(x)\big),
\end{aligned}$$

which proves (28).

To prove (29) one may assume that the scalar n is a nonnegative integer, since both sides are polynomial functions of n. We make this assumption and proceed by induction on $n \in \mathbb{N}$.

When $n = 0$ the result holds in view of (26), and when $n = 1$ in view of (27) and (28). For $n \geq 2$ we apply formula (iii) in Proposition 41 as follows.

$$\chi_I(x)(-n) = \chi_I(x)(-n+1-1) = \sum_{I=S\sqcup T} \chi_S(x|_S)(-n+1)\,\chi_T(x/_S)(-1).$$

By induction hypothesis, this equals

$$\sum_{I=S\sqcup T} \chi_S\big(\mathrm{s}_S(x|_S)\big)(n-1)\,\chi_T\big(\mathrm{s}_T(x/_S)\big)(1).$$

Changing variables $(S,T)\mapsto(T,S)$ and using commutativity of $\Bbbk$, this may be rewritten as

$$\sum_{I=S\sqcup T} \chi_S\big(\mathrm{s}_S(x/_T)\big)(1)\,\chi_T\big(\mathrm{s}_T(x|_T)\big)(n-1).$$

In view of formula (iii) in Proposition 41, applied this time to $\mathrm{s}_I(x)$, and (19), this equals

$$\chi_I\big(\mathrm{s}_I(x)\big)(1+n-1) = \chi_I\big(\mathrm{s}_I(x)\big)(n),$$

as needed. ∎

Formula (28) is a *reciprocity* result of a very general nature. It explains why one should be interested in an explicit antipode formula: such information allows for knowledge of the values of *all* polynomial invariants at negative integers. While the invariant depends on the chosen character, the antipode only depends on the Hopf monoid structure. The antipode acts as a universal link between the values of the invariants at positive and negative integers.

Remark. It can be shown that for any character χ on a connected Hopf monoid $\mathbf{H}$,

$$\chi_I(x) = \chi_I\big(\mathrm{s}_I^2(x)\big)$$

for all $x \in \mathbf{H}[I]$. If $\mathbf{H}$ is either commutative or cocommutative, then in fact $\mathrm{s}^2 = \mathrm{id}$. The general case follows since χ factors through the abelianization of $\mathbf{H}$ ($\mathbf{E}$ being commutative).

5. Generalized permutahedra and other examples

graphs, matroids, building sets

6. The braid arrangement

Antipode for Σ via combinatorics.
Braid arrangement and antipode in terms of Euler chars

7. Rings in the category of species

Appendix A. Lagrange's theorem for Hopf monoids

The analogy between groups and Hopf monoids extends to include a counterpart to Lagrange's classical theorem. This states that for any subgroup K of a group H, $H \cong K \times Q$ as (left) K-sets, where $Q = H/K$. In particular, if H is finite, then $|K|$ divides $|H|$. There exists an analogous result for finite-dimensional Hopf algebras, the Nichols-Zoeller theorem [12, Theorem 3.1.5], and one for connected Hopf monoids [2, Theorem 7]. The latter states that if $\mathbf{K}$ is a Hopf submonoid of a connected Hopf monoid $\mathbf{H}$, then

$$\mathbf{H} \cong \mathbf{K} \cdot \mathbf{Q},$$

where $\mathbf{Q}$ is a certain species canonically associated to $\mathbf{H}$ and $\mathbf{K}$. The proof of this result is beyond the scope of these notes. An interesting consequence is that the quotient power series

$$\mathbf{H}(x)/\mathbf{K}(x)$$

has nonnegative integer coefficients [2, Corollary 13]. This the quotient between the exponential generating functions of $\mathbf{H}$ and $\mathbf{K}$ (Section 1.2), which a priori is only guaranteed to have integer coefficients.

Consider the surjective morphism of Hopf monoids $\mathbf{L} \to \mathbf{E}$ (see Example 3). The dual version of Lagrange's theorem affords a species $\mathbf{Q}$ such that $\mathbf{L} \cong \mathbf{E} \cdot \mathbf{Q}$. It follows that

$$\mathbf{Q}(x) = \frac{\mathbf{L}(x)}{\mathbf{E}(x)} = \frac{e^{-x}}{1-x},$$

so $\dim \mathbf{Q}[n]$ is the number of derangements of $[n]$. In [2, Section 5.3] we construct a linear basis of $\mathbf{Q}$ indexed by derangements. ($\mathbf{Q}$ is not the linearization of the set species of derangements.)

Now consider the Hopf monoid $\boldsymbol{\Sigma}$, the Hopf submonoid $\mathbf{L}$, and the species $\mathbf{Q}$ afforded by Lagrange's theorem. It follows that

$$\mathbf{Q}(x) = \frac{\boldsymbol{\Sigma}(x)}{\mathbf{L}(x)} = \frac{1-x}{2-\exp(x)}.$$

It is known from [17, Exercise 5.4.(a)] that

$$\frac{1-x}{2-\exp(x)} = \sum_{n\geq 0} \frac{s_n}{n!}\, x^n$$

where s_n is the number of *threshold* graphs with vertex set $[n]$ and no isolated vertices.

Problem. *Construct a linear basis for* $\mathbf{Q}$ *indexed by threshold graphs.*

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