Enumeration of tableaux and plane partitions

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1 Lecture 1

1.1 Integer partitions

Partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $|\lambda| = \sum_i \lambda_i n$, we write it as $\lambda \vdash n$. By convention we let $\lambda_0 = \infty$ and there exists k such that $\lambda_i = 0$ for $i > k$, the number of distinct parts $\#\{\lambda_j > \lambda_{j+1}\}$.

Example 1.1. *Partitions of* 5 *are* $5, 41, 32, 311, 221, 2111, 11111$.

Let $p(n)$ be the number of partitions of n.

Proposition 1.2. $\sum_{n\geq 0} p(n)q^n = \prod_{i\geq 1} \frac{1}{1-q^i}$.

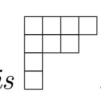
Proof. Note that $\frac{1}{1-q^i} = q + q^i + q^{2i} + q^{3i} + \cdots$. ☐

Let $p_S(n)$ be the number of partitions of n where the parts belong to a multiset S, then

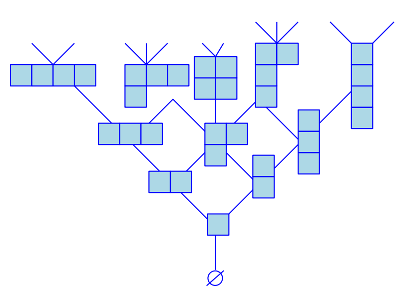
$$\sum_{n\geq 0} p_S(n)q^n = \prod_{i\in S} \frac{1}{1-q^i}.$$

1.2 Young diagrams

A Young diagram is an arrangement of boxes left justified such that there are λ_i squares/cell in the ith row.

Example 1.3. *For the partition* $\lambda = (4, 3, 1, 1)$ *the Young diagram is*  .

The partitions form a poset called *Young's lattice* consisting of partitions ordered by containment of their Young diagrams: $\lambda \lhd \mu$ if the diagram of λ is contained in the diagram of μ and $|\lambda| = |\mu| - 1$.



Our goal is to enumerate walks in Young's lattice that start at $\varnothing$ arrive at λ and use up (U) or down steps (D).

Fix a word $w \in \{U, D\}^*$ $w = A_n A_{n-1} \cdots A_2 A_1$ where A_i is U or D.

Example 1.4. $\varnothing, (1), (2), (1), (1,1), (1,1,1), (2,1,1), (1,1,1) = \lambda$ *where the steps are* U, U, D, U, U, U, D *and so* $w = DUUDUU$.

Example 1.5 (Walks of type U^n)**.**

$$\varnothing, \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

Definition 1.6 (Standard Young tableaux)**.** Let $\lambda \vdash n$, a Standard Young tableau (SYT) of shape λ is a filling of the cells of the Young diagram of λ with all the elements $\{1, 2, \ldots, n\}$ that is increases in rows and columns. The number of SYT of shape λ is denoted by f^λ. □

The number f^λ has a beautiful product formula proved in the 50s.

Theorem 1.7.

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where $h(u)$ is the length of the hook *of u: the number of cells under in the same column and to the right in the same row of u (including u).*

Example 1.8. *For the shape* $(2,2,1)$*, the hook-lengths are given by* $\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$.

$$f^{(2,2,1)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

There are several proofs of this formula including a beautiful probabilistic proof by Greene-Nijenhuis-Wilf.

Lemma 1.9. $w = D^{S_k} U^{r_k} \ldots D^{S_1} U^{r_1}$. *and $\lambda \vdash n$. There exists at least one walk from $\varnothing$ to λ of type W if*

- $\sum_{i=1}^k r_i - \sum_{i=1}^k s_i = n,$
- *for all j* $\sum_{i=1}^j (r_i - s_i) \geq 0$

We call such a word is λ-valid.

Remark 1.10. The number of valid words of length $2n$ starting at $\varnothing$ to $\varnothing$ is given by the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$. □

1.3 Linear transformations on integer partitions

We will use physics notation: $\langle \lambda |$ ("bra") and $| \lambda \rangle$ ("ket") so that

$$\langle \lambda | \mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

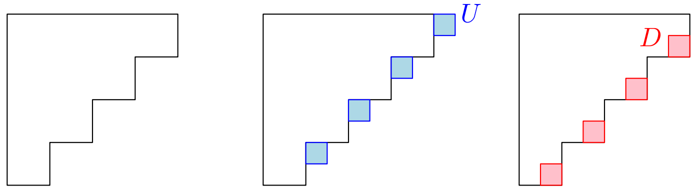
$$U|\lambda\rangle = \sum_{\mu \rhd \lambda} |\mu\rangle,$$
$$\langle\lambda| U = \sum_{\mu \rhd \lambda} \langle\mu|,$$
$$D|\lambda\rangle = \sum_{\mu \lhd \lambda} |\mu\rangle,$$
$$\langle\lambda| D = \sum_{\mu \lhd \lambda} \langle\mu|,$$

Example 1.11.

$$U|(3,2)\rangle = |(3,2,1)\rangle + |(3,3)\rangle + |(4,2)\rangle,$$
$$\langle(3,2)| D = \langle(3,2,1)| + \langle(3,3)| + \langle(4,2)|.$$

Proposition 1.12. *If λ has r distinct parts then $U|\lambda\rangle$ has $r+1$ terms and $D|\lambda\rangle$ has $r-1$ parts.*

Proof. The number of distinct parts of λ is the number of corners of λ. The operator U will add a corner to λ and the operator D will delete a corner to λ.



□

The key lemma to study our goal is the following identity.

Lemma 1.13.
$$DU = UD + I,$$
where I is the identity operator.

Proof.
$$DU|\lambda\rangle = \sum_{\mu} |\mu\rangle + (r+1)|\lambda\rangle$$
where the first sum is over μ such that there exist j,k such that $\lambda_j = \mu_j - 1$ and $\lambda_k = \mu_k + 1$.
$$UD|\lambda\rangle = \sum_{\mu} |\mu\rangle + r|\lambda\rangle.$$
where the first sum is over μ such that there exist j,k such that $\lambda_j = \mu_j - 1$ and $\lambda_k = \mu_k + 1$.

When we take the difference $(DU - UD)|\lambda\rangle$ we obtain $|\lambda\rangle$. □

Theorem 1.14. *Let λ be a partition, w be a λ-valid word. The number of walks from $\varnothing$ to λ of type w is $f^\lambda \cdot \prod_{i \in S_w}(a_i - b_i)$, where $a_i = \#\{j < i \mid A_j = U\}$, $b_i = \#\{j < i \mid A_j = D\}$ and $S_w = \{i \mid A_i = D\}$.*

Note that the second term in the product is independent of λ.

Example 1.15. *$W = U^3D^2U^2DUT3$, $S_w = \{4,7,8\}$ (positions from right to left), $a_4 = 3, a_7 = 5, a_8 = 5, b_4 = 0, b_7 = 1, b_8 = 2$. Then the number of walks from $\varnothing$ to any $\lambda \vdash 5$ is $f^\lambda \cdot (3-0)(5-1)(5-2)$.*

Proof. We want to compute $\langle\lambda|\, w\,|\varnothing\rangle$. When w see a DU by Lemma 1.13 we replace it by UD or I repeatedly and we obtain

$$\langle\lambda|\, W\,|\varnothing\rangle = \sum_{i-j=n} r_{i,j}(w)\,\langle\lambda|\, U^iD^j\,|\varnothing\rangle = \sum_{i-j=n} r_{n,0}(\varnothing)\,\langle\lambda|\, U^n\,|\varnothing\rangle\,,$$

donde $\langle\lambda|\, U^n\,|\varnothing\rangle = f^\lambda$. One can show that the coefficients $r_{i,j}(w)$ satisfy

$$\begin{aligned} r_{i,j}(w) &= 0; \quad \text{if } i<0 \text{ or } j<0 \text{ or } i-j\neq n \\ r_{0,0}(\emptyset) &= 1 \\ r_{i,j}(Uw) &= r_{i-1,j}(w); \\ r_{i,j}(Dw) &= r_{i,j-1}(w) + (i+1)r_{i+1,j}(w). \end{aligned}$$

This determines the $r_{i,j}(w)$ to be $\prod_{i\in S_w}(a_i - b_i)$. □

2 Lecture 2

Last time we looked at Young's lattice.

$$f^{(2,2,1)} = \frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5.$$

If we look at walks with just U steps from $\varnothing$ to λ we obtain a Standard Young tableaux (SYT). We can compute the number of these SYT is given by the hook-length formula **??**.

$$f^{(2,2,1)} =$$

We sketch how to prove such a formula.

2.1 Algorithm to compute random SYT

Claim 2.1.

$$f^\lambda = \sum_{\alpha \text{ corner}} f^{\lambda/\alpha},$$

where λ/α is the diagram of λ where you delete α.

Proof. In a SYT T of shape $\lambda \vdash n$, the entry n will be in a corner. □

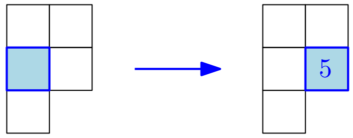
Example 2.2.

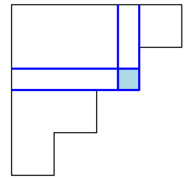
$$f^{(2,2,1)} = f^{(2,2)} + f^{(2,1,1)} = .$$

The algorithm is the following.

1. Pick a square u in the diagram of λ with probability $1/n$.

2. If u is not a corner, pick another cell in the hook of u with probability $\frac{1}{h(u)-1}$.
3. Do this until you reach a corner. In the corner put n.





Proposition 2.3. *The probability that the algorithm stops in a corner* $\alpha = (x, y)$ *is*

$$frac1n \prod_{i=1} \left(1 + \frac{1}{h(i,j)-1}\right) \prod_{j=1} \left(1 + \frac{1}{h(i,j)-1}\right).$$

With this proposition we can then prove the hooklength formula.

$$\frac{f^{\lambda/\alpha}}{f^{\lambda}} = \prod \frac{h(i,j)}{h(i,j)-1} \prod \frac{h(i,j)}{h(i,j)-1}$$

and

$$\sum_{\alpha \text{ corner}} \frac{f^{\lambda/\alpha}}{f^{\lambda}} = 1.$$

2.2 Walks

Because

$$\langle \lambda | \mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

then if w is a valid λ word then $\langle \lambda | \, w \, | \varnothing \rangle$ is the number of walks from $\varnothing$ to λ.

Example 2.4.

$$DUUDU = UUDD + 4UUD + 2U, \langle (1) | \, DUUDUD \, | \varnothing \rangle = 2 \langle (1) | \, U \, | \varnothing \rangle = 2.$$

$D \, | \varnothing \rangle = 0.$

$$W = \sum_{i-j=n} r_{i,j}(w) U^i D^j,$$

where $r_{i,j}(w)$ is a nonnegative integer. Since

$$UW = \sum_{i-j=n} r_{i,j}(w) U^{i+1} D^j,$$

then

$$r_{i,j}(UW) = r_{i-1,j}(W).$$

Since

$$DW = \sum_{i-j=n} r_{i,j}(w) DU^i D^j,$$

From Exercise **??** $DU^i = U^i D + iU^{i-1}$ then

$$r_{i,j}(DW) = r_{i,j-1}(w) + (i+1) r_{i+1,j}(w).$$

When $j = 0$ the first term disappears. This is why we get a product formula for

Example 2.5.

$$\begin{aligned} r_{2,0}(DUUDU) &= 2r_{2,0}(UUDU), \\ &= 2 \end{aligned}$$

Example 2.6. *Let* $w = D^nU^n = A_{2n}A_{2n-1}\cdots A_1$, $\langle\varnothing|\, D^nU^n\, |\varnothing\rangle$,
$S_w = \{n+1, n+2, \ldots, 2n\}$, $a_i = n$ *for* $i \in S_w$ *and* $b_i = i - n - 1$
By Theorem 1.14,

$$\langle\varnothing|\, W\, |\varnothing\rangle = \prod_{i \in S_w}(a_i - b_i) = n!.$$

Thus there is a bijection from these walks to permutations.

Remark 2.7. $\langle\lambda|\,(D+U)^n\,|\varnothing\rangle$ is the number of walks of length n that start at $\varnothing$ and end at λ. These objects are called *oscillating tableaux.* And for $\lambda = \varnothing$,

$$\langle\varnothing|\,(D+U)^n\,|\varnothing\rangle = (2n-1)!!.$$

□

Remark 2.8. The above product formula work because we start with $\varnothing$. For example $\langle\lambda|\, w\, |\mu\rangle$ for $w = u^n$, we get skew standard Young tableaux and there are no product formula for these. □

2.3 Bijection

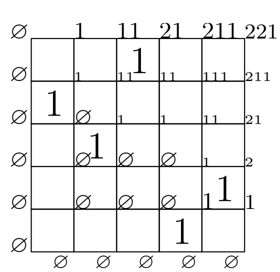
We will give bijections among three objects called the Robinson-Schensted correspondence:

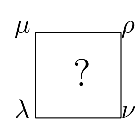
- permutations of $\{1, 2, \ldots, n\}$,
- walks of type D^nU^n that start at $\varnothing$ and end in $\varnothing$,
- paris of SYT with n cells of the same shape.

$$\varnothing, \to \ytableaushort{1} \to \ytableaushort{12} \to \ytableaushort{12,3}\,, \quad \ytableaushort{12,34}\,, \quad \ytableaushort{12,34,5}\,, \quad \ytableaushort{13,24,5}.$$

We will use the local rule. We start with a permutation matrix.

Example 2.9. *For* $\sigma = (4, 3, 5, 1, 2)$
Given λ, μ, ν *we obtain* ρ *where* $\rho_1 = \max(\mu_1, \nu_1) + ?$ *y para* $i > 1$ $\rho_i = \max(\mu_i, \nu_i) + \min(\mu_{i-1}, \nu_{i-1}) - \lambda_i$.





If we look at the partitions in the top horizontal boundary and the right vertical boundary of the matrix we obtain two walks from $\varnothing$ to some λ with only U steps. These are the pair of SYT of the same shape. In this example.

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

There is another way to obtain these pair of tableaux using *row insertion*

Claim 2.10. *If you fix μ and ν this is a bijection between $(\lambda, ?)$ and ρ where*

- $\lambda \leq \mu$, $\lambda \leq \nu$,
- $\rho \geq \mu, \rho \geq \nu$ *and*
- $|\rho| = |\mu| + |\nu| - ? - |\lambda|$.

Also if λ labels any point p, the weight of $|\lambda|$ of λ is the number of 1s in the quarter plane left and below p.

Corollary 2.11. *If $\sigma \mapsto (P, Q)$ then $\sigma^{-1} \mapsto (Q, P)$.*

Proof. With the local rule, the construction for σ^{-1} corresponds to the tranpose of the construction, so P and Q are switched. $\square$

Corollary 2.12.

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

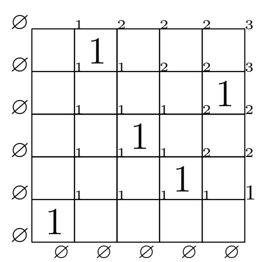
3 Lecture 3

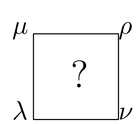
3.1 Increasing subsequences of permutations

We simplify the local rule to be $\rho_1 = \max(\mu_1, \nu_1) + ?$.

Definition 3.1. Given an increasing subsequence of $\sigma = (\sigma_1, \ldots, \sigma_n)$, an increasing subsequence is a subsequence satisfying $\sigma_{i_1} < \sigma_{i_2} < \ldots < \sigma_{i_k}$ such that $i_1 < i_2 < \ldots < i_k$. $\square$

Example 3.2. $\sigma = (1, 5, 3, 2, 4)$. *Some examples of increasing subsequences are* $134, 124, 34$.





$\rho_1 = \max(\mu_1, \nu_1) + ?$

The local rule we defined also works for other matrices of nonnegative integers not just permutation matrices. To study this we will need Schur polynomials.

3.2 Schur polynomials

Let λ be a partition of n. A semi standard Young tableau of shape λ is a filling of the Ferrers diagram with integers that is increasing in rows and strictly increasing in columns. If T is a SSYT then $x^T = x_1^{\#_1(T)} x_2^{\#_2(T)} \cdots$.

Example 3.3. *For the shape $\lambda = (6,5,3)$, the semistandard tableau* $\begin{ytableau} 1 & 1 & 1 & 3 & 4 & 4 \\ 2 & 4 & 4 & 5 & 5 \\ 6 & 9 & 9 \end{ytableau}$ *has weight* $x^T = x_1^3 x_2 x_3 x_4^4 x_5^2 x_6 x_9^2$.

The Schur polynomial is defined as

$$s_\lambda(x_1, \ldots, x_m) = \sum_{T \in SSYT(\lambda)} x^T,$$

where the sum is over SSYT T of shape λ.

Example 3.4.

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + \cdots$$

the SSYT are

$$\begin{ytableau} 2 & 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}, \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau}.$$

Theorem 3.5. *The Schur polynomial is symmetric.*

Proof. The proof uses the Bender-Knuth involution. □

Example 3.6.

$$\begin{ytableau} 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau}$$

3.3 Gelfand-Tsetlin triangles

We will use Gelfand-Tsetlin triangles when we think of SSYT.

A Gelfand-Tsetlin triangle (GTT) is a center justified array of nonnegative integers such that the ith row has i entries, each row is a partition $a_{ij} \geq a_{i,j+1}$, the rows interlace $a_{i,j} \geq a_{i+1j,j+1} \leq a_{i,j+1}$.

$$\begin{array}{lccccccccc}
\text{row } m & a_{11} & & a_{12} & & a_{13} & & \cdots & & a_{1n} \\
 & & a_{22} & & a_{22} & & \cdots & & a_{2n} & \\
\vdots & & & a_{33} & & \cdots & & a_{3n} & & \\
 & & & & \ddots & & ⋰ & & & \\
\text{row } 2 & & & & & & & & & \\
\text{row } 1 & & & & & a_{nn} & & & &
\end{array}
\qquad
\begin{array}{ccc}
a_{ij} & & a_{i,j+1} \\
 & \searrow \quad \nearrow & \\
 & a_{i+1,j+1} &
\end{array}$$

These triangles are in bijection with SSYT. Given a SSYT T the map from the ith row of the corresponding GTT are the shape of the entries less than or equal to i in the SSYT.

Example 3.7. *For $m = 5$.*

$$\begin{array}{ccccccccc}
6 & & 3 & & 2 & & 2 & & 0 \\
 & 4 & & 2 & & 2 & & 0 & \\
 & & 4 & & 2 & & 1 & & \\
 & & & 3 & & 1 & & & \\
 & & & & 3 & & & &
\end{array}
\quad \leftrightarrow \quad
\begin{ytableau} 1 & 1 & 1 & 3 & 5 & 5 \\ 2 & 3 & 5 \\ 3 & 4 \\ 5 & 5 \end{ytableau}.$$

Question 3.8. *What does the Bender-Knuth involution on SSYT translate to in GTT.*

3.4 Interlacing partitions

Let λ and μ be partitions, we say $\lambda \succeq \mu$ (λ interlaces μ) if and only if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq m_2 \geq \lambda_3 \geq \cdots$.

Example 3.9. *$\lambda = (4, 2, 2)$ and $\mu = (3, 2, 1)$ interlace since $4 \geq 3 \geq 22 \geq 2 \geq 1$.*

Note that the rows of the GTT interlace.

In Lecture 1 we defined $U|\lambda\rangle = \sum_{\mu \triangleright \lambda} \mu$, where we only add one box. Now we allow adding more boxes as long as the resulting partitions interlace. Equivalenly this means that λ/μ is a *horizontal strip*, i.e. no to boxes are on the same column.

Definition 3.10.

$$\Gamma_+ |\lambda\rangle = \sum_{\mu \succ \lambda} t^{|\mu|-|\lambda|} |\mu\rangle .$$
$$\langle\lambda| \Gamma_+(t) = \sum_{\mu \prec \lambda} t^{|\lambda|-|\mu|} \langle\mu| ,$$
$$\Gamma_- |\lambda\rangle = \sum_{\mu \prec \lambda} t^{|\mu|-|\lambda|} |\mu\rangle ,$$
$$\langle\lambda| \Gamma_-(t) = \sum_{\mu \succ \lambda} t^{|\mu|-|\lambda|} \langle\mu| .$$

□

Example 3.11.

$$\Gamma_+(t) |(3, 2, 1)\rangle = t^3 |(4, 2, 2)\rangle + \cdots$$

Proposition 3.12.

$$\begin{aligned}
&\langle\varnothing| \Gamma_+(t) = \langle\varnothing| , \\
&\Gamma_-(t) \langle\varnothing| = \langle\varnothing| , \\
&\langle\varnothing| \Gamma_-(t) = \langle\varnothing| + \langle(1)| t + \langle(2)| t^2 + \cdots
\end{aligned}$$

To a GTT we associate a weight as we illustrate with the following example.

Example 3.13. *The GTT in Example 3.7 the weight of the GTT is $x_1^3 x_2^{4-1} x_3^{7-4} x_4^{8-7} x_5^{13-8}$.*

The following result expresses the Schur function in terms of the operators $\Gamma_\pm(t)$.

Theorem 3.14.

$$\begin{aligned}
s_\lambda(x_1, \ldots, x_m) &= \langle\lambda| \Gamma_+(x_m) \cdots \Gamma_+(x_2)\Gamma_+(x_1) |,\rangle \varnothing, \\
&= \langle\varnothing| \Gamma_-(x_m) \cdots \Gamma_-(x_2)\Gamma_-(x_1) |\lambda\rangle .
\end{aligned}$$

Proof. □

Our new goal is given $w = w - 1 \cdots w_m$ a word in the letters $\{+, -\}$, we want to compute

$$\langle\lambda| \Gamma_{w_1}(x_1)\Gamma_{w_2}(x_2) \cdots \Gamma_{w_m}(x_m) |\varnothing\rangle .$$

Proposition 3.15.

$$\Gamma_-(u)\Gamma_-(v) = \Gamma_-(u)\Gamma_-(u),$$
$$\Gamma_+(u)\Gamma_+(v) = \Gamma_+(v)\Gamma_+(u),$$
$$\Gamma_-(u)\Gamma_+(v) = \frac{1}{1-uv}\Gamma_+(v)\Gamma_-(u).$$

Proof. The proof uses the local rule. Fix μ and ν.

$$\begin{aligned} \langle\mu|\,\Gamma_-(u)\Gamma_+(v)\,|\nu\rangle &= \sum_\rho \langle\mu|\,\Gamma_-(u)\,|\rho\rangle\,\langle\rho|\,\Gamma_+(v)\,|\nu\rangle\,, \\ &= \sum_{\rho\succ\mu,\rho\succ\nu} u^{|\rho|-|\mu|}v^{|\rho|-|\nu|}. \\ \langle\mu|\,\Gamma_+(v)\Gamma_-(u)\,|\nu\rangle &= \sum v^{|\mu|-|\lambda|}u^{|\nu|-|\lambda|}\sum_{k=0}^{\infty}(uv)^k. \end{aligned}$$

We multiply both equations by $u^?v^?$ and get

$$\sum_{\rho\succ\nu,\rho\succ\mu}(uv)^{|\rho|} = \sum_{\lambda\succ\nu,\lambda\succ\mu}(uv)^{|\mu|+|\nu|-|\lambda|+k}.$$

□

Proving the commutation of the operators is done with the following proposition.

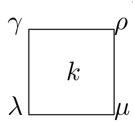
Proposition 3.16. *Fix μ and ν. There exists a bijection between partitions ρ such that (i) $\rho\succ\mu$ and $\rho\succ\nu$ and (ii) (λ,k) are such that $\lambda\prec\mu$ and $\lambda\prec\nu$ and $k\geq 0$. This is such that $|\rho| = |\mu|+|\nu|-|\lambda|+k$.*

4 Lecture 4

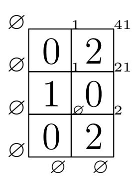
Local rule $\lambda\succ\mu$, $\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\cdots$.

If ρ is defined by $\rho_1=\max(\mu_1,\nu_1)+k$ and for $i>1$, $\rho_i=\max(\mu_i,\nu_i)+\min(\mu_{i-1},\gamma_{i-1})+\lambda_{i-1}$.

To see why ρ interlaces it suffices to note that the first minus the last term are ≥ 0 and second minus the third term are ≥ 0.

Example 4.1. 

4.1 RSK

Example 4.2. 

$\rho_1 = \max(\mu_1,\nu_1)+?$

$$P = \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2\\\cline{1-1}\end{array},\; Q = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2\\\cline{1-1}\end{array},\quad \begin{array}{ccccc} 4&&1&&0\\ &2&&1&\\ &&2&& \end{array},\quad \begin{array}{ccccc} 4&&1&&0\\ &1&&0&\\ &&0&& \end{array}$$

This gives a bijection between a matrix $A=(A_{ij})$ and pairs of SSYT of same shape such that P contains $\sum_j A_{i,j}$ entries equal to i, Q contains $\sum_i A_{i,j}$ entries equal to j.

4.2 Back to operators

We defined operators $\Gamma_{\pm}(u)$ satisfying

Theorem 4.3 (Cauchy identity)**.**

$$\sum_{\lambda} s_\lambda(x_1,\ldots,x_m)s_\lambda(y_1,\ldots,y_m) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-x_iy_j}.$$

Proof.

$$s_\lambda(x_1,\ldots,x_n) = \langle\varnothing|\,\Gamma_-(x_1)\cdots\Gamma_-(x_n)\,|\lambda\rangle\,,$$
$$s_\lambda(y_1,\ldots,y_m) = \langle\varnothing|\,\Gamma_-(y_1)\cdots\Gamma_-(y_m)\,|\lambda\rangle\,,$$

Let A_λ and B_λ denote the RHS of these equations. Then

$$\sum_{\lambda} s_\lambda(x_1,\ldots,x_m)s_\lambda(y_1,\ldots,y_m) = \sum_{\lambda} A_\lambda B_\lambda = \langle\varnothing|\,\Gamma_-(x_1)\cdots\Gamma_-(x_n)\Gamma_-(y_1)\cdots\Gamma_-(y_m)\,|\varnothing\rangle\,.$$

□

Example 4.4.

We want the Cauchy identity for the following. Let $p(n)$ be the number of partitions of n

$$\sum_{n} p(n)q^n = \prod_{i\geq 1}\frac{1}{1-q^i}.$$

we will use the Cauchy identity to show that

$$\sum_{n\geq 0} pp(n)q^n = \prod_{i\geq 1}\frac{1}{(1-q^i)^i}.$$

where $pp(n)$ is the number of *plane partitions* of n.

Given a partition $\lambda = (\lambda_1,\lambda_2,\ldots)$, a plane partition $\Pi = (\pi_{i,j})$ is an array of non negative integers such that $\pi_{i,j}\geq\pi_{i+1,j}$ and $\pi_{i,j}\geq\pi_{i,j+1}$. Its weight $|\Pi|$ is $\sum_{i,j}\pi_{i,j}$.

From now on assume $1\leq i,j\leq m$.

Example 4.5. *For* $m=3$, $\Pi = \begin{array}{|c|c|c|}\hline 3&2&2\\\hline 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$, $|\Pi| = 12$.

Proposition 4.6. *Let $\mathcal{P}_m$ be the set of plane partitions $\Pi = (\pi_{i,j})$ with $1\leq i,j\leq m$ then*

$$\sum_{\Pi\in\mathcal{P}_m} q^{|\Pi|} = \prod_{i=1}^{m}\prod_{j=1}^{m}\frac{1}{1-q^{i+j-1}}.$$

Corollary 4.7.

$$\lim_{m\to\infty}\sum_{\Pi\in\mathcal{P}_m} q^{|\Pi|} = \prod_{i=1}^{\infty}\frac{1}{(1-q^i)^i}.$$

proof of Proposition. We want to use the Cauchy identity and view $s_\lambda(x_1,\ldots,x_m)$ as the generating series for GTT with top row λ with m rows and $s_\lambda(y_1,\ldots,y_n)$ as the generating series for GTT with top row λ with n columns. □

Example 4.8.

$$\begin{array}{|c|c|c|c|}\hline 4&3&2&2\\ \hline 4&3&1&1\\ \hline 2&2&1\\ \cline{1-3} 1\\ \cline{1-1}\end{array} \leftrightarrow \begin{array}{ccccccc} &&&2&&& \\ &&2&&1&& \\ &3&&1&&0& \\ 4&&3&&1&&0 \\ &4&&2&&0& \\ &&2&&0&& \\ &&&1&&& \end{array},$$

Theorem 4.9. *There exists a bijection between plane partitions* Π *with main diagonal* λ *and pairs of* GTT (T_1, T_2) *with top row* λ *such that*

$$|\Pi| = |T_1| + |T_2| - |\lambda|.$$

With this bijection we finish the proof of the generating function since

$$\begin{aligned}\sum_{\Pi} q^{|\Pi|} &= \sum_{\lambda} \sum_{T_1 \text{ GTT}} q^{|T_1|} \sum_{T_1 \text{ GTT}} q^{|T_2|-|\lambda|} \\ &= \sum_{\lambda} s_\lambda(1, q, q^2, \ldots, q^m) s_\lambda(1, q, q^2, \ldots, q^{m-1}) \\ &= \prod_{i=1}^{m} \prod_{j=1}^{m} \frac{1}{1 - q^{i+j-1}}.\end{aligned}$$

4.3 Reverse plane partitions

fix a shape λ and fill the entries of the Young diagram with nonnegative integers such that the entries increase in rows and columns.

Example 4.10. *For* $\lambda = (3, 3, 2)$, $\begin{array}{|c|c|c|}\hline 0&0&2\\ \hline 0&3&3\\ \hline 1&3\\ \cline{1-2}\end{array}$. *For a rectangle* $\lambda = (3, 3)$, $\begin{array}{|c|c|c|}\hline 0&1&1\\ \hline 2&2&4\\ \hline\end{array}$ *by flipping this one we obtain a plane partition* $\begin{array}{|c|c|c|}\hline 4&2&2\\ \hline 1&1&0\\ \hline\end{array}$.

Let $\mathcal{R}_\lambda$ be the set of RPP of shape λ,

Theorem 4.11.

$$\sum_{\Pi \in \mathcal{R}_\lambda} q^{|\Pi|} = \prod_{u \in \lambda} \frac{1}{1 - q^{h(i,j)}}.$$

Example 4.12. *For the shape* $(3, 3, 2)$ *the hooks are* $\begin{array}{|c|c|c|}\hline 5&4&2\\ \hline 4&3&1\\ \hline 2&1\\ \cline{1-2}\end{array}$,

$$\sum_{\Pi \in \mathcal{R}_{(3,3,2)}} q^{|\Pi|} = \frac{1}{(1-q)^2(q-1^2)^2(1-q^3)(1-q^4)^2(1-q^5)}.$$

We will prove it using the vetex operators.

Example 4.13. $\varnothing \succ \beta^{(1)} \succ \beta^{(2)} \prec \beta^{(3)} \succ \beta^{(4)} \prec \cdots$, *with* $|\Pi| = \sum_i |\beta^{(i)}|$.

$$\sum_{\Pi\in\mathcal{R}_{(3,3,2)}} q^{|\Pi|} = \langle\varnothing|\,\Gamma_-(q^{-1})\Gamma_-(q^{-2})\Gamma_+(q^3)\Gamma_-(q^4)\Gamma_+(q^3)\Gamma_+(q^6)\,|\varnothing\rangle$$

$$= \frac{1}{(1-q)}\frac{1}{(1-q^2)^2}\frac{1}{(1-q)}\frac{1}{(1-q^3)}\frac{1}{(1-q^4)^2}\frac{1}{(1-q^5)}.$$

A similar proof works for all shapes. For example for shape $(4,2,1)$ *we get*

$$\langle\varnothing|\,\Gamma_-(q^{-1})\Gamma_+(q^2)\Gamma_-(q^{-3})\Gamma_+(q^4)\Gamma_-(q^{-5})\Gamma_-(q^{-6})\Gamma_+(q^7)\,|\varnothing\rangle\,.$$