Symmetric Functions and Eulerian Polynomials

Lecture 2: Symmetric and quasisymmetric functions

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Basic definitions

$\mathbb{Q}[[X]]$ denotes the ring of formal power series in the variables $X = \{x_1, x_2, \ldots, \}$.

$f(\mathbf{x}) \in \mathbb{Q}[[X]]$ is a symmetric function if for all $\sigma \in \cup_{n \geq 1} \mathfrak{S}_n$

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots).$$

A symmetric function is homogeneous of degree n if each term has degree n.
Example: $x_1x_2^2 + x_2x_1^2 + x_1x_3^2 + x_3x_1^2 + x_2x_3^2 + x_3x_2^2 + \ldots$ is a homogeneous symmetric function of degree 3.

Let Sym_n denote the vector space (over $\mathbb{Q}$) of homogeneous symmetric functions of degree n and let Sym denote the ring of symmetric functions of bounded degree.

Bases for Sym_n

We can view a partition $\lambda \vdash n$ as an infinite sequence by padding it with zeros. That is if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we can view λ as $(\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots)$.

Given an infinite sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive integers, let

$$\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

Monomial symmetric functions: For $\lambda \vdash n$, let

$$m_\lambda(\mathbf{x}) := \sum_\alpha \mathbf{x}^\alpha$$

where the sum ranges over distinct rearrangements α of λ viewed as an infinite sequence.

Example: $m_{2,1}(\mathbf{x}) := x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

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Elementary symmetric functions:

$$\begin{aligned} e_n(\mathbf{x}) &:= \sum_{i_1 < \cdots < i_n} x_{i_1} \ldots x_{i_n} \\ e_\lambda(\mathbf{x}) &:= e_{\lambda_1} \cdots e_{\lambda_k} \qquad \text{if } \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \end{aligned}$$

Basis for Sym_n

Complete homogeneous symmetric functions:

$$h_n(\mathbf{x}) := \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \ldots x_{i_n}$$

$$h_\lambda(\mathbf{x}) := h_{\lambda_1} \cdots h_{\lambda_k} \qquad \text{if } \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$$

Power-sum symmetric functions:

$$p_n(\mathbf{x}) := \sum_{i \geq 1} x_i^n$$

$$p_\lambda(\mathbf{x}) := p_{\lambda_1} \cdots p_{\lambda_k} \qquad \text{if } \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$$

Theorem

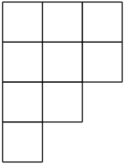
$\{m_\lambda : \lambda \vdash n\}, \{e_\lambda : \lambda \vdash n\}, \{h_\lambda : \lambda \vdash n\}, \{p_\lambda : \lambda \vdash n\}$ are all basis for Sym_n.

Thus the dimension of the vector space Sym_n equals the number of partitions of n.

Schur functions

Associate with each $\lambda \vdash n$, an array of cells with λ_i cells in row i for each i. This is called the Young diagram of shape λ.

Example: Young diagram of shape $(3, 3, 2, 1)$



A semistandard Young tableau of shape λ is a filling of the diagram λ with positive integers so that

- each row weakly increases from left to right
- each column strictly increases from top to bottom

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3	5	8
6	6	
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$$x^T = x_1 \, x_3^3 \, x_5 \, x_6^2 \, x_7 \, x_8$$

A semistandard Young tableau of shape λ is a filling of the diagram λ with positive integers so that

- each row weakly increases from left to right
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Let SST_λ be the set of semistandard Young tableaux of shape λ. For each $T \in SST_\lambda$, let $x^T = x_1^{a_1} x_2^{a_2} \cdots$, where a_i is the number of occurances of i in T.

The Schur function of shape $\lambda \vdash n$ is

$$s_\lambda(\mathbf{x}) := \sum_{T \in SST_\lambda} x^T$$

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Example: The semistandard Young tableaux of shape $\lambda = (2,1)$ with entries at most 3 are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} s_{2,1} &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \dots \\ &= m_{2,1} + 2m_{1,1,1} \end{aligned}$$

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A semistandard tableau T has type $\alpha = (\alpha_1, \alpha_2, \dots)$ if T has α_i entries equal to i for each $i \in \mathbb{P}$. We write $type(T) = \alpha$. Note that $x^T = x^{type(T)}$.

It is not obvious that the Schur functions are symmetric. To prove that they are we only need to show

$$|\{T \in SST_\lambda : type(T) = \alpha\}| = |\{T \in SST_\lambda : type(T) = \beta\}|$$

whenever α and β are related by an adjacent transposition. There is a nice involution on SST_λ that proves this (due to Bender and Knuth).

The Schur basis

Theorem (Schur basis)

$\{s_\lambda : \lambda \vdash n\}$ is a basis for Sym_n.

The Kostka numbers for $\lambda, \mu \vdash n$ are defined by

$$K_{\lambda,\mu} := |\{T \in SST_\lambda : type(T) = \mu\}|.$$

Once we establish the symmetry of the Schur functions, it is easy to see that for all $\lambda \vdash n$,

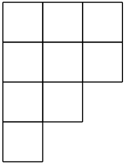
$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu$$

From this and a certain scalar product for which $\{s_\lambda\}$ is an orthonormal basis and the $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual, we get for all $\lambda \vdash n$,

$$h_\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda} s_\mu$$

The involution ω

The conjugate of a partition $\lambda \vdash n$ is the partition $\lambda' \vdash n$ whose Young diagram is the transpose of the Young diagram of λ.
Example: $\lambda = (3, 3, 2, 1)$

$\lambda =$ 

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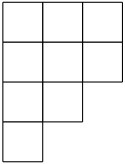
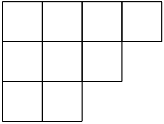
$\lambda =$ 

$\lambda' =$ 

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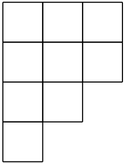
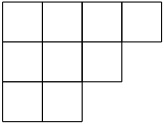
$\lambda =$ 

$\lambda' =$ 

Let $\omega : Sym_n \to Sym_n$ be the involution that takes s_λ to $s_{\lambda'}$ for all $\lambda \vdash n$.
What does ω do to other bases?

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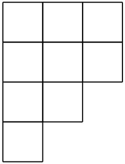
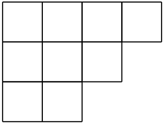
$\lambda =$ 

$\lambda' =$ 

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What does ω do to other bases?

- $\omega(h_\lambda) = e_\lambda$
- $\omega(e_\lambda) = h_\lambda$
- $\omega(p_\lambda) = (-1)^{n-l(\lambda)} p_\lambda$

Other expansions

Recall

$$K_{\lambda,\mu} := |\{T \in SST_\lambda : type(T) = \mu\}|.$$

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu$$

$$h_\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda} s_\mu$$

Let $z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$, where m_i is the number of occurrences of i in $\mu \vdash n$. One can show

$$h_n = \sum_{\mu \vdash n} z_\mu^{-1} p_\mu.$$

Applying the involution ω yields

$$e_\lambda = \sum_{\mu \vdash n} K_{\mu',\lambda} s_\mu$$

$$e_n = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} z_\mu^{-1} p_\mu.$$

Quasisymmetric functions

$f(\mathbf{x}) \in \mathbb{Q}[[X]]$ is a quasisymmetric function if

$$\mathrm{coef}(f; x_1^{a_1} \dots x_k^{a_k}) = \mathrm{coef}(f; x_{i_1}^{a_1} \dots x_{i_k}^{a_k})$$

for all $i_1 < \cdots < i_k$ and $a_1, \dots, a_k \in \mathbb{N}$.

Let $QSym_n$ denote the vector space of homogeneous quasisymmetric functions of degree n and let $QSym$ denote the ring of quasisymmetric functions of bounded degree.

Note: Every symmetric function is quasisymmetric, but not conversely.

Examples:

$$f(x) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \dots$$

$$g(x) = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_4^2 + x_2 x_4^2 + x_3 x_4^2 + \dots$$

These are examples of monomial quasisymmetric functions.

Monomial basis for $QSym_n$

Monomial quasisymmetric functions: Given $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$, let

$$M_\alpha := \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}.$$

Examples.

$$M_{2,1} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \ldots$$

$$M_{1,2} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_4^2 + x_2 x_4^2 + x_3 x_4^2 + \ldots$$

Note. $M_{2,1} + M_{1,2} = m_{2,1}$.

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Note. $M_{2,1} + M_{1,2} = m_{2,1}$.

More generally, for $\lambda \vdash n$,

$$m_\lambda = \sum_\alpha M_\alpha,$$

where the α ranges over all compositions of n whose decreasing rearrangement is λ.

Fact. $\{M_\alpha | \alpha \models n\}$ is a basis for $QSym_n$. Thus dim $QSym_n$ equals the number of compositions of n, which is 2^{n-1}

Gessel's Fundamental basis for $QSym_n$

For $S \in [n-1]$, let

$$F_S := \sum_{\substack{i_1 \geq \ldots \geq i_n \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \ldots x_{i_n}.$$

Theorem (Gessel - 1984)

$\{F_S : S \subseteq [n-1]\}$ *is a basis for* $QSym_n$

Note:

- $F_\emptyset = h_n$
- $F_{[n-1]} = e_n$

Involution ω extends to the larger space $QSym_n$ as follows.
$\omega : QSym_n \to QSym_n$ is defined on basis elements by

$$\omega(F_S) = F_{[n-1]\setminus S}.$$

For symmetric functions this reduces to the involution that was defined before. Note

$$\omega(h_n) = \omega(F_\emptyset) = F_{[n-1]} = e_n$$

Expansion of the Schur functions in F-basis.

A standard Young tableau of shape λ is a filling of the diagram $\lambda \vdash n$ with distinct entries $1, 2, \ldots, n$ so that the rows and columns (strictly) increase.

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & 9 \\ \hline 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 8 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Let SYT_λ be the set of standard Young tableaux of shape λ.
For $T \in SYT_\lambda$, let

$$DES(T) := \{i \in [n] : i \text{ is higher than i+1 in } T\}.$$

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Theorem (Gessel - 1984)

For all $\lambda \vdash n$,

$$s_\lambda = \sum_{T \in SYT_\lambda} F_{DES(T)}.$$

Specialization

For $f(x) \in \mathbb{R}[[X]]$, define the stable principal specialization as follows:

$$\mathrm{ps}(f(x_1, x_2, \dots,)) := f(1, q, q^2, \dots)$$

Lemma (Gessel)

For all $S \subseteq [n-1]$,

$$\mathrm{ps}(F_S) = \frac{q^{\sum S}}{(1-q)(1-q^2)\dots(1-q^n)},$$

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Consequently

$$\begin{aligned} \mathrm{ps}(s_\lambda) &= \sum_{T \in SYT_\lambda} \mathrm{ps}(F_{DES(T)}) \\ &= \frac{\sum_{T \in SYT_\lambda} q^{\mathrm{maj}(T)}}{(1-q)(1-q^2)\dots(1-q^n)}, \end{aligned}$$

where $\mathrm{maj}(T) = \sum_{i \in DES(T)} i$.

q-analog of hook legth formula

$$\mathrm{ps}(s_\lambda) = \frac{\sum_{T\in SYT_\lambda} q^{\mathrm{maj}(T)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

The original definition of Schur function (due to Cauchy - 1815) involved determinants. Using that definition Stanley proved

Theorem (Stanley (1971))

$$\mathrm{ps}(s_\lambda) = \frac{q^{b_\lambda}}{\prod_{x\in\lambda}(1-q^{h_x})},$$

where $b(\lambda) = \sum (i-1)\lambda_i$ *and* h_x *is the hook length at cell* x.

Equating the right hand sides of these two formulas yields:

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Corollary (q-analog of the hook-length formula)

$$\sum_{T \in SYT_\lambda} q^{\mathrm{maj}(T)} = q^{b(\lambda)} \, \frac{[n]_q!}{\prod_{x \in \lambda}[h_x]_q}.$$