

Lecture 4 Practice Problems

Michelle Wachs

- (1) Which of the following are banners. (The barred letters are blue, the others are red.)
 - (a) $884\bar{2}23\bar{3}$
 - (b) $\bar{8}\bar{8}9\bar{2}$
 - (c) $884\bar{2}2$
- (2) Recall the bijection that takes ornaments to banners. (The barred letters are blue, the others are red.)
 - (a) Find the image of the ornament $(2, \bar{2})(\bar{3}, 5, 3, \bar{3}, 7, \bar{1})(2, \bar{2})$
 - (b) Find the unique ornament that maps to the banner $3\bar{3}3\bar{1}5\bar{2}$
- (3) Let $W_n := \{w \in \mathbb{P}^n : w_i \neq w_{i+1} \ \forall i \in [n-1]\}$. Expand $\sum_{w \in W_3} x^w t^{\text{des}(w)}$ in the fundamental basis.
- (4) Let $P := P_{5,3}$ and $G := G_{5,3}$.
 - (a) Is $X_{G_{5,3}}(x, t)$ symmetric?
 - (b) Give all the P -tableaux T together with the corresponding $\text{inv}_G(T)$.
 - (c) Expand $X_G(x, t)$ in the Schur basis.
 - (d) Give 5 terms of the expansion of $X_G(x, t)$ in the F -basis.
- (5) Show that for all $\sigma \in \mathfrak{S}_n$, $\text{inv}_{<2}(\sigma) = \text{des}(\sigma^{-1})$
- (6) It is a result of Rawlings that $A_n^{(r)}(q, q) = [n]_q!$.
 - (a) Verify this for $n = 3$ and all r .
 - (b) Prove this for $r = 2$ and all n .
- (7) Let $P = P_{n,r}$ and let $G = G_{n,r}$. Show that there exists a bijection ϕ from \mathfrak{S}_n to the set of pairs (A, B) , where A is a P -tableaux and B is a semistandard tableaux of the same shape as A , and if $\phi(\sigma) = (A, B)$ then $\text{inv}_{<r}(\sigma) = \text{inv}_{<r}(A)$ and $\text{DES}_{\geq r}(\sigma) = \text{DES}(B)$.
- (8) Let $\rho : Qsym_n \rightarrow Qsym_n$ be the involution defined on the basis of monomial quasisymmetric functions by $\rho(M_\alpha) = M_{\alpha^{\text{rev}}}$ for each composition α . Extend the involution ρ to $Qsym_n[t]$. Prove that

$$\rho(X_G(\mathbf{x}, t)) = t^{|E|} X_G(\mathbf{x}, t^{-1}).$$
- (9) Prove that if $X_G(\mathbf{x}, t)$ is symmetric (in \mathbf{x}) then it is palindromic.
- (10) Prove that $X_{G_{n,r}}(\mathbf{x}, t)$ is symmetric in \mathbf{x} .