Gaussian Paragraphs of Pseudolinear Arrangements

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Content

Gauss words

Gauss words

Gauss words are finite sequences of letters associated with self-intersecting closed curves in the plane. (These curves have no triple self-intersections.)

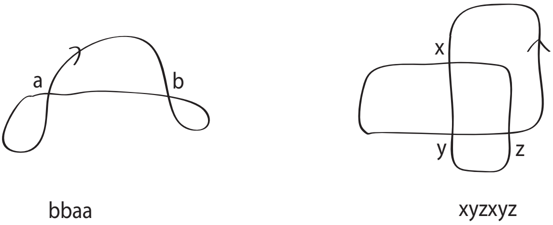

Figure: The Gauss words corresponding to these curves.

Gauss words

The following questions arise:

(i) Which words over some alphabet are realizable as Gaussian words?

(ii) Which curves can be uniquely reconstructed from their Gaussian words?

(iii) What is the common structure of curves having the same Gaussian word?

Subwords

(i) Given a word $\omega = x\alpha x\beta$, we define the *vertex split* at x to be the word $\omega_a = \alpha^{-1}\beta$.

(ii) Given a word $\omega = x\alpha x\beta$. We define the *loop removal* at x to be the word obtained by deleting x and both ocurrences of the letters in α.

(iii) A *subword* of a word ω is any word obtained by a sequence of vertex splits and loop removals.

First characterization of Gauss words:

Theorem (Lovász and Marx, 1976)

A word ω is realizable if and only if it contains no subwords of the form

$$x_1x_2\ldots x_nx_1x_2\ldots x_n, \text{ with } n \text{ even.}$$

A no realizable word:

$$xyzxwzrwyr$$

Pseudolinear arrangements

A *pseudoline* is a continuous curve ℓ in $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \ell$ consists of exactly two infinite connected regions.

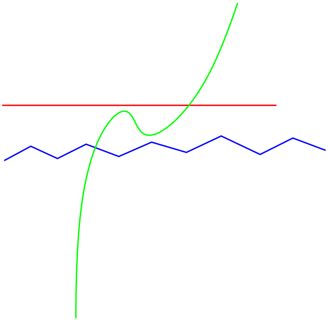

Figure: Three pseudolines.

Pseudolinear arrangements

An *arrangement of pseudolines* is a collection $L = \{p_0, p_1, \ldots, p_m\}$ of pseudolines that intersect (necessarily cross) pairwise exactly once. We will say that L is *simple if no point of* $\mathbb{R}^2$ *belongs to more than two pseudolines.*

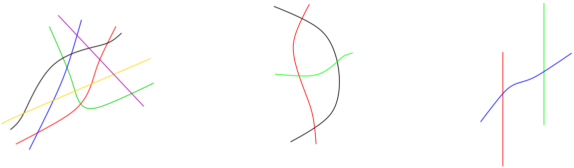

Figure: Only the left collection of pseudolines form a simple arrangement of pseudolines.

Double ocurrence paragraphs

Double Ocurrence Paragraph

Let Σ be a finite set of letters, say $a, b, c \ldots$ Any finite sequence of letters of Σ will be called a *word* over Σ. A set of n words $W = \{w_1, w_2, \ldots, w_n\}$ over Σ is a *n-double occurrence paragraph* if and olny if

(C1) Every letter appears exactly two times in $w_1 \cup w_2 \cup \ldots \cup w_n$.

(C2) No letter appears twice in any word.

(C3) There is a unique letter in every pair of words.

Double Ocurrence Paragraph

For example

$$W = \{abc, aef, cde, bdf\}$$

is a 4-DOP with

$$\Sigma = \{a, b, c, d, e, f\}$$

Let $G(W)$ be the graph with vertex set Σ in which two letters x and y are joined if and only if x and y are adjacent in some word of W.

Let $\overline{G}(W)$ be the graph obtained from $G(W)$ by adding a new vertex v^* and joining v^* to every vertex of $G(W)$ with degree at most 3.

Double Ocurrence Paragraph

For example

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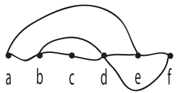


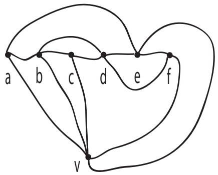


Figure: $G(W)$ and $\overline{G}(W)$

Double Ocurrence Paragraph

Proposition

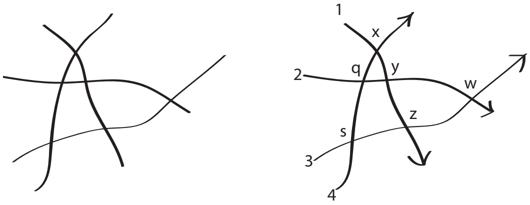
If W is n-DOP over Σ, then the number of letters in every word is exactly $n-1$ and $|\Sigma| = \binom{n}{2}$.

Double Ocurrence Paragraph

Proposition

Let $n \geq 2$ be an integer. Any n-SPA can be encoded by an n-DOP.

Sketch of the proof:



$\{ xyz,\ qyz,\ szw, sqx \}$

Double Ocurrence Paragraphs

On the other hand, not every n-DOP is the code of some n-SPA. If W is a n-DOP for which there exists a L n-SPA such that W encodes L, we say that W is *realizable*.

Double Ocurrence Paragraphs

On the other hand, not every n-DOP is the code of some n-SPA.
If W is a n-DOP for which there exists a L n-SPA such that W encodes L, we say that W is *realizable*.
For example,

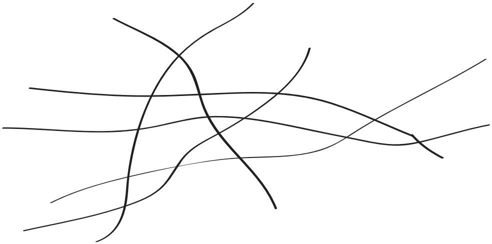
$$\{abd, afc, bec, def\}$$

is not realizable.

n-DOP's coming from n-SPA's

Properties of n-DOP's coming from n-SPA's

Throughout this section, W is a realizable n-DOP and A is the n-SPA encoded by W. A crossing of A corresponding to a vertex of degree 2 in $G(W)$ will be called a *sharp crossing*.

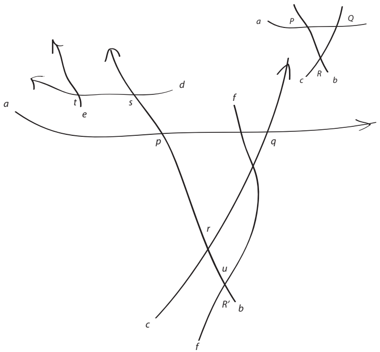


Properties of n-DOP's coming from n-SPA's

Lemma

$G(W)$ has at least three vertices of degree two.

Sketch of the proof:

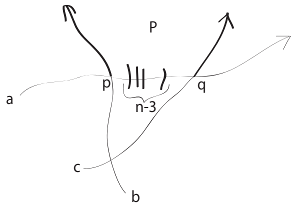


Properties of n-DOP's coming from n-SPA's

Lemma

$G(W)$ has at most $f(n) := 2\lfloor n/2 \rfloor + (-1)^{n+1}$ vertices of degree two.

Sketch of the proof: It is easy to see that $G(W)$ has at most n sharp crossings. Since $f(n) = n$ and $f(n) = n - 1$ for n odd and even, respectively, we need to check only the case n even.

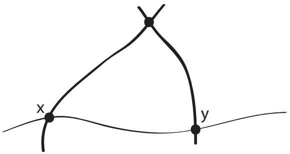


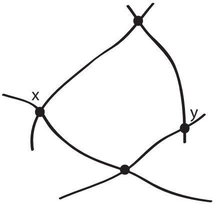
Properties of n-DOP's coming from n-SPA's

Proposition

$G(W)$ is 2-connected.

Sketch of the proof: Let x and y be two vertices of $G(W)$. We analyze two cases:



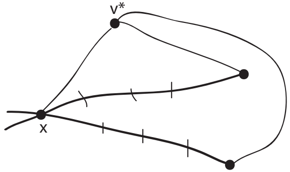


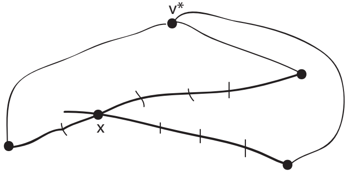
Properties of n-DOP's coming from n-SPA's

Proposition

$\overline{G}(W)$ is 3-connected.

Sketch of the proof: Since $G(W)$ is 2-connected, it is enough to show that for every $x \in V(G)$, $\overline{G}(W)$ has at least 3 internally disjoint x, v^*-paths. Again we analyze two cases:

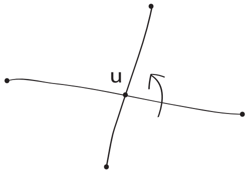




Main result

Crossing property

Let W be an n-DOP and let Λ be a planar embedding of $G := \overline{G}(W)$. Let v^* be the apex vertex of G and let $u \neq v^*$ be a vertex of G. Now consider the cyclic order in which the neighbors of u appear in Λ. We will say that the neighbors of u are *interlaced* in Λ if any two consecutive of them are not both in a single word of W. If the neighbors of every vertex of $G \setminus \{v^*\}$ are interlaced in Λ, then Λ has the *crossing property*.



Main result

Theorem

Let W, G, and v^* be as in previous paragraph. Then W is realizable as a pseudo-linear arrangement if and only if G has a planar embedding Π which has the crossing property. Moreover, if the embedding Π exists, it is unique up to a homeomorphism.

Corollary

Determining if a Gaussian paragraph is realizable can be done in time linear in the number of crossings (or quadratic in the number of lines).

Thanks for your attention!