Exercises: Algebraic Structures on Combinatorial Species

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1. Antipodes and power series

(1) Let G be the species of simple graphs. Use Takeuchi's formula to compute the antipode on the path P_3 on 3 vertices, and check that it agrees with the formula given in lecture in terms of flats.

(2) Let W be the species of weaves. Use Takeuchi's formula to compute the antipode on the path P_3 on 3 vertices, and check that it agrees with the formula given in lecture in terms of planar rooted trees.

(3) Let Π be the species of partitions, and let π_n be the trivial partition of the set $[n]$ into 1 block. Assume $\varphi, \psi \in \mathbb{X}(\Pi)$ are characters on Π, and define $a_n = \varphi(\pi_n)$, $b_n = \psi(\pi_n)$.
   (a) Assume $\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right) \cdot \left(\sum_{n\geq 0} b_n \frac{x^n}{n!}\right) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$. Check that $c_n = (\varphi * \psi)(\pi_n)$ in the case $n = 3$. Can you prove it for general n?
   (b) Suppose $\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right)^{-1} = \sum_{n\geq 0} d_n \frac{x^n}{n!}$. Check that $d_n = (\varphi \circ \mathrm{S})(\pi_n)$ in the case $n = 3$.

(4) Let H be the submonoid of W generated by paths. Let $\varphi, \psi \in \mathbb{X}(\mathrm{H})$ be characters on H, and let $a_n = \varphi(P_{n-1})$, $b_n = \psi(P_{n-1})$, where P_{n-1} denotes the path on $n-1$ vertices.
   (a) Convince yourself that H is indeed a Hopf submonoid of W.
   (b) Assume $\left(\sum_{n\geq 1} a_n x^n\right) \circ \left(\sum_{n\geq 1} b_n x^n\right) = \sum_{n\geq 1} c_n x^n$. Check that $c_n = (\varphi * \psi)(P_{n-1})$ in the case $n = 4$.
   (c) Suppose $\left(\sum_{n\geq 1} a_n x^n\right)^{\langle -1\rangle} = \sum_{n\geq 1} d_n x^n$. Check that $d_n = (\varphi \circ \mathrm{S})(P_{n-1})$ in the case $n = 4$.

(5) Let ζ be the character on the species G defined by $\zeta(g) = 1$ if the graph g has no edges, and $\zeta(g) = 0$ otherwise. Use Takeuchi's formula to prove that $\zeta(\mathrm{S}(g)) = \pm\, \mathrm{o}(g)$, where $\mathrm{o}(g)$ denotes the number of acyclic orientations of g.