Symmetric Functions and Eulerian Polynomials

Lecture 1: Permutation Statistics and Eulerian polynomials

Michelle Wachs
University of Miami

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Lectures

Lecture 1: Permutation Statistics and Eulerian Polynomials

Lecture 2: Symmetric and Quasisymmetric Functions

Lecture 3: Eulerian quasisymmetric functions

Lecture 4: Chromatic quasisymmetric functions

Binomial Coefficients and Eulerian numbers

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

Binomial Coefficients and Eulerian numbers

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$$

Binomial Coefficients and Eulerian numbers

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• coeff's of polynomial
$(t+1)^n = \sum_{j=0}^{n} \binom{n}{j} t^j$

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• coeff's of Eulerian polynomial
$A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$

Binomial Coefficients and Eulerian numbers

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- coeff's of polynomial $(t+1)^n = \sum_{j=0}^{n} \binom{n}{j} t^j$
- rows add to 2^n

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&4&&1&& \\ &1&&11&&11&&1& \\ 1&&26&&66&&26&&1 \end{array}$$

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- coeff's of Eulerian polynomial $A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$
- rows add to $n!$

Binomial Coefficients and Eulerian numbers

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- rows add to 2^n
- subsets of $\{1, 2, \ldots, n\}$ of size j

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- coeff's of Eulerian polynomial $A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$
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- permutations in $\mathfrak{S}_n$ with j descents

Binomial Coefficients and Eulerian numbers

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- coeff's of Eulerian polynomial $A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$
- rows add to $n!$
- permutations in $\mathfrak{S}_n$ with j descents

- Rows are palindromic and unimodal.

Eulerian polynomials - Euler's definition

$$\sum_{i \geq 1} t^i = \frac{t}{1-t}$$

Eulerian polynomials - Euler's definition

$$\begin{aligned} \sum_{i\geq 1} t^i &= \frac{t}{1-t} \\ \sum_{i\geq 1} i t^i &= \frac{t}{(1-t)^2} \end{aligned}$$

Eulerian polynomials - Euler's definition

$$\begin{aligned}
\sum_{i\geq 1} t^i &= \frac{t}{1-t} \\
\sum_{i\geq 1} i t^i &= \frac{t}{(1-t)^2} \\
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Leonhard Euler
(1707-1783)

Euler's triangle

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

Euler's definition

$$\sum_{i\geq 1} i^n t^i = \frac{t\, A_n(t)}{(1-t)^{n+1}}$$

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i\geq 1} it^i = \frac{t}{(1-t)^2}$$

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Euler's definition

$$\sum_{i\geq 1} i^n t^i = \frac{t\,A_n(t)}{(1-t)^{n+1}}$$

Euler's exponential generating function formula

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z}-t}$$

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Permutations

Given a set A, a permutation on A is a bijection on A.

The group of permutations on A under composition is called the symmetric group on A and is denoted by $\mathfrak{S}_A$.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n := \mathfrak{S}_{[n]}$.

Two line notation:

$$\sigma = \begin{bmatrix} 1\,2\,3\,4\,5 \\ 4\,5\,2\,1\,3 \end{bmatrix} \in \mathfrak{S}_5$$

One line notation:

$$\sigma = [\ 4\,5\,2\,1\,3\ ] \in \mathfrak{S}_5$$

Cycle notation:

$$\sigma = (1, 4)(2, 5, 3) \in \mathfrak{S}_5$$

Partitions and Compositions

A partition of $n \in \mathbb{P}$ is a weakly decreasing sequence of positive integers whose sum is n.

A composition of $n \in \mathbb{P}$ is a sequence of positive integers whose sum is n.

Partitions of 4	Compositions of 4
(4)	(4)
(3,1)	(3,1), (1,3)
(2,2)	(2,2)
(2,1,1)	(2,1,1), (1,2,1), (1,1,2)
(1,1,1)	(1,1,1,1)

If λ is a partition of n, we say $\lambda \vdash n$ and $|\lambda| = n$.

If μ is a composition of n, we say $\mu \models n$ and $|\mu| = n$.

$l(\lambda)$ denotes the length of a partition (or composition) λ.

Associate a partition of n with a permutation $\sigma \in \mathfrak{S}_n$, by writing σ in cycle form and letting $\lambda(\sigma)$ be the sequence of cycle sizes listed in weakly decreasing order. The partition $\lambda(\sigma)$ is the cycle type of σ.

Example: $\lambda((1,4),(2,7,5)(3,6)) = (3,2,2)$

Eulerian polynomials - combinatorial interpretation

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

Eulerian polynomials - combinatorial interpretation

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Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

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Excedance set: $\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i\}$

$$\sigma = 32541 \qquad \mathrm{EXC}(\sigma) = \{1, 3\}$$

Define $\mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|$. So

$$\mathrm{exc}(32541) = 2$$

Eulerian polynomials - combinatorial interpretation

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{des}(\sigma)} = 1 + 4t + t^2$$

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1 + 4t + t^2$$

Euler's triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

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Eulerian polynomial

$$A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

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MacMahon (1905) showed equidistribution of des and exc.
Carlitz and Riordin (1955) showed equals $A_n(t)$.

The characterizations of $A_n(t)$

Combinatorial

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

Euler's triangle

				1				
			1		1			
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Recurrence relation

$$\left\langle {n \atop j} \right\rangle = (n-j) \left\langle {n-1 \atop j-1} \right\rangle + (j+1) \left\langle {n-1 \atop j} \right\rangle$$

Euler's definition

$$\sum_{i \geq 1} i^n t^i = \frac{t\, A_n(t)}{(1-t)^{n+1}}$$

Euler's exponential generating function formula

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}$$

Palindromicity and Unimodality

A polynomial $f(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{R}[t]$ is

- palindromic (with center of symmetry $\frac{n}{2}$) if $a_i = a_{n-i}$ for all i
- unimodal if for some c

$$a_0 \leq a_1 \leq \cdots \leq a_c \geq \cdots \geq a_{n-1} \geq a_n$$

- positive if $a_i \geq 0$ for all i

Example: $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$

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Sum and Product Lemma

Let $A(t)$ and $B(t)$ be Positive, Unimodal, Palindromic with respective centers of symmetry c_A and c_B. Then

- $A(t)B(t)$ is PUP with center of symmetry $c_A + c_B$.
- If $c_A = c_B$ then $A(t) + B(t)$ is PUP with center of symmetry c_A.

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Example 1: $[5]_t[2]_t + [6]_t + [4]_t[3]_t$, where $[k]_t := 1 + t + \cdots + t^{k-1}$.

Example 2: Palindromicity and unimodality of rows of Pascal's triangle are consequences since the polynomial $(1+t)^n$ is product of PUP's.

Palindromicity and Unimodality

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- If $c_A = c_B$ then $A(t) + B(t)$ is PUP with center of symmetry c_A.

Theorem: The Eulerian polynomials are palindromic and unimodal.

Proof: From Euler's exponential generating formula, we can derive

$$A_n(t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\ldots,k_m \geq 2} \binom{n}{k_1 - 1, k_2, \ldots, k_m} t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$$

where

$$[k]_t := 1 + t + \cdots + t^{k-1}.$$

Center of symmetry of $t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$ is

$$m - 1 + \sum_{i=1}^{m} \frac{k_i - 2}{2} = \frac{n-1}{2}. \quad \square$$

γ-positivity

$f(t) \in \mathbb{R}[t]$ is palindromic $\iff \exists\ \gamma_k \in \mathbb{R}$ such that

$$f(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k}.$$

If $\gamma_k \geq 0$ for all k then $f(t)$ said to be γ-positive.

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If $\gamma_k \geq 0$ for all k then $f(t)$ said to be γ-positive.

γ-positive $\implies$ palindromic and unimodal

Example: $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$.

$$\begin{array}{rcl} A_5(t) & = & 1 + 26t + 66t^2 + 26t^3 + t^4 \\ 1t^0(1+t)^4 & = & 1 + 4t + \ \ 6t^2 + \ \ 4t^3 + t^4 \\ 22\, t^1(1+t)^2 & = & 22t + 44t^2 + 22t^3 \\ 16\, t^2(1+t)^0 & = & 16t^2 \end{array}$$

So

$$A_5(t) = 1t^0(1+t)^4 + 22t^1(1+t)^2 + 16t^2(1+t)^0.$$

Thus $A_5(t)$ is γ-positive.

γ-positivity

Foata & Schutzenberger (1970):

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}\, t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n : \sigma$ has no double descents, no final descent & $\mathrm{des}(\sigma) = k\}|$.

3.2.14 has a double descent. 124.3 has a final descent.

γ-positivity

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$\mathfrak{S}_3$	des
123	0
~~132~~	
213	1
~~231~~	
312	1
~~321~~	

$$\begin{aligned} A_3(t) &= 1t^0(1+t)^2 + 2t^1(1+t)^0 \\ &= 1 + 4t + t^2. \end{aligned}$$

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Another property stronger than unimodality is log-concavity; real-rootedness is still stronger. $A_n(t)$ has only real roots.

Mahonian permutation statistics and q-analogs

Let $\sigma \in \mathfrak{S}_n$.

Inversion Number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\mathrm{inv}(3142) = 3$$

Major Index:

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{maj}(3142) = \mathrm{maj}(3.14.2) = 1 + 3 = 4$$

Major Percy Alexander MacMahon
(1854 - 1929)

Mahonian Permutation Statistics - q-analogs

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$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Mahonian Permutation Statistics - q-analogs

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}| \qquad \mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

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$$\sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} = 1 + 2q + 2q^2 + q^3$$

Mahonian Permutation Statistics - q-analogs

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$\mathfrak{S}_3$	inv	maj
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$$\begin{aligned}\sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} \\ &= 1 + 2q + 2q^2 + q^3 \\ &= (1 + q + q^2)(1 + q)\end{aligned}$$

Mahonian Permutation Statistics - q-analogs

$$\text{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}| \qquad \text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i$$

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_3} q^{\text{inv}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_3} q^{\text{maj}(\sigma)} \\ &= 1 + 2q + 2q^2 + q^3 \\ &= (1 + q + q^2)(1 + q) \end{aligned}$$

Theorem (MacMahon 1905)

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

Other examples of q-analogs

Fact: The number of words of length n over the alphabet $\{1, 2\}$ with k 1's is $\binom{n}{k}$.

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q-analog: Let $\mathfrak{S}_{n,k}$ be the set of words of length n over the alphabet $\{1, 2\}$ with k 1's . Then

$$\sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\mathrm{inv}(\sigma)} = \left[\begin{array}{c} n \\ k \end{array} \right]_q$$

where

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

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where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Proof of first equality: Use Foata bijection.

Proof of second equality: Show

$$\sum_{\sigma\in\mathfrak{S}_{n,k}} q^{\mathrm{inv}(\sigma)} \; [k]_q![n-k]_q! = [n]_q!$$

Other examples of q-analogs

Proof of second equality: Show

$$\sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\mathrm{inv}(\sigma)}\ [k]_q![n-k]_q! = [n]_q!$$

Use the map $\phi : \mathfrak{S}_{n,k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k} \to \mathfrak{S}_n$ that takes (σ, α, β) to the word obtained from σ by replacing the subword of 1's by α and the subword of 2's by $\tilde{\beta}$, where $\tilde{\beta}$ is obtained from β by replacing each letter i by $i + k$. Check that

- ϕ is a bijection
- $\mathrm{inv}(\tilde{\beta}) = \mathrm{inv}(\beta)$
- $\mathrm{inv}(\phi(\sigma, \alpha, \beta)) = \mathrm{inv}(\sigma) + \mathrm{inv}(\alpha) + \mathrm{inv}(\tilde{(\beta)})$

Example: $n = 5, k = 2$, $\sigma = 12212$, $\alpha = 21$, $\beta = 231$. Then $\tilde{\beta} = 453$ and

$$\phi(\sigma, \alpha, \beta) = 24513$$

Other examples of q-analogs

Fact: The number of derangements (i.e. permutations with no fixed points) in $\mathfrak{S}_n$ is given by

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Other examples of q-analogs

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q-analog: Let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$. Then

$$\sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma)} = [n]_q! \sum_{k=0}^{n} q^{\binom{k}{2}} \frac{(-1)^k}{[k]_q!}.$$

Due independently to Gessel and MW (1989).

Doesn't work for inv.

q-analogs of Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

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Theorem (~~Carlitz 1954~~ MacMahon 1916)

$$\sum_{i\geq 1} [i]_q^n \, t^i = \frac{t A_n^{\mathrm{maj},\mathrm{des}}(q,t)}{\prod_{i=0}^n (1-tq^i)}$$

q-analogs of Euler's exp. generating function formula

Theorem (Stanley 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

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Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}$$

q-Eulerian polynomials and q-Eulerian numbers

Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj},\mathrm{exc}}(q, tq^{-1})\frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n\geq 0}\frac{z^n}{[n]_q!}$$

Proof uses symmetric function theory, which we will talk about next time.

From now on

$$A_n(q,t) := A_n^{\mathrm{maj},\mathrm{exc}}(q, tq^{-1}) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

and

$$\left\langle {n \atop j} \right\rangle_q := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}$$

Palindromicity and unimodality of the q-Eulerian numbers

$n\backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\dots$	$6+6q+11q^2+\dots$	$4+3q+5q^2+\dots$	1

Theorem (Shareshian and MW)

The q-Eulerian polynomial $A_n(q,t) = \sum_{t=0}^{n-1} \left\langle {n \atop j} \right\rangle_q t^j$ is

- *palindromic in the sense that $\left\langle {n \atop j} \right\rangle_q = \left\langle {n \atop n-1-j} \right\rangle_q$ for $0 \le j \le \frac{n-1}{2}$*
- *q-unimodal in the sense that $\left\langle {n \atop j} \right\rangle_q - \left\langle {n \atop j-1} \right\rangle_q \in \mathbb{N}[q]$ for $1 \le j \le \frac{n-1}{2}$*

Palindromicity and unimodality of the q-Eulerian numbers

Theorem (Shareshian and MW)

The q-Eulerian polynomial $A_n(q,t) = \sum_{t=0}^{n-1} \left\langle {n \atop j} \right\rangle_q t^j$ is palindromic and q-unimodal.

Proof: We use our q-analog of Euler's exponential generating function formula to prove

$$A_n(q,t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\dots,k_m \geq 2} \left[{n \atop k_1 - 1, k_2, \dots, k_m} \right]_q t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t,$$

where

$$\left[{n \atop k_1, \dots, k_m} \right]_q := \frac{[n]_q!}{[k_1]_q! \cdots [k_m]_q!}$$

Then apply the Sum & Product Lemma.

q-γ-positivity of q-Eulerian polynomials

Recall: Foata & Schutzenberger (1970):

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}\, t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n : \sigma$ has no double descents, no final descent & $\mathrm{des}(\sigma) = k\}|$.

Theorem (Shareshian and Wachs (2010))

Let $\Gamma_{n,k}$ be the set of permutations in $\mathfrak{S}_n$ with no double descents, no final descent and $\mathrm{des}(\sigma) = k$. Let

$$\gamma_{n,k}(q) := \sum_{\sigma \in \Gamma_{n,k}} q^{\mathrm{inv}(\sigma)}.$$

Then

$$A_n(q,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q)\, t^k (1+t)^{n-1-2k},$$

.

Proof uses our q-analog of Euler's exponential generating function and a symmetric function identity of Gessel.

Cycle-type Eulerian polynomials

For $\lambda \vdash n$, let $\mathfrak{S}_\lambda$ be the set of permutations of cycle type λ. Define the cycle-type Eulerian polynomial as follows

$$A_\lambda(t) := \sum_{\sigma \in \mathfrak{S}_\lambda} t^{\mathrm{exc}(\sigma)}$$

For $\lambda \vdash n$ and $i \in \mathbb{P}$, let $m_i(\lambda)$ be the number of occurrences of i in λ.

Brenti (1993): $A_\lambda(t)$ is palindromic and unimodal with center of symmetry $c = \frac{n - m_1(\lambda)}{2}$.

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Brenti (1993): $A_\lambda(t)$ is palindromic and unimodal with center of symmetry $c = \frac{n - m_1(\lambda)}{2}$.

Now define the cycle-type q-Eulerian polynomial

$$A_\lambda(q,t) := \sum_{\sigma \in \mathfrak{S}_\lambda} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}.$$

Henderson and MW (2010): $A_\lambda(q,t)$ is palindromic and q-unimodal with center of symmetry $c = \frac{n - m_1(\lambda)}{2}$.

Proof uses symmetric function theory and representation theory.

Derangements

Corollary. Let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$ and let

$$D_n(q,t) := \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}.$$

Then $D_n(q,t)$ is palindromic and q-unimodal with center of symmetry $\frac{n}{2}$.

Also a consequence of
Shareshian and MW (2010):

$$\sum_{n \geq 0} D_n(q,t) z^n = \frac{1-t}{\exp_q(tz) - t\exp_q(z)}$$

and of
Shareshian and MW (2010): Let $\Gamma_{n,k}$ be the set of permutations in $\mathfrak{S}_n$ with no double descents, no final descent, no initial descent, and $\mathrm{des}(\sigma) = k$. Let

$$\gamma_{n,k}(q) := \sum_{\sigma \in \Gamma_{n,k}} q^{\mathrm{inv}(\sigma)}.$$

Then

$$D_n(q,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q)\, t^{k+1}(1+t)^{n-2k-2},$$

Log-concavity

A sequence $(a_0, a_1, \ldots, a_n)$ is log-concave if $a_j^2 > a_{j-1}a_{j+1}$ for all j. We will say a polynomial $\sum_{j=0}^{n} a_j t^j$ is log concave if its sequence of coefficients $(a_0, a_1, \ldots, a_n)$ is log-concave.

Example. $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$.

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A sequence of polynomials $(a_0(q), a_1(q), \ldots, a_n(q))$ is q-log-concave if

$$a_j(q)^2 - a_{j-1}(q)a_{j+1}(q) \in \mathbb{N}[q],$$

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Conjecture (Shareshian and MW)

- *For all n, $A_n(q,t)$ is q-log-concave.*
- *For all n and $\lambda \vdash n$, $A_\lambda(q,t)$ is q-log-concave.*

We checked this up to $n = 8$.

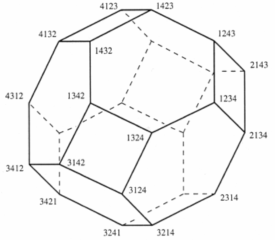
Geometric interpretation of Eulerian polynomials

The *h*-polynomial of a d-dimensional convex polytope $\mathcal{P}$ is defined by

$$h_{\mathcal{P}}(t) := \sum_{j=0}^{d} f_{d-1-j}(t-1)^j$$

where f_i is the number of faces of $\mathcal{P}$ of dimension i.

The permutohedron $\mathcal{P}_n$ is the convex hull of points in $\mathbb{R}^n$ of the form $(\sigma(1), \ldots, \sigma(n))$, where $\sigma \in \mathfrak{S}_n$. This is an $(n-1)$-dimensional polytope embedded in $\mathbb{R}^n$.



For each convex polytope $\mathcal{P}$, there is another convex polytope $\mathcal{P}^*$ called the polar dual. The number of i-dimensional faces of $\mathcal{P}^*$ equals the number of $(d-i)$-dimensional faces of $\mathcal{P}$ for each i.

Theorem: $A_n(t) = h_{\mathcal{P}_n^*}(t)$.

Geometric interpretation of Eulerian polynomials

Theorem: $A_n(t) = h_{\mathcal{P}_n^*}(t)$.

Dehn-Sommerville equations: The h-polynomial of every simplicial convex polytope is palindromic.

Stanley (1980): The h-polynomial of every simplicial convex polytope is unimodal.
This is part of the celebrated g-theorem of Billera, Lee, and Stanley.

Gal's conjecture (2005): The h-polynomial of a flag simplicial convex polytope $\mathcal{P}$ is γ-positive.

Fact: $\mathcal{P}_n^*$ is simplicial and flag.

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We will see a geometric interpretation of the q-Eulerian polynomials.

Symmetric Functions and Eulerian Polynomials

Lecture 2: Symmetric and quasisymmetric functions

Michelle Wachs
University of Miami

Cuarto Encuentro Colombiano de Combinatoria 2016

Basic definitions

$\mathbb{Q}[[X]]$ denotes the ring of formal power series in the variables $X = \{x_1, x_2, \ldots, \}$.

$f(\mathbf{x}) \in \mathbb{Q}[[X]]$ is a symmetric function if for all $\sigma \in \cup_{n \geq 1} \mathfrak{S}_n$

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots).$$

A symmetric function is homogeneous of degree n if each term has degree n.
Example: $x_1x_2^2 + x_2x_1^2 + x_1x_3^2 + x_3x_1^2 + x_2x_3^2 + x_3x_2^2 + \ldots$ is a homogeneous symmetric function of degree 3.

Let Sym_n denote the vector space (over $\mathbb{Q}$) of homogeneous symmetric functions of degree n and let Sym denote the ring of symmetric functions of bounded degree.

Bases for Sym_n

We can view a partition $\lambda \vdash n$ as an infinite sequence by padding it with zeros. That is if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we can view λ as $(\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots)$.

Given an infinite sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive integers, let

$$\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

Monomial symmetric functions: For $\lambda \vdash n$, let

$$m_\lambda(\mathbf{x}) := \sum_\alpha \mathbf{x}^\alpha$$

where the sum ranges over distinct rearrangements α of λ viewed as an infinite sequence.

Example: $m_{2,1}(\mathbf{x}) := x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

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Elementary symmetric functions:

$$\begin{aligned} e_n(\mathbf{x}) &:= \sum_{i_1 < \cdots < i_n} x_{i_1} \ldots x_{i_n} \\ e_\lambda(\mathbf{x}) &:= e_{\lambda_1} \cdots e_{\lambda_k} \qquad \text{if } \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \end{aligned}$$

Basis for Sym_n

Complete homogeneous symmetric functions:

$$
\begin{aligned}
h_n(\mathbf{x}) &:= \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \ldots x_{i_n} \\
h_\lambda(\mathbf{x}) &:= h_{\lambda_1} \cdots h_{\lambda_k} \qquad \text{if } \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n
\end{aligned}
$$

Power-sum symmetric functions:

$$
\begin{aligned}
p_n(\mathbf{x}) &:= \sum_{i \geq 1} x_i^n \\
p_\lambda(\mathbf{x}) &:= p_{\lambda_1} \cdots p_{\lambda_k} \qquad \text{if } \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n
\end{aligned}
$$

Theorem

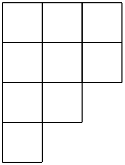
$\{m_\lambda : \lambda \vdash n\}, \{e_\lambda : \lambda \vdash n\}, \{h_\lambda : \lambda \vdash n\}, \{p_\lambda : \lambda \vdash n\}$ *are all basis for* Sym_n.

Thus the dimension of the vector space Sym_n equals the number of partitions of n.

Schur functions

Associate with each $\lambda \vdash n$, an array of cells with λ_i cells in row i for each i. This is called the Young diagram of shape λ.

Example: Young diagram of shape $(3, 3, 2, 1)$



A semistandard Young tableau of shape λ is a filling of the diagram λ with positive integers so that

- each row weakly increases from left to right
- each column strictly increases from top to bottom

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1	3	3
3	5	8
6	6	
7		

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$$x^T = x_1\, x_3^3\, x_5\, x_6^2\, x_7\, x_8$$

A semistandard Young tableau of shape λ is a filling of the diagram λ with positive integers so that

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Let SST_λ be the set of semistandard Young tableaux of shape λ. For each $T \in SST_\lambda$, let $x^T = x_1^{a_1} x_2^{a_2} \cdots$, where a_i is the number of occurances of i in T.

The Schur function of shape $\lambda \vdash n$ is

$$s_\lambda(\mathbf{x}) := \sum_{T \in SST_\lambda} x^T$$

Schur functions: $s_\lambda := \sum_{T \in SST_\lambda} x^T$

Example: The semistandard Young tableaux of shape $\lambda = (2, 1)$ with entries at most 3 are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} s_{2,1} &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \dots \\ &= m_{2,1} + 2m_{1,1,1} \end{aligned}$$

Schur functions: $s_\lambda := \sum_{T \in SST_\lambda} x^T$

Example: The semistandard Young tableaux of shape $\lambda = (2,1)$ with entries at most 3 are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} s_{2,1} &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \dots \\ &= m_{2,1} + 2m_{1,1,1} \end{aligned}$$

A semistandard tableau T has type $\alpha = (\alpha_1, \alpha_2, \dots)$ if T has α_i entries equal to i for each $i \in \mathbb{P}$. We write $type(T) = \alpha$. Note that $x^T = x^{type(T)}$.

It is not obvious that the Schur functions are symmetric. To prove that they are we only need to show

$$|\{T \in SST_\lambda : type(T) = \alpha\}| = |\{T \in SST_\lambda : type(T) = \beta\}|$$

whenever α and β are related by an adjacent transposition. There is a nice involution on SST_λ that proves this.

The Schur basis

Theorem (Schur basis)

$\{s_\lambda : \lambda \vdash n\}$ is a basis for Sym_n.

The Kostka numbers for $\lambda, \mu \vdash n$ are defined by

$$K_{\lambda,\mu} := |\{T \in SST_\lambda : type(T) = \mu\}|.$$

Once we establish the symmetry of the Schur functions, it is easy to see that for all $\lambda \vdash n$,

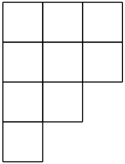
$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu$$

From this and a certain scalar product for which $\{s_\lambda\}$ is an orthonormal basis and the $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual, we get for all $\lambda \vdash n$,

$$h_\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda} s_\mu$$

The involution ω

The conjugate of a partition $\lambda \vdash n$ is the partition $\lambda' \vdash n$ whose Young diagram is the transpose of the Young diagram of λ.
Example: $\lambda = (3, 3, 2, 1)$

$\lambda =$ 

The involution ω

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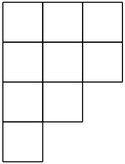
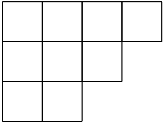
$\lambda =$ 

$\lambda' =$ 

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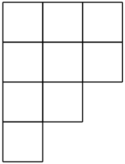
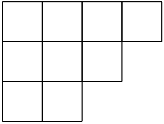
$\lambda =$ 

$\lambda' =$ 

Let $\omega : Sym_n \to Sym_n$ be the involution that takes s_λ to $s_{\lambda'}$ for all $\lambda \vdash n$.
What does ω do to other bases?

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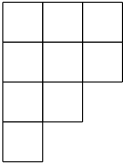
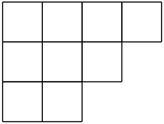
$\lambda =$ 

$\lambda' =$ 

Let $\omega : Sym_n \to Sym_n$ be the involution that takes s_λ to $s_{\lambda'}$ for all $\lambda \vdash n$.
What does ω do to other bases?

- $\omega(h_\lambda) = e_\lambda$
- $\omega(e_\lambda) = h_\lambda$
- $\omega(p_\lambda) = (-1)^{n-l(\lambda)} p_\lambda$

Other expansions

Recall

$$K_{\lambda,\mu} := |\{T \in SST_\lambda : type(T) = \mu\}|.$$

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu$$

$$h_\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda} s_\mu$$

Let $z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$, where m_i is the number of occurrences of i in $\mu \vdash n$. One can show

$$h_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} p_\mu.$$

Applying the involution ω yields

$$e_\lambda = \sum_{\mu \vdash n} K_{\mu',\lambda} s_\mu$$

$$e_\lambda = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} z_\lambda^{-1} p_\mu.$$

Quasisymmetric functions

$f(\mathbf{x}) \in \mathbb{Q}[[X]]$ is a quasisymmetric function if

$$\mathrm{coef}(f; x_1^{a_1} \dots x_k^{a_k}) = \mathrm{coef}(f; x_{i_1}^{a_1} \dots x_{i_k}^{a_k})$$

for all $i_1 < \cdots < i_k$ and $a_1, \dots, a_k \in \mathbb{N}$.

Let $QSym_n$ denote the vector space of homogeneous quasisymmetric functions of degree n and let $QSym$ denote the ring of quasisymmetric functions of bounded degree.

Note: Every symmetric function is quasisymmetric, but not conversely.

Examples:

$$f(x) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + \dots$$

$$g(x) = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_1x_4^2 + x_2x_4^2 + x_3x_4^2 + \dots$$

These are examples of monomial quasisymmetric functions.

Monomial basis for $QSym_n$

Monomial quasisymmetric functions: Given $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$, let

$$M_\alpha := \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}.$$

Examples.

$$M_{2,1} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \ldots$$

$$M_{1,2} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_4^2 + x_2 x_4^2 + x_3 x_4^2 + \ldots$$

Note. $M_{2,1} + M_{1,2} = m_{2,1}$.

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Note. $M_{2,1} + M_{1,2} = m_{2,1}$.

More generally, for $\lambda \vdash n$,

$$m_\lambda = \sum_\alpha M_\alpha,$$

where the α ranges over all compositions of n whose decreasing rearrangement is λ.

Fact. $\{M_\alpha | \alpha \models n\}$ is a basis for $QSym_n$. Thus dim $QSym_n$ equals the number of compositions of n, which is 2^{n-1}

Gessel's Fundamental basis for $QSym_n$

For $S \in [n-1]$, let

$$F_S := \sum_{\substack{i_1 \geq \ldots \geq i_n \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \ldots x_{i_n}.$$

Theorem (Gessel - 1984)

$\{F_S : S \subseteq [n-1]\}$ *is a basis for* $QSym_n$

Note:

- $F_\emptyset = h_n$
- $F_{[n-1]} = e_n$

Involution ω extends to the larger space $QSym_n$ as follows.
$\omega : QSym_n \to QSym_n$ is defined on basis elements by

$$\omega(F_S) = F_{[n-1]\setminus S}.$$

For symmetric functions this is the same involution that was defined before. Note

$$\omega(h_n) = \omega(F_\emptyset) = F_{[n-1]} = e_n$$

Expansion of the Schur functions in F-basis.

A standard Young tableau of shape λ is a filling of the diagram $\lambda \vdash n$ with distinct entries $1, 2, \ldots, n$ so that the rows and columns (strictly) increase.

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & 9 \\ \hline 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 8 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Let SYT_λ be the set of standard Young tableaux of shape λ.
For $T \in SYT_\lambda$, let

$$DES(T) := \{i \in [n] : i \text{ is higher than i+1 in } T\}.$$

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For all $\lambda \vdash n$,

$$s_\lambda = \sum_{T \in SYT_\lambda} F_{DES(T)}.$$

Specialization

For $f(x) \in \mathbb{R}[[X]]$, define the stable principal specialization as follows:

$$\mathrm{ps}(f(x_1, x_2, \ldots,)) := f(1, q, q^2, \ldots)$$

Lemma (Gessel)

For all $S \subseteq [n-1]$,

$$\mathrm{ps}(F_S) = \frac{q^{\sum S}}{(1-q)(1-q^2)\ldots(1-q^n)},$$

where $\sum S := \sum_{s \in S} s$.

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Consequently

$$\begin{aligned} \mathrm{ps}(s_\lambda) &= \sum_{T \in SYT_\lambda} \mathrm{ps}(F_{DES(T)}) \\ &= \frac{\sum_{T \in SYT_\lambda} q^{\mathrm{maj}(T)}}{(1-q)(1-q^2)\ldots(1-q^n)}, \end{aligned}$$

where $\mathrm{maj}(T) = \sum_{i \in DES(T)} i$.

q-analog of hook legth formula

$$\mathrm{ps}(s_\lambda) = \frac{\sum_{T \in SYT_\lambda} q^{\mathrm{maj}(T)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Theorem (Stanley)

$$\mathrm{ps}(s_\lambda) = \frac{q^{b_\lambda}}{\prod_{x \in \lambda}(1-q^{h_x})},$$

where $b(\lambda) = \sum (i-1)\lambda_i$.

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Corollary

$$\sum_{T \in SYT_\lambda} q^{\mathrm{maj}(T)} = q^{b(\lambda)} \frac{[n]_q!}{\prod_{x \in \lambda}[h_x]_q}.$$