Enumeration of tableaux and plane partitions

Sylvie Corteel

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1 Lecture 1

1.1 Integer partitions

Partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $|\lambda| = \sum_i \lambda_i n$, we write it as $\lambda \vdash n$. By convention we let $\lambda_0 = \infty$ and there exists k such that $\lambda_i = 0$ for $i > k$, the number of distinct parts $\#\{\lambda_j > \lambda_{j+1}\}$.

Example 1.1. *Partitions of* 5 *are* $5, 41, 32, 311, 221, 2111, 11111$.

Let $p(n)$ be the number of partitions of n.

Proposition 1.2. $\sum_{n\geq 0} p(n)q^n = \prod_{i\geq 1} \frac{1}{1-q^i}$.

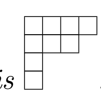
Proof. Note that $\frac{1}{1-q^i} = q + q^i + q^{2i} + q^{3i} + \cdots$. □

Let $p_S(n)$ be the number of partitions of n where the parts belong to a multiset S, then

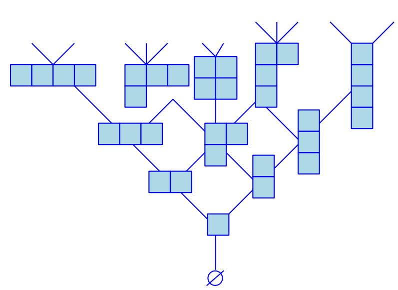
$$\sum_{n\geq 0} p_S(n)q^n = \prod_{i\in S} \frac{1}{1-q^i}.$$

1.2 Young diagrams

A Young diagram is an arrangement of boxes left justified such that there are λ_i squares/cell in the ith row.

Example 1.3. *For the partition* $\lambda = (4, 3, 1, 1)$ *the Young diagram is*  .

The partitions form a poset called *Young's lattice* consisting of partitions ordered by containment of their Young diagrams: $\lambda \lhd \mu$ if the diagram of λ is contained in the diagram of μ and $|\lambda| = |\mu| - 1$.



Our goal is to enumerate walks in Young's lattice that start at $\varnothing$ arrive at λ and use up (U) or down steps (D).

Fix a word $w \in \{U, D\}^*$ $w = A_n A_{n-1} \cdots A_2 A_1$ where A_i is U or D.

Example 1.4. $\varnothing, (1), (2), (1), (1,1), (1,1,1), (2,1,1), (1,1,1) = \lambda$ *where the steps are* U, U, D, U, U, U, D *and so* $w = DUUDUU$.

Example 1.5 (Walks of type U^n)**.**

$$\varnothing, \quad \begin{array}{|c|}\hline 1\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}.$$

Definition 1.6 (Standard Young tableaux)**.** Let $\lambda \vdash n$, a Standard Young tableau (SYT) of shape λ is a filling of the cells of the Young diagram of λ with all the elements $\{1, 2, \ldots, n\}$ that is increases in rows and columns. The number of SYT of shape λ is denoted by f^λ. $\square$

The number f^λ has a beautiful product formula proved in the 50s.

Theorem 1.7.

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where $h(u)$ *is the length of the* hook *of* u*: the number of cells under in the same column and to the right in the same row of* u *(including* u*).*

Example 1.8. *For the shape* $(2,2,1)$*, the hook-lengths are given by* $\begin{array}{|c|c|}\hline 4&2\\\hline 3&1\\\hline 1\\\cline{1-1}\end{array}$.

$$f^{(2,2,1)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

There are several proofs of this formula including a beautiful probabilistic proof by Greene-Nijenhuis-Wilf.

Lemma 1.9. $w = D^{S_k} U^{r_k} \ldots D^{S_1} U^{r_1}$. *and* $\lambda \vdash n$. *There exists at least one walk from* $\varnothing$ *to* λ *of type* W *if*

- $\sum_{i=1}^k r_i - \sum_{i=1}^k s_i = n$,
- *for all* j $\sum_{i=1}^j (r_i - s_i) \geq 0$

We call such a word is λ*-valid.*

Remark 1.10. The number of valid words of length $2n$ starting at $\varnothing$ to $\varnothing$ is given by the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$. $\square$

1.3 Linear transformations on integer partitions

We will use physics notation: $\langle\lambda|$ ("bra") and $|\lambda\rangle$ ("ket") so that

$$\langle\lambda|\mu\rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

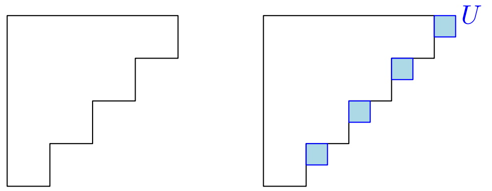
$$U|\lambda\rangle = \sum_{\mu \rhd \lambda} |\mu\rangle ,$$
$$\langle\lambda| U = \sum_{\mu \rhd \lambda} \langle\mu| ,$$
$$D|\lambda\rangle = \sum_{\mu \lhd \lambda} |\mu\rangle ,$$
$$\langle\lambda| D = \sum_{\mu \lhd \lambda} \langle\mu| ,$$

Example 1.11.

$$U|(3,2)\rangle = |(3,2,1)\rangle + |(3,3)\rangle + |(4,2)\rangle ,$$
$$\langle(3,2)| D = \langle(3,2,1)| + \langle(3,3)| + \langle(4,2)| .$$

Proposition 1.12. *If λ has r distinct parts then $U|\lambda\rangle$ has $r+1$ terms and $D|\lambda\rangle$ has $r-1$ parts.*

Proof. The number of distinct parts of λ is the number of corners of λ. The operator U will add a corner to λ and the operator D will delete a corner to λ.



□

The key lemma to study our goal is the following identity.

Lemma 1.13.
$$DU = UD + I,$$
where I is the identity operator.

Proof.
$$DU|\lambda\rangle = \sum_{\mu} |\mu\rangle + (r+1)|\lambda\rangle$$
where the first sum is over μ such that there exist j, k such that $\lambda_j = \mu_j - 1$ and $\lambda_k = \mu_k + 1$.
$$UD|\lambda\rangle = \sum_{\mu} |\mu\rangle + r|\lambda\rangle .$$
where the first sum is over μ such that there exist j, k such that $\lambda_j = \mu_j - 1$ and $\lambda_k = \mu_k + 1$.

When we take the difference $(DU - UD)|\lambda\rangle$ we obtain $|\lambda\rangle$. □

Theorem 1.14. *Let λ be a partition, w be a λ-valid word. The number of walks from $\varnothing$ to λ of type w is $f^{\lambda} \cdot \prod_{i \in S_w}(a_i - b_i)$, where $a_i = \#\{j < i \mid A_j = U\}$, $b_i = \#\{j < i \mid A_j = D\}$ and $S_w = \{i \mid A_i = D\}$.*

Note that the second term in the product is independent of λ.

Example 1.15. *$W = U^3D^2U^2DUT3$, $S_w = \{4, 7, 8\}$ (positions from right to left), $a_4 = 3, a_7 = 5, a_8 = 5, b_4 = 0, b_7 = 1, b_8 = 2$. Then the number of walks from $\varnothing$ to any $\lambda \vdash 5$ is $f^\lambda \cdot (3-0)(5-1)(5-2)$.*

Proof. We want to compute $\langle\lambda|\, w \,|\varnothing\rangle$. When w see a DU by Lemma **??** we replace it by UD or I repeatedly and we obtain

$$\langle\lambda|\, W \,|\varnothing\rangle = \sum_{i-j=n} r_{i,j}(w) \,\langle\lambda|\, U^i D^j \,|\varnothing\rangle = \sum_{i-j=n} r_{n,0}(\varnothing) \,\langle\lambda|\, U^n \,|\varnothing\rangle \,,$$

donde $\langle\lambda|\, U^n \,|\varnothing\rangle = f^\lambda$. One can show that the coefficients $r_{i,j}(w)$ satisfy

$$\begin{aligned} r_{i,j}(w) &= 0; \quad \text{if } i < 0 \text{ or } j < 0 \text{ or } i - j \neq n \\ r_{0,0}(\emptyset) &= 1 \\ r_{i,j}(Uw) &= r_{i-1,j}(w); \\ r_{i,j}(Dw) &= r_{i,j-1}(w) + (i+1)r_{i+1,j}(w). \end{aligned}$$

This determines the $r_{i,j}(w)$ to be $\prod_{i \in S_w}(a_i - b_i)$. ∎