Exercises: Algebraic Structures on Combinatorial Species

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1. Antipodes and characters

(1) * Recall that in a Hopf monoid, the antipode map satisifies:

$$\mathrm{s}_I(x \cdot y) = \mathrm{s}_T(y) \cdot \mathrm{s}_S(x).$$

Use this to recover the antipode formula for the Hopf monoid E, and also to give a formula for the antipode of the Hopf monoid L.

(2) * We say that a partition Y *refines* a partition X if each block of Y is contained in a block of X, and we write $X \leq Y$. Define

$$(Y/X)! := \prod_{B \in X} (n_B)!,$$

where $n_B = l(Y|_B)$ is the number of blocks of Y that are contained in the block B of X.

(a) Let $I = \{1, 2, 3\}$ and $X = \{\{1, 2, 3\}\} \in \Pi[I]$. Using Takeuchi's formula compute $\mathrm{s}_I(X)$, and verify that

$$S_I(X) = \sum_{Y : X \leq Y} (-1)^{l(Y)} (Y/X)!\, Y.$$

Can you prove this formula in general?

(b) A composition G *refines* a composition F if each block of F is obtained by merging a number of contiguous blocks of G, and we write $F \leq G$. The *opposite* of a composition $F = (S_1, ..., S_k)$ is the composition $\overline{F} = (S_k, \ldots, S_1)$.

Show that the antipode of the Hopf monoid Σ of set compositions is given by

$$S_I(F) = \sum_{G : \overline{F} \leq G} (-1)^{l(G)} G.$$

(c) Recover the antipode formula for L by applying Exercise 2.6 (in worksheet 2) to the inclusion $\mathrm{L} \to \Sigma$.

(3) Use Takeuchi's formula to compute the antipode of the species L.

(4) * Describe all characters $\zeta : \mathrm{L} \to \mathrm{E}$, and the character group $\mathbb{X}(\mathrm{L})$.

(5) Let G be the species of simple graphs: $\mathrm{G}[I]$ is the set of simple graphs with vertex set I. It is a Hopf monoid where: $g_1 \cdot g_2$ is the graph whose edges are those of g_1 and g_2 (the union of the graphs), $g|_S$ is the graph whose edges are those edges of g with both vertices in S (the graph induced by g on S), and $g/_S = g|_T$.

Compute the antipode on the path on 3 vertices using Takeuchi's formula.

(6) Let G denote the Hopf monoid of graphs. Let ζ be the character defined by

$$\zeta_I(G) = q^{\#\text{ of edges of } G.}$$

Compute $\chi_I(G)(n)$ for the complete graphs K_3 and K_4. Can you find a general formula for K_n?