Symmetric Functions and Eulerian Polynomials

Lecture 3: Eulerian Quasisymmetric functions

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q-Eulerian polynomials

$$A_n(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

and

$$\left\langle {n \atop j} \right\rangle_q := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}$$

Theorem (Shareshian & MW 2006)

$$\sum_{n \geq 0} A_n(q,t) \frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}$$

Lift to a Symmetric Function identity

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Lift to a Symmetric Function identity

Let

$$H(z) := \sum_{n \geq 0} h_n z^n,$$

where h_n is the nth complete homogeneous symmetric function

$$h_n : \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

$$\frac{(1-t)H(z)}{H(zt) - tH(z)}$$

$$\Big\downarrow \quad \begin{aligned} x_i &:= q^{i-1} \\ z &:= z(1-q) \end{aligned}$$

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$$\begin{array}{ccc}
{\color{red}?} & = & \dfrac{(1-t)H(z)}{H(zt) - tH(z)} \\
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\end{array}$$

Lift to a Symmetric Function identity

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each excedance.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

Define

$$\mathrm{DEX}(\sigma) := \mathrm{DES}(\bar{\sigma})$$

$$\mathrm{DEX}(531462) = \mathrm{DES}(\bar{5}.\bar{3}14.\bar{6}2) = \{1, 4\}$$

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$$\sum_{i \in \mathrm{DEX}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$$

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$$\mathrm{maj}(531462) = 1 + 2 + 5 = 8 \qquad \mathrm{exc}(531462) = 3$$

Recall

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Hence

$$F_{\mathrm{DEX}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

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Thus

$$\begin{aligned}
\mathrm{ps}(\sum_{\sigma\in\mathfrak{S}_n} F_{\mathrm{DEX}(\sigma)} t^{\mathrm{exc}(\sigma)}) &= \frac{\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)} \\
&= \frac{A_n(q,t)}{(1-q)(1-q^2)\dots(1-q^n)}
\end{aligned}$$

$$\mathrm{ps}(\sum_{\sigma\in\mathfrak{S}_n} F_{\mathrm{DEX}(\sigma)} t^{\mathrm{exc}(\sigma)})\ (1-q)^n = \frac{A_n(q,t)}{[n]_q!}$$

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Now define the Eulerian quasisymmetric function

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}$$

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Cycle-type Eulerian Quasisymmetric Functions

Eulerian quasisymmetric fcn.

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}$$

Theorem (Shareshian and MW)

$$\sum_{n \geq 0} \left(\sum_{j=0}^{n-1} Q_{n,j} t^j \right) z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}$$

In order to prove the theorem we needed to consider

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_\lambda \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}$$

Clearly

$$Q_{n,j} = \sum_{\lambda \vdash n} Q_{\lambda,j}$$

and

$$\mathrm{ps}(Q_{n,\lambda}) = \frac{A_\lambda(q,t)}{(1-q)(1-q^2)\ldots,(1-q^n)}.$$

Bicolored necklaces and ornaments

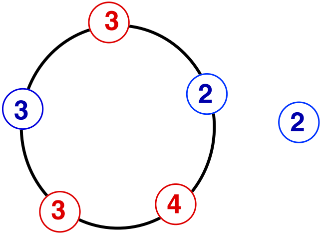
A bicolored necklace is a primitive circular word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

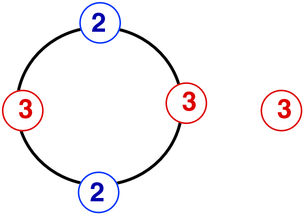
such that if size > 1

- a blue letter is followed by letter greater than or equal in value
- a red letter is followed by a letter less than or equal in value

Necklaces of size 1 are blue.



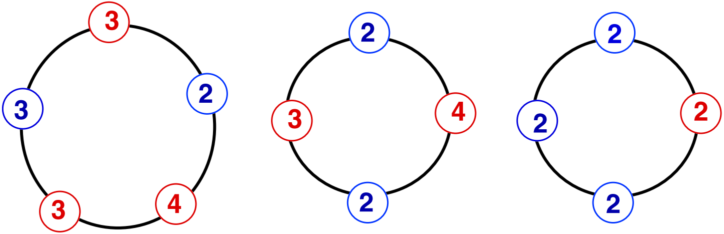
necklaces



not necklaces

Bicolored necklaces and ornaments

An ornament of type λ is a multiset of necklaces whose necklace sizes form partition λ



type $= (5, 4, 4)$ weight $= x_2^7 x_3^4 x_4^2$

Let $\mathcal{R}_{\lambda,j}$ = set of ornaments of type λ with j red letters.

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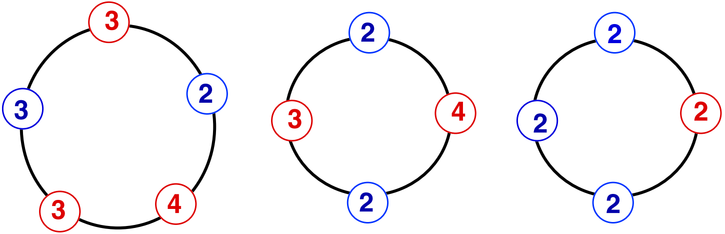

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Theorem (Shareshian and MW (2006))

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} wt(R)$$

The bijection that proves $Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} wt(R)$

$$F_{\mathrm{DEX}(\sigma)} = \sum_{\substack{s_1 \geq \cdots \geq s_n \\ i \in \mathrm{DEX}(\sigma) \Rightarrow s_i > s_{i+1}}} x_{s_1} \ldots x_{s_n}$$

Let $\sigma = 45162387$

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Let $\sigma = \bar{4}\bar{5}1.\bar{6}23.\bar{8}7 \qquad s = (7,7,7,5,5,4,2,2)$

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1. write σ in cycle form,

$$\sigma = (1,4,6,3)(2,5)(7,8).$$

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(This is a colored analog of a bijection of Gessel and Reutenauer.)

Eulerian Quasisymmetric Functions

Eulerian quasisymmetric fcn.

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Theorem (Shareshian and MW)

$$\sum_{n \geq 0} \left(\sum_{j=0}^{n-1} Q_{n,j} t^j \right) z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}$$

Note: This theorem implies that $Q_{n,j}$ is actually symmetric.

It also implies that $Q_{n,j}$ is *h*-positive (which means that when we expand it in the h-basis, all the coefficients are positive). We can use it to show that $\sum_{j=0}^{n} Q_{n,j} t^j$ is palindromic and *h*-unimodal (which means that $Q_{n,j} - Q_{n,j-1}$ is h-positive when $j < \frac{n-1}{2}$).

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Note that h-positivity implies Schur-positivity and p-positivity.

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Properties of the Eulerian quasisymmetric functions

Property	$\sum_{j\geq 0} Q_{n,j}t^j$	$\sum_{j\geq 0} Q_{\lambda,j}t^j$
palindromic	yes	
symmetric	yes	
h-positive	yes	
h-unimodal	yes	

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$$Q_{(6),3} = 2h_{(4,2)} - h_{(4,1,1)} + h_{(3,2,1)} + h_{(5,1)}$$

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palindromic	yes	yes
symmetric	yes	yes
h-positive	yes	no
h-unimodal	yes	no

$$Q_{(6),3} = 2h_{(4,2)} - h_{(4,1,1)} + h_{(3,2,1)} + h_{(5,1)}$$

$$Q_{(6),3} = 3s_{(6)} + 3s_{(5,1)} + 3s_{(4,2)}) + s_{(3,3)} + s_{3,2,1}$$

Theorem (Henderson and MW): $\sum_{j\geq 0} Q_{\lambda,j}t^j$ is Schur-positive and Schur-unimodal. Consequently $A_\lambda(q,t)$ is q-unimodal.

Theorem (Sagan, Shareshian and MW): $\sum_{j\geq 0} Q_{\lambda,j}t^j$ is p-positive.

Schur-Gamma-positivity of $Q_{n,j}$

Theorem (Gessel)

$$1 + \sum_{n \geq 1} z^n \sum_{w \in ND_{n,i}(\mathbb{P})} x_w t^i (1+t)^{n-1-2i} = \frac{(1-t)H(z)}{H(tz) - tH(z)}$$

where $x_w := x_{w_1} x_{w_2} \dots x_{w_n}$ *and*

$$ND_{n,i}(\mathbb{P}) := \{w \in \mathbb{P}^n \mid w0 \textit{ has no double descents \& } \mathrm{des}(w) = i\}$$

$$w = 779.1558.25 \;\in\; ND_{9,2}(\mathbb{P})$$

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$$T = \begin{array}{ccccccc} & & & & & \boxed{2} & \boxed{5} \\ & & \boxed{1} & \boxed{5} & \boxed{5} & \boxed{8} & \\ \boxed{7} & \boxed{7} & \boxed{9} & & & & \end{array}$$

$$x_w = x^T$$

Summing over words in $ND_{n,i}$ with the same descent set gives a Schur function of hook shape determined by the descent set.

Schur-Gamma-positivity of $Q_{n,j}$

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \Gamma_{n,i} \; t^i (1+t)^{n-1-2i}$$

where

$$\Gamma_{n,i} := \sum_{\mu \in SH_{n,i}} s_\mu$$

and $SH_{n,i}$ is the set of skew hooks of size n where

- all columns have size at most 2
- last column has size 1
- i columns have size 2

Schur-Gamma-positivity of $Q_{n,j}$

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \Gamma_{n,i} \; t^i (1+t)^{n-1-2i}$$

where

$$\Gamma_{n,i} := \sum_{\mu \in SH_{n,i}} s_\mu$$

and $SH_{n,i}$ is the set of skew hooks of size n where

- all columns have size at most 2
- last column has size 1
- i columns have size 2

Fact. Skew Schur-functions are Schur positive (Littlewood-Richardson rule is used to find the coefficients).

Thus $\Gamma_{n,i}$ is Schur-positive, which means that $\sum_{j=0}^{n-1} Q_{n,j} t^j$ is Schur-γ-positive.