On Tutte polynomial, medial graphs and links

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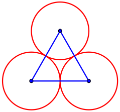
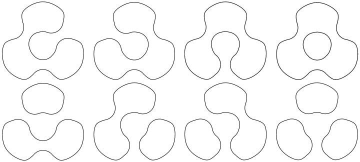
5th Encuentro Colombiano de Combinatoria ECCO
June 14 of 2016

Motivation

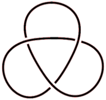
Eulerian partitions

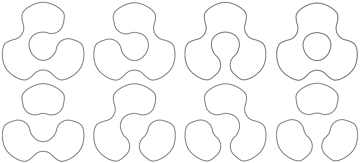




$$3 + 4z + z^2$$

Khovanov's Homology





$$t + t^3 - t^4$$

Motivation

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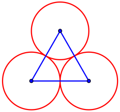
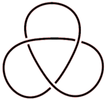
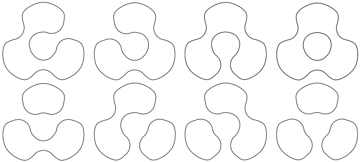
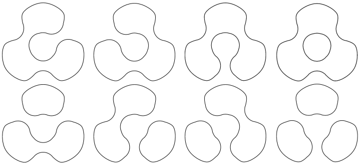
Khovanov's Homology









$$3 + 4z + z^2$$

$$t + t^3 - t^4$$

Is there any relationship?

Sketch of talk

1. Preliminaries
2. Tutte polynomial
3. Links and Jones polynomial
4. Results

1. Preliminaries

Graphs:

A pair of non-empty sets (E, V), namely E the set of edges and V the set of vertices, such that every two elements in V are connected by an edge.

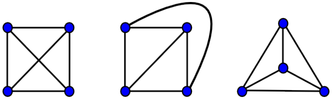


Figure: Visual representation of a graph.

Simple cycles in a graph:

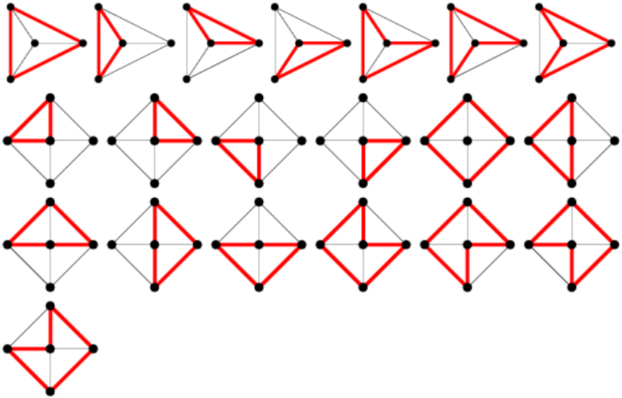


Figure: Symple cycles in a graph.

Medial graph:
Given a plane and connected graph G, its medial graph $M(G) = H$ is a 4-regular graph such that have its vertices over the edges of G and where two vertices are connected only if they're adjacent.

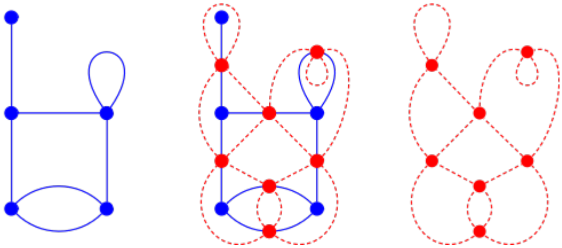
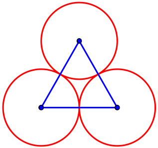
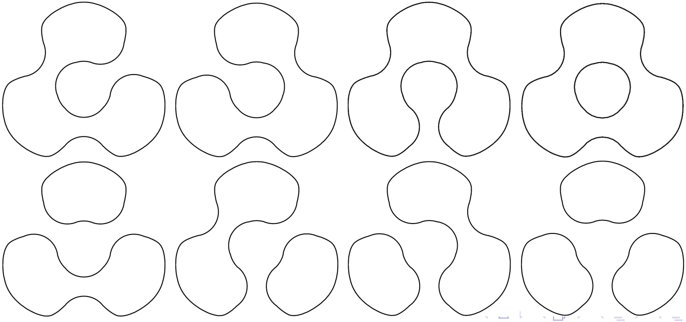


Figure: A connected graph and its medial graph.

Eulerian partitions without crossings in a graph:
An *eulerian partition* π of H, is a partition of the edges of H in to a edge-disjoint cycles. This cycles may share vertices, but they may have not any edges in common.
Eulerian partitions without crossings are eulerian partitions that hace no crossings at any vertex.

Eulerian partitions without crossings of the medial graph of a triangle:





Theorem

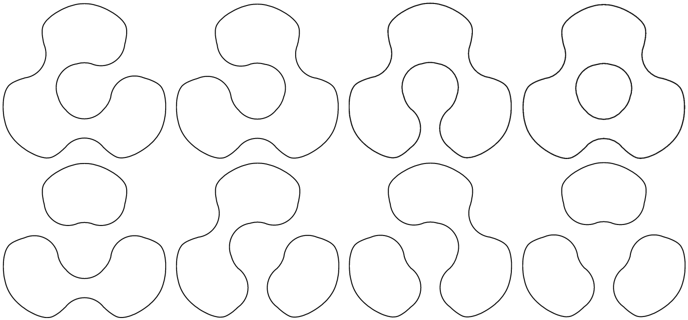
The number of eulerian partitions without crossings of the medial graph H of a given planar and connected graph G is $2^{|E|}$

Martin-Las Vergnas polynomial:
Given a planar (connected) graph G, H its medial graph and P an eulerian partition without crossings of H. Let be $\gamma(P)$ the number of simple cycles in P. The martin-lasVergnas polynomial of H is given by

$$ML(z) = \sum_{P \in \Pi_{nc}(H)} z^{\gamma(p)-1}$$

where $\Pi_{nc}(H)$ denote the set of eulerian partitions without crossings of H.

For the medial graph of a triangle we have that:



then we have $ML(z)_{K_3} = 3z^0 + 4z^1 + 1z^2 = 3 + 4z + z^2$

2. Tutte polynomial

Tutte polynomial

Tutte polynomial is one of the most important graphical invariants and usually appears in enumerative problems.
Let be G a graph, A subset of E and let be $r(A)$ the range of A, then the tutte polynomial of G is given by:

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{r-r(A)} (y-1)^{|A|-r(A)}$$

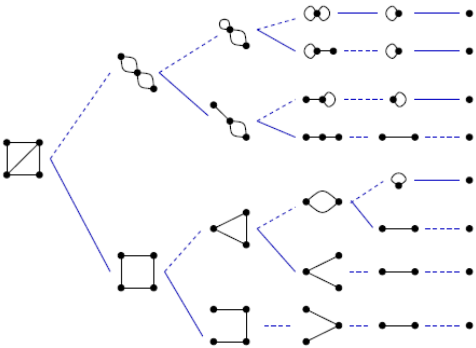
Edge deletion-contration

Deletion: From now on $G \setminus e$ is the obtained graph from G by removing keeping its end vertices.

Contraction: G/e is the obtained graph from G by collapsing the end vertices of e in just one vertex

Tutte polynomial

Example

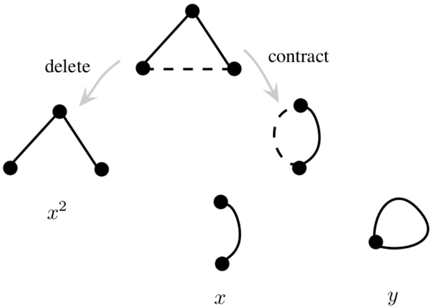


Tutte-Grothendieck invariants:

$$T_G(x,y) = \begin{cases} T_{G\setminus e}(x,y) + T_{G/e}(x,y) & \text{if e is ordinary;} \\ y \cdot T_{G\setminus e}(x,y) & \text{if e is a loop;} \\ x \cdot T_{G\setminus e}(x,y) & \text{if e is a coloop;} \\ 1 & \text{if G is the empty graph;} \end{cases}$$

Tutte polynomial

For the triangle we have:



$$T(K_3; x, y) = x^2 + x + y$$

Tutte polynomial

Theorem

Let be G a connected graph, then:

- $T_G(1,1)$ *is the number of spanning trees of G*
- $T_G(2,0)$ *is the number of acyclic orientations of G*
- *The chromatic polynomial* $\chi_G(u)$ *of G is given by :*

$$\chi_G(u) = u^{K(E)}(-1)^{r(E)}T_G(1-u,0)$$

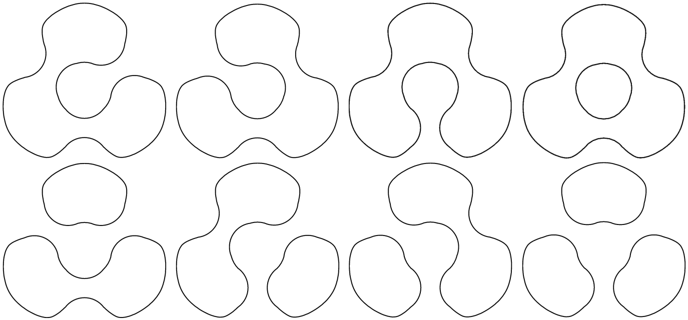
Theorem (Martin-Las Vergnas)

Let G a plane and connected graph, we denote by $\Pi_{nc}(H)$ the set of eulerian partitions without crossings of $H = M(G)$ the medial graph of G if $\gamma(P)$ is the number of simple cycles in any partition P, then:

$$T_G(z+1, z+1) = \sum_{P \in \Pi_{nc}(H)} (z)^{\gamma(P)-1} = ML(z)$$

Tutte polynomial

Example:



Thus we have:

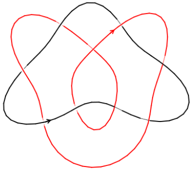
$$\begin{aligned} T(z+1, z+1) &= (z+1)^2 + (z+1) + (z+1) \\ &= z^2 + 2z + 1 + z + 1 + z + 1 \\ &= 3 + 4z + z^2 \end{aligned}$$

3. Links and Jones polynomial

Links:

A link is a smooth embedding (image) of several disjoint circles in $\mathbb{R}^3$.

An oriented link is meant a smooth mapping of the disjoint union of oriented circles.



States For a link diagram D, a state for D is a diagram obtainedby replacing each crossing if D with 0,1-smoothings. The result is a disjoint union of simple loops.

Figure: 0,1- smoothinngs on any crossing.

Figure: A link diagram and one of its states.

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Theorem

The number of states of a diagram with n crossings is 2^n.

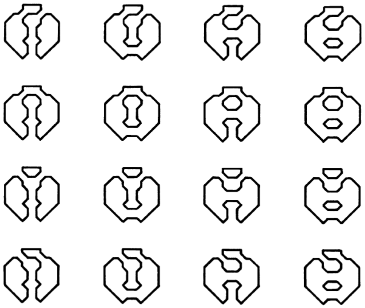


Figure: The 16 states of the diagram.

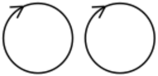
Links and Jones polynomial

The Jones polynomial is a knot polynomial discovered by Vaughan Jones in 1984. Specifically, it is an invariant of an oriented knot or link which assigns to each oriented knot or link a Laurent polynomial in the variable $t^{1/2}$ with integer coefficients and is defined by

$$V_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_P (-1)^r q^r (q + q^{-1})^{k-1}$$

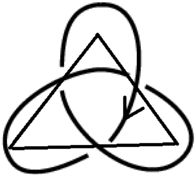
where k is the number of simple cycles and r is the number of *1-smoothings* on P

V  $= 1$

V  $= -\left(\frac{1}{\sqrt{t}} + \sqrt{t}\right)$

V  $= t + t^3 - t^4$

A link diagram is alternating if the crossings alternate under, over, under, over, as one travels along each component of the link. A link is alternating if it has an alternating diagram.



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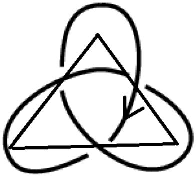


Theorem (Jones, 1985)

Given K an alternanting link and G its subjacent graph, then:

$$V_K(t) = (-1)^{W(K)} \cdot t^{\frac{1}{4}(3(WK)-2V+E+2)} \cdot T_G(-t, \frac{-1}{t})$$

, where W(K) is the writhe number of K.

Links and Jones polynomial

Example:

W(K)=3-0=3 , therefore

$$V(t) = (-1)^3 t^{\frac{(9-6+3+2)}{4}} (t^2 - t - \frac{1}{t})$$

$$= -t^2(t^2 - t - \frac{1}{t}) = t^3 + t - t^4$$

4. Results

Our polynomial!!!

Eulerian polynomial

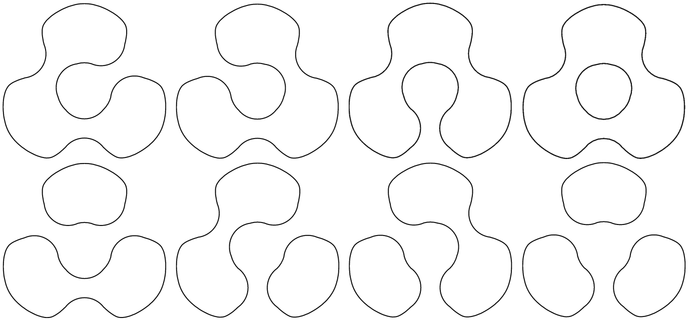
Given a graph G (plane and connected), H its medial medial and $\Pi_{nc}(H)$ the set of eulerian partitions of H, the eulerian polynomial of G si given by:

$$E_G(a,b) = \sum_{P \in \Pi_{nc}(H)} a^{K(P)} b^{\gamma(P)-1}$$

where K(P) is number of 1-smoothings on P.

Results

Example:



Therefore we have $E_{K_3}(a,b) = a^3 b^2 + 3a^2 b + 3a + b$

Results

relationship with Martin-LasVergnas

Recall that the Martin-LasVergnas polynomial was defined by:

$$ML_H(x) = \sum_{P \in \Pi_{nc}(H)} (z)^{\gamma(P)-1}$$

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Theorem

Given a plane and connected graph G, H its medial graph, then:

$$ML_H(z) = E_G(1, z)$$

Results

Example

We saw that for the triangle $ML_{M(K_3)}(z) = 3 + 4z + z^2$ but also $E_{K_3}(a, b) = a^3b^2 + 3a^2b + 3a + b$. Then:

$$\begin{aligned} E_{K_3}(1, z) &= 1^3(z)^2 + 3(1)^2(z) + 3(1) + (z) \\ &= z^2 + 3z + 3 + z \\ &= z^2 + 4z + 3 \end{aligned}$$

relationship with Jones polynomial

Remember that the jones polynomial was defined as

$$V_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_P (-1)^r q^r (q + q^{-1})^{k-1}$$

A straightforward computation show that:

$$V_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} E_G(-q, q + q^{-1})$$

Results

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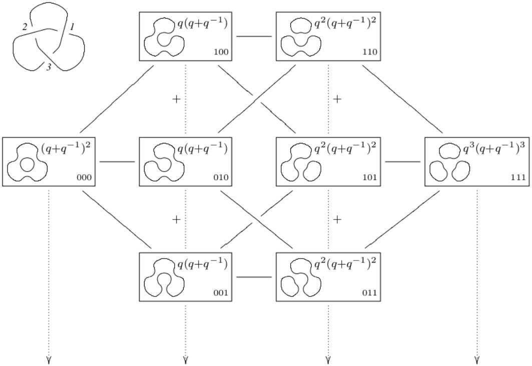
Remember that the jones polynomial for the trefoil knot is given by $V(t) = t + t^3 - t^4$. Now let see that our computation agree with this.



$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

$$q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\circledS)=q^2+q^6-q^8.$$

For the last taking $t=q^2$ we get $V(t)=t+t^3-t^4$.

Theorem (-S.,2015$^+$)

Let G a plane and connected graph, H its medial graph, E_G its eulerian polynomial is given by the recurrence

$$E_G(a,b) = \begin{cases} aE_{G\setminus e}(a,b) + E_{G/e}(a,b) & \text{if } e \text{ is ordinary;} \\ E_{G\setminus e}(a,b)(a+b) & \text{if } e \text{ is a loop;} \\ E_{G/e}(a,b)(ab+1) & \text{if } e \text{ is a coloop} \end{cases}$$

Therefore:

$$E_G(a,b) = a^{n-\gamma} T_G(ab+1, a^{-1}b+1)$$

Results

Example

The tutte polynomial of a triangle is $x^2 + x + y$, then

$$\begin{aligned} E_{K_3}(a,b) &= a((ab+1)^2 + (ab+1) + a^{-1}b + 1) \\ &= a(a^2b^2 + 2ab + 3 + ab + a^{-1}b) \\ &= a^3b^2 + 3a^2b + 3a + b, \end{aligned}$$

In loving memory of my grandma Nidia.
1954 - 2016

Thank you very much!