**CIMPA Research School:**
**Algebraic, Enumerative and Geometric Combinatorics - ECCO 2016**

# Triangulations of polytopes. Problem sheet.

## 2 Oriented matroids

1. *Show that there are only three possible oriented matroids for five points in general position (=no 3 colinear) in the plane.

2. *Write down the whole list of circuits and cocircuits of the first two point configurations of Problem 1.1. (Clue, since the configurations are in general position there should be exactly $\binom{6}{4}$ pairs of opposite circuits and $\binom{6}{2}$ pairs of opposite cocircuits in each of them).

3. Write down the circuits and cocircuits of the following *vector* configuration of rank 3: $\{(0,2,1),(0,-1,-2),(1,0,2),(-2,0,-1),(2,1,0),$ $(-1,-2,0)\}$. Conclude that it is dual to the vertices of the hexagon.

4. Show that the above list of vectors is actually a Gale transform of the set of vertices of the hexagon as given in Problem 1.1. Remark: the extra coordinate that you have (the configuration is two dimensional but it is embedded in an affine plane in $\mathbb{R}^3$) implies that you can already think of the vertices of the hexagon as vectors, without the need of an additional coordinate constantly equal to 1. The reason is that this extra coordinate would be a linear combination of the three that you have, since your points lie in the plane $(x_1 + x_2 + x_3)/4 = 1$ in $\mathbb{R}^3$.

5. *Construct a Gale transform of the m.o.a.e. (the first configuration in Problem 1.1). Same remark as above about the third coordinate.

6. *Consider versions of the m.o.a.e. where the inner triangle is rotated with respect to the outer one. (That is, you have the six vertices of two equilateral triangles with the same barycenter and one inside the other). Convince yourself of the following behavior: if the rotation is small, neither the oriented matroid nor the set of triangulations of your point set changes. As you keep rotating, there is a precise point when both the oriented matroid and the set of triangulations change. Try to describe the change. (How many and which circuits, cocircuits, and triangulations are affected?).

7. **Oriented matroid of a directed graph.** Let $G$ be a directed graph with $n$ edges and $k$ vertices: a pair $(V,E)$ where $V = \{1,\dots,k\}$ is a set of $k$ vertices and $E$ a list of $n$ directed edges (each directed edge is an ordered pair $\overrightarrow{ij} := (i,j)$ with $i,j \in V$). Consider the directed

incidence matrix $A$ of $G$: the $k \times n$ matrix whose $l$-th column is the vector $e_i - e_j$, where $i$ and $j$ are the head and tail of the $l$-th edge. Let $V$ be the set of columns of $A$ (so that the elements of $V$ are in bijection with edges of $G$, and the number of coordinates is the number of vertices of $G$). Show that:

(a) A subset of $V$ is linearly dependent if, and only if, the corresponding set of edges of $G$ contains a cycle. (Clue: for the if part, show how to get a dependence from the cycle; for the only if part, argue that if the set of edges contains no cycle then some vertex is used in a single edge, and use that to induct on the number of edges: removing that edge from the list reduces the rank of your set of vectors by one).

(b) Circuits of $V$ are in bijection to (not necessarily well-directed) cycles of $G$.

(c) For each circuit, the two sides of the corresponding oriented circuit are the edges pointing to one or the other direction along the cycle.

8. (a) *Write down all the oriented circuits and cocircuits of the point configuration $V = \{(6,0),(3,0),(2,2),(0,0),(0,3),(0,6)\}$. (Clue: there are seven circuits and seven cocircuits).

(b) *Using the description in parts (b) and (c) of the previous problem (even if you did not solve the problem), write down the oriented circuits of the oriented matroid of the graph $G = (\{1,2,3,4\}, \{\overrightarrow{12}, \overrightarrow{13}, \overrightarrow{14}, \overrightarrow{23}, \overrightarrow{24}, \overrightarrow{34}\})$. (The complete graph on 4 vertices, directed from smaller indices to bigger indices).

(c) *In both cases you get the same oriented circuits, so these oriented matroids are the same. What do the cocircuits of $V$ correspond to in $G$?