CIMPA Research School:
Algebraic, Enumerative and Geometric Combinatorics - ECCO 2016

Triangulations of polytopes. Problem sheet.

3 Flips. Graph of triangulations

1. *Show that for a convex n-gon all triangulations have exactly $n - d - 1 = n - 3$ flips, where n is the number of vertices and $d = 2$ is the dimension.

2. *Show that for the triangular prism $\Delta_2 \times \Delta_1$ all triangulations have exactly $n - d - 1 = 2$ flips, where $n = 6$ is the number of vertices and $d = 2$ is the dimension.

3. If you did problem 2.b of sheet 1, show that flips among triangulations of $\Delta_k \times \Delta_1$ correspond to permutations that differ in a transposition of consecutive elements. In particular, all triangulations have exactly $n - d - 1 = k$ flips, where $n = 2k + 2$ is the number of vertices and $d = k + 1$ is the dimension.

4. *Show that m.o.a.e. has triangulations with three and with four flips. (Here $n - d - 1$ is three).

5. *Show that the 3-cube has triangulations with four, and with six flips. (Here $n - d - 1$ is four).

6. *Draw the graphs of flips among triangulations of the configuration you chose in Problem 1.1.

7. *Draw the graph of flips among triangulations of the following six points: the five vertices of a regular pentagon, together with its center.

8. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations of the convex n-gon. Show that:

   (a) *If one of them is the triangulation that joins one vertex to all the others, then we can go from $\mathcal{T}_1$ to $\mathcal{T}_2$ in $n - 3$ or less flips.

   (b) *No matter what they are, we can go from $\mathcal{T}_1$ to $\mathcal{T}_2$ in $2n - 6$ or less flips.

   (c) In fact, we can go from $\mathcal{T}_1$ to $\mathcal{T}_2$ in $2n - 10$ or less flips.

   (d) Let n be even and suppose that $\mathcal{T}_1$ uses all the even ears[3] and $\mathcal{T}_2$ uses all the odd ears[4]. Show that it is impossible to go from $\mathcal{T}_1$ to $\mathcal{T}_2$ in less than $\frac{3}{2}n - 5$ flips. Clue: classify the possible diagonals

[3] that is, suppose that for every even i, the triangle $i - 1, i, i + 1$ is in $\mathcal{T}_1$

[4] the same, for all odd i

of an n-gon in three types: even-even, even-odd, and odd-odd. This gives you, for each triangulation, a vector of length three that counts the number of diagonals of each type. Study how that vector changes by a flip (there are several cases), and what change you need in order to go from $\mathcal{T}_1$ to $\mathcal{T}_2$.

9. Let $A = \{\pm e_i \pm e_j : i, j \in \{1,2,3\}\} \cup \{(0,0,0)\}$ (the vertices and barycenter of a regular cube-octahedron). Let $\mathcal{T}$ be a triangulation in which all tetrahedra use $(0,0,0)$ as a vertex (put differently, $\mathcal{T}$ is one of the 64 triangulations obtained triangulating the 6 quadrilateral facets of $\mathrm{conv}(A)$ and coning the boundary to $(0,0,0)$). Show that:

   (a) The number of flips in $\mathcal{T}$ equals 6 plus twice the number of vertices of degree four that you see in the triangulation of the boundary.

   (b) There are two triangulations with only 6 flips (observe that $n - d - 1 = 9$ in this example).

   (c) Are there any triangulations with 8 flips?