EXERCISES - THURSDAY- ECCO 2016

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The exercises marked * are recommended to review the understanding of the material.

1. Increasing subsequences

Exercise 1. Let S_n be the set of permutations of $\{1, \ldots, n\}$. An increasing subsequence of a permutation $w = w_1 \ldots w_n$ is a sequence $w_{i_1} < \ldots < w_{i_k}$ such that $i_1 < \ldots < i_k$.

0.* What is one of the longest increasing subsequence of $(1, 6, 4, 3, 5, 2)$?
0b.* What are all the increasing subsequences of $(3, 2, 1)$ and $(2, 4, 1, 3)$?
1. Show that if we apply the local rule algorithm to a permutation matrix the length of the longest increasing subsequence is equal to the size of the 1st part of the shape of the SYT.
2.* What are the longest decreasing subsequences of $(3, 2, 1)$ and $(2, 4, 1, 3)$? How can we read the length of the longest decreasing subsequence on the shape of the SYT?

Exercise 2. Given a permutation σ let (P, Q) be the SYT resulting from the Robinson-Schensted algorithm. Suppose that the length of the increasing subsequence of a permutation σ is equal to the size of the 1st part of the shape of the SYT P and the length of the longest decreasing subsequence is equal to the number of parts of the shape of the SYT Q.
0.* (magic trick) Each group member come up with a "random" permutation of size 10. Find the longest increasing subsequence and longest decreasing subsequence of that permutation. Is the size of one of these subsequences at least 4? If not see the T.A. to claim a prize.
1.* Show that all permutations of S_{mn+1} have an increasing subsequence of size $m + 1$ or a decreasing subsequence of length $n + 1$.
2. Deduce that the average value of the longest ineasing subsequence of the permutations of S_n is greater than or equal to $\sqrt{n}/2$.
3.* How many permutations of S_{mn} have a longest increasing subsequence of length m and a longest decreasing subsequence of length n? (Hint: Hook length formula!)

Date: June 16th, 2016.

4.(harder) Prove 1. in terms of permutations without using Robinson-Schensted (Hint: pigeonhole principle).

Exercise 3.* 0.* Find the permutations of size $n = 2, 3, 4$ with no increasing subsequence of length 3. 1. Show that the number of permutations in S_n with no increasing subsequence of size 3 is equal to the n^{th} Catalan number.

2. Semi-standard Young tableaux and Gelfand Tsetlin triangles

Recall that a semi standard Young tableau (SSYT) is a filling of the Ferrers diagram with integers that is increasing in rows and strictly increasing in columns.

A Gelfand Tsetlin triangle (GT) is a triangle where the i^{th} row contains a partition with i non negative parts and the rows interlace:

$$
\begin{array}{lccccccccc}
\text{row } m & a_{11} & & a_{12} & & a_{13} & & \cdots & & a_{1n} \\
 & & a_{22} & & a_{22} & & \cdots & & a_{2n} & \\
\vdots & & & a_{33} & & \cdots & & a_{3n} & & \\
 & & & & \ddots & & \iddots & & & \\
\text{row } 2 & & & & & & & & & \\
\text{row } 1 & & & & & a_{nn} & & & &
\end{array}
\qquad
\begin{array}{ccc}
a_{ij} & & a_{i,j+1} \\
 & \searrow\!\!\geq \quad \leq\!\!\nearrow & \\
 & a_{i+1,j+1} &
\end{array}
$$

The bijection between SSYT and GT triangles is the following: the i^{th} row of the triangle is the shape of the entries that are at most i in the tableau.

0.* What is the Gelfand Tsetlin triangle corresponding to the semi-standard young tableau (SSYT)

$$
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & 2 & 2 & 3 \\
2 & 3 & 3 & & & & \\
3 & & & & & &
\end{array}
$$

1.* What is the SSYT corresponding to the following Gelfand Tsetlin triangle.

$$
\begin{array}{ccccccccc}
5 & & 4 & & 3 & & 1 & & 0 \\
 & 5 & & 4 & & 3 & & 0 & \\
 & & 5 & & 3 & & 3 & & \\
 & & & 5 & & 3 & & & \\
 & & & & 4 & & & &
\end{array}
$$

2. Apply the Bender-Knuth involution on the SSYT in part 1. to exchange the number of 2s and 3s. Compute the corresponding GT triangle.

3.(harder) Find an involution on Gelfand Tsetlin triangles that proves the symmetry of the Schur polynomials.

3. Local rule and interlacing partitions

If we fix two partitions μ and ν. Two partitions α and β interlace if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$$

We denote this by $\alpha \succ \beta$.

Exercise 0.* Do $(3,2,2)$ and $(3,2,1,1)$ interlace? Do $(5,2,2,1,1)$ and $(4,2,1,1)$ interlace? Do $(5,2,2,1,1,1)$ and $(4,2,1,1)$ interlace?

The "local rule" is a bijection between

- (λ, x) with λ a partition such that $\lambda \prec \mu$ and $\lambda \prec \nu$ and $x \in \mathbb{N}$ and
- ρ a partition such that: $\rho \succ \mu$ and $\rho \succ \nu$ and $|\mu|+|\nu| = |\rho|+|\lambda|+x$.

It can be defined as: $\rho_1 = \max(\mu_1, \nu_1) + x$ and for $i > 1$, $\rho_i = \max(\mu_i, \nu_i) + \min(\mu_{i-1}, \nu_{i-1}) - \lambda_{i-1}$.

1.* Example

- Apply the local rule for $\mu = (3,2,2)$, $\lambda = (2,2)$, $\nu = (4,2)$ and $x = 2$
- Is ρ is partition? Are ρ and μ interlacing?
- Give the inverse bijection.

2. Now we want to prove the general statement. Show that ρ is always a partition and show that $\rho \succ \mu$.