CIMPA Research School:
Algebraic, Enumerative and Geometric Combinatorics - ECCO 2016

Triangulations of polytopes. Glossary (III)

3 Flips. Graph of triangulations

- **Triangulations of a circuit**: A circuit C with oriented circuit (C^+, C^-) has exactly two triangulations
$$T^+ = \{C \setminus \{c_i\} \mid c_i \in C^+\} \qquad \text{and} \qquad T^- = \{C \setminus \{c_i\} \mid c_i \in C^-\}.$$
- **Link:** The link of a face F of a triangulation T is $\{G \mid F \cup G \in T, F \cap G = \emptyset\}$.
- **Bistellar flips, graph of triangulations:** Let $\mathcal{T}$ be a triangulation of V. Suppose there is a circuit C in V such that $\mathcal{T}$ contains one of the two triangulations, say $\mathcal{T}^+$, of C, and suppose further that the links in $\mathcal{T}$ of all the cells of $\mathcal{T}^+$ are identical. Let $L = \{L_1, \ldots, L_k\}$ be that link.

  Then it is possible to construct a new triangulation $\mathcal{T}'$ of V as follows:
$$\mathcal{T}' = \mathcal{T} \setminus \{T \cup L_i : T \in T^+, L \in Ł\} \cup \{T \cup L_i : T \in T^-, L \in Ł\}.$$
  That is, we remove $\mathcal{T}^+$ (together with its link) from $\mathcal{T}$ and insert $\mathcal{T}^-$ (with the same link). This operation is called a **(geometric bistellar) flip** and $\mathcal{T}'$ is said to be adjacent to $\mathcal{T}$. The set of all triangulations of V, under adjacency by flips, forms the *graph of triangulations*, or *flip-graph*, of V.

  If $|C^+| = i$ and $|C^-| = j$ we say that flip is **of type** (i, j).
- **Flips via walls (or, "how to detect and count flips")** We call **walls** of a subdivision the faces of codimension one. An **interior wall** is a wall separating two cells (as opposed to a **boundary wall**, which is a wall contained in a facet of conv(V)).

  If F is an interior wall in a triangulation $\mathcal{T}$, the two cells S_1, S_2 incident to F have $d+2$ points in total. (Each is a d-simplex, with $d+1$ points, but d of the points are in their intersection F). Hence, they contain a unique oriented circuit (C^+, C^-) which we may assume oriented so that the points $S_1 \setminus S_2$ and $S_2 \setminus S_1$ are in C^+.[1] Then there are chances that $\mathcal{T}$ contains the triangulation $\mathcal{T}^+$ of that circuit and that a flip can be performed on that circuit. If this happens, we say that this flip is **witnessed by** F.

  Lemma 4 *A flip of type (i, j) and with link $L = \{L_1, \ldots, L_k\}$ is witnessed by exactly $k\binom{i}{2}$ walls. In particular:*

  1. *Every flip is witnessed by at least one wall, except those of type $(1, j)$.*
  2. *Flips of type $(1, j)$ are only possible if $\mathcal{T}$ does not use all the points in V. In fact, there is exactly one for each unused point.*

[1] Q: why is it always true that these two points are in the circuit, and in the same side of it?

Triangulations of polytopes. Glossary (IV)

4 Regular triangulations and subdivisions

- **Regular subdivision:** Let $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$ and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ be any vector. The regular subdivision of V obtained by the **lifting vector** α is defined as follows:

(i) Let $\tilde{v}_i = (v_i, \alpha_i)$ for each i and compute the facets of $\tilde{V} := \{\tilde{v}_1, \dots, \tilde{v}_n\}$.

(ii) Project the lower facets of $\tilde{V}$ onto $\mathbb{R}^d$.

Here, a **lower facet** of $Q = \mathrm{conv}(\{\tilde{v}_1, \dots, \tilde{v}_n\})$ is a facet that is visible from below. That is, a facet whose outer normal vector has its last coordinate negative.

Observe that the "projection" step is combinatorially trivial. For each lower facet $\{\tilde{v}_{i_1}, \dots, \tilde{v}_{i_k}\}$ of Q we simply make $\{v_{i_1}, \dots, v_{i_k}\}$ a cell.

For example, if $\alpha = 0$ then we get the trivial subdivision. If α has a single non-zero coordinate α_i, corresponding to a certain point $p_i \in V$, then we get the following subdivision, depending on the sign of α_i:

  - **Pulling a point** p_i**:** Taking $\alpha_i < 0$, the subdivision obtained is

$$\{F \cup \{p_i\} : F \text{ is a facet of } V \text{ not containing } p_i\}.$$

    In particular, if p_i is in the interior of $\mathrm{conv}(V)$ then the subdivision is the **cone of the boundary** of V with apex at p_i.

  - **Pushing a point** p_i**:** Taking $\alpha_i > 0$, the subdivision obtained is

$$\{V \setminus \{p\}\} \cup \{F \cup v : F \text{ is a facet of } V \setminus \{p\} \text{ visible from } p\}.$$

    Here, we say that a facet F of $V \setminus \{p_i\}$ is **visible from** p_i if p_i lies in the opposite side of the hyperplane containing) F than the rest of points of $V \setminus F$.
    In particular, if p_i is not a vertex of V then the subdivision is $\{V \setminus \{p_i\}\}$, the trivial subdivision of $V \setminus \{p_i\}$.

Another famous example of a regular subdivision is the **Delaunay subdivision** obtained taking $\alpha_i = ||v_i||$. The Delaunay subdivision does not change by translating the points, but it changes by more general affine transformations (e.g., if a coordinate is scaled up or down).

- **Lexicographic subdivision**: Let W be a subset of the points of V, given in a specific order (this order may not be their original order in V, but we denote $W = \{p_1, \ldots, p_k\}$ to simplify notation). For each of these points choose a sign $\epsilon_i \in \{+, -\}$. The lexicographic subdivision of V produced by that ordered list of points and signs is the subdivision obtained from the trivial subdivision by the sequence of pushings and pullings indicated by the list of points and the list of signs. That is: starting with the trivial subdivision, consider the points $p_1, \ldots, p_k$ in order and at each step:

  - If $\epsilon_i = +$, push the point p_i in all the cells of the previously constructed subdivision that contain p_i.
  - If $\epsilon_i = -$, pull the point p_i in all the cells of the previously constructed subdivision that contain p_i.

  In every step, the cells that do not contain p_i remain unchanged. (Remark: here it is important to consider cells as subsets of V, not as polytopes; when we say that p_i is contained in a cell S we mean $p_i \in S$, not $p_i \in \mathrm{conv}(S)$).

  Lemma 5 *The lexicographic subdivision coincides with the regular subdivision obtained choosing positive numbers $\lambda_1 \gg \cdots \gg \lambda_k \gg 0$ and taking $\alpha_i = \epsilon_i \lambda_i$ for each point in the list and $\alpha_i = 0$ for each point not in the list.*

  The actual numbers λ_i in the lemma are not important, as long as each one is sufficiently larger than the next one.

- **Secondary polytope:** It turns out that the poset of all regular subdivisions of V under refinement has the following very nice structure. The polytope $\Sigma(V)$ in this theorem is called the **secondary polytope** of the point configuration V.

  Theorem 6 (Gel'fand-Kapranov-Zeevinsky) *Let V be a point configuration with n points and dimension $n-d-1$. Then, there is a polytope $\Sigma(V)$ of dimension $n-d-1$ whose poset of (non-empty) faces is isomorphic to the refinement poset of regular subdivisions of V. In particular:*

  1. *The vertices of $\Sigma(V)$ are in bijection to the regular triangulations of V.*
  2. *Edges of $\Sigma(V)$ correspond to bistellar flips among regular triangulatons.*
  3. *Hence, every regular triangulation has at least $n-d-1$ flips, and the graph of flips among regular triangulations is connected (and $(n-d-1)$-connected).*

  These last properties ($\geq n-d-1$ flips and connected graph) hold also for non-regular triangulations in dimension two, but not in higher dimensions: triangulations with less than $n-d-1$ flips exist starting in dimension 3 (there is one in the exercise sheet), and triangulations with no flips at all exist in dimension 6. Point sets with disconnected graphs of triangulations exist in dimension 5 and higher. Whether they exist in dimensions three and four is open.

Triangulations of polytopes. Glossary (Appendix)

Appendix: Secondary cones, secondary fan, secondary polytopes

We here include a more explicit version of Theorem 6, explaining the constructon of the secondary polytope and how it relates to the Gale duality introduce in Section 2.

- **(Linear) cone, polyhedral fan**: A linear cone, or a cone is a subset $\mathbb{C}$ of the vector space $\mathbb{R}^d$ that is closed under sum and multiplication by a positive scalar. That is, if $x, y \in \mathbb{C}$ and $\lambda \geq 0$ then $x + y$ and λx are in $\mathbb{C}$. $\mathbb{C}$ is a **polyhedral cone** if it is the solution set to a finite system of linear homogeneous inequalities. That is, if
$$\mathbb{C} = \{x \in \mathbb{R}^d : Ax \leq b\},$$
where $A \in \mathbb{R}^{m\times d}$ is a matrix, $b \in \mathbb{R}^m$ is a vector, and $Ax \leq b$ means he vector in the left is coordinatewise less or equal than the one in the right. A polyhedral cone is a special type of **polyhedron** or "**unbounded polytope**". In particular every polyhedral cone has faces, a boundary, etc. A polyhedral complex whose faces are all cones is called a **polyhedral fan**. If the union of all cells is $\mathbb{R}^d$ the fan is called **complete**

- **Normal fan of a polytope P:** One example of a complete fan is the **normal fan of a polytope** P, whose faces are the **normal cones** of the faces of P. For a given face F of P, the normal cone of F is the set of linear functionals f that achieve its maximum at F. The cells (maximal faces) in the normal fan are the normal cones of vertices of P. The walls are the normal cones of edges, etc.

- **Secondary cone, secondary fan**: For a given regular subdivision $\mathcal{S}$ of a point set V with n elements, the secondary cone of $\mathcal{S}$, denoted $\mathbb{C}(V, \mathcal{S})$ is the set of lifting vectors α that produce $\mathcal{S}$ as a regular subdivision. It is a polyhedral cone in $\mathbb{R}^n$. The set of all secondary cones of regular subdivisions of V is a complete fan in $\mathbb{R}^n$, whose cells are the secondary cones of regular triangulations.

- **GKZ-vector:** Suppose $\mathcal{T}$ is a triangulation of $V = \{p_1, \ldots, p_n\}$. Define the vector $z(\mathcal{T}) = (z_1, \ldots, z_n) \in \mathbb{R}^n$ by $z_i = \sum \mathrm{vol}(F)$, where the sum is taken over all d-simplices F in $\mathcal{T}$ having p_i as a vertex. $z(T)$ is called the **GKZ-vector** of $\mathcal{T}$, to honor Gel'fand, Kapranov and Zelevinsky.

- **Secondary polytope:** The secondary polytope $\Sigma(V)$ is the convex hull of the GKZ-vectors of all triangulations of V.

Theorem 7 (Gel'fand-Kapranov-Zeevinsky) *1.* $\Sigma(V)$ *has dimension* $n-d-1$. *Its affine span is defined by the* $d+1$ *equations*

$$\sum_{i=1}^{n} z_i = (d+1)\,\mathrm{vol}(P), \quad \textit{and} \quad \sum_{i=1}^{n} z_i p_i = (d+1)\,\mathrm{vol}(P)c, \tag{1}$$

where c *is the barycenter of* $P = \mathrm{conv}(V)$.

2. The poset of (nonempty) faces of $\Sigma(V)$ *is isomorphic to the poset of all regular subdivisions of* V*, partially ordered by refinement:*

- *For a given regular subdivision* $\mathcal{S}$ *of* V*, the volume vectors of all regular triangulations that refine* $\mathcal{S}$ *are the vertices of a face* $F_{\mathcal{S}}$ *of* $\Sigma(V)$*. This face contains also the volume vectors of nonregular triangulations refining* $\mathcal{S}$*, but these are never vertices of it.*
- *The normal fan of* $\Sigma(V)$ *equals the secondary fan of* V*. That is to say, a lifting vector* $(\alpha_1, \ldots, \alpha_n)$ *produces* $\mathcal{S}$ *as a regular subdivision if and only if it lies in the normal cone of* $F_{\mathcal{S}}$ *in* $\Sigma(V)$*.*

3. The secondary fan, quotiented by the space of linear evaluations (that do not change the regular subdivision produced by a certain lifting vector), is naturally isomorphic to the chamber complex*of the Gale transform of* V*. Here, the chamber complex of a vector configuration of rank* k *is the polyhedral fan obtained using as walls all the possible* $k-1$ *cones spanned by elements of* V*. Put differently it is the common refinement of all the fans that can be constructed using the vector of* V *as rays (but notice that in this common refinement new vertices arise, since pairs of cones will typically not intersect properly).*

Further properties of the secondary polytope are:

1. The vertices of $\Sigma(V)$ are the GKZ-vectors of regular triangulations of V. Two nonregular triangulations can have the same GKZ-vector, but two regular ones or a regular and a nonregular one must have different GKZ-vectors.
2. If two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ correspond to adjacent vertices of $\Sigma(V)$ (that is, to the endpoints of an edge) then they are adjacent by a flip. (The converse is not true; in rare cases there is a flip between regular triangulations that does not correspond to an edge in the secondary polytope. These are called *non-regular flips*.