Singular metrics, convex bodies and multiple zeta values

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ECCO 2016: 5th Encuentro Colombiano de Combinatoria

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June 21 2016

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The *Riemann zeta function* is defined as the series

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s \in \mathbb{C}$ with Re > 1.
Goal: Compute $\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$ as a sum of triangle areas.
Start by rewriting (for $n \in \mathbb{Z}_{>0}$)

$$\frac{1}{n^2} = -\frac{1}{n} \cdot \frac{e^{-nx}}{n}\Big|_0^{\infty} = \int_0^{\infty} \frac{e^{-nx}}{n} dx.$$

Hence, the term $\frac{1}{n^2}$ equals the area

Figure: $\frac{1}{n^2}$ as the area below an exponential curve

Therefore,

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

can be visualized geometrically as follows

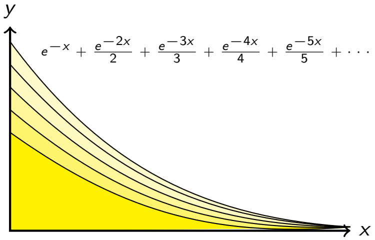


Figure: $\zeta(2)$ as a sum of areas of curved triangles

Using the Taylor expansion for the logarithm function

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{n} = -\log(1-e^{-x}),$$

we obatin

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-nx}}{n} \mathrm{d}x = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n} \mathrm{d}x$$
$$= -\int_0^{\infty} \log(1-e^{-x}) \mathrm{d}x.$$

Hence, $\zeta(2)$ equals the area of the region A determined by the curve C

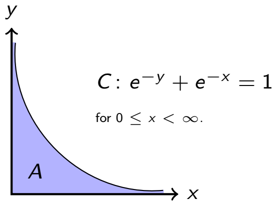


Figure: $\zeta(2)$ as the area of the region A

To compute the area of A, we make the change of variables

$$(\alpha,\beta)\mapsto(x,y)=\phi(\alpha,\beta):=\left(\log\left(\frac{\sin(\alpha+\beta)}{\sin(\alpha)}\right),\log\left(\frac{\sin(\alpha+\beta)}{\sin(\beta)}\right)\right)$$

depicted below

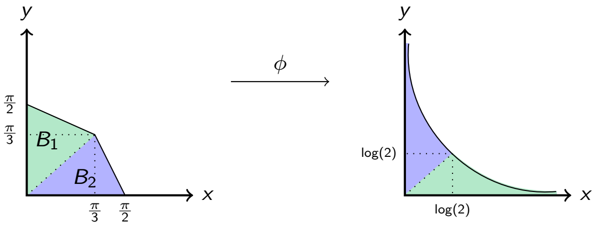


Figure: $\zeta(2)$ as the area of two triangles B_1 and B_2

$$\zeta(2)=\operatorname{area}(A)=\iint_A \mathrm{d}x\,\mathrm{d}y=\iint_B\left|\det\left(\frac{\partial\phi}{\partial\alpha},\frac{\partial\phi}{\partial\beta}\right)\right|\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$=\operatorname{area}(B)=\operatorname{area}(B_1)+\operatorname{area}(B_2)=2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{\pi}{3}=\frac{\pi^2}{6}.$$

$\zeta(2)$ as the volume of a moduli space

Consider $\mathcal{A}_1$, the *moduli space* of elliptic curves. This means that the points of $\mathcal{A}_1$ are in bijection with the isomorphism classes of elliptic curves over $\mathbb{C}$. We have

$$\begin{aligned} \mathbb{H}/\operatorname{SL}_2(\mathbb{Z}) &\simeq \mathcal{A}_1 \\ z &\mapsto \mathbb{C}/\left(\mathbb{Z}\oplus z\mathbb{Z}\right). \end{aligned}$$

A fundamental domain for this quotient space is

$$\mathcal{F} := \left\{ z = x + iy \in \mathbb{H} \,\middle|\, -\frac{1}{2} < x \leq \frac{1}{2}\,,\, x^2 + y^2 \geq 1 \right\}.$$

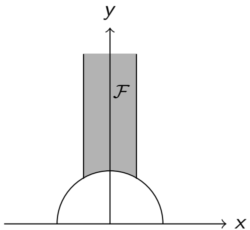


Figure: Fundamental domain $\mathcal{F}$ for $\operatorname{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}$

Hence, the volume of $\mathcal{A}_1$ with respect to the normalized hyperbolic metric $\mathrm{d}\mu$ can be computed as

$$\begin{aligned}\mathrm{vol}\,(\mathcal{A}_1) &= \int_{\mathcal{A}_1} \mathrm{d}\mu = \frac{1}{4\pi}\int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} = \frac{1}{4\pi}\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\ &= \frac{1}{4\pi}\cdot \arcsin(x)\Big|_{-1/2}^{1/2} = \frac{1}{12} = \frac{1}{2\pi^2}\zeta(2).\end{aligned}$$

Using the functional equation of the zeta function

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

the formula of the volume of $\mathcal{A}_1$ can be rewritten in the simplified form

$$\mathrm{vol}(\mathcal{A}_1) = -\zeta(-1).$$

Summary

- We have computed the special value $\zeta(2)$ by relating it to the area of two triangles.
- We have computed the volume of the moduli space $\mathcal{A}_1$ and seen it as a special value at a negative integer.

Question: What about universal objects? Can we relate their volume to values of more general Riemann zeta functions?

Consider the set of rational numbers

$$\mathbb{Q} := \left\{ \frac{a}{b} \,\middle|\, (a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}\,,\ (a,b) = 1 \right\}.$$

The *mediant* of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is given by $\frac{a}{b} \oplus \frac{c}{d} := \frac{a+c}{b+d}$.

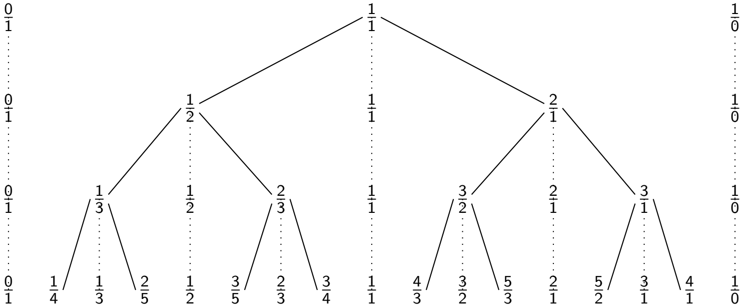


Figure: A way of enumerating the rationals: The Stern–Brocot tree

The computation of $\zeta_{\mathrm{MT}}(2,2;2)$

The *Mordell–Tornheim zeta function* is defined as the doubles series

$$\zeta_{\mathrm{MT}}(s_2, s_2; s_3) \coloneqq \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

where $s_1, s_2, s_3 \in \mathbb{C}$ with $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2) \geq \mathrm{Re}(s_3) > 1$.
We aim at computing the value

$$\zeta_{\mathrm{MT}}(2, 2; 2) \coloneqq \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2 (m+n)^2},$$

We will do this by computing the volume of a convex set.

Consider the concave function $\phi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}\,,\ (a,b) \mapsto \frac{ab}{a+b}$. In the order given by the Stern–Brocot tree, we assign values on the positive primitive vectors in $\mathbb{Z}^2$ dictated by ϕ.

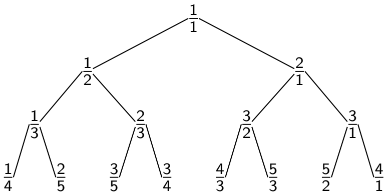


Figure: The Stern–Brocot tree

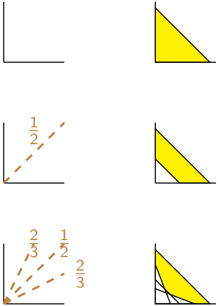


Figure: Values on primitive rays and the dual picture

In the limit, we get

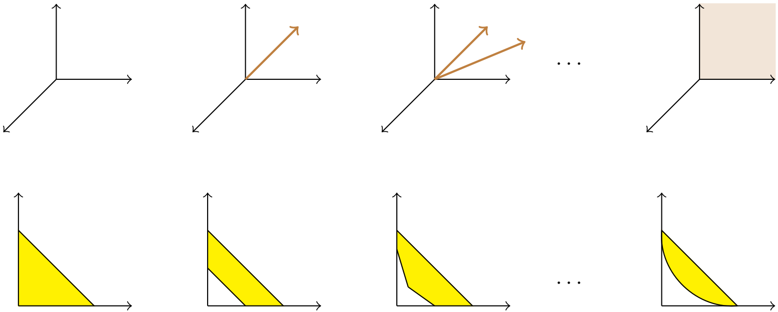
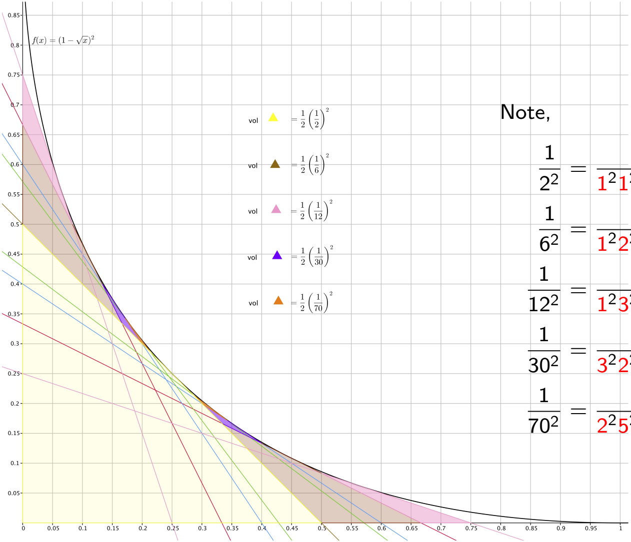


Figure: In the limit

Looking at the complement, we see



Note,

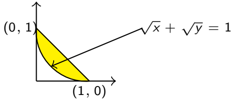
$$\frac{1}{2^2} = \frac{1}{1^2 1^2 (1+1)^2}$$
$$\frac{1}{6^2} = \frac{1}{1^2 2^2 (1+2)^2}$$
$$\frac{1}{12^2} = \frac{1}{1^2 3^2 (1+3)^2}$$
$$\frac{1}{30^2} = \frac{1}{3^2 2^2 (3+2)^2}$$
$$\frac{1}{70^2} = \frac{1}{2^2 5^2 (2+5)^2}$$

We get

$$2 \cdot \mathrm{vol}\,(\Delta) = 1 - \sum_{\substack{m,n \in \mathbb{Z}_{>\ 0} \\ (m,n)=1}} \frac{1}{m^2 n^2 (m+n)^2}\,,$$

where

$$\Delta =$$



On the other hand, we can calculate

$$\mathrm{vol}\,(\Delta) = \frac{1}{2} - \int_0^1 (1-\sqrt{x})^2 \mathrm{d}x = \frac{1}{2} - 16 = \frac{1}{3}.$$

Hence,

$$A := \sum_{\substack{m,n \in \mathbb{Z}_{>\ 0} \\ (m,n)=1}} \frac{1}{m^2 n^2 (m+n)^2} = 1 - 2 \cdot \frac{1}{3} = \frac{1}{3}.$$

Finally, we end up with

$$\zeta_{\mathrm{MT}}(2,2;2) = \zeta(6) \cdot A = \frac{\pi^6}{945} \cdot \frac{1}{3} = \frac{\pi^6}{2835}.$$

Consider now the *universal elliptic curve*

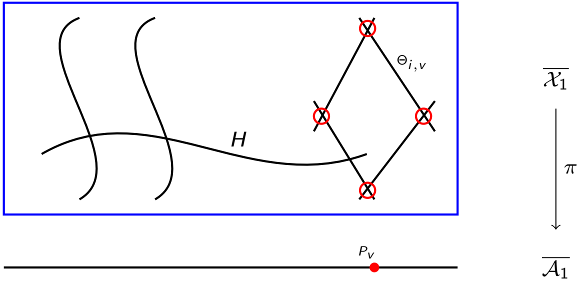


Figure: The compactified universal elliptic curve

Remark: The *volume* of $\overline{\mathcal{X}_1}$ is a numerical invariant defined in terms of *arithmetic intersection numbers*. And these in turn are defined in terms of a singular metric.
Problem: The type of singularities of the metric along the boundary $\overline{\mathcal{X}_1} \setminus \mathcal{X}_1$.

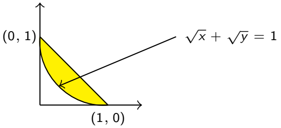
Approach (B. '15): Encode the singularity type of the metric in a *toric b-divisor* $\mathbb{D}$.

$\Rightarrow$ Metric residues = volume of a convex set (degree of toric b-divisor).

We have

$$\operatorname{vol}\left(\overline{\mathcal{X}_1}\right) = \mathcal{L}^2 - \mathrm{R} = \mathcal{L}^2 - \mathbb{D}^2 = \mathcal{L}^2 - 2 \cdot \operatorname{vol}(\Delta),$$

where $\mathcal{L}$ is a distinguished line bundle on $\mathcal{X}_1$ which carries a natural metric and

$\Delta =$ 

1. We have computed the special value $\zeta(2,2;2)$ by relating it to the volume of a convex set.
2. We have computed the volume of the universal moduli space $\mathcal{X}_1$ by seeing the needed correction term as the volume of the above convex set.

Question: What about universal moduli spaces of abelian varieties of higher dimension? (Our approach works for 2 but what about 1?)

Muchas gracias!