Karp's conjecture and graphs on 2p vertices

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Karp's conjecture on evasiveness

A *graph property* $\mathcal{P}$ is a collection of (simple) graphs on n vertices $1, 2, \ldots, n$, that is closed under isomorphism of graphs: a graph G is in $\mathcal{P}$ if and only if any graph G' isomorphic to G is also in $\mathcal{P}$.

Graph properties: being connected, being a path graph, etc.

Not graph properties: the vertex 1 has degree 2, the vertices 1 and n are adjacent.

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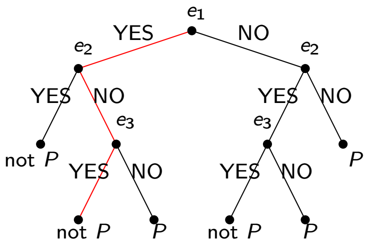
The game: there are two players; the *hider* and the *seeker*, a graph property $\mathcal{P}$ that both players know and a graph G that only the *hider* knows.

The goal of the *seeker* is to determine whether the graph G is in $\mathcal{P}$ or not. The *seeker* can ask questions "*is the edge* $\{i, j\}$ *in* G*?*". The *hider* answers *yes* or *no*.

The game ends when the *seeker* has determined if G is in $\mathcal{P}$ or not.

A *strategy* of the *seeker* is an algorithm that, depending on the answers the *hider* gives to each question, assigns an edge for asking the next question or assigns one of the answers, "G is in $\mathcal{P}$" or "G is not in $\mathcal{P}$".

Example. If $V = \{1, 2, 3\}$, then there are three edges: $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ y $e_3 = \{2, 3\}$. Let $\mathcal{P}$ be the property of having at most one edge. The figure shows a possible strategy for the *seeker*:



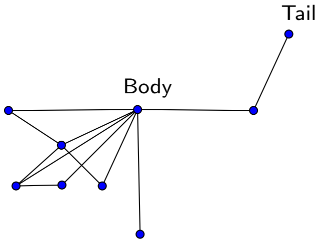
Let k be the minimal number for which there is a strategy of the *seeker* such that, regardless the graph G and the answers of the *hider*, the *seeker* can always end the game by asking at most k questions. The number k is the *complexity* $c(\mathcal{P})$ of the graph property $\mathcal{P}$.

We have that $0 \leq c(\mathcal{P}) \leq \binom{n}{2}$.

$\mathcal{P}$ is called *trivial* if it is either empty or is the family of all subsets of $\binom{V}{2}$ (all graphs), otherwise $\mathcal{P}$ is called *nontrivial*. Equivalently, $\mathcal{P}$ is trivial if $c(\mathcal{P}) = 0$.

In the extreme case that $c(\mathcal{P}) = \binom{n}{2}$, we say that the graph property $\mathcal{P}$ is *evasive*, otherwise we say $\mathcal{P}$ is *non-evasive*.

Few graph properties are known to be non-evasive. A famous example of non-evasive graph property is the property of being a *scorpion graph*, defined for $n \geq 5$ and that has complexity $\leq 6n - 13$, so that for $n \geq 11$ it is non-evasive.



A graph property $\mathcal{P}$ is *monotone* if it is closed under removal of edges.

Examples of monotone graph properties:

1. Having at most k edges, where $k < \binom{n}{2}$.
2. Not containing cycles.
3. Being disconnected.
4. Being planar.

All these graph properties are evasive.

Karp's conjecture: every nontrivial monotone graph property is evasive.

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Kahn-Saks-Sturtevant, 1984. Karp's conjecture is true if n is a power of a prime or $n = 6$.

The case $n = 10$ is still unsolved.

Simplicial Complexes

Let V be a finite set. An *(abstract) simplicial complex* on V is a collection K of subsets of V such that

(i) $\{v\} \in K$ for all $v \in V$ and
(ii) $A \in K$ and $B \subseteq A$ implies $B \in K$.

If $A \in K$ we say that A is a *face* or a *simplex* of K, and $|A| - 1$ is the *dimension* of A ($dimA$).

If K has f_i faces of dimension i, then the Euler characteristic of K is

$$\chi(K) = \sum_{i \geq 0} (-1)^i f_i.$$

The *automorphism group* of K, $Aut(K)$, is the collection of all permutations of V which leave K invariant.

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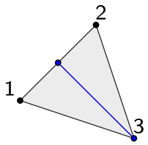
Abstract version of the fixed point set: If Γ is a subgroup of $Aut(K)$, we define a simplicial complex K^Γ as follows:

(i) the vertices of K^Γ are the orbits of the action of Γ on V that are also faces of K and
(ii) if $A_1, A_2, \ldots, A_r$ are vertices of K^Γ then $\{A_1, A_2, \ldots, A_r\}$ is a face of K^Γ if $A_1 \cup A_2 \cup \cdots \cup A_r$ is a face of K.

Example. $K = \{1, 2, 3, 12, 13, 23, 123\}$, $\Gamma = \langle (12) \rangle$.
Orbits of Γ on $\{1, 2, 3\}$: $A = \{1, 2\}, B = \{3\}$.
$K^{\Gamma} = \{A, B, AB\}$.



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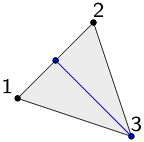
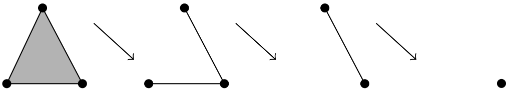


Collapsibility: A *free face* of K is a nonempty face A of K such that A is not maximal under inclusion in K, but it is contained in exactly one inclusion-maximal face B of K, where we require that $dimB = dimA + 1$.

An *elementary collapse* of K consists of the removal of a free face along with the maximal face containing it.

We say that K is *collapsible* if there is a sequence of elementary collapses that yields K to a complex consisting of a single vertex .



$\{12, 123\}$ $\quad$ $\{1, 13\}$ $\quad$ $\{2, 23\}$ $\quad$ $K' = \{3\}$

The connection with topology

A simplicial complex K can be defined to be evasive or non-evasive in a similar way to graph properties: in the game of the *hider* and the *seeker*, change $\mathcal{P}$ by K, edges by vertices of K, and the graph G by a subset of V.

We can regard a nonempty monotone graph property $\mathcal{P}$ on n vertices as an abstract simplicial complex $\Delta\mathcal{P}$ as follows:

The set of vertices of $\Delta\mathcal{P}$ is the set of two-element subsets of $V = \{1, 2, \ldots, n\}$, that is, the set of all possible edges $\{i, j\}$, $1 \leq i < j \leq n$, and the simplices of $\Delta\mathcal{P}$ are the collections of such edges that correspond to graphs belonging to $\mathcal{P}$.

If $\mathcal{P}$ is a nonempty monotone and non-evasive graph property, then $\Delta\mathcal{P}$ is a non-evasive simplicial complex.

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Kahn-Saks-Sturtevant, 1984. A non-evasive complex is collapsible.

$$\mathcal{P} \text{ non-evasive} \Rightarrow \Delta\mathcal{P} \text{ non-evasive} \Rightarrow \Delta\mathcal{P} \text{ collapsible}$$

$$\Rightarrow |\Delta\mathcal{P}| \text{ contractible} \Rightarrow |\Delta\mathcal{P}|\ \mathbb{Z}\text{-acyclic} \Rightarrow |\Delta\mathcal{P}|\ \mathbb{Z}/p\text{-acyclic}$$

$$\Rightarrow \chi(|\Delta\mathcal{P}|) = 1.$$

Euler Characteristic. Graphs on p and $2p$ vertices

From now on, $\mathcal{P}$ represents a nonempty monotone and non-evasive property of graphs on n vertices. $\Delta\mathcal{P}$ is collapsible and $\chi(\Delta\mathcal{P}) = 1$.

We can write

$$\chi(\Delta\mathcal{P}) = \sum_{[G]\subseteq\mathcal{P}} (-1)^{m_G - 1} |[G]|,$$

If there is some integer $d > 1$ that divides all the sizes $|[G]|$ for $G \in \mathcal{P}$, then d divides $\chi(\Delta\mathcal{P})$ and $\chi(\Delta\mathcal{P}) \neq 1$.

Then, $\mathcal{P}$ must contain some graphs G for which d is not a divisor of $|[G]|$.

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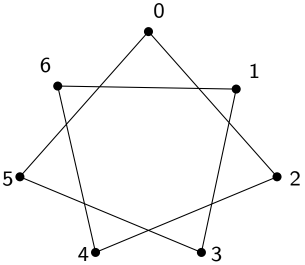
Graphs G on p vertices such that p does not divide $|[G]|$.

The p vertices will be the elements of the finite field $\mathbb{F}_p$: $0, 1, \ldots, p-1$.

For each $s \in \mathbb{F}_p$, let $C(s)$ be the graph whose edges are $\{0, s\}, \{s, 2s\}, \{2s, 3s\}, \ldots, \{(p-1)s, 0\}$.

Example. $p = 7$, $s = 2$, $C(2) = C(5)$.



If $r, s \in \{1, 2, \ldots, (p-1)/2\}$, $r \neq s$, then the p-cycles $C(s)$ and $C(r)$ do not have common edges.

If $S = \{s_1, s_2, \ldots, s_l\} \subseteq \{1, 2, \ldots, (p-1)/2\}$, $C(S)$ denotes the graph $C(s_1) \cup C(s_2) \cup \cdots \cup C(s_l)$. $C(\varnothing)$ is the graph $\overline{K}_p$.

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Proposition

If G is a graph on p vertices, then p does not divide $|[G]|$ if and only if G is isomorphic to $C(S)$ for some $S \subseteq \{1, 2, \ldots, (p-1)/2\}$.

Graphs G on $2p$ vertices such that p does not divide $|[G]|$.

Notation. If G_1 and G_2 are graphs with disjoint sets of vertices, then $G_1 \sqcup G_2$ denotes the graph with vertices $V(G_1) \cup V(G_2)$ and edges $E(G_1) \cup E(G_2)$.

The graph $G_1 + G_2$, called the *join* of G_1 and G_2, denotes the graph with vertices $V(G_1) \cup V(G_2)$ and edges

$$E(G_1) \cup E(G_2) \cup \{\{r, s\} : \ r \in V(G_1), s \in V(G_2)\}.$$

The complement graph $\overline{G}$ is the graph whose edges are those that do not belong to G.
We have $\overline{G_1 \sqcup G_2} = \overline{G}_1 + \overline{G}_2$ and $\overline{G_1 + G_2} = \overline{G}_1 \sqcup \overline{G}_2$.

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Proposition

Let G be a graph on $2p$ vertices. Then, p does not divide $|[G]|$ if and only if G is isomorphic to a graph of the form $G_1 \sqcup G_2$ or $G_1 + G_2$, where G_1 and G_2 are isomorphic to graphs of the form $C(S)$ where $S \subseteq \{1, 2, \ldots, (p-1)/2\}$.

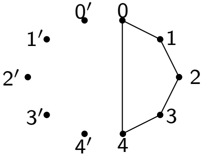
Ten Vertices

The vertices are labeled $0, 1, 2, 3, 4, 0', 1', 2', 3', 4'$.
Graphs G on the 5 vertices 0,1,2,3,4, such that 5 does not divide $|[G]|$ are

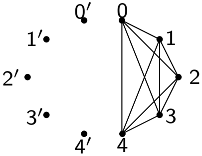
$$\overline{K}_5, C_5 \cong C(1) \cong C(2) \text{ and } K_5 = C(1, 2).$$

For the 5 vertices $0', 1', 2', 3', 4'$, we have the graphs $C(1') \cong C(2')$ and $C(1', 2')$.

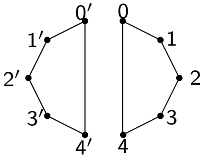
Graphs G on 10 vertices such that 5 does not divide $|[G]|$ are $\overline{K}_{10}$, K_{10}, or one of the 10 following graphs:



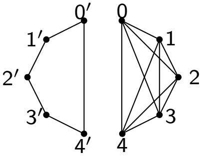
$G_1 = \overline{K}_5 \cup C(1)$



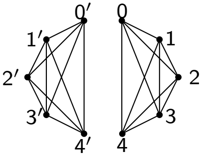
$G_2 = \overline{K}_5 \cup C(1,2)$



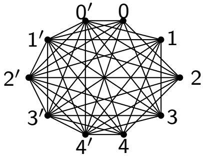
$G_3 = C(1') \cup C(1)$



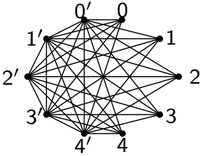
$G_4 = C(1') \cup C(1,2)$



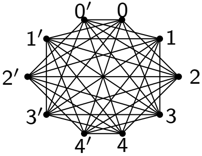
$G_5 = C(1',2') \cup C(1,2)$



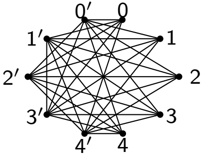
$\overline{G}_1 = C(1',2') + C(2)$



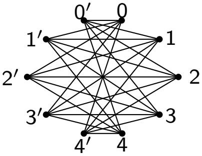
$\overline{G}_2 = C(1', 2') + \overline{K}_5$



$\overline{G}_3 = C(2') + C(2)$



$\overline{G}_4 = C(2') + \overline{K}_5$



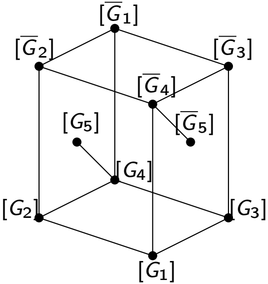
$\overline{G}_5 = K_{5,5}$

If $\mathcal{P}$ is non-evasive, then $\mathcal{P}$ must contain some of the graphs G_i or their complements.

G_i	$Aut(G_i)$	$\|[G_i]\|$	$(-1)^{m_{G_i}-1}\|[G_i]\|(mod\ 5)$
G_1	$D_{10} \times S_5$	$2^4 \cdot 3^3 \cdot 7$	+4
G_2	$S_5 \times S_5$	$2^2 \cdot 3^2 \cdot 7$	-2
G_3	$D_5 \wr S_2$	$2^5 \cdot 3^4 \cdot 7$	-4
G_4	$S_5 \times D_{10}$	$2^4 \cdot 3^3 \cdot 7$	+4
G_5	$S_5 \wr S_2$	$2 \cdot 3^2 \cdot 7$	-1

Note: $Aut(\overline{G}_i) = Aut(G_i), m_{\overline{G}_i} = 45 - m_{G_i} \equiv m_{G_i} \mod 5$.

The Hasse diagram of the isomorphism classes $[G_i]$'s and $[\overline{G_j}]$'s is the following:



The set of isomorphism classes $[G_i], [\overline{G}_j]$ contained in $\mathcal{P}$ becomes an *order ideal* I of the poset above.

Therefore,

$$\chi(\Delta\mathcal{P}) \equiv \sum_{[G]\in I} (-1)^{m_G - 1} |[G]| \mod 5.$$

Which order ideals of the the poset above can give

$$\chi(\Delta\mathcal{P}) \equiv 1 \mod 5?$$

There are exactly 9 of these order ideals:

$$
\begin{gathered}
I_1 = \{[\overline{G_5}]\} \\
I_2 = \{[G_1], [\overline{G_4}], [\overline{G_5}]\} \\
I_3 = \{[G_1], [G_3], [\overline{G_5}]\} \\
I_4 = \{[G_1], [G_2], [\overline{G_2}], [\overline{G_4}], [\overline{G_5}]\} \\
I_5 = \{[G_1], [G_2], [G_3], [G_4], [G_5]\} \\
I_6 = \{[G_1], [G_3], [\overline{G_3}], [\overline{G_4}], [\overline{G_5}]\} \\
I_7 = \{[G_1], [G_2], [G_3], [G_4], [\overline{G_2}], [\overline{G_4}], [\overline{G_5}]\} \\
I_8 = \{[G_1], [G_2], [G_3], [\overline{G_2}], [\overline{G_3}], [\overline{G_4}], [\overline{G_5}]\} \\
I_9 = \{[G_1], [G_2], [G_3], [G_4], [\overline{G_1}], [\overline{G_2}], [\overline{G_3}], [\overline{G_4}], [\overline{G_5}]\}
\end{gathered}
$$

We will say that $\mathcal{P}$ is of *type* k if, from the G_i's and $\overline{G}_j$'s, $\mathcal{P}$ contains the isomorphism classes of graphs in I_k and no more.

We try to prove Karp's conjecture in the 10 vertices case by trying to show that none of the 9 types can happen.

Oliver groups

We say that a finite group Γ is an *Oliver group* if Γ has a normal p-subgroup Γ_1 such that the quotient Γ/Γ_1 is cyclic. For example, all finite p-groups are Oliver groups.

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Oliver. Let K be a simplicial complex, Γ be a finite subgroup of $Aut(K)$ and p be a fixed prime. If $|K|$ is $\mathbb{Z}/p$-acyclic and Γ is an Oliver group, then $\chi(|K|^{\Gamma}) = 1$. In particular, $|K|^{\Gamma}$ is nonempty.

In the abstract version of $|K|^\Gamma$, K^Γ, Oliver's theorem says that some of the orbits of Γ acting on the set of vertices of K are actually faces of K and, moreover, $\chi(K^\Gamma) = 1$.

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If Γ is a subgroup of S_n, then Γ can be regarded as a subgroup of $Aut(\Delta\mathcal{P})$. $\Delta\mathcal{P}$ is collalsible (then $\mathbb{Z}/p$-acyclic) and Oliver's theorem can be applied to $\Delta\mathcal{P}$ and Γ, if Γ is an Oliver group. Some of the orbits of Γ acting on $\binom{V}{2}$, $V = \{1, 2, \ldots, n\}$, are graphs belonging to $\mathcal{P}$ and $\chi(\mathcal{P}^{\Gamma}) = 1$.

Example. Let Γ be the group generated by the permutations $(1\ p+1), (2\ p+2), \ldots, (p\ 2p)$ and $\alpha = (1\ 2 \cdots\ p)(p+1\ p+2 \cdots 2p)$.
The subgroup H of Γ generated by $(1\ p+1), (2\ p+2), \ldots, (p\ 2p)$ is a normal 2-subgroup with quotient isomorphic to the subgroup of Γ generated by α, which is cyclic. Γ is an Oliver group.

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Each orbit of Γ acting on the two-element subsets of $\{1, 2, \ldots, 2p\}$ contains a *perfect matching*. One of such orbits is the perfect matching $\{1, p+1\}, \{2, p+2\}, \ldots, \{p, 2p\}$.

Proposition

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on $2p$ vertices, where p is prime. Then, all perfect matchings belong to $\mathcal{P}$.

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Example. Consider two disjoint copies of the finite field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$, the second copy of $\mathbb{F}_p$ will be labeled $\mathbb{F}'_p = \{0', 1', \ldots, (p-1)'\}$. Let Γ be the group generated by the permutations

$\alpha = (0\ 0')(1\ 1') \cdots (p-1\ (p-1)')$,
$\beta = (01 \cdots p-1)$,
$\gamma = (0'1' \cdots (p-1)')$.

The subgroup of Γ generated by β and γ is a normal *p*-subgroup of Γ whose quotient is cyclic isomorphic to $\langle\alpha\rangle$. Γ is an Oliver group.

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There are $(p+1)/2$ orbits of Γ on the vertices of $\Delta\mathcal{P}$. One of them consists of all edges of the form $\{x, y'\}$, $x, y \in \mathbb{F}_p$, isomorphic to the complete bipartite graph $K_{p,p}$. The remaining $(p-1)/2$ are the graphs $C(t) \sqcup C(t') \cong 2C_p$ for $t = 1, 2, \ldots, (p-1)/2$.

Proposition

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on 2p vertices, where p is and odd prime. Then at least one of $2C_p$, $K_{p,p}$ belongs to $\mathcal{P}$.

Corollary

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on 2p vertices, where $p > 3$ is prime. Then, $dim\Delta\mathcal{P} \geq 4p - 1$.

10 vertices

Proposition

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive property of graphs on 10 vertices. Then, $\mathcal{P}$ is not of type $1, 3, 7, 9$.

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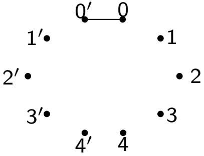
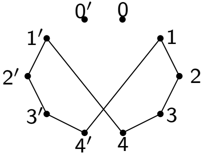
The 10 vertices are labeled $0, 1, 2, 3, 4, 0', 1', 2', 3', 4'$.

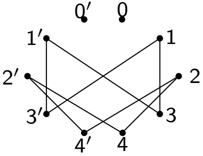
If $\mathcal{P}$ is of type 1, we use $\Gamma = \langle (00'), (12341'2'3'4') \rangle$ (a 2-group).
The (potential) vertices of $\mathcal{P}^{\Gamma}$ are:


0′ 0
1′ 1
2′ 2
3′ 3
4′ 4

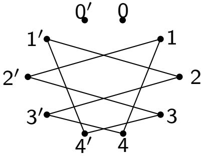
A


0′ 0
1′ 1
2′ 2
3′ 3
4′ 4

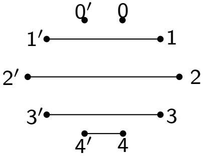
B


0′ 0
1′ 1
2′ 2
3′ 3
4′ 4

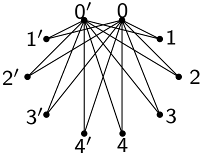
C


0′ 0
1′ 1
2′ 2
3′ 3
4′ 4

D


0′ 0
1′ 1
2′ 2
3′ 3
4′ 4

E


0′ 0
1′ 1
2′ 2
3′ 3
4′ 4

F

Smith. If Γ is a p-group acting on a $\mathbb{Z}/p$-acyclic complex K, then $|K|^\Gamma$ is also $\mathbb{Z}/p$-acyclic. In particular, $|K|^\Gamma$ is connected.

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Since $\mathcal{P}$ is of type 1, $\mathcal{P}$ contains $\overline{G}_5$ but does not contain G_1. The graphs in the following list are in $\mathcal{P}$ because each of them is isomorphic to a subgraph of $\overline{G}_5$:

$$A, B, C, D, E, A \cup B, A \cup C, A \cup D, A \cup E, B \cup D. \qquad (1)$$

The graphs in the following list are not in $\mathcal{P}$ because each of them contains a subgraph isomorphic to G_1:

$$B \cup C, B \cup E, B \cup F, C \cup D, C \cup F, D \cup E, D \cup F, E \cup F. \qquad (2)$$

The vertex A is one of the vertices of the simplicial complex $\Delta\mathcal{P}$ and A is also one of the fixed points of Γ.
This implies that Γ acts on the *link of A*:

$$lk_{\Delta\mathcal{P}}(A) = \{G : A \notin G, G \cup \{A\} \in \mathcal{P}\}.$$

Besides, $lk_{\Delta\mathcal{P}}(A)$ is a non-evasive complex. The fixed point set of the action of Γ on $lk_{\Delta\mathcal{P}}(A)$ is given by $lk_{\Delta\mathcal{P}}(A)^{\Gamma} = lk_{\Delta\mathcal{P}^{\Gamma}}(A)$. By Smith's theorem, $lk_{\Delta\mathcal{P}}(A)^{\Gamma}$ is connected.

From (1) we see that B, C, D, E are vertices of $lk_{\Delta\mathcal{P}}(A)^{\Gamma}$.

The graph F cannot be a vertex of the simplicial complex $lk_{\Delta\mathcal{P}}(A)^\Gamma$ because, on the contrary, F would be an isolated vertex of $lk_{\Delta\mathcal{P}}(A)^\Gamma$.

The vertices of $lk_{\Delta\mathcal{P}}(A)^\Gamma$ are B, C, D, E. The only other possible faces of $lk_{\Delta\mathcal{P}}(A)^\Gamma$ are $\{B, D\}$ and $\{C, E\}$. In any case $lk_{\Delta\mathcal{P}}(A)^\Gamma$ results to be non-connected. This contradiction proves that $\mathcal{P}$ cannot be of type 1.

¡MUCHAS GRACIAS!