Applications of finite spaces to the study of the asphericity of LOT complexes

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Problem: Asphericity of ribbon disc complements.

Ribbon discs are a generalization of classical knot theory.

Definitions

- A *ribbon disc* is a "good" immersion

$$i : D^2 \hookrightarrow D^4$$

- A *ribbon disc complement* is the complement $D^4 - i(D^2)$ of such an immersion.

Problem: Asphericity of ribbon disc complements.

Knots conjecture (proved in 1957 by Papakyriakopoulos)

Knot complements are aspherical.

Ribbon discs conjecture (open)

Ribbon discs complements are aspherical.

Connection to the Whitehead conjecture.

Whitehead conjecture (open) 1941

If K is an aspherical 2-dimensional polyhedron and $L \subseteq K$, then L is aspherical.

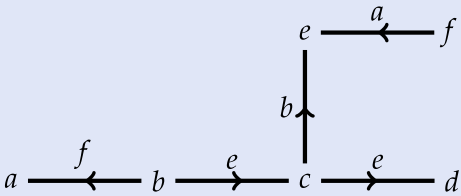
Andrews-Curtis + RDC conjecture
$\Downarrow$ (Howie)
Whitehead conjecture
(compact case)

$RDC \simeq L \hookrightarrow K^2 \simeq * \quad \forall$ RDC
Whitehead conjecture $\Rightarrow$ RDC conjecture

Ribbon discs complements are considered test cases for the Whitehead conjecture.

LOTs: Labeled Oriented Trees

A LOT $\Gamma = (E, V, s, t, \lambda)$ consists of two sets E, V of edges and vertices, and three maps $s, t, \lambda : E \to V$ source, target and label, such that the underlying graph is a tree.



LOTs, group presentation and 2-complexes

To every LOT Γ we associate:

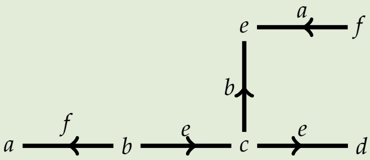
A group presentation $P(\Gamma)$

- one generator for every vertex in Γ
- one relation $s\lambda = \lambda t$ ($\lambda t \lambda^{-1} s^{-1}$) for every edge $s \xrightarrow{\lambda} t$

A 2-complex K_Γ, for which π_1 is the presented group

- one 0-cell
- one 1-cell for every vertex
- one 2-cell for every edge

Example



$$P = \langle a, b, c, d, e, f \mid faf^{-1}b^{-1}, ece^{-1}b^{-1}, ede^{-1}c^{-1}, beb^{-1}c^{-1}, aea^{-1}f^{-1} \rangle$$

Ribbon disc complements and LOTs

J. Howie associates a LOT Γ to every ribbon disc, satisfying

$$K_\Gamma \simeq D^4 - i(D^2).$$

Every LOT can be associated to a ribbon disc.

Asphericity of RDC $\Leftrightarrow$ Asphericity of LOT complexes.

Finite spaces and posets

Given X a **finite topological space**, we define

$$x \leq y \Leftrightarrow x \in U \ \forall \text{ open } U \ni y$$

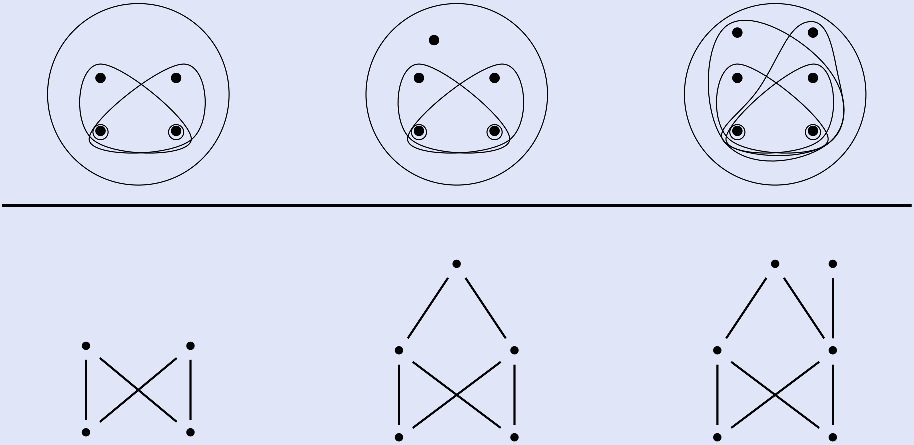
reflexivity and transitivity
antisymmetry $\Leftrightarrow T_0$

If $(X, \leq)$ is a **finite poset**

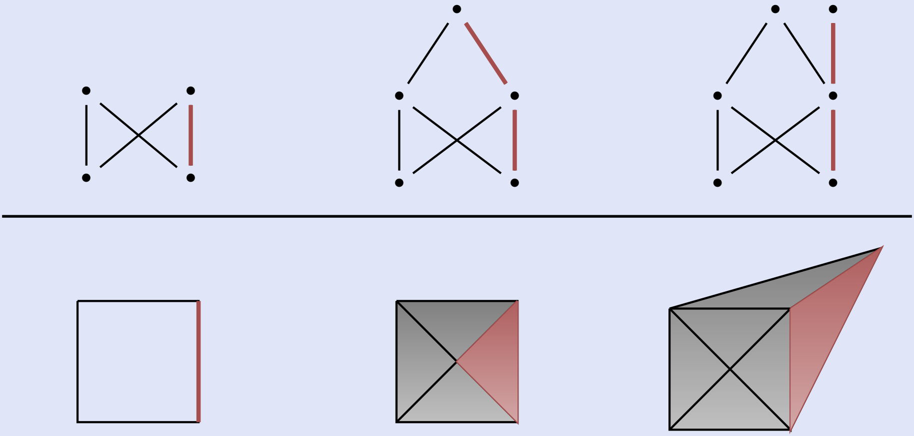
$$U_x := \{y \in X : \ y \leq x\} \subseteq X$$

are a basis for a T_0 toplogy in X

Examples: finite spaces and posets



Examples: posets and their associated complexes



Colorings

Fundamental tool

Results of Barmak and Minian on homotopy groups of 2-complexes in terms of colorings of the Hasse diagram of the corresponding poset.

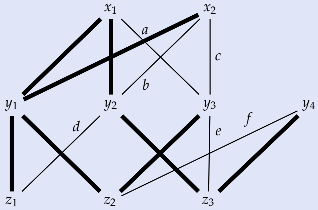
A *G-coloring* of X is a map $c : \mathcal{E}(X) \to G$.

- D a subdiagram $\mathcal{H}(X)$ that contains all the elements of X and such that the corresponding finite poset is 1-connected.
- $\pi_1(X)$:
  - One generator for every edge $\notin D$.
  - One relator for every *simple digon*.
- $\pi_2(X) \subseteq \mathbb{Z}[G]$-module generated by $\{x : \deg x = 2\}$. $(G = \pi_1(X))$

$$\left\{ \sum_{\substack{\deg x = 2 \\ g \in G}} n_g^x g x : \textstyle\sum_{x > y} \epsilon(x,y) n^x_{gc(x,y)} = 0 \ \ \forall y : \deg y = 1, \forall g \in G \right\}.$$

Colorings

The space X is a model of $S^2 \vee S^1$.



The digons induce the relations $d = 1, a = 1, ea = 1, db = 1, c = 1, ec = b$, therefore $\pi_1(X) \simeq \mathbb{Z}$, generated by f.
The generators of $\pi_2(X)$ are x_1, x_2 and the equations are

$(y_1) \quad n_g^{x_1} + n_g^{x_2} = 0 \;\; \forall\, g \in \mathbb{Z}$

$(y_2) \quad -n_g^{x_1} - n_g^{x_2} = 0 \;\; \forall\, g \in \mathbb{Z}$

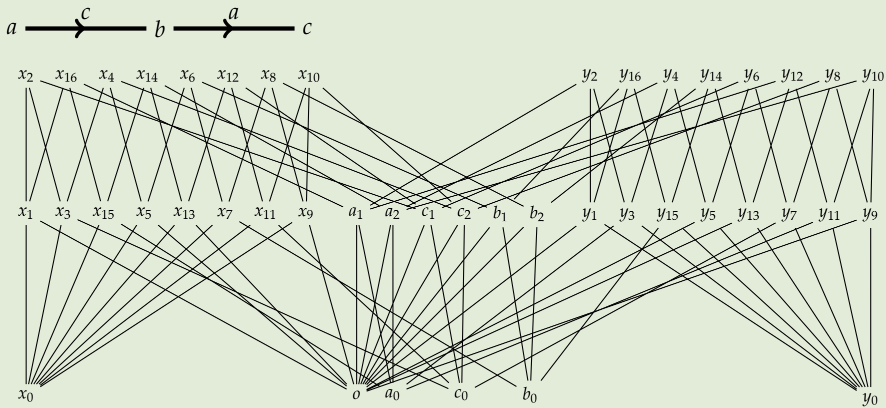
$(y_3) \quad n_g^{x_1} + n_g^{x_2} = 0 \;\; \forall\, g \in \mathbb{Z}$

Therefore $\pi_2(X) = \{\sum_{g \in \mathbb{Z}} n_g^{x_1} g(x_1 - x_2) : n_g^{x_1} \in \mathbb{Z}\} \simeq \mathbb{Z}[\mathbb{Z}]$.

LOT poset

To a given LOT Γ, we associate a LOT poset $\mathcal{X}(\Gamma) \approx K_\Gamma$.
That is, $\pi_n(K_\Gamma) = \pi_n(\mathcal{X}(\Gamma)) \forall n$.

Example



π_2 of LOTs

If X is a LOT-poset, and fixing a certain subdiagram, we obtain

- $\pi_1(\Gamma)$-presentation: equivalent to the LOT presentation $P(\Gamma)$.
- $\pi_2(\Gamma) \subseteq$ free-$\mathbb{Z}[G]$-module generated by $\{x\}_{x \in E}$, given by the equations

$$\{-\sum_{s_x=v} n^x_{gv^{-1}} + \sum_{t_x=v} n^x_{gv^{-1}\lambda_x^{-1}} + \sum_{\lambda_x=v} (n^x_{gv^{-1}} - n^x_{gt_x^{-1}v^{-1}}) = 0\}_{v\in V, g\in G}$$

Example

$a \xrightarrow{c} b \xrightarrow{a} c$ $\pi_2(\Gamma)$ *has generators x, y and equations:*

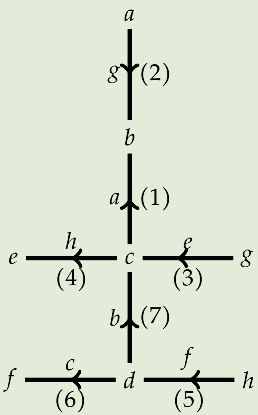
(a) $-n^x_{ga^{-1}} + n^y_{ga^{-1}} - n^y_{gc^{-1}a^{-1}} = 0 \ \forall g \in G$

(b) $-n^y_{gb^{-1}} + n^x_{gb^{-1}c^{-1}} = 0, \forall g \in G$

(c) $n^y_{gc^{-1}a^{-1}} + n^x_{gc^{-1}} - n^x_{gb^{-1}c^{-1}} = 0, \forall g \in G$

Results

Example



$$(a) \quad -n^{g}_{\gamma a^{-1}} + n^{a}_{\gamma a^{-1}} - n^{a}_{\gamma b^{-1} a^{-1}} = 0 \quad \forall \gamma \in G$$

$$(g) \quad -n^{e}_{\gamma g^{-1}} + n^{g}_{\gamma g^{-1}} - n^{g}_{\gamma b^{-1} g^{-1}} = 0 \quad \forall \gamma$$

$$(e) \quad n^{h}_{\gamma e^{-1} h^{-1}} + n^{e}_{\gamma e^{-1}} - n^{e}_{\gamma c^{-1} e^{-1}} = 0 \quad \forall \gamma$$

$$(h) \quad -n^{f}_{\gamma h^{-1}} + n^{h}_{\gamma h^{-1}} - n^{h}_{\gamma e^{-1} h^{-1}} = 0 \quad \forall \gamma$$

$$(f) \quad n^{c}_{\gamma f^{-1} c^{-1}} + n^{f}_{\gamma f^{-1}} - n^{f}_{\gamma d^{-1} f^{-1}} = 0 \quad \forall \gamma$$

$$(d) \quad -n^{b}_{\gamma d^{-1}} + n^{c}_{\gamma d^{-1}} - n^{f}_{\gamma d^{-1} f^{-1}} = 0 \quad \forall \gamma$$

$$(c) \quad -n^{a}_{\gamma c^{-1}} - n^{h}_{\gamma c^{-1}} + n^{e}_{\gamma c^{-1} e^{-1}} - n^{b}_{\gamma c^{-1} b^{-1}} = 0 \quad \forall \gamma$$

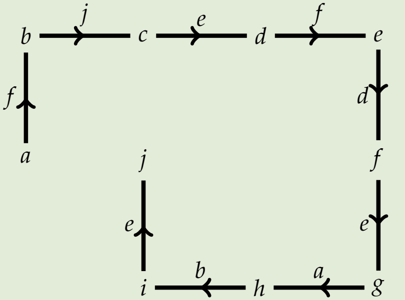
$$(b) \quad n^{g}_{\gamma b^{-1} g^{-1}} + n^{a}_{\gamma b^{-1} a^{-1}} + n^{b}_{\gamma b^{-1}} - n^{b}_{\gamma c^{-1} b^{-1}} = 0 \quad \forall \gamma$$

Results

Theorem

Let Γ *be a LOT admitting a "good enumeration", and such that* $G(\Gamma)$ *satisfies UPP. Then* Γ *is aspherical.*

Example



If $\pi_1(\Gamma)$ is UPP $\Rightarrow \Gamma$ is aspherical.

Muchas gracias.