CIMPA Research School:
Algebraic, Enumerative and Geometric Combinatorics - ECCO 2016

Triangulations of polytopes. Problem sheet.

5 Secondary polytopes

1. Draw the secondary fan of the third configuration in Exercise 1.1. (Since $n = 5$ and $d = 2$ you can draw this as a 2-dimensional fan, choosing any two coordinates for which the points corresponding to the other three coordinates form a triangle in V. Put differently, you are looking only at lifting vectors that give height zero to that triangle, which is no loss of generality).

2. Let S be a polyhedral subdivision of a point configuration A with n points. Let $\mathcal{C}(A, S) \subset \mathbb{R}^n$ be the set of all lifting vectors $\alpha \in \mathbb{R}^n$ that produce S as a regular subdivision. Show that:

   (a) $\mathcal{C}(A, S)$ is full-dimensional if, and only if, S is a triangulation.

   (b) $\mathcal{C}(A, S)$ is a convex, relatively open, polyhedral cone. That is, it is the set of solutions to a set of linear equalities and strict linear inequalities.

   (c) β is in the closure of $\mathcal{C}(A, S)$ if and only if the regular subdivision produced by β is refined by S.

3. Draw the graph of flips among *regular* triangulations of the first two configurations in Exercise 1.1 and understand how they are the graphs of two 3-dimensional polytopes (the corresponding secondary polytopes).

4. Let $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ be a configuration of n points in dimension d. The secondary polytope $\Sigma(A)$ of A lives in $\mathbb{R}^n$ but it has dimension $n-d-1$. To show that $\dim \Sigma(A) \geq n-d-1$, show that pushing points one by one you can get a chain of length $n-d-1$ in the poset of regular subdivisions of A, which coincides with the poset of faces of $\Sigma(A)$. To show that $\dim \Sigma(A) \leq n - d - 1$ show that the following $d + 1$ affine equations are independent and are satisfied by the GKZ-vector of every triangulation. Let $c = (c_1, \dots, c_d)$ be the barycenter of $P = \mathrm{conv}(A)$ and let $z = (z_1, \dots, z_n)$ be the GKZ-vector of a triangulation. Then:

$$\begin{aligned} \sum_{i=1}^{n} z_i &= (d+1)\,\mathrm{vol}(P), \\ \sum_{i=1}^{n} z_i a_i^j &= = (d+1)\,\mathrm{vol}(P) c_j, \forall j \in \{1, \dots, d\}. \end{aligned}$$

Remark: the equations admit the following matrix form:

$$\begin{pmatrix} a_1 & \dots & a_n \\ 1 & \dots & 1 \end{pmatrix} z = (d+1)\,\mathrm{vol}(P) \begin{pmatrix} c \\ 1 \end{pmatrix}$$

5. In the first configuration of Exercise 1.1, consider the GKZ-vectors of the eight triangulations that use all points. (Since the secondary polytope is three dimensional, you can look at only three coordinates; put differently, you are projecting $\mathbb{R}^6 \to \mathbb{R}^3$ forgetting three coordinates. For the sake of symmetry, use the coordinates of the three outer points). With or without doing any computations show that:

   (a) These eight points lie in an affine plane.

   (b) The two non-regular triangulations produce the same point.

   (c) The mid-point of any two "opposite" triangulations is the same.

   By symmetry, this implies the six regular triangulations produce the vertices of a regular hexagon and the two non-regular ones both produce the barycenter of it.

6. Now consider the same configuration but with one of the two concentric triangles in it slightly rotated with respect to the other one.

   (a) Are the eight GKZ-vectors still in an affine plane?

   (b) Do the two (formerly) non-regular triangulations still produce the same point?

   (c) Are the mid-points of any two "opposite" triangulations still the same?

   By answering these (or other) questions argue that the eight points now are the vertices of a 3-dimensional parallelepiped. What effect has this on the secondary polytope, and on the set of regular triangulations?