EXERCISES - ECCO 2016

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The exercises with a * are supposed to be the easier ones.

1. Partitions, corners and hooks

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of positive integers such that

$$\lambda_1 \geq \ldots \geq \lambda_k.$$

It is a partition of n if $\sum_i \lambda_i = n$. We denote this by $\lambda \vdash n$.

The Young diagram of λ is the diagram made of cells that are left justified such that there are λ_i cells in row i.

Exercise 0.*
Draw the Ferrers diagrams of the partitions of 5.

Exercise 1.* Given a partition λ, let $c(\lambda)$ the number of corners of λ (i.e. the number of is such that $\lambda_i > \lambda_{i+1}$). Let $p(n)$ be the number of partitions of n.

Show that

$$\sum_{\lambda \vdash n} c(\lambda) = \sum_{i=0}^{n-1} p(i).$$

Exercise 2.*
A triangular number is a number of the form $k(k+1)/2$ with $k \in \mathbb{N}$. The hook of a cell is the number of cells to its right and below (including the cell).

Show that the number of odd hooks minus the number of even hooks in a Young diagram is always a triangular number.

2. Counting walks in the Young lattice

Young's lattice is the infinite graph whose vertices are integer partitions ordered by containment of their Young diagram.

Date: June 13th, 2016.

Recall that U and D are linear transformations that act on integer partitions and are such that :

$$U(\lambda) = \sum_{\mu \triangleright \lambda} \mu, \quad D(\lambda) = \sum_{\mu \triangleleft \lambda} \mu,$$

where $\mu \triangleright \lambda$ means that μ and λ are partitions and the diagram of μ is the diagram of λ with one extra cell.

A walk of type $w = A_n \ldots A_i$ from $\emptyset$ to λ is a sequence $\lambda^{(0)}, \ldots, \lambda^{(n)}$ such that $\lambda^{(i-1)} \triangleright \lambda^{(i)}$ if $A_i = D$ and $\lambda^{(i-1)} \triangleleft \lambda^{(i)}$ if $A_i = U$.

Exercise 0.*
Enumerate the walks of type $UUUUU$ from $\emptyset$ to $(2,2,1)$. Enumerate the walks of type $UUDUU$ from $\emptyset$ to $(2,1)$.

Exercise 1.* Show that for any $i > 0$,

$$DU^i = U^i D + iU^{i-1}.$$

Show by induction that for any $i > 0$

$$U^n D^n = (UD - (n-1)I) \ldots (UD - I)UD.$$

Exercise 2 Let $w = A_m \ldots A_1$ be a word in the alphabet $\{U, D\}$ such that the number of such that the number of Us minus the number of Ds is n.

Suppose that

$$\begin{aligned}
r_{i,j}(w) &= 0; \text{ if } i < 0 \text{ or } j < 0 \text{or } i - j \neq n \\
r_{0,0}(\emptyset) &= 1 \\
r_{i,j}(Uw) &= r_{i-1,j}(w); \\
r_{i,j}(Dw) &= r_{i,j-1}(w) + (i+1) r_{i+1,j}(w).
\end{aligned}$$

Let $S_w = \{i \mid A_i = D\}$ and $a_i = \#\{j < i \mid A_j = U\}$ and $a_i = \#\{j < i \mid A_j = D\}$. Show that

$$r_{n,0}(w) = \prod_{i \in S_w} (a_i - b_i).$$

*Recall that $\alpha(w, \lambda)$ is the number of walks of type w in Young's lattice that start at the empty partition and end at the partition λ. Use this to prove that

$$\alpha(w, \lambda) = f^\lambda \prod_{i \in S_w} (a_i - b_i).$$

Exercise 3.* Use the previous exercise to prove that

$$\alpha(D^n U^n, \emptyset) = n!.$$

Compute

$$\alpha((DU)^n, \emptyset).$$

Exercise 4.* Recall that $w = A_n \dots A_1$ is a λ-valid word in Young's lattice if there is at least a walk for $\emptyset$ to λ of type w. Show that the number of $\emptyset$-valid words of length $2n$ is equal to $C_n = \binom{2n}{n}/(n+1)$ the n^{th} Catalan number.

Exercise 5.

We want to compute $\beta(\ell)$ the number of walks of length ℓ in young's lattice that start at the empty partition and end at the emptypartition. That is :

$$\beta(\ell) = \langle \emptyset | \, (U + D)^\ell \, | \lambda \rangle \, .$$

1.* Show that $\beta(\ell)$ is equal to zero if ℓ is odd.

2.* Apply $DU = UD + I$ and show that if $(U+D)^\ell = \sum_{i,j} B_{i,j}(\ell) U^i D^j$ then

$$B_{i,j}(\ell + 1) = B_{i,j-1}(\ell) + (i + 1)B_{i+1,j}(\ell) + B_{i-1,j}(\ell).$$

for $i, j, \ell \geq 0$ with $B_{0,0}(0) = 1$ and $B_{i,j}(\ell) = 0$ if $i, j < 0$ or $\ell < i + j$.

3. Show that $B_{i,0}(\ell) = \binom{\ell}{i}(\ell - i - 1)!!$ with $(2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdot \ldots 3 \cdot 1$.

4.* Deduce that $\beta(\ell) = (\ell - 1)!!$.

5. Recall that a perfect matching on ℓ is an involution on ℓ with no fixed points. Give a bijection between the walks counted by $\beta(\ell)$ and perfect matchings on ℓ.

Exercise 6.

Let $\beta(\ell, \lambda)$ the number of walks of length ℓ in Young's lattice that start at the empty partition and end at λ with λ partition of n.

1.* Show that $\beta(\ell, \lambda)$ is zero if $\ell - n$ is odd.

2. Show that $\beta(\ell, \lambda) = \binom{\ell}{n}(\ell - n - 1)!! f^\lambda$.

3. Hook length formula

Let λ be a partition of n. A standard Young tableau (SYT) is a filling of the diagram of λ with the number $1, 2, \ldots, n$ that is increasing in rows and columns. The number f^λ of SYT of shape λ is

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)};$$

where $h(u)$ is the number of cells in the hook of u.

Exercise 0.*
1. Draw all the SYT of shape $(2,2,1)$. Check that the hook length formula is correct for $(2,2,1)$.
2. Check that the hook length formula is correct for $\lambda = (n-k, 1^k)$ for $n > k \geq 0$.

Exercise 1.* Using the hook-length formula, show that the number of SYT of shape (n,n) is equal to C_n the n^{th} Catalan number. Give a bijective proof by using a bijection between SYT of shape (n,n) and Dyck path of length $2n$.

Exercise 2. The goal of this exercise is to prove the hook length formula.

1.* Show that if P is a SYT of shape λ then n must be in a corner of the diagram of λ.

2.* If α is a corner of the diagram of λ, let λ/α be the diagram of λ where α was erased. Show that

$$\sum_{\alpha \text{ corner}} \frac{f^{\lambda/\alpha}}{f^{\lambda}} = 1.$$

3. Show that if $\alpha = (x,y)$ is a corner and that we suppose the hook length formula true then

$$\frac{f^{\lambda/\alpha}}{f^{\lambda}} = \frac{1}{n} \prod_{1 \leq i < x} \left(1 + \frac{1}{h(i,y)-1}\right) \prod_{1 \leq j < y} \left(1 + \frac{1}{h(x,j)-1}\right).$$

4. Given a partition λ. Let us apply this algorithm
   - Choose uniformly a cell u of the diagram of λ (i.e. with probability $1/n$)
   - While this cell u is not a corner, then choose uniformly a cell in its hook (i.e. with probability $1/(h(u)-1)$)
   - Now the cell is a corner and then put n in the corner.

Take off the corner and apply the algorithm to the new shape.
Show that this algorithm will produce a SYT.

5. Let $R = \{r_1, \ldots, r_m\}$ and $C = (c_1, \ldots, c_\ell)$ be sets of rows and columns and let $\mathcal{P}(R,C)$ be the probability that the algorithm visits exactly cells in rows of R and in columns of C.

Prove that

$$\mathcal{P}(R,C) = 0; \quad \text{if } R \text{ or } C = \emptyset;$$
$$\mathcal{P}(R,C) = \frac{1}{n}; \quad \text{if } m = \ell = 1;$$

and that

$$\mathcal{P}(R,C) = \frac{1}{h(r_1,c_1)-1}[\mathcal{P}(R\backslash r_1, C)\mathcal{P}(R, C\backslash c_1)]$$

Show then by induction that :

$$\mathcal{P}(R,C) = \frac{1}{n}\prod_{i\in R\backslash\{r_m\}}\frac{1}{h_{i,c_\ell}-1}\prod_{j\in C\backslash\{c_\ell\}}\frac{1}{h_{r_m,j}-1}.$$

6. Let $p(x,y)$ the probability that the algorithm stops in the corner (x,y). Show that

$$p(x,y) = \sum_{\substack{A\subseteq\{1,\dots x-1\}\\ B\subseteq\{1,\dots y-1\}}}\mathcal{P}(A\cup\{x\}, B\cup\{y\}).$$

Deduce from this that :

$$p(x,y) = \frac{1}{n}\prod_{1\leq i<x}\left(1+\frac{1}{h(i,y)-1}\right)\prod_{1\leq j<y}\left(1+\frac{1}{h(x,j)-1}\right).$$

7. Show that this proves the hook length formula.

4. Robinson-Schensted algorithm

If we fix two partitions μ and ν. The "local rule" is a bijection between

- (λ, x) with λ a partition such that $\lambda = \mu$ or $\lambda \lhd \mu$ and $\lambda = \nu$ or $\lambda \lhd \nu$ and $x \in \{0,1\}$ and
- ρ a partition such that: $\rho = \mu$ or $\rho \rhd \mu$ and $\rho = \nu$ or $\rho \rhd \nu$ and $|\mu| + |\nu| = |\rho| + |\lambda| + x$.

If can be defined as: $\rho_1 = \max(\mu_1,\nu_1) + x$ and for $i > 1$ $\rho_i = \max(\mu_i,\nu_i) + \min(\mu_{i-1},\nu_{i-1}) - \lambda_i$.

We apply this local rule to a permutation matrix (i.e a matrix with zeros and ones and exactly one 1 in each row and column). We initialize the points to the left and below the matrix to $\emptyset$ and apply the local rule to each entry.

When we read the points on the top and to the right. We get two sequences $(\lambda^{(0)},\dots,\lambda^{(n)})$ and $(\mu^{(0)},\dots,\mu^{(n)})$ such that $\lambda^{(0)} = \mu^{(0)} = \emptyset$ and $\lambda^{(n)} = \mu^{(n)}$.

Exercise 1.* Show that when we apply the local rule to a permutation matrix : If λ labels any point p, the weight $|\lambda|$ of λ is the number of 1s in the quarter plane left and below p. Show that the local rule is bijective.

Exercise 2.* Show that when we apply the local rule to a permutation matrix, then $\lambda^{(0)} \lhd \ldots \lhd \lambda^{(n)}$ and $\mu^{(0)} \lhd \ldots \lhd \mu^{(n)}$. Show that this is a bijection between permutations of $\{1, \ldots, n\}$ and pairs of SYT with n cells of the same shape.

Exercise 3.* Show that there exists a bijection between young tableaux with n cells and involutions on $\{1, 2, \ldots, n\}$.

5. Increasing subsequences

Exercise 4. Let S_n be the set of permutations of $\{1, \ldots, n\}$. An increasing subsequence of a permutation $w = w_1 \ldots w_n$ is a sequence $w_{i_1} < \ldots < w_{i_k}$ such that $i_1 < \ldots < i_k$.

1. Show that we apply the local rule algorithm to a permutation matrix the length of the longest increasing subsequence is equal to the size of the 1st part of the shape of the SYT.
2. How can we read the length of the longest decreasing subsequence on the shape of the SYT?
3.* Show that all permutations of S_{mn+1} has an increasing subsequence of size $m+1$ or a decreasing subsequence of length $n+1$.
4.* Deduce that the average value of the longest ineasing subsequence of the permutations of S_n is greater than or equal to $\sqrt{n}/2$.
5.* How many permutations of S_{mn} have a longest increasing subsequence of length m a longest decreasing subsequence of length n?

6. Plane partitions

A plane partition $\Pi = (\pi_{i,j})$ with $1 \leq i, j \leq n$ is an array of non negative integers such that $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$. Its weight $|\Pi|$ us $\sum_{i,j} \pi_{i,j}$ and its trace $\mathrm{tr}(\Pi)$ is $\sum_i \pi_{i,i}$. Let $\mathcal{P}_n$ be the set of those plane partition.

We want to compute the generating function :

$$P_n(q,t) = \sum_{\Pi \in \mathcal{P}_n} q^{|\Pi|} t^{\mathrm{tr}(\Pi)}.$$

1. Use the decomposition of plane partitions into two SSYT of the same shape, to show that

$$P_n(q,t) = \sum_\lambda q^{-|\lambda|} t^{|\lambda|} s_\lambda(q,\ldots,q^n) s_\lambda(q,\ldots,q^n).$$

4. Show that

$$x^{|\lambda|} s_\lambda(q,\ldots,q^n) = s_\lambda(xq,\ldots,xq^n).$$

5. Use the Cacuchy identity

$$\sum_\lambda s_\lambda(x_1,\ldots,x_n) s_\lambda(y_1,\ldots,y_n) = \prod_{i,j=1}^n \frac{1}{1-x_i y_j}$$

to compute $P_n(q,t)$.

6.1. **Reverse plane partitions.** A reverse plane partition of shape λ is a filling of the cells of λ by nonnegative integers. The entries are increasing in rows and columns. Its weight is the sum of the entries.

For example

$$\begin{matrix} 0 & 0 & 7 \\ 1 & 4 & 7 \\ 1 & 6 & \end{matrix}$$

is a reverse plane partition of shape $(3,3,2)$.

1.* Let $\lambda = (3,3,2)$. Show that there exists a weight preserving bijection between reverse plane partitions of shape λ and sequences of partitions $\Lambda = (\lambda^{(0)},\ldots,\lambda^{(6)})$ such that $\lambda^{(0)},\ldots,\lambda^{(6)} = \emptyset$ et

$$\lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \succ \lambda^{(3)} \prec \lambda^{(4)} \succ \lambda^{(5)} \succ \lambda^{(6)}.$$

Recall that $\mu \succ \lambda$ if and only of $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots$.

For example $\Lambda = (\emptyset, (1), (6,1), (4,0), (7,0), (7), \emptyset)$ is such a sequence.

2.* Deduce that the generating function of such reverse plane partitions of shape $\lambda = (3,3,2)$ can be written as :

$$\langle \emptyset | \, \Gamma_-(q^{-1})\Gamma_-(q^{-2})\Gamma_+(q^3)\Gamma_-(q^{-4})\Gamma_+(q^5)\Gamma_+(q^6) \, | \emptyset \rangle \,.$$

3.* Show that this series is equal to :

$$\frac{1}{(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)^2(1-q^5)}.$$

4. Compute this series for any partition λ.