Lecture 4 Practice Problems

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(1) Which of the following are banners. (The barred letters are blue, the others are red.)
  (a) $884\bar{2}23\bar{3}$
  (b) $\bar{8}\bar{8}9\bar{2}$
  (c) $884\bar{2}2$

(2) Recall the bijection that takes ornaments to banners. (The barred letters are blue, the others are red.)
  (a) Find the image of the ornament $(2,\bar{2})(\bar{3},7,5,3,\bar{3},7,\bar{1})(2,\bar{2})$
  (b) Find the unique ornament that maps to the banner $3\bar{3}3\bar{1}5\bar{2}$

(3) Let $W_n := \{w \in \mathbb{P}^n : w_i \neq w_{i+1} \ \forall i \in [n-1]\}$. Expand $\sum_{w \in W_3} x^w t^{\mathrm{des}(w)}$ in the fundamental basis.

(4) Let $P := P_{5,3}$ and $G := G_{5,3}$.
  (a) Is $X_{G_{5,3}}(x,t)$ symmetric?
  (b) Give all the P-tableaux T together with the corresponding $\mathrm{inv}_G(T)$.
  (c) Expand $X_G(x,t)$ in the Schur basis.
  (d) Give 5 terms of the expansion of $X_G(x,t)$ in the F-basis.

(5) Show that for all $\sigma \in \mathfrak{S}_n$, $\mathrm{inv}_{<2}(\sigma) = \mathrm{des}(\sigma^{-1})$

(6) It is a result of Rawlings that $A_n^{(r)}(q,q) = [n]_q!$.
  (a) Verify this for $n = 3$ and all r.
  (b) Prove this for $r = 2$ and all n.

(7) Let $P = P_{n,r}$ and let $G = G_{n,r}$. Show that there exists a bijection ϕ from $\mathfrak{S}_n$ to the set of pairs (A, B), where A is a P-tableaux and B is a standard tableaux of the same shape as A, and if $\phi(\sigma) = (A, B)$ then $\mathrm{inv}_{<r}(\sigma) = \mathrm{inv}_{<r}(A)$ and $\mathrm{DES}_{\geq r}(\sigma) = \mathrm{DES}(B)$.

(8) Let $\rho : Qsym_n \to Qsym_n$ be the involution defined on the basis of monomial quasisymmetric functions by $\rho(M_\alpha) = M_{\alpha^{\mathrm{rev}}}$ for each composition α. Extend the involution ρ to $Qsym_n[t]$. Prove that

$$\rho(X_G(\mathbf{x},t)) = t^{|E|} X_G(\mathbf{x},t^{-1}).$$

(9) Prove that if $X_G(\mathbf{x},t)$ is symmetric (in $\mathbf{x}$) then it is palindromic.

(10) Prove that $X_{G_{n,r}}(\mathbf{x},t)$ is symmetric in $\mathbf{x}$ for all $r \in [n]$