HW2B Modern Control Theory

Master of Science in Electrical Engineering

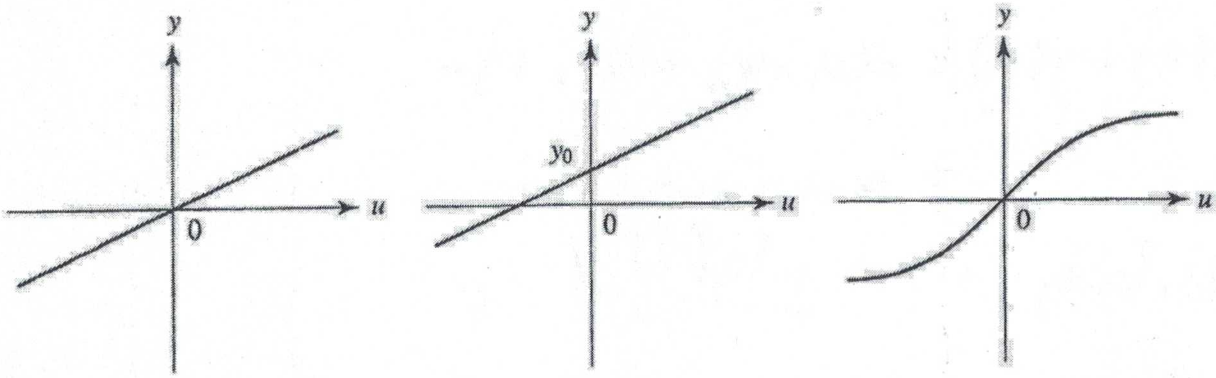
Davalos Gonzalez Erick Christopher

85%

Problem 1)

10 de

1. (10%) Consider the memoryless systems with the characteristic shown in figure below, un which u denotes the input and y the output. Which of them is a linear system? Is it possible to introduce a new output so that the system in Fig. (b) is linear?



(a) (b) (c)

a) For any system to be linear we need to satisfy

* Additivity and * Homogeneity.

The relation between y and u for a) is

* $y(u) = \alpha u$

* Additivity

$y(u_1+u_2) = \alpha(u_1+u_2) = \alpha u_1 + \alpha u_2$

$y(u_1) + y(u_2) = \alpha u_1 + \alpha u_2$

* Homogeneity $\rightarrow y(\alpha u) = \alpha y(u)$

$y(Ku) = \alpha(Ku) = (\alpha K)\cdot u$

$K\,y(u) = K(\alpha u) = (K\alpha)\underline{u}$

a) Satisfies both additivity and homogeneity. Thus its linear.

Page 1

b) In this case the relation between y and u is:

$$y(u) = \alpha u + y_0 \text{ ; } \alpha \text{ and } y_0 \text{ are constants.}$$

- $y(u_1+u_2) = \alpha(u_1+u_2) + y_0 = \alpha u_1 + \alpha u_2 + y_0$

$$y(u_1) + y(u_2) = \alpha u_1 + y_0 + \alpha u_2 + y_0$$

$$= \alpha u_1 + \alpha u_2 + 2y_0$$

Additivity is not satisfied.

- Homogeneity

$$y(ku_1) = \alpha k u_1 + y_0$$

$$k\,y(u_1) = k(\alpha u_1 + y_0) = k\alpha u_1 + k y_0$$

Does not satisfies Homogeneity. It's not linear.

To make it linear we introduce a new output y'

$$y' = y - y_0$$

$$y' = (\alpha u + y_0) - y_0 = \alpha u$$

So now the system becomes linear

c) The input-output relation is:

$$y = \sin(u).$$

- Additivity

  - $y(u_1+u_2) = \sin(u_1+u_2) =$

$= \sin(u_1+u_2) = \sin(u_1)\cos(u_2) + \cos(u_1)\sin(u_2)$

- $y(u_1) + y(u_2) = \sin(u_1) + \sin(u_2).$

$$\sin(u_1+u_2) \neq \sin(u_1) + \sin(u_2)$$

Additivity is not satisfied.

- Homogeneity.

$y(ku) = \sin(ku)$

$\underbrace{k}_{\text{Amplitude}}\, y(u) = k\sin(u).$

frequency

So $\sin(ku) \neq k\sin(u)$

The system isn't linear

10%

Problem 2)

(10%) Consider a system whose input u and output y are related by

$$y(t) = (P_\alpha u)(t) := \begin{cases} u(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

where α is a fixed constant. The system is called a *truncation operator*, which chops off the input after time α. Is the system linear? Is it time-invariant? Is it causal?

We need to check if the system is:

1. Linear. 2.- Time invariant 3.- Causal.

For linearity; $n = 1, 2.$

$$y_n(t) = (P_\alpha u_n)(t) = \begin{cases} u_n(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$ ✓

- Additivity.

$$(P_\alpha(u_1+u_2))(t) = \begin{cases} u_1(t)+u_2(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

$$y_1(t) + y_2(t) = \begin{cases} u_1(t)+u_2(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha. \end{cases}$$ ✓

The system satisfies additivity.

- Homogeneity.

$$(P_\alpha(k \cdot u))(t) = \begin{cases} k \cdot u(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

$$k y(t) = k \begin{cases} \cancel{k} \cdot u(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$ ✓

Also satisfies homogeneity. It's linear.

Page 4.

To check invariance we need to see if a time shift in the input results in an equivalent time shift in the output.

* Applying time shift to the input $u(t)$. ①

$$(P_\alpha u)(t-t_0) = \begin{cases} u(t-t_0) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

* Applying the system and shifting the output. ②

$$y(t-t_0) = \begin{cases} u(t-t_0) & \text{for } t-t_0 \le \alpha \\ 0 & \text{for } t-t_0 > \alpha \end{cases}$$

$$y(t-t_0) = \begin{cases} u(t-t_0) & \text{for } t \le \alpha+t_0 \\ 0 & \text{for } t > \alpha+t_0 \end{cases}$$

In ① the truncation occurs at $t=\alpha$ and in ② the truncation occurs at $t=\alpha+t_0$ so the system is not time invariant

2

* Causality

The system to be causal needs to depend only in present and past values of the input.

In the case:

$$y(t) = (P_\alpha u)(t) = \begin{cases} u(t) & \text{for } t \leq \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

α is a fixed time point, the output $y(t)$ is equal to $u(t)$ when $t = \alpha$ and 0 for $t > \alpha$

- For $t \leq \alpha$ the output depends on the input at the same time so it does not require future values of the input.
- For $t > \alpha$ the output is 0.

There is no dependency on future input values.
So the system is causal.

Summary for Problem 2.

Yes, system is linear.
No, the system is not time invariant.
Yes, the system is causal.

10%

Problem 3)

(15%) Consider a system with input u and output y. Three experiments are performed on the system using input $u_1(t)$, $u_2(t)$, and $u_3(t)$ for $t \geq 0$. In each case, the initial state $x(0)$ at time $t=0$ is the same. The corresponding outputs are denoted by y_1, y_2 and y_3. Which of the following statements are correct if $x(0) \neq 0$?

a) If $u_3 = u_1 + u_2$, then $y_3 = y_1 + y_2$

b) If $u_3 = 0{,}5(u_1 + u_2)$, then $y_3 = 0{,}5(y_1 + y_2)$

c) If $u_3 = u_1 - u_2$, then $y_3 = y_1 - y_2$

Which are the correct if $x(0) = 0$?

Vease definición del estado en diapositiva 9/31

¿Por que la suma?

For the $x(0) \neq 0$ case:

a) $y_1 = f(u_1) + x(0)$ and $y_2 = f(u_2) + x(0)$

$y_3 = y(3) = f(u_1+u_2) + x(0) = f(u_1) + f(u_2) + x(0)$ ①

$y_1 + y_2 = [F(u_1) + x(0)] + [f(u_2) + x(0)] =$

$= F(u_1) + F(u_2) + 2x(0).$ ②

$f(u_1) + f(u_2) + x(0) \neq f(u_1) + f(u_2) + 2x(0)$

The statement is not correct for $x(0) \neq 0$.

b) $y(u_3) = f(0.5(u_1+u_2)) + x(0) = 0.5 f(u_1+u_2) + x(0).$

$0.5(y_1+y_2) = 0.5(f(u_1) + x(0) + f(u_2) + x(0)) = \ldots$

$= \underbrace{0.5(f(u_1) + f(u_2)) + x(0)}$ $= 0.5(y_1+y_2) + x(0)$

Statement b) is correct ~~for~~ $x(0) \neq 0$

c) $y_1 = f(u_1) + x(0)$. Revisar definición del estudio.

$y_2 = f(u_2) + x(0)$.

$y_3 = f(u_3) = \underline{f(u_1 - u_2) + x(0)}$;

$y_1 - y_2 = [f(u_1) + x(0)] - [f(u_2) + x(0)]$

$y_1 - y_2 = f(u_1) - f(u_2)$

$y_3 \neq y_1 - y_2$ The statement does not hold for $x(0) \neq 0$.

Now lets suppose $x(0) = 0$.

a) $y_3 = f(u_3) = f(u_1 + u_2) = f(u_1) + f(u_2) = y_1 + y_2$

$f(u_1 + u_2) = f(u_1) + f(u_2) = y_1 + y_2$

The statement is correct for a) with $x(0) = 0$.

b) $y_3 = f(u_3) = f(0.5(u_1+u_2)) = 0.5 f(u_1+u_2)$

$= 0.5[f(u_1) + f(u_2)] = \underline{0.5(y_1+y_2)}$

The statement is correct for $x(0)=0$. ✓

c) $y_3 = f(u_3) = f(u_1-u_2) = f(u_1) - f(u_2) = y_1 - y_2$.

The statement is correct for $x(0)=0$. ✓

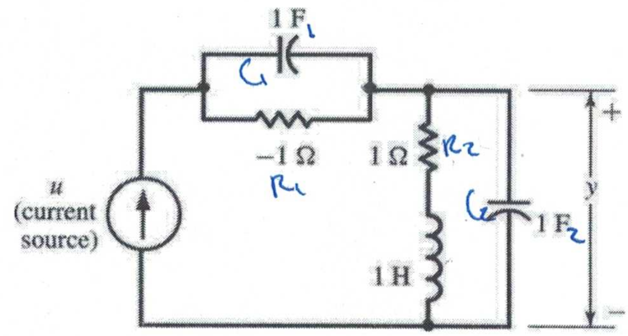
Summary: * All are correct when $x(0)=0$. ✓

* Statement is not correct for (a) when $x(0) \neq 0$. neither for (c).

20/20

Problem 4)

(20%) Find the transfer function by using the dynamical-equation description of the network in the below figure obtained in the previous homework. Do you think the transfer function is a good description of this system?



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = v_{c_1}$

$x_2 = V_{c2}$

$x_3 = i_{L_3}$

$v_L = L\frac{di(t)}{dt}$; $i_c = C\frac{dv(t)}{dt}$

$i_L = \frac{1}{L}\int v\,dt \quad v_c = \frac{1}{C}\int i\,dt$

KCL:

① $u(t) - C_1\dot{x}_1 - \frac{x_1}{R_1} = 0$ ✓

② $C_1\dot{x}_1 + \frac{x_1}{R_1} - x_3 - C_2\dot{x}_2 = 0$

$V_{R2} + V_L = V_{c2}$

③ $x_3(R_2) + L\dot{x}_3 = x_2 \rightarrow \dot{x}_3 = \frac{x_2}{L} - \frac{x_3 R_2}{L}$

① $\dot{x}_1 = \frac{-x_1}{R_1 C_1} + \frac{u(t)}{C_1}$

Using ① and ② to get $\dot{x}_2$ equation

$$C_1\left(\frac{-x_1}{R_1C_1}+\frac{u(t)}{C_1}\right)+\frac{x_1}{R_1}-x_3-C_2\dot{x}_2=0$$

$$\dot{x}_2=\frac{u(t)}{C_2}-\frac{x_3}{C_2}$$

Summary and replacing values.

① $\dot{x}_1=\frac{-x_1}{(-1)(1)}+\frac{u(t)}{1}=\underline{x_1+u(t)}.$

② $\dot{x}_2=\underline{u(t)-x_3}$

③ $\dot{x}_3=x_2-x_3$

④ $u(t)=\dot{x}_1-x_1$

$$u(t)=\frac{-x_1}{R_1C_1}+\frac{u(t)}{C_1}\rightarrow u(t)\left(1-\frac{1}{C_1}\right)=\frac{-x_1}{R_1C_1}$$

$$u(t)\left(\frac{C_1-1}{C_1}\right)=\frac{-x_1}{R_1C_1}\rightarrow u(t)=\frac{-x_1}{R_1C_1}\left(\frac{C_1}{C_1-1}\right)$$

$$u(t)=\frac{-x_1}{R_1(C_1-1)}$$

⑤ $y=x_2.$

$\dot{x} = Ax + Bu$

$y = Cx + Du.$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{B} u(t) \quad \checkmark$$

$$[y] = \overset{1\times 3}{\underset{C}{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}} \overset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

For the transfer function

$G(s) = C(sI-A)^{-1}B + D.$

$$sI - A = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s & -1 \\ 0 & 1 & s+1 \end{bmatrix} \begin{matrix} s-1 & 0 \\ 0 & s \\ 0 & 1 \end{matrix}$$

$\det(sI-A) =$

$s\cdot(s-1)(s+1) -$

$+ s-1 =$

$(s-1)[s(s+\;)+1] =$

$=(s-1)(s^2+s+1)$

$$\det(sI-A) = (s-1)\left[s(s+1) - (1)(-1)\right] = (s-1)\left[s^2+s+1\right] \quad \checkmark$$

$$= s^3 + \cancel{s^2} + \cancel{s} - \cancel{s^2} - \cancel{s} - 1 = \underline{s^3 - 1}$$

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)} \underbrace{\operatorname{adj}(sI-A)}$$

$$(sI-A)^{-1} = \frac{1}{s^3-1}\begin{bmatrix} s^2+s+1 & 0 & 0 \\ 0 & s^2-1 & 0 \\ 0 & 0 & s^2-s \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s^2+s+1}{s^3-1} & 0 & 0 \\ 0 & \frac{s^2-1}{s^3-1} & 0 \\ 0 & 0 & \frac{s^2-s}{s^3-1} \end{bmatrix}$$

$$G(s) = \underset{1\times3}{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} \frac{s^2+s+1}{s^3-1} & 0 & 0 \\ 0 & \frac{s^2-1}{s^3-1} & 0 \\ 0 & 0 & \frac{s^2-s}{s^3-1} \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + 0.$$

$$G(s) = \begin{bmatrix} 0 & \frac{s^2-1}{s^3-1} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{s^2-1}{s^3-1} = \frac{\cancel{(s-1)}(s+1)}{\cancel{(s-1)}(s^2+s+1)}$$

Due the transfer function assume linear behavior and time-invariance and our circuit has those properties the transfer function obtained provides a good description of the system.

Problems Linear Dynamical Systems

Ejercicio 2.5 *Encuentre los eigenvalores y eigenvectores de la matriz A y muestre que $B = P{-}1AP$ es una matriz diagonal. Resuelva el sistema lineal $\dot{y} = By$ y luego resuelva $\dot{x} = Ax$ usando el Corolario 2.2 anterior. Esboce ambos retratos fase en el plano x y en el plano y.*

$$a)\quad A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad b)\quad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad c)\quad A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Ejercicio 2.6 *Encuentre los eigenvalores y eigenvectores de la matriz A, resuelva el sistema lineal $\dot{x} = Ax$, determine los subespacios estable e inestable para el sistema lineal, y esboce el retrato fase para*

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} x.$$

Ejercicio 2.7 *Escriba las siguientes ecuaciones diferenciales lineales con coeficientes constantes en la forma del sistema linea (0.1) y resuelva:*

$$a)\quad \ddot{x} + \dot{x} - 2x = 0 \quad b)\quad \ddot{x} + \dot{x} = 0 \quad c)\quad \dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$$

Nota: Haga $x_1 = x$, $x_2 = \dot{x_1}$, etc.

Ejercicio 2.8 *Usando el Corolario 2.2 de esta sección, resuelva el problema de valor inicial:*

$$\dot{x} = Ax, \qquad x(0) = x_0$$

para (b) *la matriz* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ *y* $x_0 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

Ejercicio 2.10 *Sea A una matriz real de orden 2 con eigenvalores distintos λ y μ. Suponga que un eigenvector para λ es $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ y un eigenvector para μ es $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$. Esboce el retrato fase para $\dot{x} = Ax$ para los siguientes casos:*

$$a)\quad 0 < \lambda < \mu \qquad b)\quad 0 < \mu < \lambda \qquad c)\quad \lambda < \mu < 0$$

$$d)\quad \lambda < 0 < \mu \qquad e)\quad \mu < 0 < \lambda \qquad f)\quad \lambda = 0, \mu > 0$$

Exercise 2.5 $\frac{9}{10}$

b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$; $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8$

$(\lambda - 4)(\lambda + 2) \longrightarrow \underbrace{\lambda_1 = 4} \quad \underbrace{\lambda_2 = -2}$

Getting eigenvectors.

$\lambda_1 = 4$

$A - \lambda_1 I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\quad -3x + 3y = 0 \rightarrow 3y = 3x$

$3x - 3y = 0 \rightarrow 3x = 3y.$

If $x = 1$

$\underbrace{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

λ_2

$A - \lambda_2 I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\quad 3x + 3y = 0 \rightarrow 3x = -3y$

If $x = 1$

$\underbrace{\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$P = \{v_1, v_2 \ldots v_n\}$

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

From Corollary $x(t) = P\,E(t)P^{-1}x(0)$.

$\lambda_1 = 4$

$\lambda_2 = -2$ $\quad E(t) = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$; $\quad y = E(t)\overbrace{P^{-1}x(0)}^{y(0)}$

$\dot{y} = By$ ✗

$\boxed{B = P^{-1}AP = \operatorname{diag}[\lambda_1, \lambda_2]}$ ok $\Rightarrow \boxed{e^{Bt}}$ Revisar

$B = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$; $\dot{y} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\dot{y} = \begin{bmatrix} 4e^{4t} & 0 \\ 0 & -2e^{-2t} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2e^{4t} & 2e^{4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\dot{y} = \underbrace{2c_1e^{4t} + 2c_2e^{4t}}_{\dot{y}_1} \underbrace{- c_1e^{-2t} + c_2e^{-2t}}_{\dot{y}_2}$ ✗ Revisar
Es un vector de 2x1.

No comprendí esto.

$\dot{y}_1 = 4y_1$; $\dot{y}_2 = -2y_2$

$$x(t)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{4t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} =$$

$$x(t)=\begin{bmatrix}e^{4t} & e^{-2t}\\ e^{4t} & -e^{-2t}\end{bmatrix}\begin{bmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

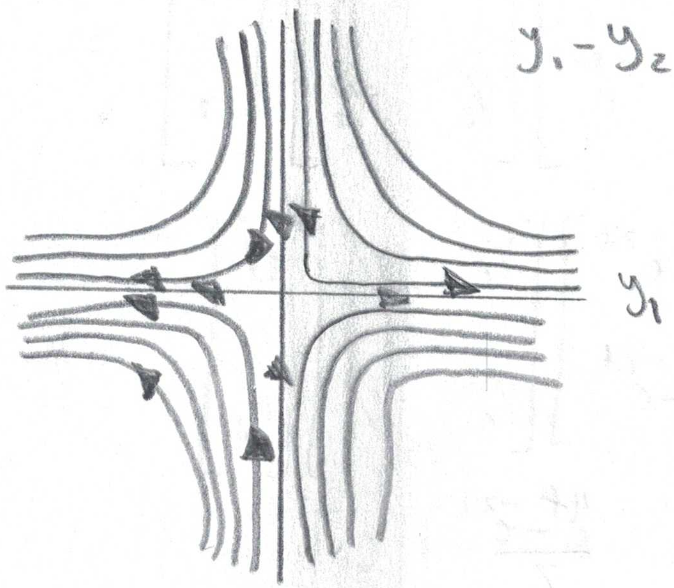
$$x(t)=\begin{bmatrix}\frac{e^{4t}+e^{-2t}}{2} & \frac{e^{4t}-e^{-2t}}{2}\\ \frac{e^{4t}-e^{-2t}}{2} & \frac{e^{4t}+e^{-2t}}{2}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

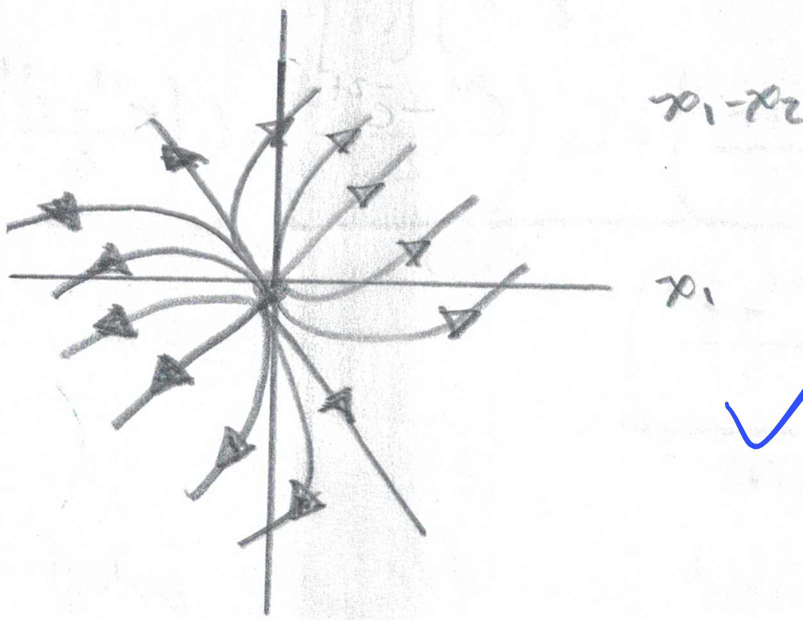
$$= \underbrace{c_1\left(\frac{e^{4t}+e^{-2t}}{2}\right)+c_2\left(\frac{e^{4t}-e^{-2t}}{2}\right)}_{x_1(t)}+\underbrace{c_1\left(\frac{e^{4t}-e^{-2t}}{2}\right)}$$

$$\underbrace{+c_2\left(\frac{e^{4t}+e^{-2t}}{2}\right)}_{x_2(t)}$$

Es un vector de $\underset{\sim}{2\times 1}$

In the next page the phase portrait is shown.


y_2
$y_1 - y_2$
y_1


x_2
$x_1 - x_2$
x_1

a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} =$

$(3-\lambda)(3-\lambda) - (1)(1) = 9 - 3\lambda - 3\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0 \quad (\lambda-4)(\lambda-2) ; \lambda_1 = 4 \quad \lambda_2 = 2.$ ✓

$\lambda = 4$

$A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\quad -x + y = 0 \rightarrow x = y$

$x - y = 0 = x = y$

$x = 1$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

$\lambda_2 = 2$

$A - \lambda_2 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\quad x + y = 0 \quad x = -y$

$x + y = 0$

$x = 1$ ✓

$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$E(t) = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix}$; $B = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

$$\dot{y} = \underbrace{\begin{bmatrix} 4e^{4t} & 0 \\ 0 & 2e^{2t} \end{bmatrix}}_{B \cdot E(t)} \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{y(0)}$$

¿Porque?

$y = E(t)P^{-1}x(0)$

$\dot{y} = By$

$$\dot{y} = \begin{bmatrix} 2e^{4t} & 2e^{4t} \\ e^{2t} & -e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Revisar

$$\dot{y} = \underbrace{2e^{4t} \cdot c_1 + 2e^{4t} c_2}_{\dot{y}_1} + \underbrace{e^{2t} \cdot c_1 - e^{2t} c_2}_{\dot{y}_2}$$

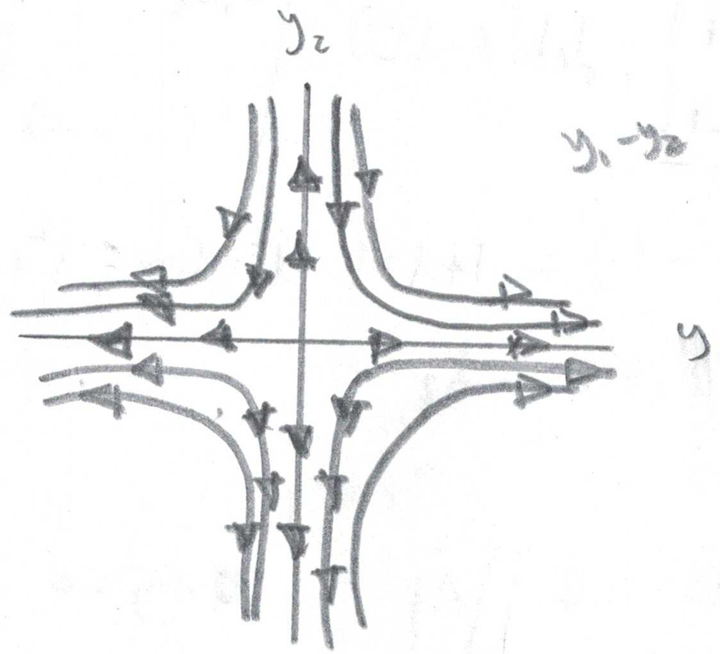
Es un vector de 2x1

$\dot{y}_1 = 4y_1$; $\dot{y}_2 = 2y_2$; $x(t) = PE(t)P^{-1}x(0)$

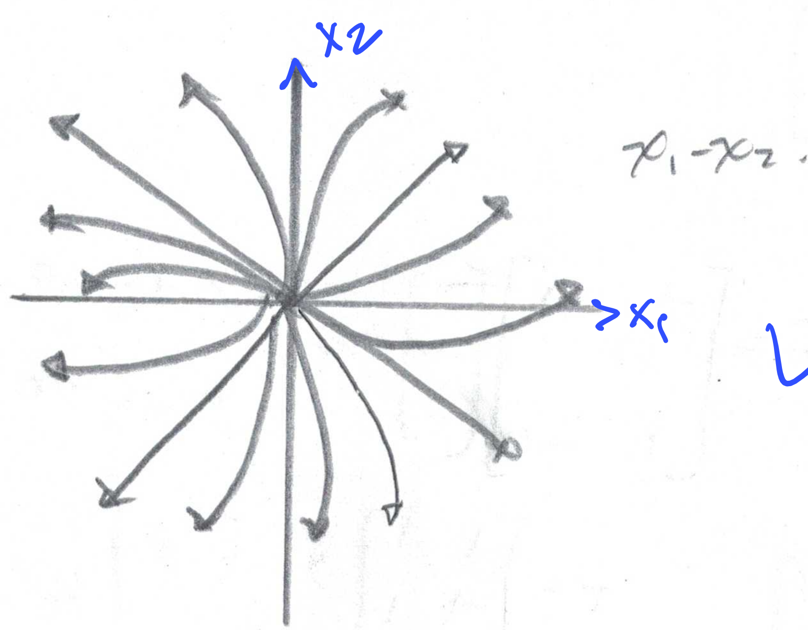
$$x(t) = \underbrace{\begin{bmatrix} e^{4t} & e^{2t} \\ e^{4t} & -e^{2t} \end{bmatrix}}_{P \cdot E(t)} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{e^{4t}+e^{2t}}{2} & \frac{e^{4t}-e^{2t}}{2} \\ \frac{e^{4t}-e^{2t}}{2} & \frac{e^{4t}+e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Phase portrait in the next page.



y_2
y_1
$y_1 - y_2$



x_2
x_1
$x_1 - x_2$.

c) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$

$(-1-\lambda)(-1-\lambda) - (1) = 1 + \lambda + \lambda + \lambda^2 - 1 = \lambda^2 + 2\lambda = 0$

$\lambda(\lambda+2) \quad \underline{\lambda_1 = -2, \lambda_2 = 0}$

$\lambda_1 = -2$

$A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad x + y = 0$

$x = -y$

$x = 1$

$\underbrace{\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$\lambda_1 = 0$

$A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad \begin{matrix} -x + y = 0 \\ x - y = 0 \end{matrix} \quad x = y$

$x = 1$

$\underbrace{\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$E(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$$\dot{y} = \begin{bmatrix} -2e^{-2t} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\boxed{B \cdot E(t).}$ α

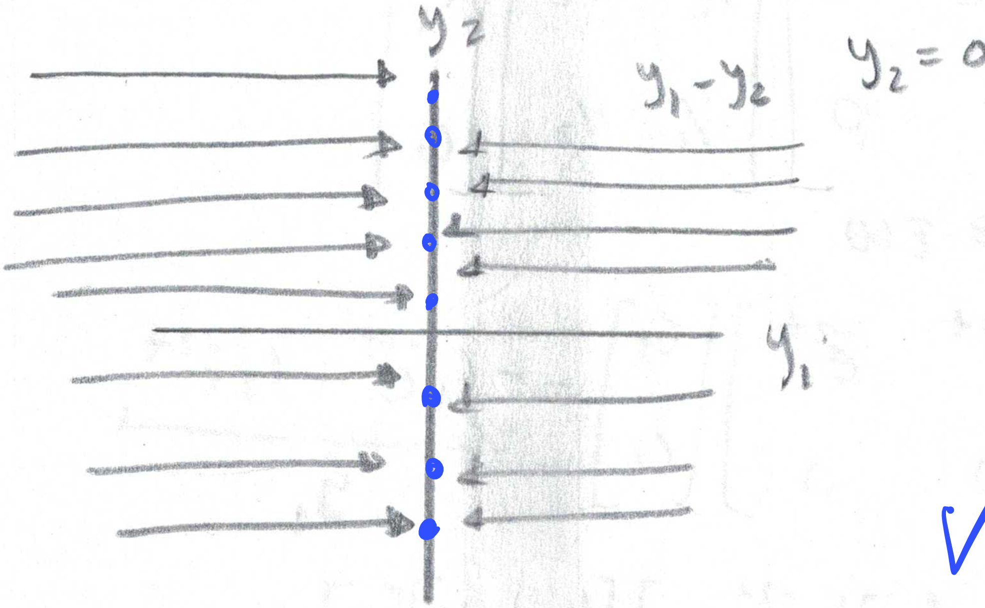
Revisar, es un vector

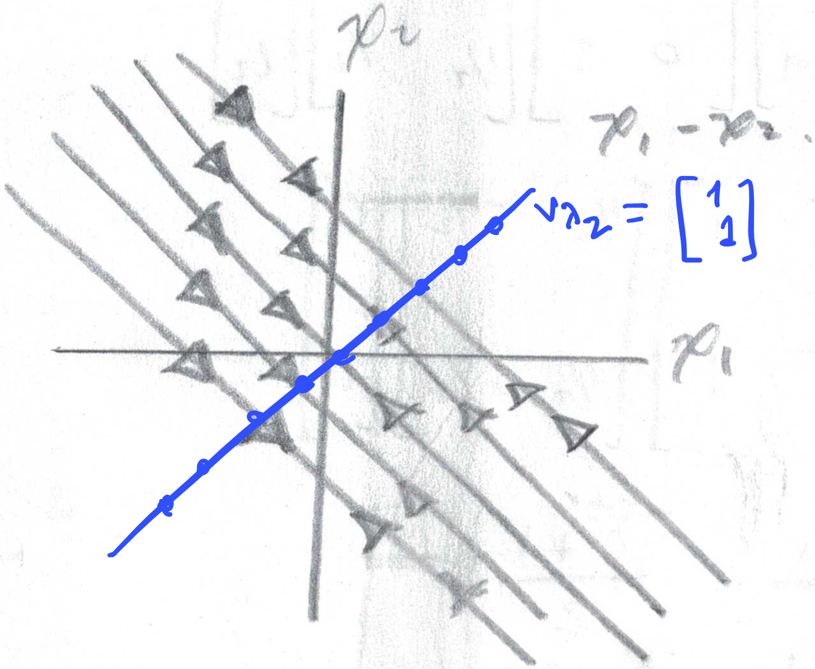
$$\dot{y} = \begin{bmatrix} -e^{-2t} & e^{-2t} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \underbrace{-c_1 e^{-2t} + c_2 e^{-2t}}_{\cancel{\dot{y}_1}} \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{-2t} & 1 \\ -e^{-2t} & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{e^{-2t}+1}{2} & \frac{-e^{-2t}+1}{2} \\ \frac{-e^{-2t}+1}{2} & \frac{e^{-2t}+1}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$





¡Revisar!

2020

Exercise 2.6.

$$A=\begin{bmatrix}1&0&0\\1&2&0\\1&0&-1\end{bmatrix};\ \det(A-\lambda I)=.$$

$$\left[\begin{array}{ccc|cc}1-\lambda&0&0&1-\lambda&0\\1&2-\lambda&0&1&2-\lambda\\1&0&-1-\lambda&1&0\end{array}\right]=-\lambda^3+2\lambda^2+\lambda-2.$$

$\lambda_1=1,\ \lambda_2=-1,\ \lambda_3=2.$

$\lambda=1$

$$A-\lambda_1 I=\begin{bmatrix}0&0&0\\1&1&0\\1&0&-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&0\\1&0&-2\\0&0&0\end{bmatrix}R_2-R_1=\begin{bmatrix}1&1&0\\0&-1&-2\\0&0&0\end{bmatrix}R_2+R_1$$

$$\begin{bmatrix}1&0&-2\\0&-1&-2\\0&0&0\end{bmatrix}$$

$x_1-2x_3=0 \to x_1=2x_3$

$-x_2-2x_3=0 \quad x_2=-2x_3$

$x_3=x_3$

$$\vec{V}_1=\begin{bmatrix}2x_3\\-2x_3\\x_3\end{bmatrix}\quad x_3=1\quad \vec{V}_1=\begin{bmatrix}2\\-2\\1\end{bmatrix}$$

$\lambda_2 = -1$

$$A - \lambda_2 I = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_1/2 \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = x_3$

$$\underbrace{\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$\lambda_3 = 2.$

$$A - \lambda_3 I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_3 + R_1 \end{matrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\underbrace{\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} ; \; P = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} ; \; P^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$x(t) = P E(t) P^{-1} x(0).$$

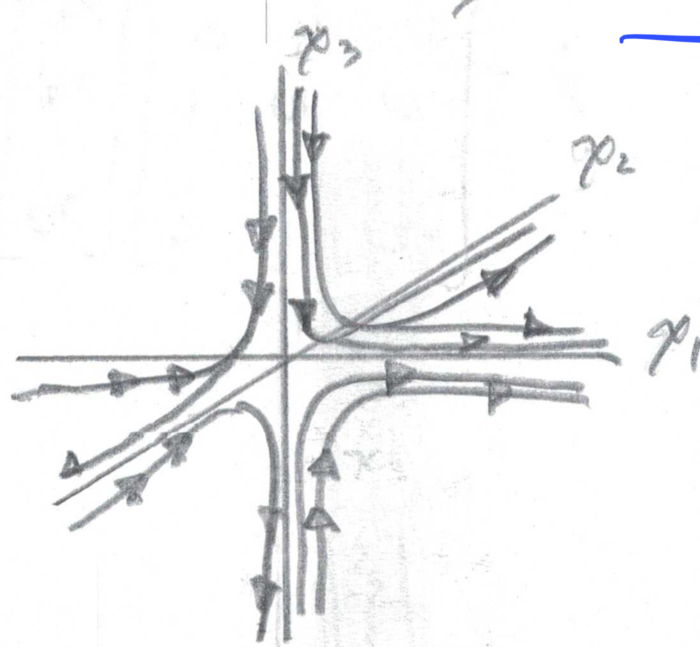
$$x(t) = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 2e^{t} & 0 & 0 \\ -2e^{t} & 0 & e^{2t} \\ e^{t} & e^{-t} & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{t} & 0 & 0 \\ -e^{t} + e^{2t} & e^{2t} & 0 \\ \frac{e^{t} - e^{-t}}{2} & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$E_{stable} = \{\vec{V}_3\}$$

$$E_{unstable} = \{\vec{V}_1, \vec{V}_2\}$$



3%

Exercis 2.7.

a) $\ddot{x} + \dot{x} - 2x = 0$

$\dot{x}_1 = x_2$

$\dot{x}_2 = 2x_1 - x_2$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ ✓

b) $\dddot{x} + \ddot{x} = 0$

$\dot{x}_2 = -x_2$

$\dot{x}_1 = x_2$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ ✓

c) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$

$\dot{x}_3 = 2x_3 + x_2 - 2x_1$

$\dot{x}_2 = x_3$

$\dot{x}_1 = x_2$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ ✓

Falto resolver los casos.

10%

2.8

From Exercise 2.6 we have the general form of matrix A, we now evaluate the $x(0)$ conditions:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

From page 27:

$$x(t) = \begin{bmatrix} e^{t} & 0 & 0 \\ -e^{t}+e^{2t} & e^{2t} & 0 \\ \frac{e^{t}-e^{-t}}{2} & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1(t) = e^{t}$$

$$x_2(t) = -e^{t} + e^{2t} + 2e^{2t}$$

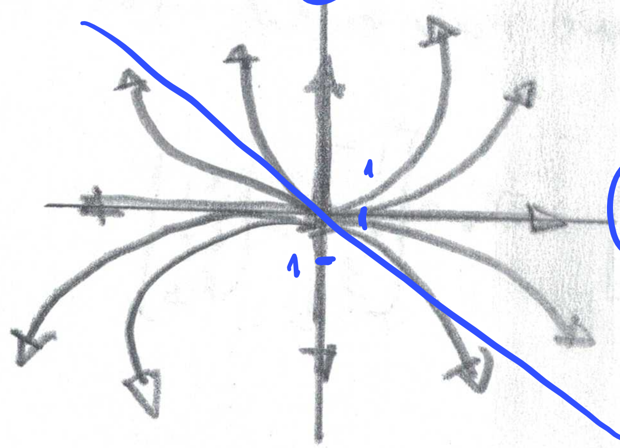
$$x_3(t) = \frac{e^{t}-e^{-t}}{2} + 3e^{-t}$$

5%

2.10

a)

No es V_μ, $V_\mu = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T$



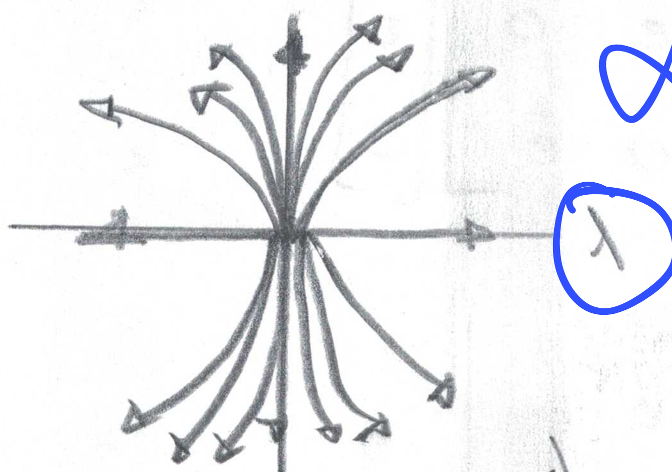
λ

V_λ

V_μ

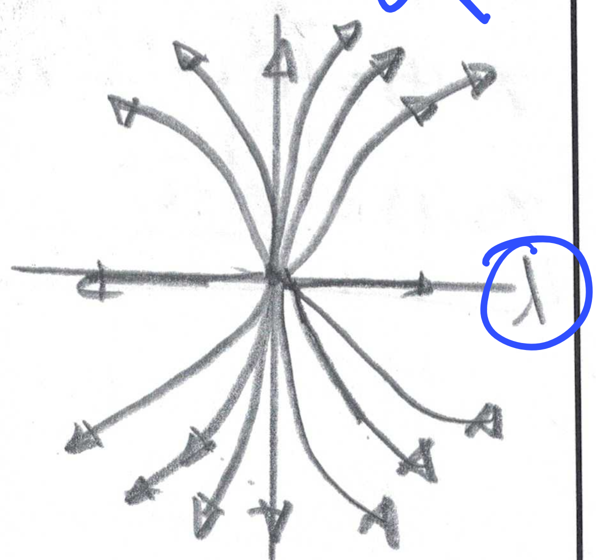
μ, λ son eigenvalores, no eigenvectores

b)



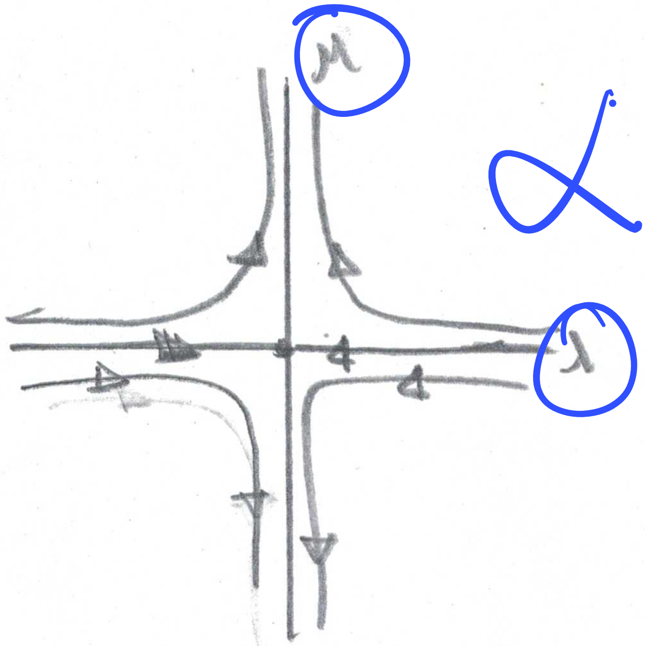
λ

c)

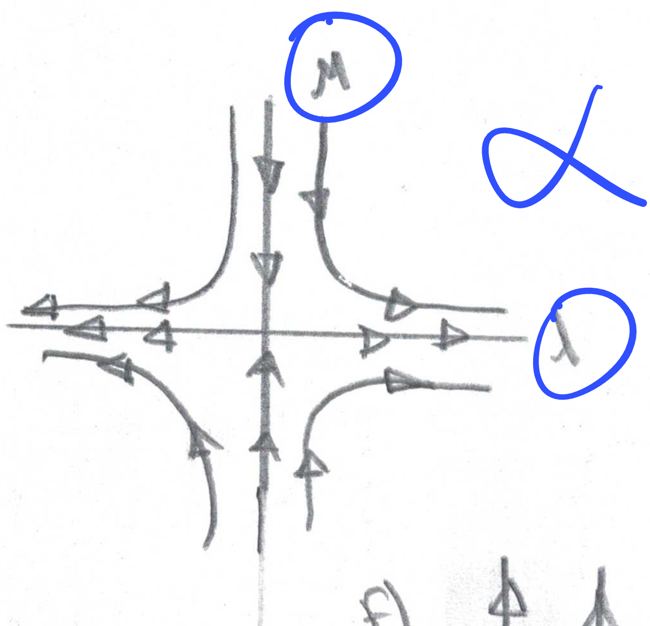


λ

d)



e)



f)