Analyzing Aliasing in Discrete-Time Signal Processing

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***Abstract*—Aliasing is a significant problem in digital signal processing, where continuous signals are sampled to create discrete signals. This paper explores the effects of aliasing through the analysis of sinusoidal signals sampled at various frequencies. By examining the relationships between the signal frequency, sampling frequency, and resulting alias frequencies, we illustrate the importance of adhering to the Nyquist rate.**

Keywords— Aliasing, Nyquist Rate, Sampling Frequency, Digital Signal Processing.

I. Nomenclature

- $x(n)$: Sampled signal
- $x_a(nT) = x_a(t)$: Continuous-time signal
- F_0: Frequency of the continuous-time signal
- F_s: Sampling rate
- F_N: Nyquist frequency
- T: Sampling period
- N: Discrete period
- t: Time
- f_0: Frequency of the sampled signal
- n: Sample index

II. Introduction

A. *Why does Aliasing Occurs*

Aliasing occurs when a signal is sampled at a rate that is not sufficient to capture its frequency content accurately. This happens when the sampling rate is lower than the Nyquist rate, which is twice the highest frequency present in the signal. As a result, high-frequency components of the signal are misrepresented as lower frequencies in the sampled version. When trying to reconstruct the original signal from these samples, the aliasing effect makes it impossible to recover the original high-frequency components, leading to distortion or a completely different signal.

B. *Sampling Theorem and Nyquist Frequency*

The Nyquist frequency is a key concept in signal processing, very important for preventing aliasing when converting a continuous signal into a digital one. It represents half of the sampling rate used in a digital system. For example, if you sample a signal at 1000 Hz, the Nyquist frequency is 500 Hz. This means that any frequency component in the original signal higher than 500 Hz cannot be accurately captured and will appear as a lower frequency. This is:

$$F_N = \frac{F_s}{2} \tag{1}$$

The Nyquist-Shannon sampling theorem states that you must sample at a rate that is at least twice the highest frequency in the signal. If a signal contains frequencies above this, it will overlap with lower frequencies in the digital representation, causing errors and distortions when trying to reconstruct the original signal.

$$F_s > 2F_{max} \tag{2}$$

Consider the following sinusoidal continuous-time signal:

$$x_a(t) = \sin(2\pi F_0 t), \quad -\infty < t < \infty \tag{3}$$

When this digital signal is sampled at discrete time intervals, the sampled signal can be expressed as:

$$x(n) = \sin\left(2\pi \frac{F_0}{F_s} n\right) = \sin(2\pi f_0 n) \quad -\infty < n < \infty \tag{4}$$

Where $F_s = \frac{1}{T}$.

III. Case study: Sampling of sinusoidal signals

A. *Plot the signal x(n) in eq.4 given the following parameters:*

$$0 \le n \le 99, F_s = 5kHz \ and \ F_0 = 0.5, 2, 3, 4.5 \ kHz \tag{5}$$

As we discussed, the sampling theorem tells us that the maximum frequency that can be sampled without aliasing is the sampling rate over two. In this case the sampling rate is fixed at 5kHz, so the maximum frequency that can be sampled without aliasing is 2.5kHz, any sample above 2.5kHz will experience aliasing.

We expect aliasing in the 3 and 4.5kHz signals.

*Sampling and plotting the signal with 0.5kHz fundamental frequency:

$$f_0 = \frac{F_0}{F_s} = \frac{0.5kHz}{5kHz} = \frac{1}{10} < \frac{1}{2} \tag{6}$$

$$x_1(n) = \sin\left(2\pi \frac{0.5}{5} n\right), \quad 0 \le n \le 99 \tag{7}$$

The fundamental discrete period of the 0.5kHz signal sampled at a sampling rate of 5kHz is N = 10 samples per cycle. Then plotting (7):

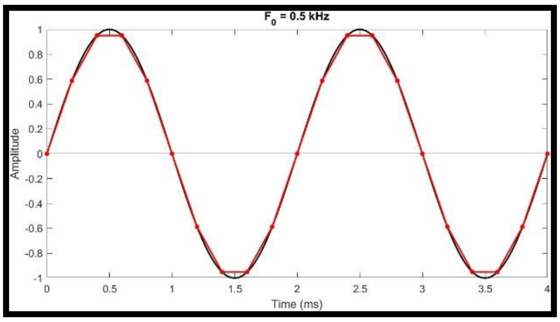

Fig. 1. 0.5kHz Original Signal vs Sampled Signal

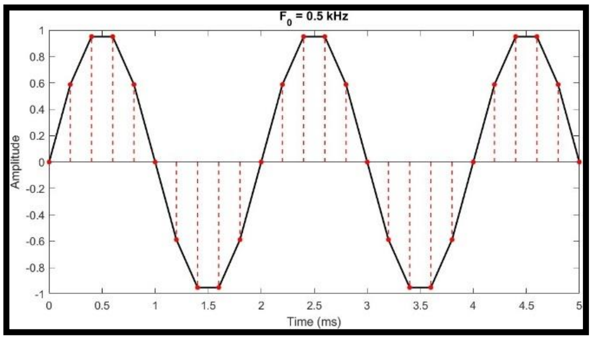

Fig. 2. 0.5kHz Sampled Signal and Its Samples

In Fig. 1 and Fig. 2, we can appreciate the original sinusoid in black, and the sampled signal with 10 samples per cycle in red. The signal is sampled correctly $F_S > 2F_0$ so we can reconstruct the signal without any problem.

*Sampling and plotting the signal with 2kHz fundamental frequency:

$$f_0 = \frac{F_0}{F_S} = \frac{2kHz}{5kHz} = \frac{2}{5} < \frac{1}{2} \tag{8}$$

$$x_2(n) = \sin\left(2\pi \frac{2}{5} n\right), \qquad 0 \leq n \leq 99 \tag{9}$$

The fundamental discrete period of the 2kHz signal sampled at a sampling rate of 5kHz is N = 5 samples per two cycles. We can see that the numerator of the discrete frequency tells us the cycles and the denominator the samples in these cycles. Then plotting (9):

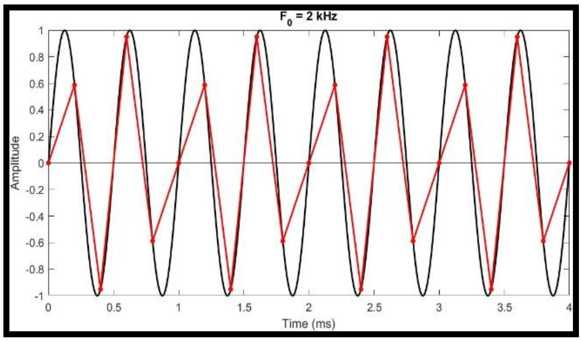

Fig. 3. 2kHz Original Signal vs Sampled Signal

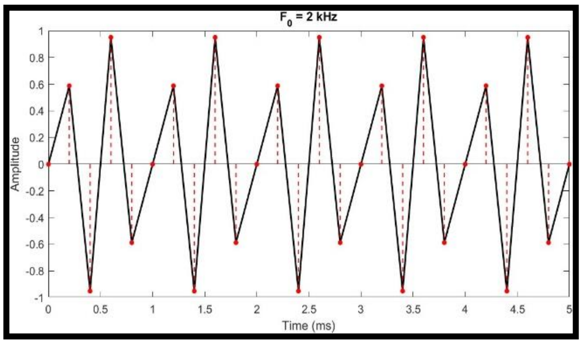

Fig. 4. 2kHz Sampled Signal and Its Samples

In this case, the sampling rate is slightly above the Nyquist rate (4kHz) for this signal. However, this rate is not a direct multiple of its fundamental frequency (2kHz). This led to some interesting effects as shown in Fig. 3 and Fig. 4.

The sampled waveform seems to have captured the main characteristic of the original signal, though some distortions near the peaks. While the signal can be reconstructed, some distortion is inevitable due to the sampling rate used.

*Sampling and plotting the signal with 3kHz fundamental frequency:

$$f_0 = \frac{F_0}{F_S} = \frac{3kHz}{5kHz} = \frac{3}{5} > \frac{1}{2} \tag{10}$$

$$x_3(n) = \sin\left(2\pi \frac{3}{5} n\right), \qquad 0 \leq n \leq 99 \tag{11}$$

The highest rate of oscillation in a discrete-time sinusoid is attained when $f = \frac{1}{2}\ or\ f = -\frac{1}{2}$ [1].

As we discussed in the beginning of this chapter we expected aliasing when sampling this sinusoid because the sampling rate is less than the Nyquist rate (6kHz) for this signal. Mathematically we can see that $f_0 = \frac{3}{5}$ and $\frac{3}{5} > \frac{1}{2}$ so it indicates it is an aliasing of another signal, so let's figure out which is the fundamental frequency of this alias.

$$x_3(n) = \sin\left(2\pi \frac{3}{5} - 2\pi\right) n = \sin(-2\pi n \tfrac{2}{5}) \tag{12}$$

$$f_0 = -\frac{2}{5} \tag{13}$$

When sampling the 3kHz sinusoid with a sampling rate of 5kHz you get an alias of the 2kHz sinusoid. We can see this in more detail in the following plots. Then plotting (11):

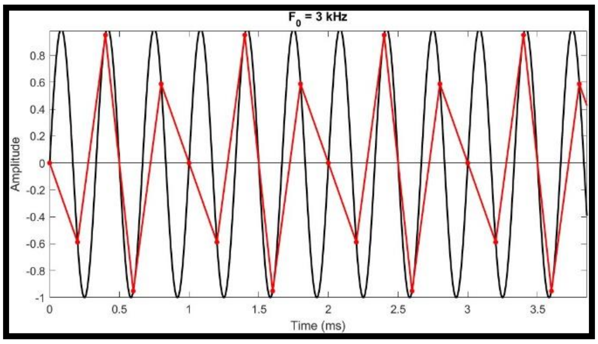

Fig. 5. 3kHz Original Signal vs Sampled Signal

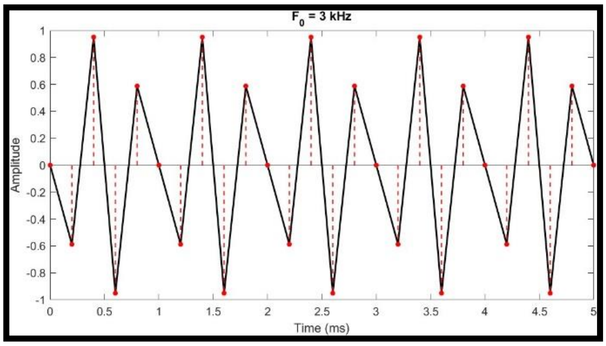

Fig. 6. 3kHz Sampled Signal and Its Samples

Both, visually and mathematically we can see that the sampled 3kHz signal with a 5kHz sampling rate is an alias of the 2kHz signal. Hence, the analog signal obtained after reconstructing the digital signal will have ambiguities, thus the sampling process isn't reliable.

*Sampling and plotting the signal with 4.5kHz fundamental frequency:

$$f_0 = \frac{F_0}{F_S} = \frac{4.5kHz}{5kHz} = \frac{9}{10} > \frac{1}{2} \tag{14}$$

$$x_4(n) = \sin\left(2\pi \frac{9}{10} n\right), \qquad 0 \le n \le 99 \qquad (15)$$

This signal will experience aliasing when sampling because the sampling rate is less than its Nyquist rate (9kHz).

$$x_4(n) = \sin\left(2\pi \frac{9}{10} - 2\pi\right) n = \sin(-2\pi n \frac{1}{10}) \qquad (16)$$

$$f_0 = -\frac{1}{10} \qquad (17)$$

When sampling the 4.5kHz signal with a sampling rate of 5kHz you get an alias of the 0.5kHz signal. We can see this in more detail in the following plots. Then plotting (15):

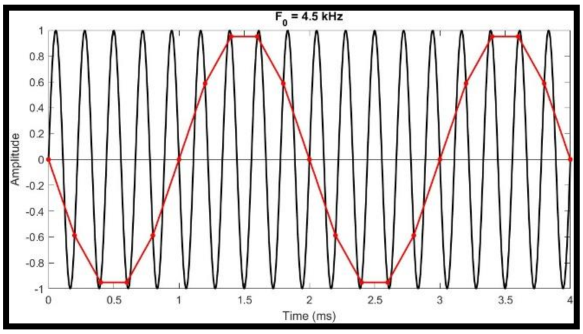

Fig. 7. 4.5kHz Original Signal vs Sampled Signal

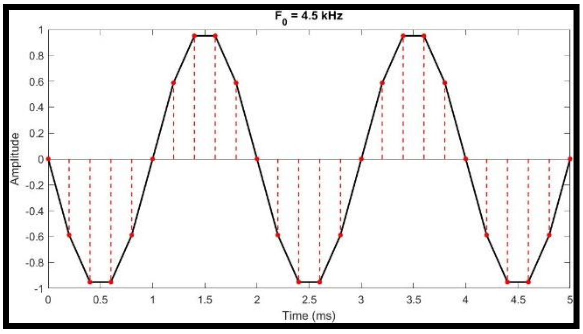

Fig. 8. 4.5kHz Sampled Signal and Its Samples

The analog signal resulting from reconstructing the sampled signal will have ambiguities, thus the sampling process isn't reliable.

B. *Suppose that Fo = 2kHz and Fs = 50kHz.*
Answer the following:

- Plot the signal x(n). ¿What is the frequency f_0 of the signal x(n)?
- Plot the signal y(n) created by taking the even-numbered samples of x(n). ¿Is this a sinusoidal signal? ¿Why? If so, ¿what is its frequency?

* Plot the signal x(n). ¿What is the frequency f_0 of the signal x(n)?

$$f_0 = \frac{F_0}{F_S} = \frac{2kHz}{50kHz} = \frac{2}{50} = \frac{1}{25} \qquad (18)$$

$$x_5(n) = \sin\left(2\pi \frac{1}{25} n\right), \qquad 0 \le n \le 99 \qquad (19)$$

The fundamental discrete period of the 2kHz signal sampled at a sampling rate of 50kHz is N = 25 samples per cycle and its discrete frequency is 0.04 cycles per sample. The Nyquist rate for this signal is (4kHz), so the sampling rate is almost 13 times more than the required to sample this signal. We can expect to have a good sampled signal. We proceed to plot (18):

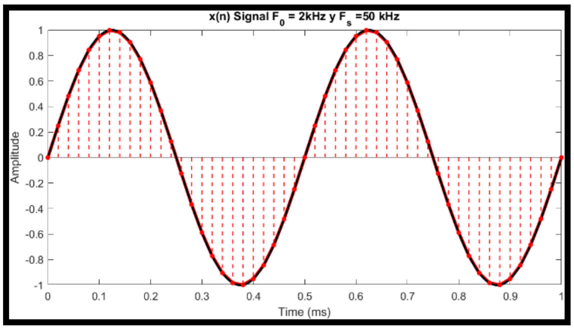

Fig. 9. 2kHz Original Signal vs Sampled Signal with Fs = 50kHz

* Plot the signal y(n) created by taking the even- numbered samples of x(n). ¿Is this a sinusoidal signal? ¿Why? If so, ¿what is its frequency?

$$f = \frac{F_0}{F_S} = \frac{2kHz}{50kHz/2} = \frac{2}{25} \qquad (20)$$

The discrete period of the 2kHz signal sampled at a sampling rate of 50kHz but only taking the even samples is N= 25 samples each two cycles and its discrete frequency is 0.08 cycles per sample.

Even though the continuous frequency remains at 2kHz, the discrete frequency of y(n) is different due to the change in sampling rate.

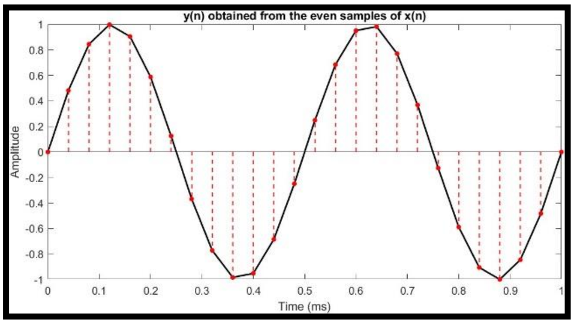

Fig. 10. 2kHz Sampled Signal y(n) and Its Even Samples with Fs = 50kHz

In Fig. 9 we have the original signal in black and the sampled signal in red. It's easy to see that the sampling process at high sampling rates gives us a good approximation of what the original signal is.

When removing the other samples except the even ones (down sampling) we get Fig. 10. This signal is still sinusoidal because the periodic nature of the sine wave:

$$y(n) = \sin\left(\frac{2\pi f_0 n}{F_S/2} + \varphi\right) \qquad (21)$$

Aliasing is a fundamental preoccupation in digital signal processing, occurring when continuous signals are sampled below the Nyquist rate, which can lead to frequency distortion and the potential loss of information. This phenomenon was demonstrated through the analysis of sinusoidal signals sampled at different rates, illustrating how inadequate sampling can cause the original signal to be misrepresented in the digital domain. Proper sampling techniques, including adherence to the Nyquist criterion, are essential to accurately capture and reconstruct continuous signals without introducing aliasing.

IV. Appendix

```
% Universidad de Guadalajara - Master of Science in
Electrical Engineering
% Erick Christopher Davalos Gonzalez

% Digital Signal Processing
% Project 1. Aliasing
%Consider the following signal:
% xa(t) = sin(2*pi*Fo*t)

% The above signal can be described in it's sampling form:
% x(n) = xa(nT) = sin(2*pi*(Fo/Fs)*n) where Fs = 1/T

% a)
% Sampling the signal in the interval 0<= n <=99, Fs =
5kHz Fo = [0.5,
% 2,3,4.5] kHz

Fs = 5;  % Sampling Frequency in KHz
F0_values = [0.5, 2, 3, 4.5];  % Fundamental Frequencies
n = 0:99;  % Samples
t = n / Fs;  % Time for each sample
t0 = linspace(0,(max(t))/4,1000);

% Creating a figure to contain subpolts
figure;
for i = 1:length(F0_values)
    F0 = F0_values(i);
    y = sin(2*pi*(F0/Fs)*n);  % Sampled signal
    subplot(2, 2, i);  % 2x2 plot
    % Graph
    plot(t, y, 'k', 'LineWidth', 1.5);
    hold on;
    % Point and line for each sample
    stem(t, y, '--r', 'LineWidth', 1, 'Marker', '.','MarkerSize',
15,'Color','r');
    xlim([0 5]);

    title(['F_0 = ', num2str(F0), ' kHz']);
    xlabel('Time (ms)');
    ylabel('Amplitude');
    hold off;

end
sgtitle('Sampled Signal and Samples');  % Title

% Creating other figure to contain plot original signal vs
sampled signal.
figure;
for i = 1:length(F0_values)

    F0 = F0_values(i);
    y0 = sin(2*pi*(F0)*t0);  % Original Signal
    y = sin(2*pi*(F0/Fs)*n);  % Sampled signal

    subplot(2, 2, i);  % 2x2 plot

    % Graph
    plot(t0, y0, 'k', 'LineWidth', 1.5);
    hold on;
    plot(t,y, 'r', 'LineWidth', 1.5);
    stem(t, y,'LineStyle','none','Marker', '.','MarkerSize',
15,'Color','r');
    xlim([0 4])

    title(['F_0 = ', num2str(F0), ' kHz']);
    xlabel('Time (ms)');
    ylabel('Amplitude');
    hold off;

 end
 sgtitle('Original Signal vs Sampled Signal');  % Title


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%
% b)
Fs = 50;  % Sampling Frequency in KHz
F0 = 2;   % Fundamental Frequency in Khz
n = 0:99;  % Samples
t = n / Fs;  % Time

% Graph
x = sin(2*pi*(F0/Fs)*n);
figure;

subplot(2, 2, 1);
y0 = sin(2*pi*(F0)*t0); %original signal
plot(t, x, '.-r', 'LineWidth', 3);
hold on;
plot(t0, y0, 'k', 'LineWidth', 2); %plotting original signal
xlim([0 1])
stem(t, x,'--r', 'LineWidth', 1, 'Marker', '.','MarkerSize',
15,'Color','r');
title(['x(n) Signal F_0 = ',num2str(F0),'kHz y F_s
=',num2str(Fs),' kHz']);

xlabel('Time (ms)');
ylabel('Amplitude');
hold off;

% taking even samples
y = x(1:2:end);  % taking even samples
t_y = t(1:2:end);  % time of even samples
% Graph y(n)
subplot(2, 2, 2);
plot(t_y, y, 'k', 'LineWidth', 1.5);
hold on;
stem(t_y, y,'--r', 'LineWidth', 1, 'Marker', '.','MarkerSize',
15,'Color','r');
xlim([0 1])
title('y(n) obtained from the even samples of x(n)');
xlabel('Time (ms)');
ylabel('Amplitude');
hold off;
sgtitle('Analysis of x(n) even samples');
```

V. References

[1] J. G. Proakis and D. K. Manolakis, Digital Signal Processing (4th ed.). New York: Prentice Hall, 2007