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Homework IA
Control Modern Theory

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Problem 1] Given the set $\{a, b\}$ with $a \neq b$. Define the rules of addition and multiplication such that the set $\{a, b\}$ forms a field. What are the 0 and 1 elements? 9%

To form a field we need to have: $\underline{\overset{+}{\text{CANI}}\ \overset{\cdot}{\text{CANI}}\ \overset{+/\cdot}{\text{D}}}$

* Addition (+)

Commutative ← CANI → Inverse

Associative, Neutral

(+/.) D → Multiplication distributivity respect to addition.

* Multiplication (·)

Commutative ← CANI → Inverse

Associative, Neutral

+	a	b
a	a	b
b	b	a

Table 1

·	a	b
a	a	a
b	a	b

Table 2

$$\boxed{\{a, b\} = \{0, 1\}}$$

We can see from table 1 that for addition:

1) $a+b=b+a$ * Commutativity

2) $(a+b)+b = a+(b+b)$ * Associativity

3) $b+(-b)=a$ * Additive Inverse

4) $b+a=b$ * Neutral additive

5) $(a+b)b = ab+bb$ * Distributivity

From Table 2 for multiplication we have:

6) $a \cdot b = b \cdot a$ * Commutativity

7) $(a \cdot b)b = a(b \cdot b)$ * Associativity

8) $b \cdot b = b$ * Neutral / we can see b is it's own neutral multiplicative

$b \cdot b^{-1} = b$

9) $b \cdot b^{-1} = 1$ * Inverse Definir -1/0

10) $(a \cdot b)(b+a) = abb + aba$ * Distributivity (Previously Demonstrated in Addition)

Summary:

$\{a,b\} = \{0,1\}$ It forms a field such that:

$b+b=a$. Thus $1+1=0$.

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Problem 2 | Let $\mathbb{R}(s)$ denote the set of all rational functions with real coefficients. Show that $(\mathbb{R}(s), \mathbb{R}(s))$ and $(\mathbb{R}(s), \mathbb{R})$ are linear spaces.

$$V = \left\{ f \mid f = \frac{P}{Q} \in \mathbb{R}(s), Q \neq 0 \right\}$$ ✓

For the $(\mathbb{R}(s), \mathbb{R}(s))$ case, we can see that the field is the same as the set so we can conclude:

$$\overset{(+)}{\text{CANI}} \quad \overset{(\cdot)}{\text{CANI}} \quad \overset{+/\cdot}{\text{D}}$$

* Properties of a field.

We talked about the meaning of CANI CANI D in problem 1. Now we will indicate the properties that forms a vector space:

$\overset{(+)}{\text{CANI}}$ $\overset{(\cdot)}{\text{ADDU}}$

Commutativity ← C

A → Associativity

N → Nutral

I → Inverse

A → Associativity

D → Distributivity − (scalar + scalar)(vector)

D → Distributivity − (scalar)(vector + vector)

U → Unity $(1x = 1)$ ← Part of the field.

If $\mathbb{R}(s)$ is said to be a field then the addition and multiplication properties are also in the $\mathbb{R}(s)$ set. Hence we can say $(\mathbb{R}(s), \mathbb{R}(s))$ is a vector space.

* A field $\mathbb{K}$ over itself always forms a space $(\mathbb{K}, \mathbb{K})$.

Referencia

Se pide mostrar que es un espacio vectorial, no que argumente o justifique.

¿Is $(\mathbb{R}(s), \mathbb{R})$ a linear space?

$$V = \left\{ x \mid x = \frac{P}{Q} \in \mathbb{R}(s),\ Q \neq 0 \right\}$$

⊞ Addition properties for vector spaces.

Let x, y, z be rational functions with real coefficients.

1) Associativity

To prove: $x + (y+z) = (x+y) + z$

- $x + (y+z) = \frac{P}{Q} + \left(\frac{K}{S} + \frac{R}{U}\right) = \frac{P}{Q} + \left(\frac{KU + SR}{SU}\right) = \frac{PSU + Q(KU+SR)}{QSU}$

$$= \frac{P\cancel{SU}}{Q\cancel{SU}} + \frac{\cancel{Q}K\cancel{U}}{\cancel{Q}S\cancel{U}} + \frac{\cancel{QS}R}{\cancel{QS}U} = \boxed{\frac{P}{Q} + \frac{K}{S} + \frac{R}{U}}$$

- $(x+y) + z = \left(\frac{P}{Q} + \frac{K}{S}\right) + \frac{R}{U} = \left(\frac{PS + QK}{QS}\right) + \frac{R}{U} = \frac{U(PS+QK) + QSR}{QSU}$

$$= \frac{\cancel{U}PS}{QS\cancel{U}} + \frac{\cancel{UQ}K}{\cancel{Q}S\cancel{U}} + \frac{\cancel{QS}R}{\cancel{QS}U} = \boxed{\frac{P}{Q} + \frac{K}{S} + \frac{R}{U}}$$ It satisfies Associativity.

2) Commutativity.

To prove $x + y = y + x$

- $x + y = \frac{P}{Q} + \frac{K}{S} = \boxed{\frac{PS + QK}{QS}}$

- $y + x = \frac{K}{S} + \frac{P}{Q} = \boxed{\frac{KQ + SP}{SQ}}$ It satisfies Commutativity.

¿Por que usar la notación de conjunto?

3) Addition Neutral Element.

$\{ \forall x \in \mathbb{R}(s) : \exists 0 \in \mathbb{R}(s) \mid x+0 = x = 0+x \}$

To prove: $x+0=x$

$$x+0 = \frac{P}{Q} + \frac{0}{1} = \frac{P(1)+Q(0)}{Q(1)} = \boxed{\frac{P}{Q}}$$ ✓

4) Addition Inverse Element.

To prove $x+(-x)=0$

$$x+(-x) = \frac{P}{Q} + \left(-\frac{P}{Q}\right) = \frac{P}{Q} - \frac{P}{Q} = \frac{PQ-PQ}{Q \cdot Q} = \frac{0}{Q \cdot Q} = \boxed{0}$$ ✓

- Multiplication properties for linear spaces.

Let $a,b \in \mathbb{R}$ and $x,y \in \mathbb{R}(s)$

1) Associativity

To prove: $(ab)x = a(bx)$

- $ab(x) = ab\left(\frac{P}{Q}\right) = \boxed{\frac{abP}{Q}}$
- $a(bx) = a\left(\frac{bP}{Q}\right) = \boxed{\frac{abP}{Q}}$

It is a rational function so it satisfies the associative property for multiplication.

2) Distributivity - scalar(vector+vector)

To prove: $a(x+y) = ax + ay$

- $$a(x+y) = a\left(\frac{P}{Q} + \frac{K}{S}\right) = a\left(\frac{PS+QK}{QS}\right) = \boxed{\frac{a(PS+QK)}{QS}}$$ ✓

$\bullet\ ax + ay = a\left(\frac{P}{Q}\right) + a\left(\frac{k}{S}\right) = \frac{aP}{Q} + \frac{ak}{S} = \frac{aPS + Qak}{QS}$

$= \boxed{\frac{a(PS + Qk)}{QS}}$ ✓

3) Distributivity - (scalar+scalar) vector.

To prove: $(a+b)x = ax + bx$

$\bullet\ (a+b)x = (a+b)\frac{P}{Q} = \frac{(a+b)P}{Q} = \boxed{\frac{aP+bP}{Q}}$ ✓

$\bullet\ ax + bx = a\left(\frac{P}{Q}\right) + b\left(\frac{P}{Q}\right) = \frac{aP}{Q} + \frac{bP}{Q} = \frac{(aP+bP)\cancel{Q}}{Q\cdot\cancel{Q}}$

$= \boxed{\frac{aP+bP}{Q}}$ ✓

4) Unity

$\forall x \in \mathbb{R}(s) : 1x = x$ where $1 \in \mathbb{R}$

$\frac{1}{1}\cdot x = \frac{1}{1}\left(\frac{P}{Q}\right) = \frac{(1)P}{(1)Q} = \boxed{\frac{P}{Q}}$

Along pages 3 to 6 we have discussed and demonstrated that $(\mathbb{R}(s), \mathbb{R}(s))$ and $(\mathbb{R}(s), \mathbb{R})$ are linear spaces.

In the case of $(\mathbb{R}(s), \mathbb{R})$ all the results derived from the 8 demonstrations are rational functions so the closure propertie for addition and multiplication is satisfied.

Summary: $(\mathbb{R}(s), \mathbb{R}(s))$ and $(\mathbb{R}(s), \mathbb{R})$ are linear spaces. ✓

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• Problem 3 Let V be the ordered pairs (a,b) of real numbers with addition in V and scalar multiplication on V defined by

$$(a,b)+(c,d)=(a+c,b+d)\ ,\ k(a,b)=(ka,0)$$

Show that V satisfies all the axioms except $1u=u$.

□ Addition properties for linear spaces

1) Commutative.

To prove : $(a,b)+(c,d)=(c,d)+(a,b)$

• $(a,b)+(c,d)=\boxed{(a+c,b+d)}$

• $(c,d)+(a,b)=(c+a,d+b)=\boxed{(a+c,b+d)}$

2) Associativity.

To prove: $[(a,b)+(c,d)]+(h,f)=(a,b)+[(c,d)+(h,f)]$

• $[(a,b)+(c,d)]+(h,f)=(a+c,b+d)+(h,f)=\boxed{(a+c+h,b+d+f)}$.

• $(a,b)+[(c,d)+(h,f)]=(a,b)+(c+h,d+f)$

$=\boxed{(a+c+h,b+d+f)}$

3) Neutral Element

$\forall (a,b) \in V : \exists (0,0) \in V \mid (a,b) + (0,0) = (a,b)$

- $(a,b) + (0,0) = (a+0, b+0) = (0+a, 0+b) = \boxed{(a,b)}$ ✓

4) Inverse Element

To prove: $(a,b) + (-a,-b) = (0,0)$

- $(a,b) + (-a,-b) = (a+(-a), b+(-b)) = (a-a, b-b) = \boxed{(0,0)}$ ✓

□ Multiplication properties for linear Spaces

$k(a,0) = (ka,0)$ This means $k \cdot b = 0$ so because $k \cdot a \neq 0$, k cant be 0 in fact $b=0$.

Let $(a,b) \in V$ and k, q be part of the field.

1) Associativity

T.P $(k \cdot q) \cdot (a,b) = k \cdot (q \cdot (a,b))$

- $(k \cdot q)(a,b) = (kqa, kqb) = \boxed{(kqa, 0)}$

- $k(q(a,b)) = k \cdot (qa, qb) = (kqa, kqb) = \boxed{(kqa, 0)}$ ✓

Next page

2) Distributivity - (scalar + scalar) vector

To prove: $(k+q)(a,b) = k(a,b) + q(a,b)$

- $(k+q)(a,b) = ((k+q)\cdot a, (k+q)\cdot b) = (ka+qa, kb+qb)$
$= \boxed{(ka,kb) + (qa,qb)}$

- $k(a,b) + q(a,b) = \boxed{(ka,kb) + (qa,qb)}$

Revisar la definición.

-1

3) Distributivity - (scalar)(vector + vector)

-1

To prove $k[(a,b) + (c,d)] = k(a,b) + k(c,d)$

Revisar

- $k[(a,b) + (c,d)] = k(a+c, b+d) = (k(a+c), \underline{k(b+d)})$
$= (ka+kc, kb+kd) = \boxed{(ka,kb) + (kc,kd)}$

Revisar

- $k(a,b) + k(c,d) = \boxed{(ka,kb) + (kc,kd)}$

4) Unity $(1x = x)$

To prove: $\forall (a,b) \in V : 1(a,b) = (a,b)$ with 1 from the field.

$k(a,b) = (ka, 0)$ so there isn't any scalar that gives us the same pair of numbers because under multiplication $b=0$.

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Problem 4 Determine the range and basis of the image and the kernel for each of the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad m \times n = 3 \times 3$$

- $\ker(A_1) = \{(x,y,z) : A(x,y,z) = 0\}$
- $\mathrm{Im}(A_1) = \{ y \in \mathbb{R}^3 \mid y = Ax \text{ such that } x \in \mathbb{R}^3 \}$
- Basis:

Let V be a subspace of $\mathbb{R}^n$. A basis of V is the set of vectors $\{v_1, v_2 \ldots v_n\}$ such that:

* $V = \text{span}\{v_1, v_2 - v_n\}$
* the set $\{v_1, v_2 - v_n\}$ is linearly independent

- Image of A_1

input vector

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ 0 \\ z \end{bmatrix}$$

$$\mathrm{Im}(A_1) = \left\{ \begin{bmatrix} y \\ 0 \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\}$$

- Kernel of A_1

$$\ker(A_1) = \{(x,y,z) : (y,0,z) = (0,0,0)\}$$

$$\begin{bmatrix} y \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} y=0 \\ 0=0 \\ z=0 \end{matrix} \qquad \ker(A_1) = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ -1%

After transformation we "lost" information related to x.

- Basis of the A_1 image.

$Ax = y \longrightarrow Ax = y(I) \longrightarrow$ Identity Matrix

$$[\text{illegible}] \begin{bmatrix} y \\ 0 \\ z \end{bmatrix} = y \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + \cancel{0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + z \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : [\text{illegible}] \right\}$$

$$A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- Kernel of A_2

Row transformation does not change the kernel.

↳ Row 3 times -3 then we add it to row 2.

$$\begin{bmatrix} 4 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x + y - z = 0$

$0x - y + 0z = 0 \rightarrow y = 0$

$x + y + 0z = 0 \rightarrow x = 0$

$4x - z = 0 \rightarrow z = 0$

$y = 0$

$x = 0$

$$\boxed{\ker(A_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \rightarrow \text{Dimension } 0$$

Es un conjunto, no un vector

- Image of A_2

Becaus it's kernel is the vector 0 and it's a square matrix it's colum space are the actual columns.

$$\operatorname{Im}(A_2) = \text{gen}\left\{ x\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, y\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, z\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

↑

Revisar la definición del conjunto.

• Basis of the image (A_2)

To find a basis of the image we need to identify the linearly independent columns.

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} F_1(\frac{-3}{4})+F_2 \\ F_1(\frac{-1}{4})+F_3 \end{matrix} \begin{bmatrix} 4 & 1 & -1 \\ 0 & \frac{5}{4} & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} F_2(\frac{4}{5}) \begin{bmatrix} 4 & 1 & -1 \\ 0 & 1 & \frac{3}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} F_2(\frac{-3}{4})+F_3$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & \frac{-1}{5} \end{bmatrix}$$

This shows there are pivots in the three colums. Hence the three colums are linearly independant.

$$= \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 : \mathbb{R}^3 \to \mathbb{R}^3$$

Column 1 times -2 and add to column 2.
Column 1 times -3 and add to column 3
Column 1 times -4 and add to column 4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Column 2 times -2 and add to column 3.
Column 2 times 2 and add to column 4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} = \begin{bmatrix} x \\ y \\ h \end{bmatrix}$$

We see 3 independent columns.

$$\mathrm{Im}(A_3) = \underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \cancel{x, y, h \in \mathbb{R}} \right\}}_{\text{Revisar}}$$

* Basis of the image.

Above we can see the basis of the image.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$$

- Kernel of A_3

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_2(2)+R_1$$

$$\begin{bmatrix} 1 & 0 & -1 & 8 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} = \begin{bmatrix} x-z+8h \\ -y-2z+2h \\ h. \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x-z+8h&=0 \\ -y-2z+2h&=0 \\ h&=0 \end{aligned} \longrightarrow \begin{aligned} x-z&=0 \\ -y-2z&=0 \\ h&=0 \end{aligned}$$

$$\boxed{\operatorname{Ker}(A_3) = \alpha \left\{ \begin{bmatrix} z \\ -2z \\ z \\ 0 \end{bmatrix} \right\}}$$

Dimension 1

el kernel es un espacio o solo un vector. Revisar la definición de un conjunto.

solo

Problem 5 Given $\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

How many solution does it have? What happens if $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ -3 & 3 & 2 \\ -1 & 2 & 1 \end{array}\right] \begin{array}{l} R_1(\frac{1}{2}) \\ R_1(3)+R_2 \\ R_1+R_3 \end{array} \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{13}{2} \\ 0 & \frac{3}{2} & \frac{5}{2} \end{array}\right] \begin{array}{l} R_2(\frac{1}{2})+R_1 \\ R_2(\frac{2}{3}) \\ R_2(-\frac{3}{2})+R_3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{11}{3} \\ 0 & 1 & \frac{13}{3} \\ 0 & 0 & -4 \end{array}\right] \quad \begin{array}{l} x_1 = \frac{11}{3} \\ x_2 = \frac{13}{3} \\ \boxed{0 \neq -4} \end{array}$$

Does not have any solutions. $0 \neq 4$.

- $y = [1 \ 1 \ 1]^T$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -3 & 3 & 1 \\ -1 & 2 & 1 \end{array}\right] \begin{array}{l} R_1(\frac{1}{2}) \\ R_1(3)+R_2 \\ R_1+R_2 \end{array} \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} \end{array}\right] \begin{array}{l} R_2(\frac{1}{2})+R_1 \\ R_2(\frac{2}{3}) \\ R_2(-\frac{3}{2})+R_3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & \frac{4}{3} \\ 0 & 0 & \frac{5}{3} \\ 0 & 0 & -1 \end{array}\right] \quad \begin{array}{l} x_1 = \frac{4}{3} \\ x_2 = \frac{5}{3} \\ \boxed{0 \neq -1} \end{array}$$

Does not have any solutions. $0 \neq 1$.

Problem 6) $F = \{f \mid f: \mathbb{R} \to \mathbb{R}\}$

*Exercise 2.18 Show that W is a subspace of F.

20%

1) W is all the bounded functions
($f: \mathbb{R} \to \mathbb{R}$ is bounded if $\exists M \in \mathbb{R}^+ \ni |f(x)| \leq M, \forall x \in \mathbb{R}$)

Let $f, g \in W$ if f and g are bounded exist $M \geq 0$

$$|f(x)| \leq M_1$$

$$|g(x)| \leq M_2 \quad \forall x \in \mathbb{R}$$

$$|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2$$

$$M_1 + M_2 = M_3 \geq 0$$

$$|(f+g)(x)| \leq M_1 + M_2$$

$$|(f+g)(x)| \leq M_3 \quad \forall x \in \mathbb{R}$$

f+g is a bounded function

Subspace test

- Addition of vectors $\longrightarrow$ Completed.
- Multiplication by a scalar

Let $f \in W$, $\lambda \in \mathbb{R}$ $\exists M \geq 0$ $|f(x)| \leq M, \forall x \in \mathbb{R}$

$$|(\lambda f)(x)| = |\lambda \cdot f(x)| = |\lambda| \cdot |f(x)|$$

$$* \; |\lambda| \cdot |f(x)| \leq |\lambda| \cdot M$$

$$|(\lambda f)(x)| \leq |\lambda| \cdot M \quad , \forall x \in \mathbb{R}$$

$$M_2 = |\lambda| \cdot M \geq 0$$

$$|(\lambda f)(x)| \leq M_2$$

$\lambda f \in W$.

Addition of vectors and multiplication by a scalar are satisfied. Thus W is a subspace of F.

2) W is all the even functions ($f:\mathbb{R}\to\mathbb{R}$ is even if $f(-x)=f(x)$, $\forall x\in\mathbb{R}$).

To show that W is a subspace of F we need to verify:

* The set W contains the zero vector.

$f(x)=0$ for all $x\in\mathbb{R}$ is even because:

$$f_0(-x)=0=f_0(x)$$

W contains the zero vector.

* Addition closure

$\{f,g\in W, x\in\mathbb{R}\}$ $f(-x)=f(x)$ and $g(-x)=g(x)$

$$(f+g)(-x)=f(-x)+g(-x)=f(x)+g(x)=(f+g)(x)$$

* Closed under scalar multiplication.

$\{f\in W, \lambda\in\mathbb{R}\}$

$$(\lambda f)(-x)=\lambda f(-x)=\lambda f(x)=(\lambda f)(x)$$

Since W satisfies the three necessary properties we can conclude that W is a subspace of F.

5) W is all the integrable functions in $0 \le x \le 1$

If $f, g \in W$ are integrable the integrals of f and g exist then $f + \lambda g$ integral does exist because:

$$\int_0^1 (f+\lambda g)(x)\,dx = \int_0^1 f(x)\,dx + \lambda \int_0^1 g(x)\,dx$$

f and g are integrables on the interval $0 \le x \le 1$:

$$* \int_0^1 f(x)\,dx < \infty, \quad \int_0^1 g(x)\,dx < \infty$$

It's value must be finite.

*Exercise 2.19 $V = \mathbb{R}^n[t]$ $a_0 + a_1 t + a_2 t^2 + \dots a_n t^n$ with real coefficients $a_i \in \mathbb{R}$. Prove W is a subspace of V.

20/20

1).

W is all the polynomial with integers as coefficients.

To prove: $p(t) + q(t) \in W$

$$p(t) + q(t) = (a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n) + (b_0 + b_1 t + b_2 t^2 + \dots b_n t^n)$$

$$= (a_0 + b_0 + a_1 t + b_1 t + a_2 t^2 + b_2 t^2 + \dots a_n t^n + b_n t^n)$$

$$= ((a_0 + b_0) + (a_1 + b_1) t + (a_2 + b_2) t^2 + \dots (a_n + b_n) t^n)$$

$$= (C_0 + (C_1) t + (C_2) t^2 + \dots (C_n) t^n) = h(x)$$

h(x) has integers as coefficients thus its closed.

Page 20.

If $p(t) \in W$, $\alpha \in F$ then $\alpha p(t) \in W$

Let

$p(t) = 3+2t$, $3,2 \in \mathbb{Z}$

Let $\alpha = \frac{1}{2} \in \mathbb{R}$

$\alpha \cdot p(t) = \frac{1}{2}(3+2t) = \frac{3}{2} + t$

$\alpha p(t)$ isn't necessarily a polynomial with integers as coefficients. Hence W is not a subspace of F.

2) W is all the polynomials with degree ≤ 3

Let $p(x)$ and $q(x) \in W$

$p(x) + q(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3)$

$= (a_0 + b_0) + (a_1 x + b_1 x) + (a_2 x^2 + b_2 x^2) + (a_3 x^3 + b_3 x^3)$

$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$

$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 = h(x)$ degree ≤ 3

* To prove $0 \in W$ $a_i \in \mathbb{R}$

$p(x) = 0 + 0x + 0x^2 + 0x^3 = 0$

• $\alpha \in F$, $p(x) \in W$

To demonstrate : $\alpha p(x) \in W$

$$\alpha p(x) = \alpha(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$$

$$= \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3$$

It's still degree 3.

Summary : W is a subspace of V ✓

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Exercise 2.20 $Ax=b$ is a non-homogeneous system (linear) with n unknowns over a field ($\mathbb{K}$). Show that the set of solutions (solution set) is not a subspace of $\mathbb{K}^n$.

$$V = \{x \in \mathbb{K}^n : Ax = b, b \neq 0\}$$

$$A(0) = b$$

$\boxed{0 \neq b}$ V is not a subspace of $\mathbb{K}^n$

$$0 \notin V \Rightarrow A \cdot 0 = b$$