Analysis of Limited and Non-Limited Bandwidth Signal Using the Discrete Fourier Transform

Erick Christopher Davalos Gonzalez
Universidad de Guadalajara (CUCEI)
Master of Science in Electrical Engineering
Guadalajara, Jalisco, México
erick.davalos2937@alumnos.udg.mx

***Abstract* — This paper presents an analysis of limited and non-limited bandwidth signals using the Discrete Fourier Transform (DFT). Specifically, the effects of varying the sampling rate and observation duration (Tobs) on the spectrum are studied. The limited bandwidth signal has a DC component and multiple cosine components of different frequencies. The non-limited bandwidth signal is represented by a periodic square wave. By altering sampling rates and observation time, the document evaluates the spectral representation accuracy, aliasing phenomena, and the implications of signal truncation. The results provide insights into the practical considerations for sampling and spectral analysis of signals with diverse bandwidth characteristics.**

Keywords* — *Discrete Fourier Transform (DFT), Limited Bandwidth Signal, Non – Limited Bandwidth Signal, Sampling Rate, Observation Time, Signals, Signal Reconstruction, Frequency domain.

I. Nomenclature

- T_0: Fundamental Period
- T_{obs}: Observation Time
- F_s: Sampling Frequency
- f_0: Fundamental Frequency
- ω_0: Fundamental Angular Frequency
- $x(n)$: Discrete Time Signal
- $x(t)$: Continuous Time Signal
- $X(\omega)$: Fourier Transform
- C_n: Modulus of Compact Fourier Series
- θ_n: Angle Displacement of Fourier Series
- D_n: Coefficient for Exponential Fourier Series

II. Introduction

A. Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a tool in digital signal processing, enabling the transition from time-domain to frequency-domain representations of discrete signals. The accuracy of the DFT relies heavily on the sampling rate and observation duration (Tobs) used to acquire the signal. In practice, these parameters dictate the resolution and fidelity of the spectral representation, impacting the analysis of signals.

To perform frequency analysis on a discrete-time signal x(n), we convert the time-domain sequence to an equivalent frequency-domain representation. We know that the Fourier gives such a representation transform X(ω) of the sequence x(n). However, X(ω) is a continuous function of frequency and therefore it is not a computationally convenient representation of the sequence x(n).

The limited bandwidth signal is composed of a DC component and cosine terms with frequencies ranging from 20 Hz to 1000 Hz, while the non-limited bandwidth signal is a periodic square wave. These two examples are chosen to highlight the effects of spectral leakage, aliasing, and truncation in signals with different bandwidth properties.

For a sequence x(n) of length N, the DFT samples the Fourier transform at N equidistant points in the frequency domain.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \tag{1}$$

B. Inverse Discrete Fourier Transform

The original sequence x(n) can be reconstructed from the DFT samples X(k) using the IDFT (Inverse Discrete Fourier Transform) shown in (2).

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \tag{2}$$

III. Case Study

A. Limited Bandwidth Signal

Limited bandwidth signals consist of a finite set of frequency components, ensuring that their spectrum is within a specific range.

The case study signal consists in 4 cosine terms and a DC component, the signal can be represented as follows:

$$\begin{aligned} x(t) = 0.2 + 0.4\cos(2\pi * 120t) + 0.3\cos(2\pi * 240t) \\ + 0.2\cos(2\pi * 600t) \\ + 0.3\cos(2\pi * 1200t) \end{aligned} \tag{3}$$

$$f_s > 2\, f_{max} \tag{4}$$

In this subsection, we examine the spectral representation of a limited bandwidth signal using a sampling frequency of 3.2kHz to avoid aliasing and an observation time (Tobs) of 1/40, then a 2kHz sampling frequency is used to analyze the spectrum, after this, the frequency of the 240Hz and 1200Hz is changed to 245Hz and 1205Hz. The analysis explores how the sampling frequency under and above the Nyquist rate and the observation time influence the accuracy of the frequency-domain representation, especially in terms of spectral resolution and aliasing.

$$dt = \frac{1}{Fs} = \frac{1}{3.2kHz} = 0.3125ms \tag{5}$$

$$N = \frac{Tobs}{dt} = \frac{0.025s}{0.3125ms} = 80\ samples \tag{6}$$

$$df = \frac{Fs}{N} = \frac{3.2kHz}{80} = 40Hz \tag{7}$$

Throughout Fig. 1 to Fig. 3, using a 3.2kHz sampling frequency the sampled signal is close to look like the original signal, the maximum frequency in the signal is 1200Hz, thus, a minimum of 2400Hz sampling frequency is required. In this case the sampling frequency is 33% above the sampling frequency needed so we will be able to extract all the spectrum information without any aliasing. Moreover, the full recovery of the signal can be achieved.

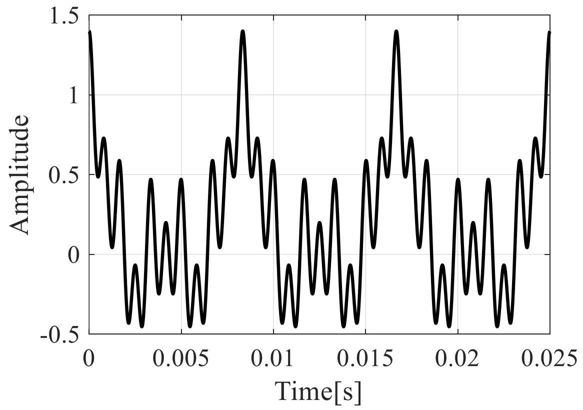

Fig. 1. Case Study Analog Signal

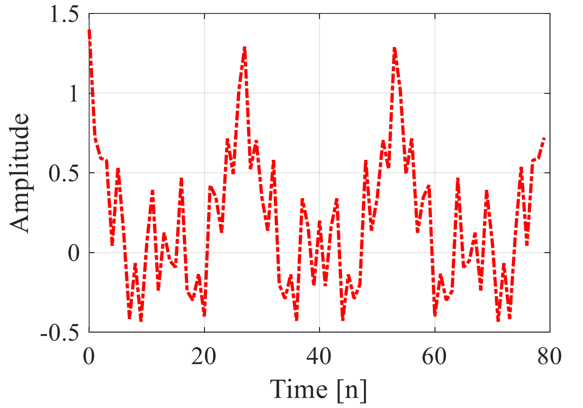

Fig. 2. Sampled Signal at 3.2kHz with 1/40 Tobs

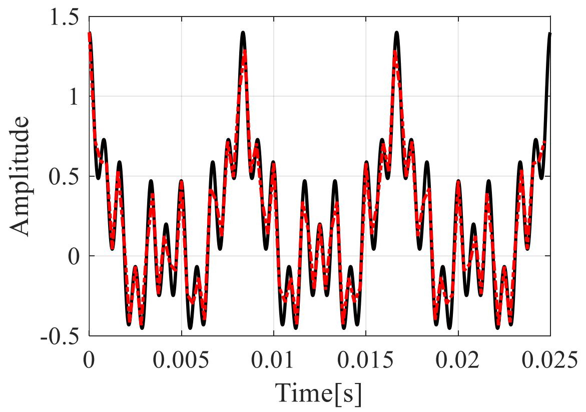

Fig. 3. Analog Signal and Sampled Signal (3.2kHz 1/40)

Now, the computation is conducted using equation (1) and the previously stated parameters.

The Discrete Fourier Transform (DFT) of the x(n) signal is presented to highlight its discrete spectral nature, as shown in Fig. 4. In Fig. 5, the spectrum is plotted using a continuous line for visual clarity; however, it is essential to understand that this appearance is purely a result of how MATLAB generates the plot. The true nature of the spectrum remains discrete, this discrete nature arises from the finite length of the time-domain signal and the periodicity assumed in the DFT.

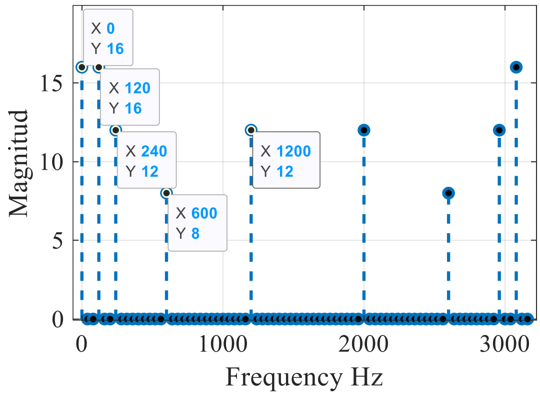

Fig. 4. DFT of the Sampled Signal at 3.2kHz with 1/4 Tobs (stem)

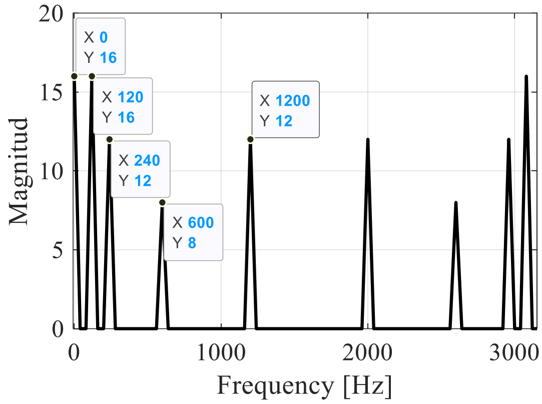

Fig. 5. DFT of the Sampled Signal at 3.2kHz with 1/4 Tobs (plot)

The analysis confirms that the frequencies present in the signal are precisely at 0Hz, 120Hz, 240Hz, 600Hz and 1200Hz, with no energy at other frequencies. This clean spectrum representation indicates no spectral leakage or aliasing is observed. The absence of leakage spectra components is due to important factors:

- Observation time: the chosen Tobs = (1/40Hz) ensures that the observation window aligns well with the signal's periodic components.
- Sampling frequency: Using a sampling frequency above the Nyquist rate criterion ensures that all frequencies in the signal are correctly sampled without aliasing.

With the spectra obtained, the IDFT stated in equation (2) is used to recover the sampled signal x(n).

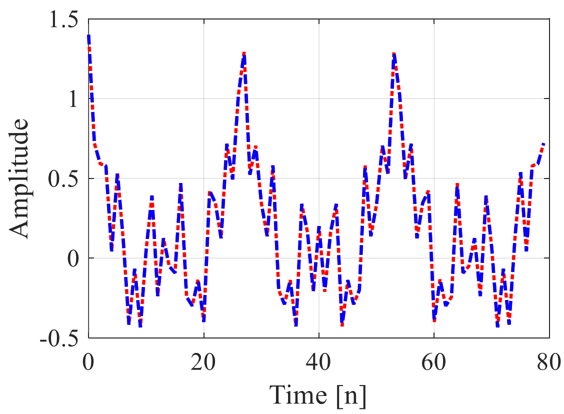

Fig. 6. Recovered signal (blue) and sampled signal (red)

In Fig. 6 the recovered signal is the same as the sampled signal, thus, can be concluded that if the sampling rate meets the Nyquist criteria the signal can be fully recovered.

For the next case the signal is modified as follows:

$$x(t) = 0.2 + 0.4\cos(2\pi * 120t) + 0.3\cos(2\pi * 245t) + 0.2\cos(2\pi * 600t) + 0.3\cos(2\pi * 1205t) \quad (8)$$

So, the spectrum is:

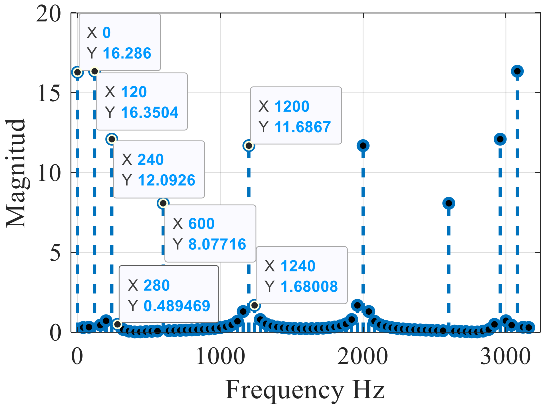

Fig. 7. DFT with Modified Frequencies and 3.2kHz Fs (stem)

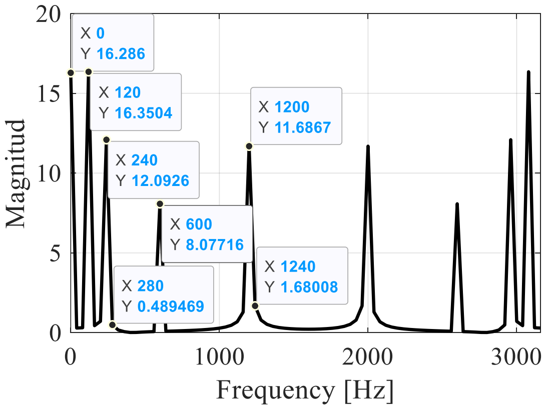

Fig. 8. DFT with Modified Frequencies and 3.2kHz Fs (plot)

The DFT assumes that the signal is periodic within the observation window (Tobs). If the signal is not perfectly periodic in this interval, discontinuities at the boundaries introduce additional frequency artifacts. The cosine term with 245 and 1205 frequencies are not multiples of 40Hz as stated in equation (7). The recovered signal is shown in Fig. 9.

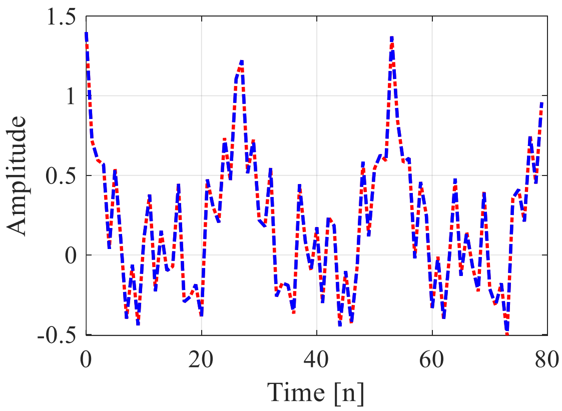

Fig. 9. Recovered modified signal (blue) and sampled signal (red)

In Fig. 9 it can be shown that the Nyquist criterion ensures replicas don't overlap, keeping the baseband spectrum intact. The only way to lose signal information during sampling is by violating the Nyquist criterion, which cancels out original frequencies due to spectral overlap.

Now, for study purposes, using the original signal in (3) but modifying the sampling rate to 2kHz assuming the Nyquist rate equal to 2.4kHz an analysis is conducted.

The spectrum of Fig.1 but using a 2kHz sampling frequency is shown in Fig.10 and Fig.11.

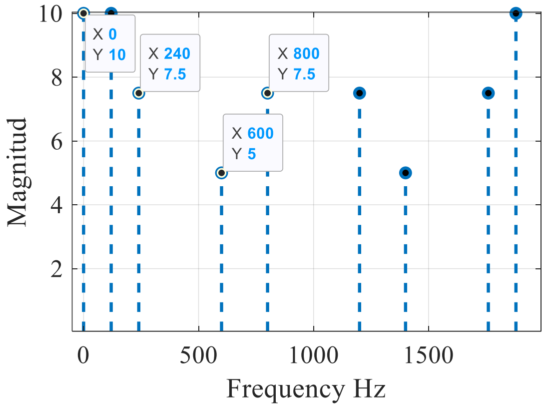

Fig. 10. DFT with 2kHz Modified Sampling Rate (stem)

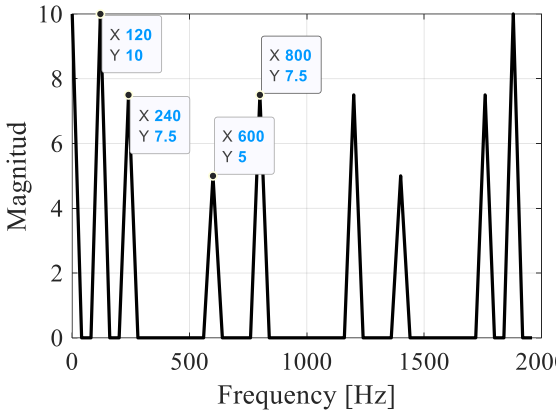

Fig. 11. DFT with 2kHz Modified Sampling Rate (plot)

In Fig. 10 and Fig. 11, an aliased frequency component at 800 Hz is observed, which does not exist in the original signal. This component is an alias of the 1200 Hz frequency due to the violation of the Nyquist criterion, as the sampling frequency was insufficient to capture frequencies above the Nyquist limit.

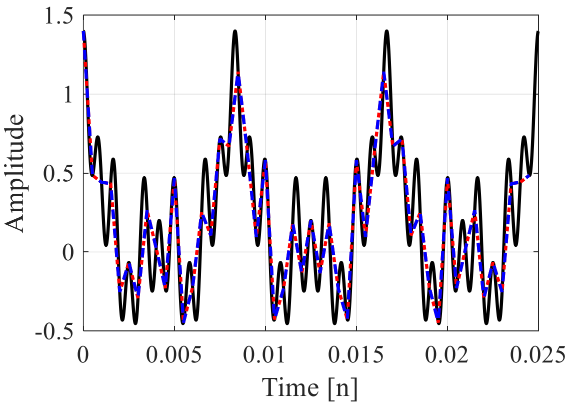

Fig. 12. Recovered signal (blue), sampled signal (red) and original signal (black)

An important fact to notice is that the IDFT will give us the original sampled signal, but originally the sampled signal introduced an alias into the spectrum. So, in conclusion, it's important to do a sampling following the Nyquist criterion.

¿What happens if an ideal interpolator is used to reconstruct an aliased signal?

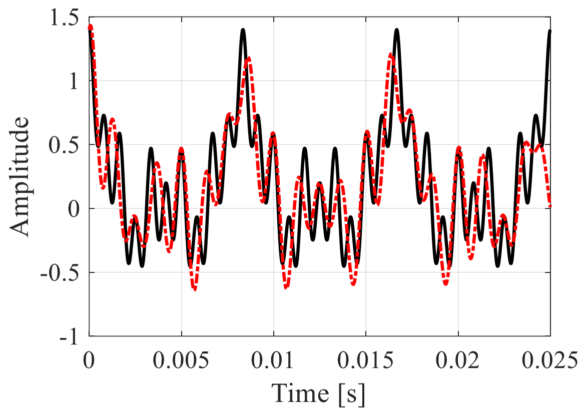

Fig. 13. Original signal (black) and interpolated aliased signal (red) using a sampling frequency of 2kHz.

As summary it can be concluded that higher sampling frequency allows the spectrum to display higher frequency components, and a higher observation time increases the resolution of the spectrum enabling closer-spaced signals to be found but still the maximum frequency is limited by the Nyquist frequency of Fs/2.

Using the MATLAB function fft to compute the spectrum of the sampled signal with a sampling frequency of 3.2kHz.

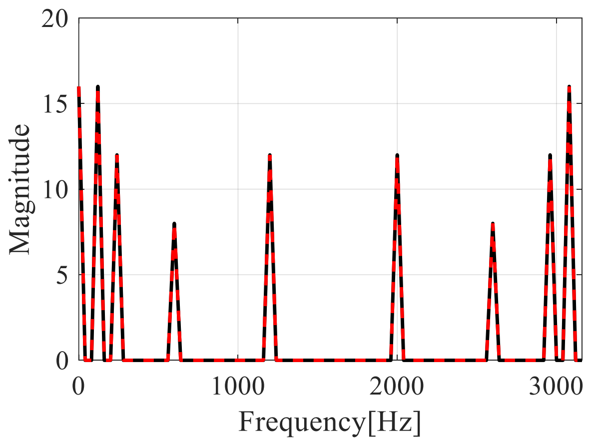

Fig. 14. Spectrum of the original sampled signal (black) and spectrum obtained using the fft function for the same signal (red).

Both spectra are the same but the difference between the computation time is around 1 order of magnitude, 3.11ms for the manual iteration computing and 0.67ms for the fft function.

B. Non - Limited Bandwidth Signal

A non-limited bandwidth signal is a signal whose frequency components extend over an infinite range, without a strict cutoff or boundary in the frequency domain. This means that the signal contains energy at frequencies across the entire spectrum.

The case study for this subsection will be a periodic square wave, the signal can be mathematically described as follows:

$$x(t) = \begin{cases} 1, & 0 \leq t < \frac{T}{2} \\ 0, & \frac{T}{2} \leq t < T \end{cases}, \qquad T = 2\pi \qquad (8)$$

The Fourier transform of (8) reveals that it contains energy at all odd harmonics of the fundamental frequency 1/T.

$$f_o = \frac{1}{T} = \frac{1}{2\pi} = 0.15915Hz \qquad (9)$$

The expectations for the further analysis are that there will be Gibbs phenomena when trying to reconstruct the signal due to discontinuities, the DFT will show nonzero components only at odd samples of the spectrum and odd harmonics of the fundamental frequency shown in (9).

The spectrum will theoretically extend to infinity, but in practice, it will be limited by the sampling frequency and the number of samples. Hence, frequencies above Fs/2 will fold back into lower frequencies, interfering with the true harmonic components

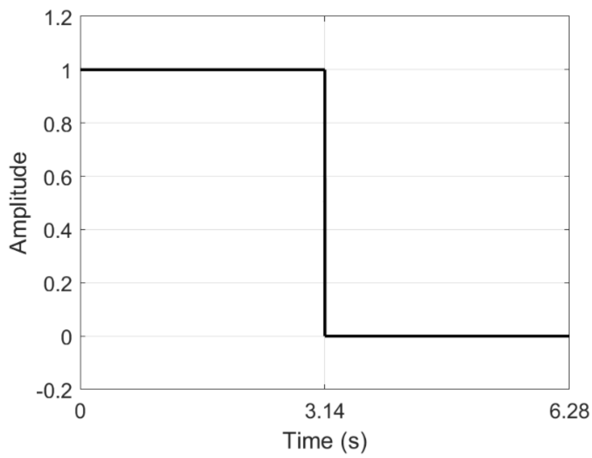

Fig. 15. Case Study Signal for Non – Limited Bandwidth analysis.

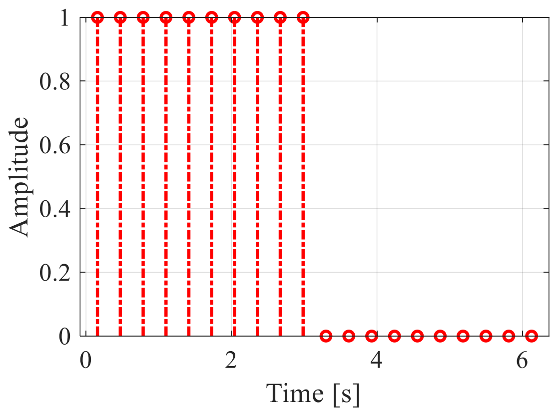

Fig. 16. Sampled Non - Limited Bandwidth Signal

The signal was sampled at a rate of 10/pi Hz with an observation time (Tobs) of 2pi this gives 20 samples throughout the whole period.

In Fig. 17, each odd sample contains a peak, indicating the presence of odd frequency components in the signal. This behavior occurs because the periodic square wave contains only odd harmonics of the fundamental frequency.

It is important to note that the straight peaks in each odd sample result from the fact that the sampling frequency is a multiple of the fundamental frequency. This ensures that the harmonics of the square wave align perfectly with the DFT, avoiding spectral leakage and making the odd harmonics clearly visible in the magnitude spectrum.

Additionally, the magnitude of each peak decreases as the harmonic order increases. This reflects the diminishing contribution of higher-frequency components to the overall signal.

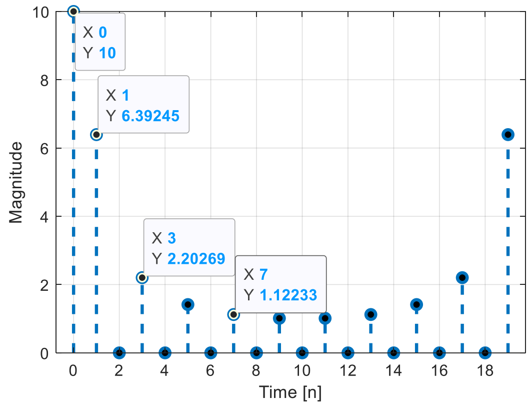

Fig. 17. Spectrum of the Non - Limited Bandwidth Signal (stem) and sample index in the x axis.

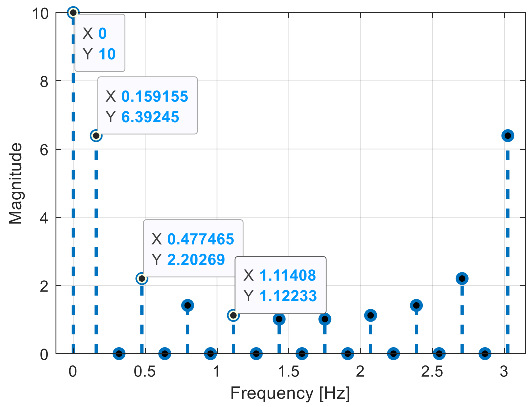

Fig. 18. Spectrum of the Non - Limited Bandwidth Signal (stem) and frequency (Hz) in the x axis.

From equation (9) and Fig. 18 can be observed that each peak occurs in odd multiples of the value stated in (9).

Using a continuous representation with the MATLAB plot function:

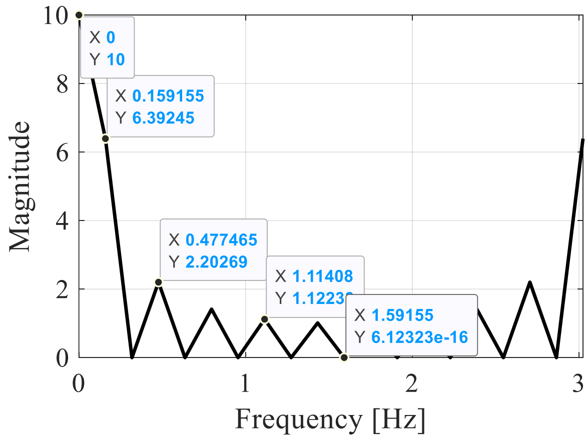

Fig. 19. Spectrum of the Non - Limited Bandwidth Signal (plot) and frequency (Hz) in the x axis.

$$f = \frac{k}{T} = \frac{k}{2\pi} = k(0.15915Hz) \qquad (10)$$

As analyzed previously the maximum observable frequency is limited to Fs/2, for this case the sampling frequency is 10/pi that is:

$$Fs = \frac{10}{\pi} = 3.1831Hz \qquad (11)$$

$$F_{obs} = \frac{Fs}{2} = \frac{3.1831}{2} = 1.59155Hz \qquad (12)$$

It can be clearly seen in Fig. 19, after 1.59155Hz the samples repeats, it's the nature of the tool because the nature of the exponential function, it introduces “imaginary” frequencies, to get the actual magnitude of each frequency component below Fs/2 a multiplication by a factor of 2 is needed, then this value needs to be divided by the number of samples.

Using the fft function to plot the spectrum of the sampled square wave:

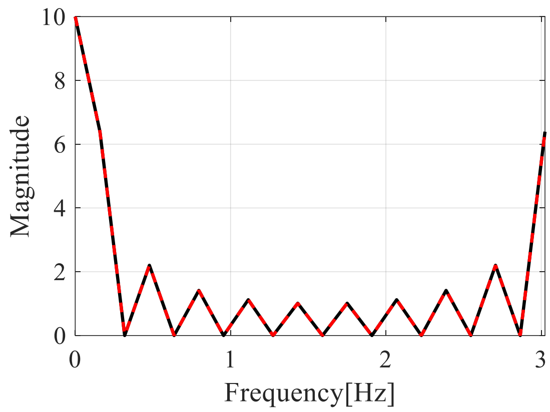

Fig. 20. Spectrum of the original sampled signal (black) and spectrum obtained using the fft function for the same signal (red).

Both spectra are the same but the difference between the computation time is around 2 orders of magnitude, 2.229ms for the manual iteration computing and 0.091ms for the fft function.

Now, using the IDFT to get the original sampled signal:

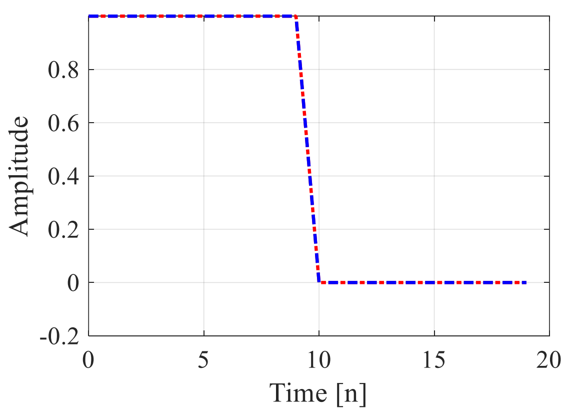

Fig. 21. Recovered signal (blue) and sampled signal (red)

$$x_{interp}(t) = \sum_{n=0}^{N-1} x(n) sinc\left[F_s\left(t - \frac{n}{F_s}\right)\right] \qquad (13)$$

$$x_{interp}(t) = \sum_{n=0}^{19} x(n) sinc\left[\frac{10}{\pi}\left(t - \frac{\pi n}{10}\right)\right] \quad (14)$$

The computation of (14) is conducted to obtain the analog reconstruction of the square wave.

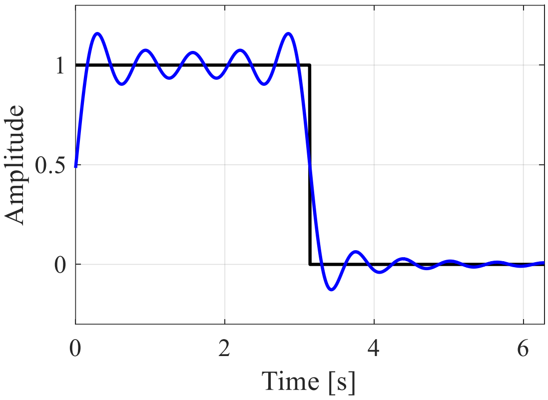

Fig. 22. Original signal (black) and interpolated signal (blue) using a sampling frequency of 10/pi.

The interpolation of a periodic square wave, a non-bandwidth-limited signal, using ideal sinc interpolation is limited by the sampling frequency. In this analysis, the square wave was sampled at 10/pi, and the reconstructed signal exhibits a manifestation of the Gibbs phenomenon. These distortions occur because the square wave contains infinite odd harmonics, while the sampling process only retains components below the Nyquist frequency.

Higher harmonics are either aliased to lower frequencies or lost entirely, resulting in a band-limited approximation of the original signal.

To improve reconstruction fidelity requires increasing the sampling frequency to include more harmonics, an example is shown in Fig. 23 where a sampling frequency of 100/pi is used.

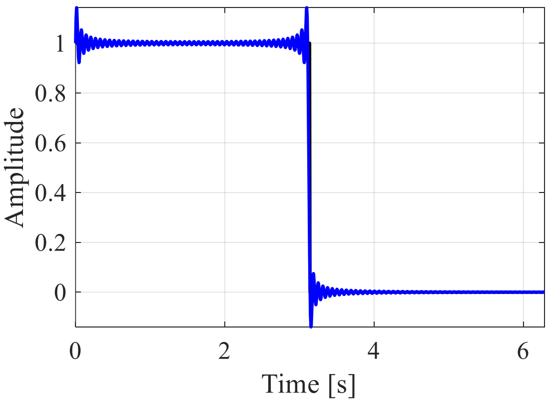

Fig. 22. Original signal (black) and interpolated signal (blue) using a sampling frequency of 100/pi.

C. Relation between the DFT and the Fourier Series.

The Discrete Fourier Transform (DFT) and the Fourier Series are closely related. While the Fourier Series represents a continuous periodic signal as a sum of sinusoidal components with discrete frequencies, the DFT operates on sampled data, providing a discrete representation of the signal's spectrum. The DFT can be seen as a sampled version of the Fourier Series, where the periodic signal is truncated to a finite duration and evaluated at discrete points in time. This section explores the connection between these two transforms, highlighting how the DFT approximates the Fourier Series.

TABLE 1. KEY DIFFERENCES BETWEEN THE DFT AND FOURIER SERIES

Aspect	**Fourier Series**	**DFT**
Signal	Continuous and periodic x(t)	Discrete and finite x(n)
Spectrum	Discrete frequencies, continuous-valued	Discrete frequencies, discrete-valued
Computation	Coefficients involve integrals	Coefficients involve summations
Periodicity	Signal is periodic	Assumes input is periodic in N samples

The DFT and Fourier Series are connected when the periodic signal being analyzed is sampled.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \quad (14)$$

$$c_n = \frac{1}{T_o}\int_{T_o} x(t) e^{-jn\omega t}\, dt, \qquad \omega = \frac{2\pi}{T} \quad (15)$$

If the signal x(t) is sampled at a rate Fs (sampling frequency), the continuous periodic signal becomes a discrete sequence.

$$x(n) = x(t)|_{t=nTs}\,, \qquad T_s = \frac{1}{F_s} \quad (16)$$

The sampled signal x(n) has N = T/Ts samples per period and the Fourier coefficients of (15) are approximated by the sampled data.

The DFT coefficients X(k) are related to the Fourier Series coefficients as follows:

$$X(k) \approx c_n N\,, \qquad N = T * F_s \quad (17)$$

The Fourier Series calculation is carried out using X(k) calculated previously for the square wave using a sampling frequency of 10/pi as indicated at the beginning of the section.

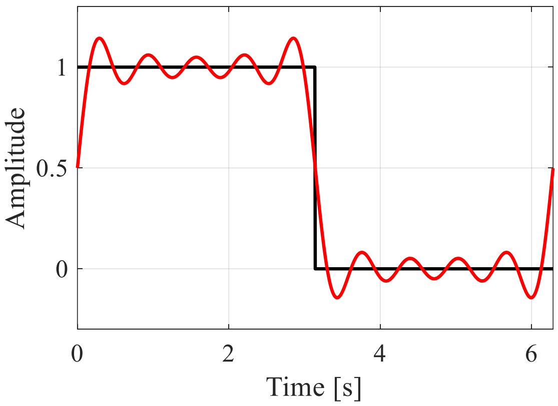

Fig. 23. Original signal (black) and its reconstruction using Fourier Series with 10 harmonics (red).

$$N_{harmonics} = \frac{f_{nyquist}}{f_o} = \frac{\frac{Fs}{2}}{f_o} = \frac{\frac{10}{2\pi}}{\frac{1}{2\pi}} = 10 \quad (18)$$

The approximation with Fourier Series in Fig. 23 consists in 10 harmonics.

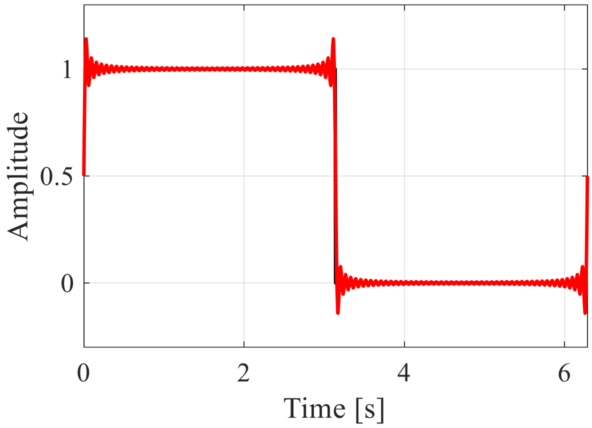

Fig. 24. Original signal (black) and its reconstruction using Fourier Series with 100 harmonics (red).

$$N_{harmonics} = \frac{f_{nyquist}}{f_o} = \frac{\frac{Fs}{2}}{f_o} = \frac{\frac{100}{2\pi}}{\frac{1}{2\pi}} = 100 \qquad (19)$$

IV. Conclusions

Ensuring the Nyquist criterion is fundamental for accurate signal reconstruction. The analysis demonstrates that when the sampling frequency is sufficiently higher than twice the maximum frequency of the signal, aliasing is avoided, and the original signal can be fully recovered. This principle is validated in the case of limited bandwidth signals sampled at 3.2 kHz, where the spectral components are preserved without distortion.

Properly aligning the observation window with the signal's periodicity minimizes spectral leakage and ensures a clean frequency-domain representation.

For non-limited bandwidth signals, such as square waves with infinite harmonics, the finite nature of sampling introduces challenges. While higher sampling frequencies improve the inclusion of more harmonics and reduce reconstruction artifacts, limitations remain due to the band-limited nature of the sampling process.

The application of the Fast Fourier Transform (FFT) highlights its computational efficiency in spectral analysis. By reducing computation time by approximately two orders of magnitude compared to manual DFT calculations, FFT proves to be a practical tool for real-world signal processing tasks without compromising accuracy.

Finally, the relationship between the Discrete Fourier Transform (DFT) and the Fourier Series underscores the importance of sampling considerations. The DFT serves as a discrete approximation of the Fourier Series, influenced by the sampling rate and the number of harmonics captured.

V. References

[1] J. G. Proakis and D. K. Manolakis, Digital Signal Processing (4th ed.). New York: Prentice Hall, 2007