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# HW3A Modern Control Theory

Master of Science in Electrical Engineering

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**Ejercicio 3.14** *Calcule las exponenciales de las siguientes matrices:*

(a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We want to find $e^{At}$ matrix for ⓐ, ⓑ and ⓒ.

* Obtaining eigenvalues for ⓐ

$\det(A - \lambda I)$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} = (2-\lambda)(-3-\lambda) = -6-2\lambda+3\lambda+\lambda^2$$

$$= \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$$

$\lambda_1 = -3$, $\lambda_2 = 2$

$$e^{\bar{A}t} = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

To find $e^{At}$ we know that: $e^{At} = P e^{\bar{A}t} P^{-1}$ wher P is the eigenvectors matrix.

$\lambda_1 = -3$

$$A - \lambda_1 I = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} 5x_1 = 0 \\ 0 = 0 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda_2 = 2$

$$A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P^{-1} = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$e^{At} = P e^{Bt} P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} \\ e^{-3t} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

b) $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$; $\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 2 \\ 0 & -1-\lambda \end{bmatrix} \right|$

$(1-\lambda)(-1-\lambda) - 0(2) = -1 - \lambda + \lambda + \lambda^2 = (\lambda^2 - 1)$

$\lambda = \pm\sqrt{1} \Rightarrow \underline{\lambda_1 = 1};\ \underline{\lambda_2 = -1}$

$\boxed{e^{Bt} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}}$ $\quad \underline{\lambda_1 = 1}$ $\quad A - \lambda_1 I = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$ $\begin{cases} 2x_2 = 0 \\ -2x_2 = 0 \end{cases}$ $\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$

$A - \lambda_2 I = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow 2x_1 + 2x_2 = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\quad x_1 = -x_2$

$P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$; $P^{-1} = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]$ $\begin{matrix} R_2 + R_1 \\ R_2(-1) \end{matrix}$ $\underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right]}$

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Calcular B

$$e^{At} = P e^{Bt} P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \underline{\begin{bmatrix} e^{t} & e^{t} - e^{-t} \\ 0 & e^{-t} \end{bmatrix}}$$

(c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, $\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 0 \\ 5 & 1-\lambda \end{bmatrix} \right|$

$\det(A - \lambda I) = (1-\lambda)(1-\lambda) - 0 = 1 - \lambda - \lambda + \lambda^2 = \underline{\lambda^2 - 2\lambda + 1}$

$(\lambda - 1)(\lambda - 1) = \underline{\lambda_1 = \lambda_2 = 1}$

Obtaining $e^{Bt}$ and P matrices

**calcular B**

$\boxed{e^{Bt} = \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}}$ To form P.

$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} \rightarrow \vec{V}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

To obtaine the generalized vector

$(A - I)\vec{V}_2 = \vec{V}_1 \rightarrow \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $0 = 0$

$5x_1 + 0x = 1 \rightarrow x_1 = \frac{1}{5}$

$\vec{V}_{2G} = \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}$ set $x_2 = 0$.

$P = \begin{bmatrix} 0 & 1/5 \\ 1 & 0 \end{bmatrix} \rightarrow P^{-1} = \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$

$$e^{At} = P e^{Jt} P^{-1} = \begin{bmatrix} 0 & 1/5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 0 & \frac{e^t}{5} \\ e^t & te^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} e^t & 0 \\ 5te^t & e^t \end{bmatrix}}$$ ✓

**Ejercicio 4.13** *Encuentre la solución del sistema lineal $\dot{x} = Ax$ donde*

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(a) $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

d

$x(t) = e^{At} x_0$

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^3 = 0$$

$-\lambda - 2 = 0 \rightarrow \underline{\lambda = -2}, \quad \underline{\lambda_1 = \lambda_2 = \lambda_3 = -2}$ ✓

There will be 2 generalized vectors.

$\underline{\lambda = -2}$

$$(A-\lambda I)x = 0 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x_3 = x_3 \\ x_1 = 0 \\ x_2 = 0 \end{matrix} \quad \vec{V}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ ✓

$\vec{V}_{2G}$

$$(A-\lambda I)\vec{V}_{2G} = \vec{V}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_1^2 = 0 \; ; \; x_2^2 = 1 \; ; \; x_3^2 = x_3^2 \longrightarrow \vec{V}_{2G} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ ✓

set $x_3^2 = 0$

$\vec{V}_{3G}$

$$(A-\lambda I)\vec{V}_{3G} = \vec{V}_{2G} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{V}_{3G} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ ✓

$x_1^3 = 1, \; x_2^3 = 0 \quad x_3^3 = x_3^3 \quad$ set $x_3^3 = 0$

$$P=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix};\ P^{-1}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix};\ e^{Bt}=\underbrace{\begin{bmatrix}e^{-2t}&te^{-2t}&t^2e^{-2t}\\0&e^{-2t}&te^{-2t}\\0&0&e^{-2t}\end{bmatrix}}_{\text{Calcular B}}$$

$$e^{At}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}e^{-2t}&te^{-2t}&t^2e^{-2t}\\0&e^{-2t}&te^{-2t}\\0&0&e^{-2t}\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$e^{At}=\begin{bmatrix}0&0&e^{-2t}\\0&e^{-2t}&te^{-2t}\\e^{-2t}&te^{-2t}&t^2e^{-2t}\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}e^{-2t}&0&0\\te^{-2t}&e^{-2t}&0\\t^2e^{-2t}&te^{-2t}&e^{-2t}\end{bmatrix}$$

$$x(t)=e^{At}x_0=\underbrace{C_1e^{-2t}+C_2\left[e^{-2t}(t+1)\right]+C_3\left[e^{-2t}(t^2+t+1)\right]}$$

→ Revisar, este es un vector.

Pase 6.

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**Ejercicio 5.5** *Para las siguientes matrices A, resuelva el problema de valor inicial, Teorema 5.1, determine los subespacios estable e inestable y esboce su retrato fase*

*a)* $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{bmatrix}$ *d)* $\begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

d

$$\begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (-1) = (-1)^2 + (-\lambda)^2 + 2(-1)(-\lambda) + 1$$

$$= \lambda^2 + 2\lambda + 2 = \underline{(-1+j)(-1-j)}$$ ✓

$$\begin{vmatrix} -\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - (-2)(1) = -2\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 2\lambda + 2 = \underline{(1+j)(1-j)}$$ ✓

$\lambda = -1+j$

$$\begin{bmatrix} -1-(-1+j) & -1 & 0 & 0 \\ 1 & -1-(-1+j) & 0 & 0 \\ 0 & 0 & -(-1+j) & -2 \\ 0 & 0 & 1 & 2-(-1+j) \end{bmatrix} = \begin{bmatrix} -j & -1 & 0 & 0 \\ 1 & -j & 0 & 0 \\ 0 & 0 & 1-j & -2 \\ 0 & 0 & 1 & 3-j \end{bmatrix}$$

$-jx_1 - x_2 = 0 \rightarrow x_2 = -jx_1$

$x_1 - jx_2 = 0 \rightarrow x_1 = jx_2$ $\Big\rangle$ $x_1 = j$ ; $x_2 = 1$

$(1-j)x_3 - 2x_4 = 0$

$x_3 + (3-j)x_4 = 0 \rightarrow x_3 = -(3-j)x_4$

set $x_4 = 0$

$x_3 = 0$

$$\underline{\vec{V}_1 = \begin{bmatrix} j \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$ ✓

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For $\vec{V}_2$ we change the imaginary part to be the opposite sign since $\lambda_1 = \bar{\lambda}_2$

$\vec{V}_2 = \begin{bmatrix} -j \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ✓ $\vec{V}_3 = \begin{bmatrix} 0 \\ 0 \\ -1+j \\ 1 \end{bmatrix}$ ✓ $\vec{V}_4 = \begin{bmatrix} 0 \\ 0 \\ -1-j \\ 1 \end{bmatrix}$ ✓

For $\lambda_3 = 1+j$,

$$\begin{bmatrix} -1-(1+j) & -1 & 0 & 0 \\ 1 & -1-(1+j) & 0 & 0 \\ 0 & 0 & -(1+j) & -2 \\ 0 & 0 & 1 & 2-(1+j) \end{bmatrix} = \begin{bmatrix} -2-j & -1 & 0 & 0 \\ 1 & -2-j & 0 & 0 \\ 0 & 0 & -1-j & -2 \\ 0 & 0 & 1 & 1-j \end{bmatrix}$$

$\vec{V}_3$

$(-2-j)x_1 - x_2 = 0 \rightarrow x_2 = -x_1(2+j)$

$x_1 + (-2-j)x_2 = 0 \rightarrow x_1 = -x_2(2+j)$

set $x_1 = 0$ then $x_1 = x_2 = 0$

$(-1-j)x_3 - 2x_4 = 0 \rightarrow x_3 = \dfrac{2}{-1-j}x_4$

$x_3 + (1-j)x_4 = 0$

$\rightarrow x_3 = \dfrac{2(-1+j)}{(-1-j)(-1+j)}x_4 = (-1+j)x_4$

$(-1+j)x_4 + (1-j)x_4 = 0 \rightarrow x_4(0) = 0 \rightarrow$ set $x_4 = 1$.

Using $\vec{v}_1$ and $\vec{v}_3$ the other two are their conjugate

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Imaginary part

$$P = [\vec{v}_1 \; v_1 \; \vec{v}_2 \; v_2]$$

Real.

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$; \; \underbrace{P^{-1}AP}_{B} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$e^{Bt} = \begin{bmatrix} e^{-t}\cos t & -e^{-t}\sin t & 0 & 0 \\ e^{-t}\sin t & e^{-t}\cos t & 0 & 0 \\ 0 & 0 & e^{t}\cos t & -e^{t}\sin t \\ 0 & 0 & e^{t}\sin t & e^{t}\cos t \end{bmatrix}$$

$$x(t) = \underbrace{P \cdot e^{Bt} \cdot P^{-1}}_{= e^{At}}$$

$$\boxed{x(t) = e^{At} x_0}$$

* Stable Subspace
Any perturbation within the subspace spanned by $(-1 \pm j)$ will decay so it's stable.

* Unstable Subspace.
The subspace spanned by the eigenvectors associated to $1 \pm j$ will be unstable.

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**Ejercicio 6.13** *Resuelva el problema de valor inicial (6.1) cuando la matriz A es:*

$$a)\ \begin{bmatrix}0&0&0&0\\1&0&0&1\\1&0&0&1\\0&-1&1&0\end{bmatrix}\quad b)\ \begin{bmatrix}2&0&0&0\\1&2&0&0\\0&1&2&0\\0&0&1&2\end{bmatrix}\quad c)\ \begin{bmatrix}1&1&1&1\\2&2&2&2\\3&3&3&3\\4&4&4&4\end{bmatrix}$$

*y*

$$d)\ \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}\quad e)\ \begin{bmatrix}1&-1&0&0\\1&1&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}\quad f)\ \begin{bmatrix}1&-1&1&0\\1&1&0&1\\0&0&1&-1\\0&0&1&1\end{bmatrix}$$

F

$$A-\lambda I = \begin{bmatrix}1-\lambda&-1&1&0\\1&1-\lambda&0&1\\0&0&1-\lambda&-1\\0&0&1&1-\lambda\end{bmatrix}$$

By cofactors.

$$\det(A-\lambda I) = (1-\lambda)^2\cdot\det\begin{bmatrix}1-\lambda&0&1\\0&1-\lambda&-1\\0&1&1-\lambda\end{bmatrix}$$

$$-(-1)\det\begin{pmatrix}1&0&1\\0&1-\lambda&-1\\0&1&1-\lambda\end{pmatrix}+1\cdot\det\begin{pmatrix}1&1-\lambda&1\\0&0&-1\\0&0&1-\lambda\end{pmatrix}$$

$$\det(A-\lambda I) = \lambda^4-4\lambda^3+8\lambda^2-8\lambda+4$$

$$=(\lambda^2-2\lambda+2)^2 \Rightarrow \lambda^2-2\lambda+2 = \underbrace{(1-j)}_{\lambda_2}\underbrace{(1+j)}_{\lambda_1}$$

So we have $\lambda_1=\lambda_3$ and $\lambda_2=\lambda_4$ where $\lambda_1=\bar{\lambda}_2$ and $\lambda_3=\bar{\lambda}_4$. Each root with multiplicity 2.

Complex pairs with multiplicity 2 ← Case

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$S = \text{diag}[\lambda]$ ; $N = A - S$

$$x(t) = P\,\text{diag}[e^{\lambda_j t}]P^{-1}\left[I + Nt + \dots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right]x_0$$

$$x(t) = P\,\text{diag}\left\{e^{a_j t}\begin{bmatrix}\cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t\end{bmatrix}\right\}P^{-1}\left[I + Nt + \dots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right]x_0$$

$\lambda = 1 - j$

$$(A - (1-j)I)w = \begin{bmatrix} j & -1 & 1 & 0 \\ 1 & j & 0 & 1 \\ 0 & 0 & j & -1 \\ 0 & 0 & 1 & j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$jx_1 - x_2 + x_3 = 0$

$x_1 + jx_2 + x_4 = 0$

$jx_3 - x_4 = 0 \rightarrow j(-jx_4) - x_4 = 0 \rightarrow x_4 - x_4 = 0$ set $x_4 = 0$

$x_3 + jx_4 = 0 \rightarrow x_3 = -jx_4$ $\quad x_3 = 0$

$jx_1 = x_2$ $\quad x_2 = j$

$x_1 + j(jx_1) = 0$ set $x_1 = 1$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ j \\ 0 \\ 0 \end{bmatrix}$$

✓

$$(A-\lambda I)^2 w_2 = \begin{bmatrix} -2 & -2j & 2j & 2 \\ 2j & -2 & 2 & 2j \\ 0 & 0 & -2 & -2j \\ 0 & 0 & 2j & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$-2x_3 - 2jx_4 = 0 \longrightarrow \underline{x_3 = -jx_4}$ set $\underline{x_3 = 1}$

$2jx_3 - 2x_4 = 0 \longrightarrow jx_3 = x_4$ $\underline{x_4 = j}$

$-2x_1 - 2jx_2 + 2j + 2j = 0$ $x_1 = 0$

$x_2 = 2.$

$-2x_1 - 2jx_2 = -4j \longrightarrow x_1 + jx_2 = 2j$

$$\underline{w_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ j \end{bmatrix}}$$ ✓

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ✓

$$P^{-1} = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$S = P \cdot \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{B?} P^{-1}$$

$$S = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad ; N = A - S$$

$$Nt = \begin{bmatrix} 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 2t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x(t) = P \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix} P^{-1} [I + Nt] x_0$$

$$x(t) = \begin{bmatrix} \cos(t) & -\sin t & 2\sin t & 0 \\ \sin t & \cos t & 0 & -2\sin t \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} [I + Nt] x_0$$

$$x(t) = \begin{bmatrix} \cos t & -\sin t & -t\cos t + 2\sin t & -2t\sin t \\ \sin t & \cos t & -t\sin t & 2t\cos t - 2\sin t \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} x_0$$

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**Ejercicio 7.10** *Encuentre la forma canónica de Jordan de las siguientes matrices A y resuelva el problema de valor inicial (7.4):*

a) $\begin{bmatrix}1&0&0&0\\1&2&0&0\\1&2&3&0\\1&2&3&4\end{bmatrix}$ b) $\begin{bmatrix}1&0&0&0\\1&2&0&0\\1&0&2&0\\1&1&0&2\end{bmatrix}$ c) $\begin{bmatrix}2&1&4&0\\0&2&1&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$ d) $\begin{bmatrix}2&1&4&0\\0&2&1&-1\\0&0&2&1\\0&0&0&2\end{bmatrix}$

First, we get the eigenvalues for matrix d.

$$\det(A-\lambda I) = \lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda-2)^2(\lambda-2)^2 = 0.$$

$\lambda = 2$, $ma(\lambda) = 4$.

Eigenvector

$\lambda = 2 \rightarrow (A-2I)x = 0.$

$$(A-2I)x = \begin{bmatrix}0&1&4&0\\0&0&1&-1\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$x_2 + 4x_3 = 0 \rightarrow x_2 = 0$

$x_3 - x_4 = 0 \rightarrow x_3 = 0$

$x_4 = 0$

$x_1 \rightarrow$ free

$\vec{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $mg(\lambda) = 1.$ ✓

Finding the generalized vectors by Jordan chains.

* $(A-2I)\vec{v}_2 = \vec{v}_1$
* $(A-2I)\vec{v}_3 = \vec{v}_2.$
* $(A-2I)\vec{v}_4 = \vec{v}_3$

After finding $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ we form the Jordan Canonical form matrix. J. ✓

$(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $x_2 + 4x_3 = 1 \longrightarrow \underline{x_2 = 1}$
- $x_3 - x_4 = 0 \longrightarrow \underline{x_3 = 0}$
- $\underline{x_4 = 0}$

$x_1 \to$ free

set $x_1 = 0$.

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

✓

$(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- $x_2 + 4x_3 = 0 \longrightarrow \underline{x_2 = -4}$
- $x_3 - x_4 = 1 \longrightarrow \underline{x_3 = 1}$
- $\underline{x_4 = 0}$

$x_1 \to$ free.

set $x_1 = 0$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

✓

$(A-\lambda I)\vec{v}_4 = \vec{v}_3$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

- $x_2 + 4x_3 = 0 \to \underline{x_2 = 12}$
- $x_3 - x_4 = -4 \to \underline{x_3 = -3}$

$\underline{x_4 = 1}$ $\quad x_1$ free

Set $x_1 = 0$

$$\vec{v}_4 = \begin{bmatrix} 0 \\ 12 \\ -3 \\ 1 \end{bmatrix}$$

So $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$J = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\underbrace{x(t) = e^{At}x_0 = Pe^{Jt}P^{-1}x_0} \; ; \; e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

Nilpotent

$$e^{Jt} = \underbrace{e^{\lambda t}} \sum_{k=0}^{\infty} \frac{(Nt)^k}{k!} \; ; \; \sum_{k=0}^{3} \frac{(Nt)^k}{k!} = I + Nt + \frac{N^2t^2}{2!} + \frac{N^3t^3}{3!}$$

$$e^{Jt} = e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P e^{Jt} = e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t-4 & (t^2-8t+24)/2 \\ 0 & 0 & 1 & t-3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{x(t) = \underbrace{e^{2t}\begin{bmatrix} 1 & t & \frac{t^2+8t}{2} & \frac{t^3+9t^2}{6} \\ 0 & 1 & t & \frac{t^2-2t}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Pe^{Jt}P^{-1}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}}_{x_0}}$$

10/10

**Ejercicio 8.14** *¿Para qué valoers de los parámetros a y b tiene el sistema lineal $\dot{x} = Ax$ un sumidero en el origen?* Negative real part.

Complex value

$$A = \begin{bmatrix} a & -b \\ b & 2 \end{bmatrix}$$

To make the origin a sink we need both eigenvalues to have negative real part.

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} a-\lambda & -b \\ b & 2-\lambda \end{vmatrix}$$

$(a-\lambda)(2-\lambda) - (-b)(b) = 0$

$2a - a\lambda - 2\lambda + \lambda^2 + b^2 = 0$

$\underset{a}{\lambda^2} - \underset{b}{\lambda(2+a)} + \underset{c}{(2a+b^2)} = 0$

$$\lambda_{1,2} = \frac{(2+a) \pm \sqrt{(2+a)^2 - 4(1)(2a+b^2)}}{2}$$ ✓

To have a sink at the origin:

$2+a < 0$ ✓

$\boxed{a < -2}$ Negative Real Part.

¿Es necesaria esta condición?

$(2+a)^2 - 4(2a+b^2) < 0$

$4 + 4a + a^2 - 8a - 4b^2 < 0$

$a^2 - 4a - 4b^2 + 4 < 0$

$\sqrt{(a-2)^2 < 4b^2}$

$\boxed{|a-2| < 2|b|}$

Complex

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Sol

**Ejercicio 9.17** *Encuentre los subespacios estable, inestable y centro para el sistema lineal $\dot{x} = Ax$ con $A$ dada por*

$$a)\ \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\quad b)\ \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\quad c)\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\quad \boxed{d)\ \begin{bmatrix}-1 & -3\\0 & 2\end{bmatrix}}\quad e)\ \begin{bmatrix}2 & 3\\0 & -1\end{bmatrix}$$

*y*

$$f)\ \begin{bmatrix}2 & 4\\0 & -2\end{bmatrix}\quad g)\ \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}\quad h)\ \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\quad i)\ \begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix}$$

*También esboce el retrato fase de cada uno. ¿Cuáles de estas matrices definen un flujo hiperbólico $e^{At}$?*

For matrix d the characteristic polynomial is found using:

$a_{11}$ value (-)

$$\lambda^2 - \lambda \cdot tr(A) + det(A) = 0$$

$tr(A) = -1 + 2 = \boxed{1}$ ; $det(A) = (-1)(2) - (-3)(0) = \boxed{-2}$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)\ (\lambda + 1)$$

$$\lambda_1 = 2, \quad \lambda_2 = -1$$

Eigenvectors generated with this eigenvalue are in an unstable subspace

Stable subspace.

Since there are only real eigenvalues, there is no center subspace.

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For $\lambda_1 = 2$ ✓

$(A-2I)v_1 = 0 \longrightarrow \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

• $-3x_1 - 3x_2 = 0 \longrightarrow x_1 = -x_2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

$P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, P^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

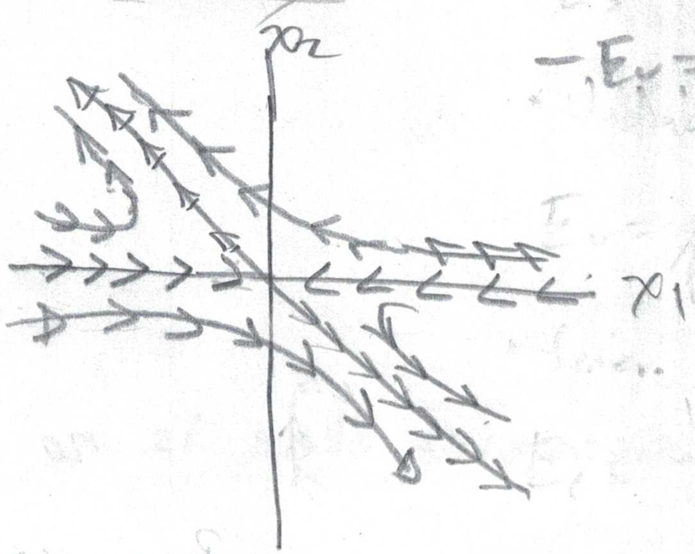
$x(t) = \begin{bmatrix} e^{-t} & -e^{2t} + e^{-t} \\ 0 & +e^{2t} \end{bmatrix}$

$(A-(-1)I)v_2 = 0$ $\quad \lambda = -1$

$\begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ • $-3x_2 = 0$, $3\boxed{x_2 = 0}$, $x_1 = x_1$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓

* Stable
  - $E_s = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$; ✓
* center
  - $E_c = \{0\}$ ✓
* Unstable
  - $E_u = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ ✓



2016

**Ejercicio 9.18** *Lo mismo que para el ejercicio anterior pero para las matrices*

$$a)\ \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 3\end{bmatrix}\quad b)\ \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\quad c)\ \begin{bmatrix}-1 & -3 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}\quad d)\ \begin{bmatrix}2 & 3 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

(b) $\det(A-\lambda I)=0$ ; $\det(A-\lambda I) = (\lambda+1)(\lambda-i)(\lambda+i)=0$

$\lambda_1=-1,\ \lambda_2=j,\ \lambda_3=-j.$

$\lambda=j$ $\quad (A-jI)x=0$

$$\begin{bmatrix} -j & -1 & 0\\ 1 & -j & 0\\ 0 & 0 & -1-j\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

*① $-jx_1 - x_2 = 0$ ; *② $x_1 - jx_2 = 0 \rightarrow x_1 = jx_2$

③ $(-1-j)(x_3)=0 \longrightarrow x_3=0$, set $x_1=1$; $x_2=-j$

$\vec{V_1}=\begin{bmatrix}1\\-j\\0\end{bmatrix}$ ; for $\lambda=-j$ $\vec{V_2}=\begin{bmatrix}1\\j\\0\end{bmatrix}$

$\lambda=-1$

$$\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1 - x_2 = 0 \quad x_1=x_2 \quad x_2=0.$

$x_1=0$

$x_1+x_2=0,\ x_1=-x_2$

$x_3=x_3$, set $x_3=1$

$\vec{V_3}=\begin{bmatrix}0\\0\\1\end{bmatrix}$

✓

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; P^{-1} = P.$$

$$B = P^{-1}AP = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right]$$

$a + bj$ ; where $a = 0$ and $b = -1$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$x(t) = P \begin{bmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} P^{-1} \cdot x_0$$

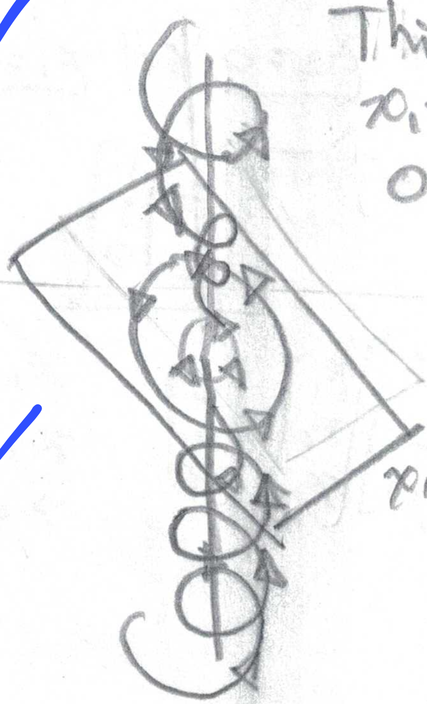
* Stable

$$E_s = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

* Center

$$E_c = \left\{ \begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \right\}$$

This rotates in trajectories $x_1$-$x_2$ and converge to 0 in the $x_3$ axis.



(d) $\begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\det(A-\lambda I)=0$

$\det(A-\lambda I) = (2-\lambda)(-1-\lambda)(-1-\lambda) = 0.$

$\lambda_1 = 2$ , $\lambda_2 = \lambda_3 = -1$

$\underline{\lambda = 2}$ $A - 2I = \begin{bmatrix} 0 & 3 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ , $3x_2 = 0$ , $-3x_2 = 0$ , $-3x_3 = 0$ $\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\underline{\lambda = -1}$

$A - (-1)I = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$3x_1 + 3x_2 = 0$

$\underline{x_1 = -x_2}$ $\underline{x_3 \text{ free}}$ $\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for $x_1 = 1$ and $x_3 = 0$.

and $\vec{V}_{2-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for $x_1 = 0$, $x_3 = 1$.

$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ; $P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\boxed{x(t) = Pe^{Bt}P^{-1}x_0}$

$e^{Bt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$

$$x(t)=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0 & 0\\ 0 & e^{-t} & 0\\ 0 & 0 & e^{-t}\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}x_0$$

$$x(t)=\begin{bmatrix}e^{2t} & e^{2t}-e^{-t} & 0\\ 0 & e^{-t} & 0\\ 0 & 0 & e^{-t}\end{bmatrix}x_0$$

✓

* Unstable

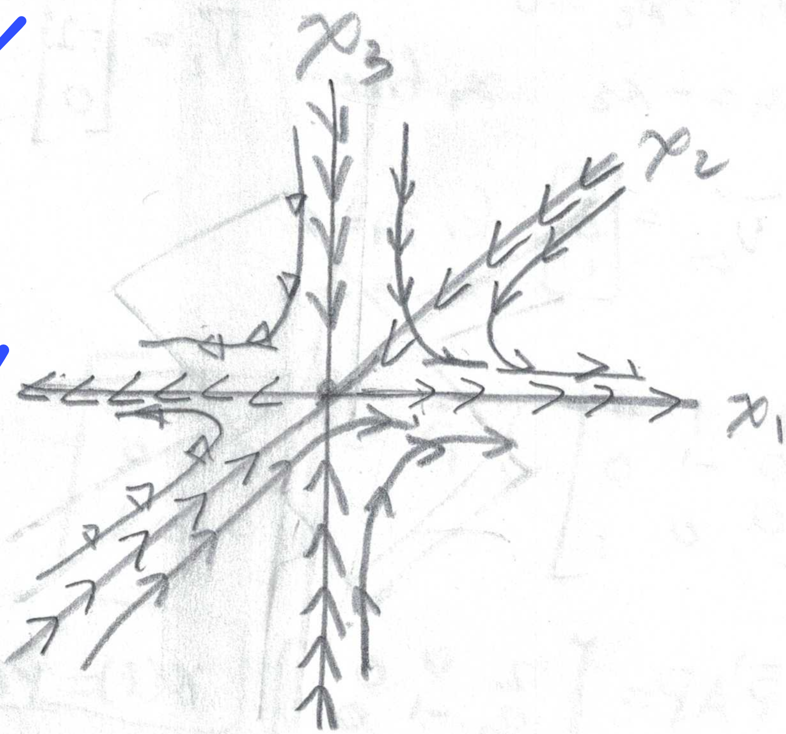
$E_U=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ ✓

* Stable

$E_S=\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ ✓

* Center

$E_C=\{0\}$. ✓






Falto 10-10

15%

**Ejercicio 3.17** *Encuentre matrices $A$ y $B$ de orden $2 \times 2$ tales que $e^{A+B} \neq e^A e^B$.*

To find matrices A and B of order 2x2 such that

$e^{A+B} \neq e^A e^B$, A and B must not commute $AB \neq BA$.

We choose

$$A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$ ✓

$$BC = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & x^2 \end{bmatrix} \quad (2)$$ ✓

From (1) and (2) we can see A and B does not commute.

$$A+B = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$$ ✓

$$e^{A+B} = \begin{bmatrix} 0 & e^{x} \\ e^{x} & 0 \end{bmatrix} \qquad e^{A \cdot B} = \begin{bmatrix} e^{x^2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{A+B} \neq e^{A \cdot B}$$ ✓

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**Ejercicio 9.20** *Muestre que las únicas líneas invariantes para el sistema $\dot{x} = Ax$ con $x \in \mathbb{R}^2$ son las líneas $ax_1 + bx_2 = 0$ donde $v = (-b, a)^T$ es un eigenvector de $A$.*

Solo

A subspace W of V is meant to be invariant if $\forall v \in W : Tv \in W$.

$v = \begin{bmatrix} -b \\ a \end{bmatrix}$; $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$Av = \lambda v$$

The vector v will be scaled "$\lambda$" under the transformation A. Any vector on the line $ax_1 + bx_2 = 0$ can be expressed as $x = \alpha v$.

scaler

$$\dot{x} = Ax = A(\alpha v) = \alpha Av = \alpha \lambda v = \lambda x.$$

This shows $\dot{x} = \lambda x$ the rate of change of $\dot{x}$ is just scaled but the direction is the same.

* the only line invariant lines for $\dot{x} = Ax$ are those aligned with eigenvectors of A.

Usar la definición de espacio invariante para la solución del problema.