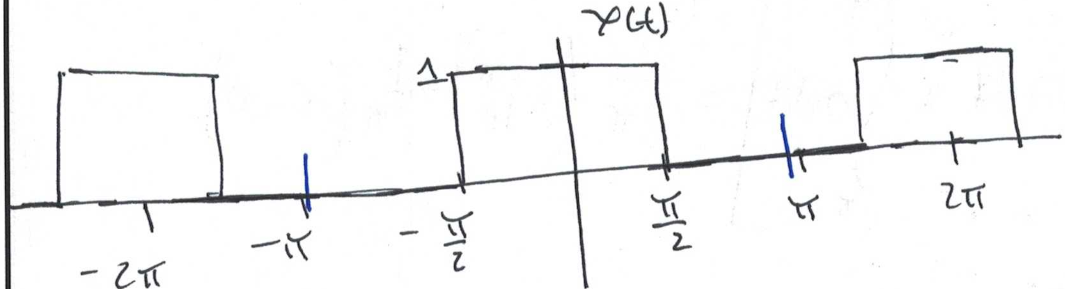
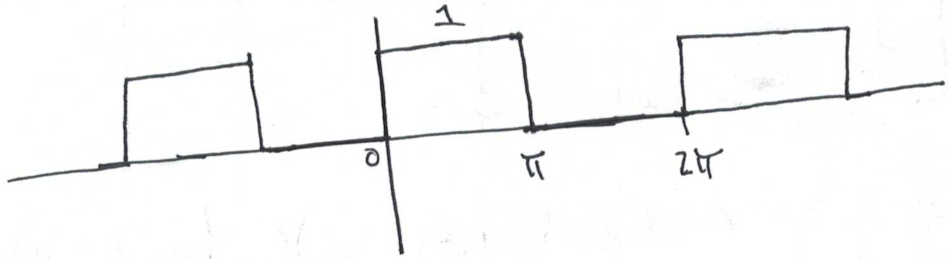
Project 3. Fourier Series and Fourier Transform.



The signal is periodic on the interval $-\pi, \pi$. We can set the interval to be $0, 2\pi$



$$x(t) = \begin{cases} 0 \le t < \pi & = 1 \\ \pi \le t < 2\pi & = 0 \end{cases}$$

Trigonometric Fourier Series.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega n t) + b_n \sin(\omega n t)$$

$$^{*}a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \; ; \; ^{*}a_n = \frac{2}{T_0}\int_{T_0} x(n) \cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(n) \sin(n\omega_0 t)\,dt$$

Calculating Coefficients-

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi}(1)\,dt + \int_{\pi}^{2\pi} 0\cdot dt\right] = \frac{1}{2\pi}\left(t\Big|_0^{\pi}\right) = \frac{1}{2\pi}(\pi - 0)$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{2}{2\pi}\left[\int_0^{\pi}(1)\cos(n\omega t)\,dt + \int_{\pi}^{2\pi} 0\cdot\cos(n\omega t)\,dt\right]$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi}\cos(\omega n t)\,dt\right] = \frac{1}{\pi n\omega}\left(\sin(\omega n t)\Big|_0^{\pi}\right)$$

$$a_n = \frac{1}{\pi n\omega}\left(\sin(\omega n\cdot\pi)\right) = 0$$

$$b_n = \frac{2}{2\pi}\left[\int_0^{\pi}(1)\sin(n\omega t)\,dt\right] = \frac{1}{\pi}\cdot\int_0^{\pi}\sin(n\omega t)\,dt$$

$$= \frac{1}{\pi n\omega}\left(-\cos(n\omega t)\Big|_0^{\pi}\right) = \frac{1}{\pi n\omega}\left(-\left[\cos(n\omega\pi) - \cos(0)\right]\right)$$

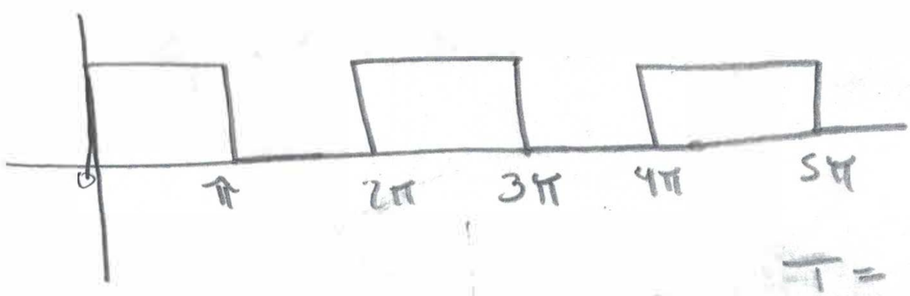
$(-1)^n$

$$= \frac{1}{\pi n\omega}\left(-(-1-1)\right) = \frac{2}{\pi n\omega} = \boxed{\frac{2}{\pi n}}$$

$\omega = 1 \rightarrow T = 2\pi;\ T = \frac{2\pi}{\omega}$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} 0 + \frac{2}{\pi n}\sin(nt)$$

$$\boxed{x(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ \text{odd numbers}}}^{\infty} \frac{2}{\pi n}\sin(nt)}$$



$2\pi f = \omega$

$f = \frac{\omega}{2\pi}$

$T = \frac{1}{f} = \frac{2\pi}{\omega}$

$2\pi = \frac{2\pi}{\omega}$

$\omega = \frac{2\pi}{2\pi} = 1$

* Compact form of Fourier Series.

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$C_0 = a_0$, $C_n = \sqrt{a_n^2 + b_n^2}$, $\theta = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$.

$C_0 = \frac{1}{2}$, $a_n = 0$ and $b_n = \frac{2}{\pi n}$. Then

$C_n = \sqrt{\left(\frac{2}{\pi n}\right)^2} = \boxed{\frac{2}{\pi n}}$, $\boxed{\theta = \frac{-\pi}{2}}$

$$x(t) = \frac{1}{2} + \sum_{\substack{n=1,3,5,\ldots 7 \\ \text{odd}}}^{\infty} \frac{2}{\pi n}\cos\left(nt - \frac{\pi}{2}\right)$$

* Exponental Forier Series.

* $$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

* $$D_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$, * $D_0 = a_0$

From the trigonometric series we have $a_0 = \frac{1}{2}$, $a_n = 0$

$b_n = \frac{2}{\pi n}$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt - \frac{j}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt = \frac{1}{2}(a_n - jb_n)$$

$$D_n = \frac{1}{2}\left(0 - j\left(\frac{2}{\pi n}\right)\right) = \boxed{\frac{-j}{\pi n}} = \frac{1}{j\pi n}$$ This is for $n>0$ and "n" odd.

Now for minus infinity values $n<0$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt + \frac{j}{T_0}\int_{T_0} x(t)\sin \omega_n t\,dt = \frac{1}{2}(a_n + jb_n)$$

$$D_n = \frac{1}{2}\left(0 + j\frac{(2)}{\pi n}\right) = \boxed{\frac{j}{\pi n}}$$

$$a_n - jb_n = \sqrt{a_n^2 + b_n^2}\; e^{j\tan^{-1}\left(\frac{b_n}{a_n}\right)} = C_n e^{j\theta_n}$$

$$a_n - j b_n = \frac{2}{\pi n} e^{j\frac{-\pi}{2}} = \frac{2}{j\pi n}$$

$$|D_n| = |D_{-n}| = \frac{1}{2} C_n = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{\pi n}\right) = \boxed{\frac{1}{\pi n}}$$

$$\angle D_n = \theta_n \quad \text{and} \quad \angle D_{-n} = -\theta_n$$

$$\boxed{\angle D_n = \frac{-\pi}{2}} \quad \text{and} \quad \boxed{\angle D_{-n} = +\frac{\pi}{2}}$$

$$D_n = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{\pi n}\right) e^{j\frac{-\pi}{2}} = \boxed{\frac{e^{j\frac{-\pi}{2}}}{\pi n}}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{e^{j\frac{-\pi}{2}}}{\pi n} \cdot e^{jnt}$$

n = odd numbers $0 < n < 0$

$$\boxed{x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{e^{j\frac{-\pi}{2}}}{\pi n} e^{jnt} + \sum_{n=-1}^{-\infty} \frac{e^{j\frac{\pi}{2}}}{\pi n} e^{-jnt}}$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{j\pi n} e^{jnt} + \sum_{n=-1}^{-\infty} \frac{1}{j\pi n} e^{-jnt}$$

when n is negative it becomes positive equal to the other term. then

$$\boxed{x(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{2}{jn\pi} e^{jnt}}$$

* Fourier Transform.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$D_n = \frac{1}{T_0} X(n\omega_0)$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$X(\omega) = \int_0^{\pi} (1)\, e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^{\pi} = \frac{e^{-j\omega\pi} - 1}{-j\omega}$$

$$X(\omega) = \frac{\cos(\omega\pi) - j\sin(\omega\pi) - 1}{-j\omega}$$

$$X(\omega) = \frac{1 - \cos(\omega\pi) + j\sin(\omega\pi)}{j\omega} \quad ; \omega = 1 = \frac{2\pi}{T} = \frac{2\pi}{2\pi}$$

$$X(1) = \frac{2}{j} = \underbrace{2e^{-\frac{\pi}{2}}}, \text{ fundamental}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-e^{-j\omega\pi}}{j\omega}\right)e^{j\omega t}\,d\omega$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{e^{j\omega t}}{j\omega}\,d\omega-\int_{-\infty}^{\infty}\frac{e^{j\omega(t-\pi)}}{j\omega}\,d\omega\right]$$

On the two integrals we have a non define value in $\omega=0$ so we apply Cauchy Principal Value

$$=\frac{1}{2\pi}\left[\pi\cdot u(t)-\pi u(t-\pi)\right]=\underline{\frac{1}{2}\left[u(t)-u(t-\pi)\right]}$$

$t=0$

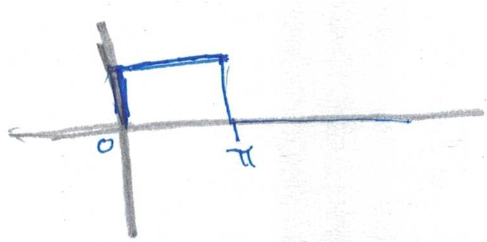
$\frac{1}{2}\left(1-u(-\pi)\right)=\underline{\frac{1}{2}}$ we need to multiply by 2 to adjust to the original signal

$$\boxed{x(t)=u(t)-u(t-\pi)}$$

$x(\pi)=1-(1)=\underline{0}$

$x(0)=1-0=\underline{1}$

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$



We can see in page 7 that $u(t)-u(t-\pi)$ only captures one cycle of the function.

$$u(t)-u(t-\pi)=\begin{cases}1 & 0\le t<\pi \\ 0, & \text{otherwise}\end{cases}$$

$$x(t)=\begin{cases}0\le t<\pi & =1 \\ \pi\le t\le 2\pi & =0\end{cases}$$

We can use Dirac delta functions to account for repetitions with

$$X(\omega)=\sum_{k=-\infty}^{\infty} C_k\,\delta(\omega-k)$$

where $C_k=\left(\frac{1-(-1)^k}{jk}\right)$ and δ is the impulse function.

For practical purposes we analyze only one period.

For periodic signals Fourier Series is recommended

Discrete Fourier Series (DFS)

$$x(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1, 0, 0, 0, 0, 0\} \; ; \; T_0 = 2\pi$$

$N = 10$ samples.

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}. \qquad x(n) = x_p(n) \quad 0 \le n \le N-1$$

$$c_k = \frac{1}{10} \sum_{n=0}^{9} x(n) \cdot e^{\frac{-j2\pi kn}{10}}$$ - from 5 to N−1 c_k is 0.

$$c_k = \frac{1}{10} \sum_{n=0}^{4} (1) e^{\frac{-2\pi kn}{10}} = \underbrace{\frac{1}{10} \sum_{n=0}^{4} e^{\frac{-\pi kn}{5}}} \quad k = 0, 1, \ldots N-1$$

Compute for $n = 0, 1, 2, 3, 4$ and $k = 0, 1, \ldots, N-1$

Getting $x_p(n)$.

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{10}} = \underbrace{\sum_{k=0}^{N-1} c_k e^{\frac{j\pi kn}{5}}}$$

Transform DFT

$$\underbrace{X(\omega) = \sum_{n=0}^{9} x(n) e^{-j\omega n}.}$$