85%

HW2A Modern Control Theory

Master of Science in Electrical Engineering

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Problem 1)

(25 %) Find the (a) loop, (b) node, and (c) state equations for the circuit shown in Figure. For points (a) and (b) write the matrix integro-differential equations, respectability.

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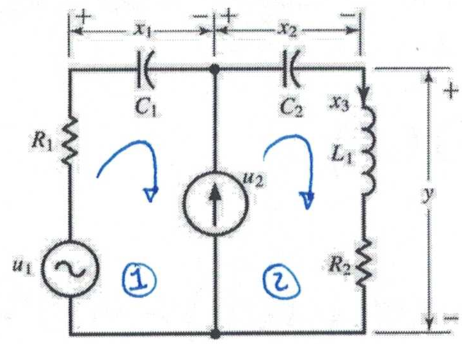


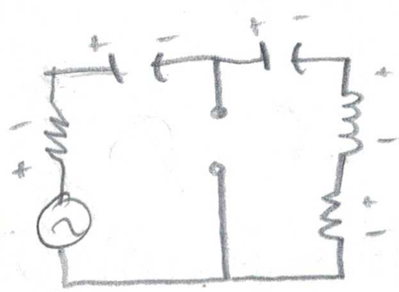
a) $\sum V = 0$

① $i_2(t) - i_1(t) = u_2(t)$

$i_1(t) = i_2(t) - u_2(t)$

$i_2(t) = u_2(t) + i_1(t)$

② Using the supermesh



$V_{R1} + V_{C1} + V_{C2} + V_{L1} + V_{R2} = u_1(t)$

$$R_1 i_1(t) + \frac{1}{C_1}\int i_1(t)dt + \frac{1}{C_2}\int i_2(t)dt + L\frac{di_2(t)}{dt} + R_2 i_2(t) = u_1(t)$$

Deriving Both sides $\left(\frac{d}{dt}\right)$

$$R_1\frac{di_1(t)}{dt} + \frac{i_1(t)}{C_1} + \frac{i_2(t)}{C_2} + L\frac{d^2 i_2(t)}{dt} + R_2\frac{di_2(t)}{dt} = \frac{du_1(t)}{dt}.$$

Let $\frac{df(t)}{dt} = D(f)$, using $i_2(t) = v_2(t) + i_1(t)$

* Grouping for i_1

* $R_1 \cdot D(i_1) + \frac{i_1(t)}{C_1} + \frac{v_2(t)}{C_2} + \frac{i_1(t)}{C_2} + LD^2(v_2 + i_1) + \ldots$

$+ R_2 \cdot D(v_2 + i_1) = D(v_1)$

* $i_1(t) \cdot \left(R_1 \cdot D + \frac{1}{C_1} + \frac{1}{C_2} + LD^2 + R_2 \cdot D\right) = v_1(t) \cdot D - \ldots$

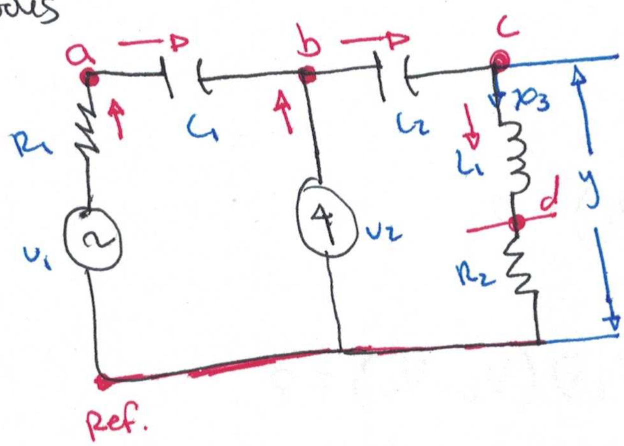
$- v_2(t)\left(\frac{1}{C_2} + LD^2 + R_2 D\right)$

Matrix form of a)

$$\underbrace{\overset{5\times1}{\begin{bmatrix} R_1 D \\ \frac{1}{C_1} \\ \frac{1}{C_2} \\ LD^2 \\ R_2 D \end{bmatrix}}}\overset{1\times1}{\begin{bmatrix} i_1(t) \end{bmatrix}} = \underbrace{\begin{bmatrix} v_1(t) \end{bmatrix}\begin{bmatrix} D \end{bmatrix} - \overset{5\times1}{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{C_2} \\ LD^2 \\ R_2 D \end{bmatrix}}\overset{1\times1}{\begin{bmatrix} v_2(t) \end{bmatrix}}}$$

Esta operación no es correcta.

(b) Nodes



$$\sum_{n=1}^{N} i_n = 0.$$

(a) $i_{R1} - i_{C1} = 0 \longrightarrow \frac{V_a - v_1(t)}{R_1} - C_1 \frac{d(V_a - V_b)}{dt} = 0$

(b) $i_{C1} + v_2(t) - i_{C2} = 0 \longrightarrow C_1 \frac{d(V_b - V_a)}{dt} + v_2(t) - C_2 \frac{d(V_b - V_c)}{dt} = 0$

(c) $i_{C2} - i_{L1} = 0 \longrightarrow C_2 \frac{d(V_c - V_b)}{dt} - \frac{1}{L_1}\int (V_c - V_d)\,dt = 0$

(d) $i_{L1} - i_{R2} = 0 \longrightarrow \frac{1}{L_1}\int (V_d - V_c)\,dt - \frac{V_d}{R_2} = 0$

* Deriving (c) and (d) to have only differential functions.

(2c) $C \frac{d^2(V_c - V_b)}{dt} - \frac{1}{L}(V_c - V_d) = 0$

(2d) $\frac{1}{L}(V_d - V_c) - \frac{d(V_d)}{dt} \cdot \frac{1}{R_2} = 0$

Let $\frac{d(f(t))}{dt} = D(f)$

(a) $\frac{V_a - v_1(t)}{R_1} - C_1 D(V_a - V_b) = 0$

(b) $C_1 \cdot D(V_b - V_a) + v_2(t) - C_2 D(V_b - V_c) = 0$

(c) $C_2 D^2 (V_c - V_b) - \frac{1}{L_1}(V_c - V_d) = 0$

(d) $\frac{1}{L_1}(V_d - V_c) - \frac{D(V_d)}{R_2} = 0$

Grouping similar terms

(a) $V_a\left(\frac{1}{R_1} - C_1 D\right) + V_b(C_1 \cdot D) = \frac{v_1(t)}{R_1}$

(b) $V_a(-C_1 D) + V_b(C_1 D - C_2 D) + V_c(C_2) = -v_2(t)$

(c) $V_b(-C_2 D^2) + V_c\left(C_2 D^2 - \frac{1}{L_1}\right) + V_d\left(\frac{1}{L_1}\right) = 0$

(d) $V_c\left(\frac{-1}{L_1}\right) + V_d\left(\frac{1}{L_1} - \frac{D}{R_2}\right) = 0$

In the next page the matrix form is shown.

Matrix form of b)

$$\begin{bmatrix} \frac{1}{R_1}-C_1D & C_1D & 0 & 0 \\ -C_1D & C_1D-C_2D & C_2 & 0 \\ 0 & -C_2D^2 & C_2D-\frac{1}{L_1} & \frac{1}{L_1} \\ 0 & 0 & -\frac{1}{L_1} & \frac{1}{L_1}-\frac{D}{R_2} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \end{bmatrix} = \begin{bmatrix} \frac{u_1(t)}{R_1} \\ -u_2(t) \\ 0 \\ 0 \end{bmatrix}$$

c) State equations.

Solving with KVL (Mesh analysis)

Using the loop equations in a) let $x_1 = v_{c1}$

$x_2 = v_{c2}$, $x_3 = i_{L1}$

① $x_3 - i_1(t) = u_2(t) \longrightarrow i_1(t) = x_3 - u_2(t)$

② $V_{R1} + V_{C1} + V_{C2} + V_{L1} + V_{R2} = u_1(t)$

$R_1(x_3 - u_2(t)) + x_1 + x_2 + L\dot{x}_3 + R_2x_3 = u_1(t)$

$L\dot{x}_3 + x_3(R_1+R_2) + x_1 + x_2 - R_1u_2(t) = u_1(t)$

$\dot{x}_3 = \frac{1}{L}u_1(t) + \frac{R_1}{L}u_2(t) - \frac{1}{L}x_1 - \frac{1}{L}x_2 - x_3\left(\frac{R_1+R_2}{L}\right)$

In page 3 from node equations (b) and (c)

(b) $i_{c1} + u_2(t) - i_{c2} = 0$

(c) $i_{c2} - i_{L1} = 0$

From (b)

$$C_1 \dot{x}_1 + u_2(t) - C_2 \dot{x}_2 = 0$$

$$\dot{x}_1 = \frac{C_2 \dot{x}_2 - u_2(t)}{C_1} = \frac{C_2}{C_1}\dot{x}_2 - \frac{u_2(t)}{C_1} \quad (1)$$

From (c)

$$C_2 \frac{dv_{c2}}{dt} = i_{L1} \longrightarrow C_2 \dot{x}_2 = x_3 \longrightarrow \boxed{\dot{x}_2 = \frac{x_3}{C_2}} \quad (2)$$

Using (1) and (2)

$$\dot{x}_1 = \frac{\cancel{C_2}}{C_1}\left(\frac{x_3}{\cancel{C_2}}\right) - \frac{u_2(t)}{C_1} = \boxed{\frac{x_3}{C_1} - \frac{u_2(t)}{C_1}}$$

$$\dot{x} = Ax + Bu.$$

$$\underset{\dot{x}}{\begin{bmatrix} \dot{v}_{c1} \\ \dot{v}_{c2} \\ \dot{i}_L \end{bmatrix}} = \underset{A}{\begin{bmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L} & -\frac{1}{L} & \frac{-(R_1+R_2)}{L} \end{bmatrix}} \underset{x}{\begin{bmatrix} v_{c1} \\ v_{c2} \\ i_L \end{bmatrix}} + \underset{B}{\begin{bmatrix} 0 & -\frac{1}{C_1} \\ 0 & 0 \\ \frac{1}{L} & \frac{R_1}{L_1} \end{bmatrix}} \underset{u}{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}$$

✓

$$y = V_{L1} + V_{R2} = L\frac{di_2(t)}{dt} + i_2(t)\cdot R_2$$

$$y = L\dot{x}_3 + x_3 \cdot R_2$$

$$y = L\left(\frac{1}{L}u_1(t) + \frac{R_1}{L}u_2(t) - \frac{1}{L}x_1 - \frac{1}{L}x_2 - x_3\left(\frac{R_1+R_2}{L}\right)\right) + x_3 \cdot R_2$$

$$y = u_1(t) + R_1 u_2(t) - x_1 - x_2 - x_3 \cdot R_1$$

$$y = Cx + Du$$

$$y = \underset{1\times 3}{\begin{bmatrix} -1 & -1 & -R_1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} V_{c1} \\ V_{c2} \\ i_L \end{bmatrix}} + \underset{1\times 2}{\begin{bmatrix} 1 & R_1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}$$

✓

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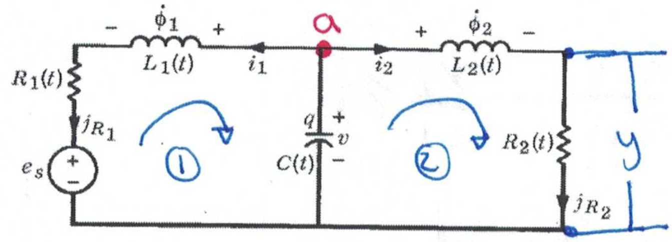
(25 %) Write the state equations for the following linear time-invariant networks in Fig.2, assuming that all the R's, C's, and L's are time-invariant, and (a) using charges and fluxes as state variables and (b) using capacitor voltage and inductor currents as the state variables.

Problem 2



a) Using KVL we set an arbitrary current for each Loop.

① $V_{R1} + V_{L1} + V_{C1} = V_{es}$ ✓

② $V_{L2} + V_{R2} - V_{C1} = 0$ ✓

$Q_c = x_1 = C \cdot V(t)$ ✓

$\phi_{L1} = x_2 = L_1 i_1(t)$

$\phi_{L2} = x_3 = L_2 i_2(t)$

$V_L = L \frac{di}{dt}$ ✓

$i_c = C \frac{dv_c}{dt}$ ✓

$\frac{x_2}{L_1} = i_1(t)$; $\frac{x_3}{L_2} = i_2(t)$

$\left(\frac{\dot x_2}{L_1} = \frac{di}{dt}\right) \cdot L_1 = \underbrace{\dot x_2 = L_1 \frac{di_1}{dt} = v_{L1}}$

① $\boxed{\frac{x_2}{L_1} R_1 + \dot x_2 + \frac{x_1}{C_1} = V_{es}}$

② $\boxed{\dot x_3 + \frac{x_3}{L_2} \cdot R_2 - \frac{x_1}{C_1} = 0}$

Analyzing node a we have $i_{L1} + i_{L2} + i_c = 0$

$i_c = -i_{L1} - i_{L2}$; $i_c = \frac{dQ_c}{dt}$

$\boxed{\dot x_1 = -\frac{x_2}{L_1} - \frac{x_3}{L_2}}$ ③

From ①, ② and ③

①
$$\dot{x}_2 = V_{cs} - \frac{x_1}{C_1} - \frac{x_2}{L_1} R_1$$

② $$\dot{x}_3 = \frac{x_1}{C_1} - \frac{x_3}{L_2} \cdot R_2$$

③ $$\dot{x}_1 = \frac{-x_2}{L_1} - \frac{x_3}{L_2}$$

$$\begin{bmatrix} \dot{q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

$\dot{x} = Ax + Bu$

$$\begin{bmatrix} \dot{x} \\ V_{L1} \\ V_{L2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L_1} & -\frac{1}{L_2} \\ -\frac{1}{C_1} & -\frac{R_1}{L_1} & 0 \\ \frac{1}{C_1} & 0 & -\frac{R_2}{L_2} \end{bmatrix}}_{3\times3} \underbrace{\begin{bmatrix} Q_c \\ \phi_{L1} \\ \phi_{L2} \end{bmatrix}}_{3\times1} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{3\times1} \underbrace{[V_{cs}]}_{1\times1}$$

$y = Cx + Du$; $y = V_{R2} = i_1(t) \cdot R_2 = \frac{x_3}{L_2} \cdot R_2$ ✓

$$y = \begin{bmatrix} 0 & 0 & \frac{R_2}{L_2} \end{bmatrix} \begin{bmatrix} Q_{C1} \\ \phi_{L1} \\ \phi_{L2} \end{bmatrix} + [0][V_{cs}]$$

b) Capacitor and inductor currents as state variables.

$x_1 = V_c = \frac{1}{C_1}\int i_c\,dt + i_0(t)$ $\qquad \dot{x}_1 = \frac{i_c(t)}{C_1}$

$x_2 = i_{L1} = \frac{1}{L_1}\int v_{L1}\,dt + v_0(t)$ $\qquad \dot{x}_2 = \frac{V_{L1}(t)}{L_1}$

$x_3 = i_{L2} = \frac{1}{L_2}\int v_{L2}\,dt + v_0(t)$ $\qquad \dot{x}_3 = \frac{V_{L2}(t)}{L_2}$

$\sum V = 0$. From ① and ② of a) in page 8.

① $V_{R1} + V_{L1} + V_{C1} = V_{cs}$ ② $V_{L2} + V_{R2} - V_{C1} = 0$

Revisar

$i_1 + i_2 + i_c = 0$

① $x_2 R_1 + L_1\dot{x}_2 + x_1 = V_{cs}$

② $L_2\dot{x}_3 + x_3 \cdot R_2 - x_1 = 0$

$i_c(t) = (i_1 - i_2)$

$i_c(t) = (x_2 - x_3)$

$$\dot{x}_1 = \frac{-x_2 - x_3}{C_1}$$

✓

$$\dot{x}_2 = \frac{V_{cs}}{L_1} - \frac{x_1}{L_1} - \frac{x_2 R_1}{L_1}$$

✓

$$\dot{x}_3 = \frac{x_1}{L_2} - \frac{x_3 R_2}{L_2}$$

✓

$\dot{x} = Ax + Bu$

$$\begin{bmatrix} \frac{dv}{dt} \\ \frac{di_1}{dt} \\ \frac{di_2}{dt} \end{bmatrix}$$

Revisar

$$\begin{bmatrix} \dot{v}_{C1} \\ \dot{i}_{L1} \\ \dot{i}_{L2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{R_1}{L_1} & 0 \\ \frac{1}{L_2} & 0 & -\frac{R_2}{L_2} \end{bmatrix} \begin{bmatrix} V_{C1} \\ i_{L1} \\ i_{L2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [V_{CS}]$$

$\frac{1}{L_1}$

$y = V_{R2} = i_2(t) \cdot R_2 = x_3 \cdot R_2$ ✓

$$y = \begin{bmatrix} 0 & 0 & R_2 \end{bmatrix} \begin{bmatrix} V_{C1} \\ i_{L1} \\ i_{L2} \end{bmatrix} + [0][V_{CS}]$$

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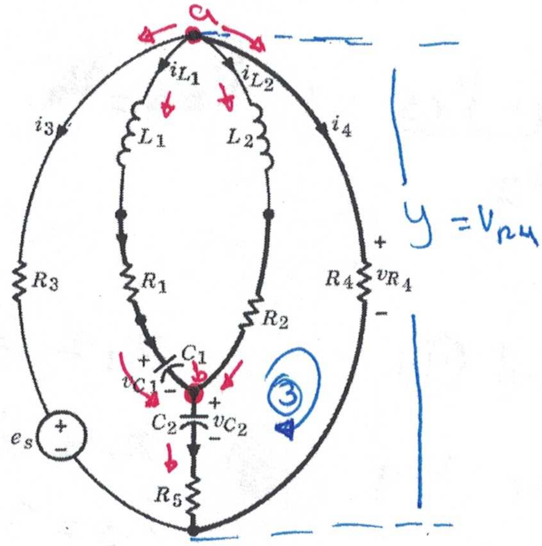
Problem 3

(25 %) Write the state equations for the following linear time-invariant networks in Fig.3, assuming that all the R's, C's, and L's are time-invariant and using capacitor voltage and inductor currents as the state variables.



Statement ①

Parallel circuits have the same voltage. the two branches between nodes a and b have the same voltage.

① $V_{L1} + V_{R1} + V_{C1} = V_{L2} + V_{R2}$ ✓

Statement ②

Currents flowing in a node is the same as leaving the node.

In node b two currents in an 1 current out.

*node b *Node a

② $i_{C2} = i_{L1} + i_{L2}$ ✓ ; (2.1) $i_{R3} + i_{L1} + i_{L2} + i_{R4} = 0$ ✓

Statement ③

The sum of voltage drops along a closed loop is 0.

For loop 3

③ $V_{R4} \underbrace{+ V_{R5}} - V_{C2} \underbrace{+ V_{R2} +} V_{L2} = 0$

check

Statement (4) the current through a resistor is the voltage difference in its terminals divided by its resistance.

(4) $$i_{R3} = \frac{V_{R4} - V_{es}}{R_3} \qquad V_{R4} = V_a$$

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From (2.1) and (4)

$$i_{R3} = -i_{L1} - i_{L2} - i_{R4}$$

$$x_1 = i_{L1}(t)$$
$$x_2 = i_{L2}(t)$$
$$x_3 = V_{C1}(t)$$
$$x_4 = V_{C2}(t).$$

$$\frac{V_{R4} - V_{es}}{R_3} = -x_1 - x_2 - \frac{V_{R4}}{R_4}$$

$$V_{R4}\underbrace{\left(\frac{1}{R_3} \overset{+}{-} \frac{1}{R_4}\right)}_{} = -x_1 - x_2 \overset{+}{-} \frac{V_{es}}{R_3}$$

$$V_{R4} = (R_3 - R_4)\left(-x_1 - x_2 - \frac{V_{es}}{R_3}\right) \quad (5)$$

Esto no es posible

From (3) and (5)

$$V_{R4} + V_{R5} - V_{C2} + V_{R2} + V_{L2} = 0$$

$$(R_3 - R_4)\left(-x_1 - x_2 - \frac{V_{es}}{R_3}\right) + C_2 \dot{x}_4 R_5 - x_4 + x_2 R_2 + L_2 \dot{x}_2 = 0 \quad (6)$$

From (2)

$$i_{C2} = i_{L1} + i_{L2}$$

$$C_2 \dot{x}_4 = x_1 + x_2 \longrightarrow \dot{x}_4 = \frac{x_1}{C_2} + \frac{x_2}{C_2} \quad (7)$$

Using ⑥ and ⑦ to find $\dot{x}_2$

$$(R_3-R_4)\left(-x_1-x_2-\frac{V_{cs}}{R_3}\right)+R_5\left(\frac{x_1}{\ldots}+\frac{x_2}{\ldots}\right)\cdot R_5-x_4+x_2R_2+L_2\dot{x}_2=0$$

$$L_2\dot{x}_2=x_4-x_1R_5-x_2R_5-(R_3-R_4)\left(-x_1-x_2-\frac{V_{cs}}{R_3}\right)-x_2R_2$$

$$L_2\dot{x}_2=x_4-x_1R_5-x_2R_5+(R_3-R_4)\left(x_1+x_2+\frac{V_{cs}}{R_3}\right)-x_2R_2$$

⑧

$$\boxed{\dot{x}_2=\frac{x_4}{L_2}+\frac{x_1(R_3-R_4-R_5)}{L_2}+\frac{x_2(R_3-R_4-R_5-R_2)}{L_2}+\frac{V_{cs}}{L_2R_3}(R_3-R_4)}$$

We know that circuits in series the current of all elements is the same.

⑨

$$i_{C1}=i_{L1}\longrightarrow C_1\dot{x}_3=x_1\longrightarrow \boxed{\dot{x}_3=\frac{x_1}{C_1}}$$

From equation ①

① $V_{L1}+V_{R1}+V_{C1}=V_{L2}+V_{R2}$

⑩

$$L_1\dot{x}_1+x_1\cdot R_1+x_3=L_2\dot{x}_2+x_2\cdot R_2$$

Using ⑩ and ⑧ to find $\dot{x}_1$

$$L_1\dot{x}_1+x_1R_1+x_3=x_4+x_1(R_3-R_4-R_5)+x_2(R_3-R_4-R_5-R_2)+\frac{V_{cs}}{R_3}(R_3-R_4)\ldots$$
$$+x_2\cdot R_2$$

⑪

$$\boxed{\dot{x}_1=\frac{x_4}{L_1}-\frac{x_3}{L_1}+\frac{x_1(R_3-R_4-R_5-R_1)}{L_1}+\frac{x_2(R_3-R_4-R_5)}{L_1}+\frac{V_{cs}}{L_1}\left(1-\frac{R_4}{R_3}\right)}$$

We have $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4$ from (11), (8), (9) and (4)

¡Revisar!

$$\begin{bmatrix} V_{L1} \\ V_{L2} \\ \dot{i}_{C1} \\ \dot{i}_{C2} \end{bmatrix} = \begin{bmatrix} \frac{\cancel{R_3-R_4-R_5-R_1}}{L_1} & \frac{\cancel{R_3-R_4-R_5}}{L_1} & -\frac{1}{L_1} & \frac{1}{L_1} \\ \frac{\cancel{R_3-R_4-R_5}}{L_2} & \frac{\cancel{R_3-R_4-R_5-R_2}}{L_2} & 0 & \frac{1}{L_2} \\ \frac{1}{C_1} & 0 & 0 & 0 \\ \frac{1}{C_2} & \frac{1}{C_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{i}_{L1} \\ \dot{i}_{L2} \\ V_{C1} \\ V_{C2} \end{bmatrix}$$

$$+ \begin{bmatrix} \overbrace{\frac{1}{L_1} - \frac{\cancel{R_4}}{L_1 R_3}} \\ \frac{1}{L_2} - \frac{\cancel{R_4}}{\cancel{L_2 R_3}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} V_{es} \end{bmatrix}$$

$$\frac{1}{L_1}\left(\frac{R_4}{R_4+R_3}\right)$$

$$L_2\left(\frac{R_4}{R_4+R_3}\right)$$

From (5) $V_{R4} = (R_3 - R_4)\left(-x_1 - x_2 - \frac{V_{es}}{R_3}\right)$

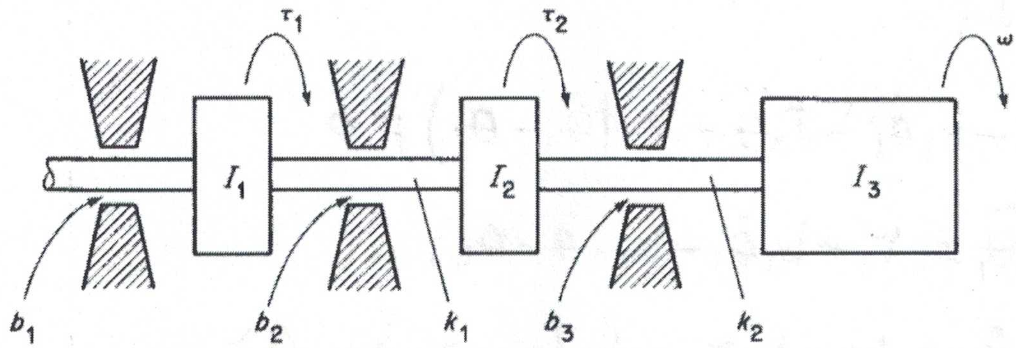
$$y = \begin{bmatrix} \cancel{(-R_3+R_4)} & \cancel{(R_3-R_4)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{i}_{L1} \\ \dot{i}_{L2} \\ V_{C1} \\ V_{C2} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 + \frac{R_4}{R_3} \end{bmatrix}}_{\frac{R_4}{R_3+R_4}} \begin{bmatrix} V_{es} \end{bmatrix}$$

$$-\frac{R_3 R_4}{R_3+R_4}$$

25%

Problem 4

(25 %) A two-stage turbine can be modelled, as shown in Fig. 4, by two inertias I_1 abd I_2 representing the rotational mass of stage. The stages are coupled by a shaft of stiffness k_1, and the effective load inertial I_3 is coupled to the turbine by a shaft of stiffness k_2. The bearing which support the turbine shaft have effective linear frictional coefficients b_1, b_2, b_3 as shown. Assume that the stiffness of the interstage shaft and the turbine load shaft can be split equally either side of the bearing. Write the system dynamical-equation description between the torque inputs stage τ_1, τ_2; and the angula velocity of the load ω.



* Inertias I_1, I_2 and I_3 are the loads.
* Shaft stiffness can be modeled as a spring.
* Friction coefficients b_1, b_2, b_3 represents a damping.
* Torques τ_1 and τ are the inputs.
* Netons Second Law

$$\tau = I\alpha \longrightarrow \ddot{\theta} \text{ angular acceleration}$$

State variables

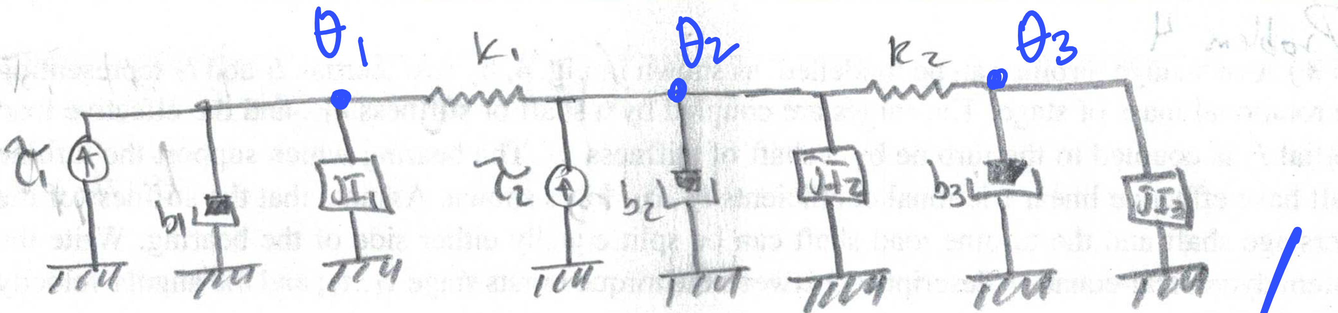
$x_1 = \theta_1$ angular position of I_1

$x_2 = \dot{\theta}_1$ angular velocity of I_1

$x_3 = \theta_2$

$x_4 = \dot{\theta}_2$

$x_5 = \theta_3$; $x_6 = \dot{\theta}_3$



For stage I_1

$$\tau_1 - b_1\dot{\theta}_1 - \underbrace{I_1\ddot{\theta}_1} - k_1(\theta_1 - \theta_2) = 0$$

$$I_1\ddot{\theta}_1 = \tau_1 + b_1\dot{\theta}_1 - k_1(\theta_1 - \theta_2)$$

$$\dot{x}_2 = \frac{\tau_1}{I_1} - \frac{b_1 x_2}{I_1} - \frac{k_1(x_1 - x_3)}{I_1} \quad (1)$$

For stage I_2

$$\tau_2 - b_2\dot{\theta}_2 - \underbrace{I_2\ddot{\theta}_2} - k_1(\theta_2 - \theta_1) - k_2(\theta_2 - \theta_3) = 0$$

$$I_2\ddot{\theta}_2 = \tau_2 - b_2\dot{\theta}_2 - k_1(\theta_2 - \theta_1) - k_2(\theta_2 - \theta_3)$$

$$\dot{x}_4 = \frac{\tau_2}{I_2} - \frac{b_2 x_4}{I_2} - \frac{k_1(x_3 - x_1)}{I_2} - \frac{k_2(x_3 - x_5)}{I_2} \quad (2)$$

Stage I_3

$$I_3\ddot{\theta}_3 + b_3\dot{\theta}_3 + k_2(\theta_3 - \theta_2) = 0$$

$$I_3\ddot{\theta}_3 = -b_3\dot{\theta}_3 - k_2(\theta_3 - \theta_2)$$

$$\dot{x}_6 = -\frac{b_3 x_6}{I_3} - \frac{k_2(x_5 - x_3)}{I_3} \quad (3)$$

$\dot{x} = Ax + Bu$; $\dot{x}_1 = x_2$; $\dot{x}_3 = x_4$; $\dot{x}_5 = x_6$

$$
\begin{bmatrix} \omega_1 \\ \alpha_1 \\ \omega_2 \\ \alpha_2 \\ \omega_3 \\ \alpha_3 \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{-k_1}{I_1} & \frac{-b_1}{I_1} & \frac{k_1}{I_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{k_1}{I_2} & 0 & -\frac{k_1+k_2}{I_2} & \frac{-b_2}{I_2} & \frac{k_2}{I_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{k_2}{I_3} & 0 & -\frac{k_2}{I_3} & \frac{-b_3}{I_3}
\end{bmatrix}
\begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \\ \theta_3 \\ \omega_3 \end{bmatrix}
$$

$$
+ \begin{bmatrix}
0 & 0 \\
\frac{1}{I_1} & 0 \\
0 & 0 \\
0 & \frac{1}{I_2} \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}
$$