1 Chapter 3: Linear Analysis of a Single Machine Infinite Bus System Forced by a Periodic Excitation.

1.1 Chapter Summary

This chapter develops the analytical formulation of a forced Single Machine Infinite Bus (SMIB) system. Its forcing input is a periodic mechanical excitation represented as a Fourier series operator, revealing how the system responds to multiple periodic frequencies simultaneously. The study begins by deriving the forced non-linear swing equation that governs the SMIB model. Then, a small signal linearization is conducted to obtain closed-form solutions through the Laplace transform showing the pole structure and its interaction with the forcing. This approach enables the analytical characterization of the instantaneous envelope modulation and resonance effects due to forced harmonics, revealing how the input waveform and system parameters affect the forced system response.

1.2 Introduction

In most of the literature, forced oscillations in power systems are studied from a linear point of view and with a single frequency forcing. That setting is useful because it let us see very clearly the frequency response of the electromechanical modes and the resonance mechanism. However, it is also a very restricted picture. Field records and several of the cases reported in recent years show that the excitation is not always a clean sinusoid: it can be a periodic but non-sinusoidal waveform coming from mechanical ripple, control limit cycles or converter modulation [1] [2] [3]. If we keep assuming a single cosine, we are forcing the model to answer a question that is simpler than the real phenomenon.

A natural way to tackle that limitation is to build the input as a Fourier series. Any periodic perturbation that appears in a generator, whether it is closer to a square wave, has a DC offset or contains several harmonics, it can be decomposed into a constant term and a sum of sinusoidal components. This signal decomposition is quite practical in the small-signal (linearized) SMIB model, where each forced harmonic component can be processed independently, this makes the system to behave as a frequency-selective filter with amplification near its natural mode. By the other hand, for large disturbances when the small-signal model is no longer valid, the non-linear

SMIB shall be used.

To handle the multi-harmonic forcing cleanly, it is more convenient to use the Laplace transform than methods such as undetermined coefficients. This approach offers several advantages: it naturally separates the zero-input and zero-state responses, making it possible to distinguish the effects of the initial conditions from those produced by the external forcing. Furthermore, the Laplace framework reveals the complete pole structure of the system, showing explicitly how each harmonic introduces a pair of complex conjugate poles in addition to the system's natural poles. This makes it easier to study modal interactions and evaluate how different frequency components excite the system's mode. By contrast, the method of undetermined coefficients, while simpler for purely periodic forcing, does not clearly separate the transient contributions: when non-zero initial conditions are applied, the homogeneous solution becomes "contaminated" by terms dependent on the particular solution to fit the transient. It yields the same response but does not clearly separate the forced transient contribution from the natural part.

1.3 The Forced Single Machine Infinite Bus (FSMIB) Model

A single machine connected to an infinite bus through a transmission line (SMIB) is one of the most fundamental and widely studied models in power systems. Despite its simplicity, it captures the essential electromechanical interaction between a synchronous generator and a large interconnected grid capable of absorbing or delivering power without modifying its operating conditions. In this case, the synchronous generator is represented by its classical model (swing equation) and assumes that all the electromagnetic power transmitted through the air-gap is converted to electric power. This electric power is coupled to the grid through the transmission line, here is where the non-linearity appears. To analyze how a multi-frequency input shapes the system response, equation (1) is proposed. It is a non-linear FSMIB system which consist of a SMIB model with a transmission line modeled as an admittance and a forcing $\Delta P_m f(t)$, where ΔP_m is the disturbance amplitude and $f(t)$ is a periodic waveform of period T represented by its

Fourier series.

$$\underbrace{\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2}+D\frac{d\delta}{dt}-|E'||V||Y|\cos(\delta-\theta-\varphi)=-(E')^2G+P_m}_{\text{Classic SMIB model with admittance}} \quad + \quad \dots$$
$$\dots \quad + \quad \Delta P_m \underbrace{\left[\frac{a_0}{2}+\sum_{k=1}^{N}\left[a_k\cos(k\omega t)+b_k\sin(k\omega t)\right]\right]}_{f(t)\text{ - Fourier series}} \tag{1}$$

The state variable $\delta(t)$ in (1) denotes the rotor power angle of the synchronous generator measured with respect to the infinite bus, the inertia constant H is expressed in seconds and ω_s is the synchronous electrical angular frequency. The coefficient D represents the damping torque, accounting for mechanical losses and load–frequency effects.

The electrical coupling between the generator and the infinite bus is represented through an equivalent network admittance $Y = G + jB$, where G and B denote the conductance and susceptance of the transmission line, respectively. The magnitude of the admittance is given by $|Y|$ and its phase angle is denoted by $\varphi = \angle Y$. The internal transient voltage magnitude of the synchronous machine is $|E'|$, while $|V|$ is the constant voltage magnitude of the infinite bus. The angle θ represents the voltage phase angle of the infinite bus reference. Together, these quantities define the non-linear electrical air-gap power exchange through the cosine term $\cos(\delta-\theta-\varphi)$, which explicitly accounts for both reactive and resistive components of the transmission network.

The constant term $(E')^2G$ represents the steady electrical power dissipation associated with the resistive component of the network, while P_m denotes the steady-state mechanical input power at the operating point. Superimposed on this equilibrium, there is a general periodic time-varying mechanical disturbance $\Delta P_m f(t)$, its coefficients a_0, a_k and b_k represent the scaling of the DC component, the cosine and sine harmonics of the disturbance, respectively, these can be calculated using equation (2). The fundamental forcing angular frequency is represented as ω, k indexes the harmonic order, N is the number of harmonics considered in the series and $T = 2\pi/\omega$ is the input's period. This representation allows the mechanical torque disturbance to capture a wide class of periodic excitations.

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt, \qquad a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt, \qquad b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt \tag{2}$$

Overall, (1) constitutes a non-linear second-order differential equation that combines inertial, damping, electrical coupling, network losses and externally imposed multi-frequency forcing. This formulation provides a framework to analyze how different mechanical disturbances interact with the intrinsic electromechanical dynamics of the system. Next section focuses on the linearization of the FSMIB model around an equilibrium point to analyze modal interactions between the system eigenstructure and the forcing harmonics using the Laplace transform theory.

1.4 Linearization of the Forced Single Machine Infinite Bus (FSMIB) Model

To analyze how the system behaves around a steady operating point $[\delta_0, \omega_s]$ we shall introduce a small angle perturbation $\delta = \Delta\delta + \delta_0$ in equation (1).

$$\begin{aligned} \frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} - |E'||V||Y|\cos\big(\Delta\delta + \delta_0 - \theta - \varphi\big) = -(E')^2 G + P_m \quad + \quad \dots \\ \dots \quad + \quad \Delta P_m\left[\frac{a_0}{2} + \sum_{k=1}^{N}\Big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\Big)\right] \end{aligned} \tag{3}$$

Now, to linearize equation (3) a Taylor series expansion of the non-linear term must be conducted. The Taylor series expansion of $\cos\big(\Delta\delta + \delta_0 - \theta - \varphi\big)$ around $\Delta\delta = 0$ is:

$$\begin{aligned} \underbrace{\cos\big(\Delta\delta + \delta_0 - \theta - \varphi\big)}_{f_{NL}(\Delta\delta)} \approx \underbrace{\cos\big(\delta_0 - \theta - \varphi\big)}_{f_{NL}(0)} + \underbrace{\Big[-\Delta\delta\,\sin\big(\delta_0 - \theta - \varphi\big)\Big]}_{\Delta\delta\, f'_{NL}(0)} \quad + \quad \dots \\ \dots \quad + \quad \underbrace{\left[-\frac{(\Delta\delta)^2}{2}\cos\big(\delta_0 - \theta - \varphi\big)\right]}_{(\Delta\delta)^2 f''_{NL}(0)/2!} + \underbrace{\left[\frac{(\Delta\delta)^3}{6}\sin\big(\delta_0 - \theta - \varphi\big)\right]}_{(\Delta\delta)^3 f^{(3)}_{NL}(0)/3!} \end{aligned} \tag{4}$$

Taking only the constant $f_{NL}(0)$ and the linear term $\Delta\delta\, f'_{NL}(0)$ of equation (4) and substituting them into equation (3), it yields:

$$\begin{aligned} \frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} - |E'||V||Y|\Big[\cos(\delta_0 - \theta - \varphi) - \Delta\delta\,\sin(\delta_0 - \theta - \varphi)\Big] = \dots \\ \dots = -(E')^2 G + P_m + \Delta P_m\left[\frac{a_0}{2} + \sum_{k=1}^{N}\Big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\Big)\right] \end{aligned} \tag{5}$$

Rearranging the constant terms, the terms with $\Delta\delta$ and considering that $-\sin(x) = \sin(-x)$, equation (5) can be rewritten as follows:

$$
\begin{aligned}
\frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} + \underbrace{|E'|\,|V|\,|Y|}_{K}\sin(\delta_0-\theta-\varphi)\,\Delta\delta \quad &= \quad \dots \\
= -(E')^2G + P_m + |E'|\,|V|\,|Y|\cos(\underbrace{\delta_0-\theta-\varphi}_{\alpha}) + \Delta P_m\left[\frac{a_0}{2} + \sum_{k=1}^{N}\Big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\Big)\right]
\end{aligned} \tag{6}
$$

Lets define the differentiation operator with an upper dot ($d\Delta\delta/dt = \dot{\Delta\delta}$) and consider $\Delta P_m = 0$. At steady state, the angle deviation is zero just like all its derivatives: $\Delta\delta = 0$, $\dot{\Delta\delta} = 0$ and $\ddot{\Delta\delta} = 0$, this let us obtain the operating-point balance equation shown in (7).

$$
K\cos(\alpha) = (E')^2G - P_m \tag{7}
$$

Replacing equation (7) into (6) removes the constant bias that arises from the linearization process around an operating point, leaving the pure linear time-varying forced perturbation model as presented in equation (9).

$$
\frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} + K\sin(\alpha)\,\Delta\delta = \Delta P_m\left[\frac{a_0}{2} + \sum_{k=1}^{N}\Big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\Big)\right] \tag{8}
$$

Multiplying equation (9) by $\omega_s/2H$ and defining the damping ratio ζ, the natural frequency ω_n and the scaled forcing amplitude γ, it yields:

$$
\begin{gathered}
\ddot{\Delta\delta} + 2\zeta\,\omega_n\,\dot{\Delta\delta} + \omega_n^2\,\Delta\delta = \underbrace{\gamma\left[\frac{a_0}{2} + \sum_{k=1}^{N}\Big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\Big)\right]}_{F_S(t)} \\
2\zeta\,\omega_n = \frac{D\,\omega_s}{2H} \qquad \omega_n^2 = \frac{K\sin(\alpha)\,\omega_s}{2H} \qquad \gamma = \Delta P_m\frac{\omega_s}{2H}
\end{gathered} \tag{9}
$$

Equation (9) is the canonical forced, damped, second-order model of the power angle deviation $\Delta\delta(t)$ around a steady operating condition. The term $\ddot{\Delta\delta}$ captures the inertial resistance of the rotor to changes in motion, $2\zeta\,\omega_n\,\dot{\Delta\delta}$ represents the net dissipative mechanisms that remove oscillation energy and $\omega_n^2\Delta\delta$ is the restoring (synchronizing) action that tends to bring the power angle back toward equilibrium.

A key quantity embedded in the restoring term is the synchronizing coefficient, defined as the small-signal slope of the electrical air-gap power with respect to the power angle at the operating point:

$$K_s = \left. \frac{dP_e}{d\delta} \right|_{\delta_0} \tag{10}$$

In (9), this slope appears through the product $K\sin(\alpha)$, so $K\sin(\alpha)$ plays the role of the effective synchronizing stiffness: when it is large, the restoring torque is strong and the system pulls back more aggressively after a displacement; when it is small, the restoring torque is weak and the dynamics become slow. In particular, as the operating condition approaches the power angle limit ($\alpha \to \pi/2$), the slope of the power transfer characteristic collapses, the synchronizing coefficient decreases and the restoring term weakens. This is the small-signal meaning of "loss of stiffness" near the stability boundary: the system becomes easier to displace and slower to return.

The dependence of ω_n and ζ on operating condition follows directly from how the synchronizing coefficient enters the model. Increasing electrical coupling increases ω_n, producing faster natural oscillations and a stronger restoring action. However, the relative damping ζ is tied to the ratio between dissipation and stiffness: if the synchronizing coefficient increases while the physical damping mechanisms represented by D do not increase proportionally, the system can become stiffer while becoming less damped, meaning oscillations may take more cycles to decay once excited. By the other hand, increasing H slows the natural oscillations and reduces the acceleration produced by a given mechanical disturbance through the gain γ, so higher inertia reduces sensitivity to the same disturbance amplitude.

Because the dynamics are linear with constant coefficients, each harmonic in the right hand-side excites the same second-order dynamics and the total response is the superposition of the individual harmonic responses plus the contribution of the DC term.

1.5 Power Angle Solution of the Linear FSMIB Model

To analyze the forced response one might use time or frequency domain methods, in this case the Laplace transform is applied to study the system presented in (9), resulting in equation (11).

$$[s^2\Delta\delta(s) - s\Delta\delta_0 - \Delta\omega_0] + 2\zeta\omega_n[s\Delta\delta(s) - \Delta\delta_0] + \omega_n^2\Delta\delta(s) = F_S(s) \tag{11}$$

Then, solving for the power angle $\Delta\delta(s)$ we get:

$$\Delta\delta(s) = \underbrace{\frac{\Delta\delta_0(2\zeta\omega_n + s) + \Delta\omega_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{\text{Zero Input Response (ZIR)}} + \underbrace{\frac{F_S(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{\text{Zero State Response (ZSR)}} \tag{12}$$

Look how clearly the transformation separates the intrinsic system response (ZIR) from all the forced dynamics (ZSR) regarding the input $F_S(s)$. One interesting point here is that both denominators are the same, this means that the system poles (eigenvalues) are excited through initial conditions $[\Delta\delta_0, \Delta\omega_0]$ or merely by the forcing $F_S(s)$. To find the poles of each term in equation (12) the denominator must be factorized and solve for each root.

$$\Delta\delta(s) = \underbrace{\frac{\Delta\delta_0(2\zeta\omega_n + s) + \Delta\omega_0}{(s - r_1)(s - r_2)}}_{\text{Zero Input Response (ZIR)}} + \underbrace{\frac{F_S(s)}{(s - r_1)(s - r_2)}}_{\text{Zero State Response (ZSR)}} \qquad \begin{aligned} r_1 &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ r_2 &= -\zeta\omega_n - j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d} \end{aligned} \tag{13}$$

1.5.1 Zero Input Response (ZIR)

The numerator of the first term in equation (13) contains two physically meaningful insights about the zero input dynamics: the term $s\Delta\delta_0$ shows how the initial rotor angle creates an immediate dynamic forcing equivalent to imparting an initial modal velocity, while the constant terms $2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0$ represent the net effective initial "impulse" that excites the system's natural modes. Here, $\Delta\omega_0$ is the actual initial speed and $2\zeta\omega_n\Delta\delta_0$ arises because the damping force at $t = 0$ depends on the initial displacement. Together, this constant terms scale the impulse response of the system.

To get the ZIR time-domain solution, partial fraction expansion is utilized to express the ZIR as a sum of simple poles at the system roots r_1 and r_2, so it is easily invertible using a transform pair.

$$\Delta\delta_{\text{ZIR}}(s) = \frac{s\Delta\delta_0 + 2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0}{(s - r_1)(s - r_2)} = \frac{A}{s - r_1} + \frac{B}{s - r_2} = \frac{A(s - r_2) + B(s - r_1)}{(s - r_1)(s - r_2)}. \tag{14}$$

Since the denominators are equal, we can equate numerators by multiplying both sides by $(s-r_1)(s-r_2)$, from which A and B can be isolated.

$$\Delta\delta_{\text{ZIR}}(s) = s\Delta\delta_0 + 2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0 = A(s-r_2) + B(s-r_1). \tag{15}$$

The residue theorem let us solve for A:

$$A = \lim_{s\to r_1} \frac{1}{(s-r_2)}\left[s\Delta\delta_0 + 2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0 - B(s-r_1)\right] = \frac{r_1\Delta\delta_0 + 2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0}{r_1 - r_2} \tag{16}$$

Similarly, evaluating at $s = r_2$ gives us the value of B:

$$B = \lim_{s\to r_2} \frac{1}{(s-r_1)}\left[s\Delta\delta_0 + 2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0 - A(s-r_2)\right] = \frac{r_2\Delta\delta_0 + 2\zeta\omega_n\Delta\delta_0 + \Delta\omega_0}{r_2 - r_1} \tag{17}$$

Considering $r_1 - r_2 = 2j\omega_d$ and $r_2 - r_1 = -2j\omega_d$, the residues can be expressed explicitly in terms of the damped natural frequency ω_d as:

$$A = \frac{\Delta\delta_0}{2} - j\frac{\Delta\omega_0 + \zeta\omega_n\Delta\delta_0}{2\omega_n\sqrt{1-\zeta^2}}, \qquad B = \frac{\Delta\delta_0}{2} + j\frac{\Delta\omega_0 + \zeta\omega_n\Delta\delta_0}{2\omega_n\sqrt{1-\zeta^2}} \tag{18}$$

Observe that A and B are complex conjugates ($A = B^*$). This conjugacy ensures the time-domain response is real-valued, as the corresponding modes are also conjugates. The coefficients reveal how initial conditions shape the system's transient behavior: the real part depends solely on the initial angle deviation $\Delta\delta_0$, while the imaginary part combines both initial angular velocity $\Delta\omega_0$ and angle, scaled by the inverse of ω_d.

By taking the simple pole sum representation in equation (14) and applying the inverse Laplace transform to the corresponding r_1 and r_2 terms, we get the time domain solution in equation (19).

$$\Delta\delta_{\text{ZIR}}(t) = \mathscr{L}^{-1}\left(\frac{A}{s-r_1} + \frac{B}{s-r_2}\right) = Ae^{r_1 t} + Be^{r_2 t} = e^{-\zeta\omega_n t}(Ae^{j\omega_d t} + Be^{-j\omega_d t}) \tag{19}$$

The expression for the imaginary part in modal coefficients A and B shown in equation (18), indicates that higher damping ζ (reduced ω_d) amplifies the magnitude of the imaginary component. This amplification reflects a stronger initial excitation of the oscillatory mode in the complex plane. However, in the time domain, this effect is counterbalanced by the faster exponential decay $e^{-\zeta\omega_n t}$ as presented in equation (19), which suppresses the oscillation more rapidly. These coefficients directly determine the magnitude and phase of the decaying sinusoidal response at frequency ω_d, linking initial conditions to the system's natural oscillatory behavior.

1.5.2 Zero State Response (ZSR)

It is worth noting that the second term in equation (13) corresponding to the ZSR naturally recovers the traditional transfer function representation $\Delta\delta(s)/F_S(s)$.

$$G(s) = \frac{\Delta\delta(s)}{F_S(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s-r_1)(s-r_2)} \tag{20}$$

This highlights that the same electromechanical modes govern both the free dynamics and the forced response. Forced oscillations arise from the persistent excitation of these intrinsic modes through the external input $F_S(s)$.

To get the complete time-domain ZSR which includes the transient and steady-state forcing contribution it is necessary to expand the second term in (13) and express the whole term in a sum of simple poles. The main challenge here is to obtain the modal contribution (residues) of every pole involved. In this case, the forcing has sinusoids, the Laplace representation of every sinusoid consist of a complex conjugate pole and the constant term (DC) is basically a step function multiplied by that constant, knowing this we expect 11 poles where 5 are complex conjugates, this results in 10 simple poles due to conjugacy. These poles are: 4 poles due to the cosine term and its interaction with the system, 4 more poles of the sine term and lastly 3 poles (2 of them are conjugates) appear due to the interaction of the constant term with the system. All of this can be easily seen in equation (21).

$$\Delta\delta_{ZS}(s) = \frac{F_S(s)}{(s-r_1)(s-r_2)} = \frac{\gamma}{(s-r_1)(s-r_2)}\left[\underbrace{\frac{a_0}{2}\left(\frac{1}{s}\right)}_{\text{constant}} + \sum_{k=1}^{N}\left(\underbrace{a_k\frac{s}{s^2+(k\omega)^2}}_{\text{cosine}} + \underbrace{b_k\frac{k\omega}{s^2+(k\omega)^2}}_{\text{sine}}\right)\right] \tag{21}$$

Multiplying γ and the natural poles $(s-r_1)(s-r_2)$ let us clearly separate each component contribution,thus, the ZSR can be expressed as a sum of solutions as expressed in equation (22).

$$\Delta\delta_{ZS}(s) = \Delta\delta_{ZS}^{DC}(s) + \Delta\delta_{ZS}^{(cos)}(s) + \Delta\delta_{ZS}^{(sin)}(s) \tag{22}$$

This is a handy representation because each term can be solved separately. Lets start with $\Delta\delta_{ZS}^{DC}(s)$ which is the DC contribution to the ZSR.

$$\Delta\delta_{ZS}^{DC}(s) = \frac{\gamma a_0}{2}\frac{1}{s(s-r_1)(s-r_2)}. \tag{23}$$

Expanding through partial fractions yields:

$$\Delta\delta_{ZS}^{DC}(s) = \frac{\gamma a_0}{2}\left[\frac{A_0}{s} + \frac{B_0}{s-r_1} + \frac{C_0}{s-r_2}\right] = \frac{\gamma a_0}{2}\frac{A_0(s-r_1)(s-r_2) + B_0\,s(s-r_2) + C_0\,s(s-r_1)}{s(s-r_1)(s-r_2)} \tag{24}$$

Since the denominators of (23) and (24) are equal, we can equate numerators by multiplying both sides by $s(s-r_1)(s-r_2)$, obtaining:

$$1 = A_0(s-r_1)(s-r_2) + B_0\,s(s-r_2) + C_0\,s(s-r_1) \tag{25}$$

The coefficients are residues at the simple poles $s = 0, r_1, r_2$. Evaluating each pole, rationalizing the denominator and simplifying gives:

$$A_0 = \lim_{s\to 0}\frac{1}{(s-r_1)(s-r_2)} = \frac{1}{r_1 r_2} = \frac{1}{\omega_n^2} \tag{26}$$

$$B_0 = \lim_{s\to r_1}\frac{1}{s(s-r_2)} = \frac{1}{r_1(r_1-r_2)} = \frac{1}{r_1(2j\omega_d)} = \frac{1}{2\omega_n^2}\left[-1 + j\frac{\zeta}{\sqrt{1-\zeta^2}}\right] \tag{27}$$

$$C_0 = \lim_{s\to r_2}\frac{1}{s(s-r_1)} = \frac{1}{r_2(r_2-r_1)} = \frac{1}{r_2(-2j\omega_d)} = \frac{1}{2\omega_n^2}\left[-1 - j\frac{\zeta}{\sqrt{1-\zeta^2}}\right] \tag{28}$$

Finally, applying the inverse Laplace transform term by term gives the DC zero-state response in the time domain as shown in equation (29).

$$\Delta\delta_{ZS}^{DC}(t) = \frac{\gamma a_0}{2}\left[A_0 + B_0 e^{r_1 t} + C_0 e^{r_2 t}\right] \tag{29}$$

The next term to be analyzed corresponds to the cosine contribution in (22). For each harmonic index k, its Laplace-domain contribution to the ZSR is:

$$\Delta\delta_{ZS}^{(\cos)}(s) = \gamma \sum_{k=1}^{N} a_k \, \frac{s}{\left[s^2+(k\omega)^2\right](s-r_1)(s-r_2)} \tag{30}$$

Factoring $s^2+(k\omega)^2 = (s+jk\omega)(s-jk\omega)$, each term admits a simple-pole expansion:

$$\frac{s}{(s+jk\omega)(s-jk\omega)(s-r_1)(s-r_2)} = \frac{A_{k,r_1}}{s-r_1} + \frac{A_{k,r_2}}{s-r_2} + \frac{B_{k,1}}{s+jk\omega} + \frac{B_{k,2}}{s-jk\omega} \tag{31}$$

Therefore, the cosine contribution can be written as a simple pole expansion with 4 simple poles and 4 unknown modal amplitudes:

$$\Delta\delta_{ZS}^{(\cos)}(s) = \gamma \left\{ \sum_{k=1}^{N} a_k \left(\frac{A_{k,r_1}}{s-r_1} + \frac{A_{k,r_2}}{s-r_2} + \frac{B_{k,1}}{s+jk\omega} + \frac{B_{k,2}}{s-jk\omega} \right) \right\} \tag{32}$$

Then, multiplying (31) by $(s+jk\omega)(s-jk\omega)(s-r_1)(s-r_2)$ and equating the numerators, yields:

$$\begin{aligned} s = A_{k,r_1}(s+jk\omega)(s-jk\omega)(s-r_2) + A_{k,r_2}(s+jk\omega)(s-jk\omega)(s-r_1) \quad + \dots \\ \dots + B_{k,1}(s-jk\omega)(s-r_1)(s-r_2) + B_{k,2}(s+jk\omega)(s-r_1)(s-r_2) \end{aligned} \tag{33}$$

The coefficients are residues at the simple poles $s = r_1, r_2, \pm jk\omega$. Evaluating each pole gives:

$$A_{k,r_1} = \lim_{s \longrightarrow r_1} \frac{s}{\left(s^2+(k\omega)^2\right)(s-r_2)} = \frac{r_1}{\left(r_1^2+(k\omega)^2\right)(r_1-r_2)} = \frac{1}{2j\omega_d} \frac{r_1}{r_1^2+(k\omega)^2} \tag{34}$$

$$A_{k,r_2} = \lim_{s \longrightarrow r_2} \frac{s}{\left(s^2+(k\omega)^2\right)(s-r_1)} = \frac{r_2}{\left(r_2^2+(k\omega)^2\right)(r_2-r_1)} = -\frac{1}{2j\omega_d} \frac{r_2}{r_2^2+(k\omega)^2} \tag{35}$$

$$B_{k,1} = \lim_{s \longrightarrow -jk\omega} \frac{s}{(s-jk\omega)(s-r_1)(s-r_2)} = \frac{1}{2\left(-jk\omega-r_1\right)\left(-jk\omega-r_2\right)} \tag{36}$$

$$B_{k,2} = \lim_{s \longrightarrow jk\omega} \frac{s}{(s+jk\omega)(s-r_1)(s-r_2)} = \frac{1}{2\left(jk\omega-r_1\right)\left(jk\omega-r_2\right)} \tag{37}$$

To simplify the input-frequency modal weights we introduce the normalized forcing frequency Ω_k in equation (38).

$$\Omega_k = \frac{k\omega}{\omega_n} \tag{38}$$

Using $r_{1,2} = -\zeta\omega_n \pm j\omega_d$ with $\omega_d = \omega_n\sqrt{1-\zeta^2}$, the residue in (34) becomes:

$$A_{k,r_1} = \frac{1}{2j\omega_d}\frac{r_1}{r_1^2+(k\omega)^2} = \frac{1}{2j\omega_d}\frac{-\zeta\omega_n+j\omega_d}{\big(-\zeta\omega_n+j\omega_d\big)^2+(k\omega)^2} \tag{39}$$

Expanding the square in (39) and using $\omega_d^2 = \omega_n^2(1-\zeta^2)$ gives:

$$\begin{aligned} r_1^2+(k\omega)^2 &= \big(-\zeta\omega_n+j\omega_d\big)^2+(k\omega)^2 = \Big(\zeta^2\omega_n^2-\omega_d^2+(k\omega)^2\Big)-j2\zeta\omega_n\omega_d \\ &= \omega_n^2\Big[(\Omega_k^2+2\zeta^2-1)-j2\zeta\sqrt{1-\zeta^2}\Big] \end{aligned} \tag{40}$$

Substituting (40) into (39) yields the compact normalized form:

$$A_{k,r_1} = \frac{1}{2j\omega_n^2\sqrt{1-\zeta^2}}\frac{-\zeta+j\sqrt{1-\zeta^2}}{(\Omega_k^2+2\zeta^2-1)-j2\zeta\sqrt{1-\zeta^2}} \tag{41}$$

After rationalization, the denominator becomes real, which transforms the expression into the standard complex rectangular form, making the amplitude and phase shift accessible for analysis.

$$A_{k,r_1} = \frac{1}{2\omega_n^2}\left[\frac{(\Omega_k^2-1)+j\dfrac{\zeta}{\sqrt{1-\zeta^2}}(\Omega_k^2+1)}{(1-\Omega_k^2)^2+(2\zeta\Omega_k)^2}\right] \tag{42}$$

Similarly, for A_{k,r_2} in (35), substitute $r_2 = -\zeta\omega_n - j\omega_d$ and follow the same expansion, normalization ($\Omega_k = k\omega/\omega_n$) and rationalization steps. Since $r_2 = r_1^*$ and all coefficients are real, the result satisfies $A_{k,r_2} = A_{k,r_1}^*$.

$$A_{k,r_2} = \frac{1}{2\omega_n^2}\left[\frac{(\Omega_k^2-1)-j\dfrac{\zeta}{\sqrt{1-\zeta^2}}(\Omega_k^2+1)}{(1-\Omega_k^2)^2+(2\zeta\Omega_k)^2}\right] \tag{43}$$

Likewise, the coefficients $B_{k,1}$ and $B_{k,2}$ at the forcing frequency $k\omega$ (particular solution) determine the steady-state amplitude and phase after the transient decays. The harmonic coefficients $B_{k,1}$,

$B_{k,2}$ in (36) and (37) are calculated as follows:

$$\begin{aligned} B_{k,1} &= \frac{1}{2\big(-jk\omega - r_1\big)\big(-jk\omega - r_2\big)} = \frac{1}{2\big(-j\Omega_k\omega_n - r_1\big)\big(-j\Omega_k\omega_n - r_2\big)} \\ &= \frac{1}{2\Big(\zeta\omega_n - j(\Omega_k\omega_n + \omega_d)\Big)\Big(\zeta\omega_n - j(\Omega_k\omega_n - \omega_d)\Big)} \end{aligned} \tag{44}$$

Expanding the product in (44) and using $\omega_d^2 = \omega_n^2(1-\zeta^2)$ yields

$$\big(-jk\omega - r_1\big)\big(-jk\omega - r_2\big) = \omega_n^2\Big[(1-\Omega_k^2) - j\,2\zeta\Omega_k\Big] \tag{45}$$

Substituting (45) into (44) gives the compact normalized form:

$$B_{k,1} = \frac{1}{2\omega_n^2}\left[\frac{1}{(1-\Omega_k^2) - j\,2\zeta\Omega_k}\right] \tag{46}$$

Rationalizing the complex denominator:

$$B_{k,1} = \frac{1}{2\omega_n^2}\left[\frac{(1-\Omega_k^2) + j\,2\zeta\Omega_k}{(1-\Omega_k^2)^2 + (2\zeta\Omega_k)^2}\right] \tag{47}$$

Similarly, for the pole at $s = jk\omega$ in (37), substitution of r_1, r_2 and rationalization of the denominator provides the equation for $B_{k,2}$. Note that $B_{k,1}^* = B_{k,2}$ because $\pm jk\omega$, sinusoids in the frequency domain comes in complex conjugates.

$$B_{k,2} = \frac{1}{2\omega_n^2}\left[\frac{(1-\Omega_k^2) - j\,2\zeta\Omega_k}{(1-\Omega_k^2)^2 + (2\zeta\Omega_k)^2}\right] \tag{48}$$

Finally, applying the inverse Laplace transform term by term yields in equation (49) which is the time-domain contribution of the forcing cosine to the zero-state response.

$$\Delta\delta_{ZS}^{(\cos)}(t) = \gamma\sum_{k=1}^{N} a_k\left[A_{k,r_1}e^{r_1 t} + A_{k,r_2}e^{r_2 t} + B_{k,1}e^{-jk\omega t} + B_{k,2}e^{jk\omega t}\right] \tag{49}$$

The procedure to get the last term $\Delta\delta_{ZS}^{(\sin)}(t)$ is similar to the previously conducted for the cosine contribution. First, the simple pole expansion regarding $\Delta\delta_{ZS}^{(\sin)}(s)$ is stated in equation (50).

$$\Delta\delta_{ZS}^{(\sin)}(s) = \gamma\left\{\sum_{k=1}^{N}\left[b_k\left(\frac{C_{k,r_1}}{s-r_1}+\frac{C_{k,r_2}}{s-r_2}+\frac{D_{k,1}}{s+jk\omega}+\frac{D_{k,2}}{s-jk\omega}\right)\right]\right\} \tag{50}$$

Then, the unknown residues are determined by evaluating at each simple pole:

$$C_{k,r_1} = \lim_{s\to r_1}\frac{k\omega}{\left(s^2+(k\omega)^2\right)(s-r_2)} = \frac{k\omega}{\left(r_1^2+(k\omega)^2\right)(r_1-r_2)} = \frac{1}{2j\omega_d}\frac{k\omega}{r_1^2+(k\omega)^2} \tag{51}$$

$$C_{k,r_2} = \lim_{s\to r_2}\frac{k\omega}{\left(s^2+(k\omega)^2\right)(s-r_1)} = \frac{k\omega}{\left(r_2^2+(k\omega)^2\right)(r_2-r_1)} = -\frac{1}{2j\omega_d}\frac{k\omega}{r_2^2+(k\omega)^2} \tag{52}$$

For the forcing poles at $\pm jk\omega$, the residues are:

$$D_{k,1} = \lim_{s\to -jk\omega}\frac{k\omega}{(s-jk\omega)(s-r_1)(s-r_2)} = \frac{k\omega}{(-2jk\omega)(-jk\omega-r_1)(-jk\omega-r_2)} \tag{53}$$

$$D_{k,2} = \lim_{s\to jk\omega}\frac{k\omega}{(s+jk\omega)(s-r_1)(s-r_2)} = \frac{k\omega}{(2jk\omega)(jk\omega-r_1)(jk\omega-r_2)} \tag{54}$$

To simplify these expressions, we use the normalized frequency Ω_k and substitute the system poles $r_{1,2} = -\zeta\omega_n \pm j\omega_d$, where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Beginning with C_{k,r_1}:

$$C_{k,r_1} = \frac{1}{2j\omega_d}\frac{k\omega}{r_1^2+(k\omega)^2} = \frac{1}{2j\omega_n^2\sqrt{1-\zeta^2}}\frac{\Omega_k}{(\Omega_k^2+2\zeta^2-1)-j2\zeta\sqrt{1-\zeta^2}} \tag{55}$$

After rationalization, C_{k,r_1} becomes:

$$C_{k,r_1} = \frac{\Omega_k}{2\omega_n^2}\left[\frac{2\zeta - j\dfrac{\Omega_k^2+2\zeta^2-1}{\sqrt{1-\zeta^2}}}{(1-\Omega_k^2)^2+(2\zeta\Omega_k)^2}\right] \tag{56}$$

For the pole at $s = r_2$, its coefficient $C_{k,r2}$ is just the conjugate of $C_{k,r1}$ since $r_2 = r_1^*$.

$$C_{k,r_1}^* = C_{k,r_2} = \frac{\Omega_k}{2\omega_n^2}\left[\frac{2\zeta + j\dfrac{\Omega_k^2+2\zeta^2-1}{\sqrt{1-\zeta^2}}}{(1-\Omega_k^2)^2+(2\zeta\Omega_k)^2}\right] \tag{57}$$

Expanding the modal coefficient definition of $D_{k,1}$ in (53) to obtain a representative form:

$$\begin{aligned} D_{k,1} &= \frac{k\omega}{(-2jk\omega)(-jk\omega - r_1)(-jk\omega - r_2)} = -\frac{1}{2j\big(-j\Omega_k\omega_n - r_1\big)\big(-j\Omega_k\omega_n - r_2\big)} \\ &= -\frac{1}{2j\Big(\zeta\omega_n - j(\Omega_k\omega_n + \omega_d)\Big)\Big(\zeta\omega_n - j(\Omega_k\omega_n - \omega_d)\Big)} \end{aligned} \tag{58}$$

Calculating the product in (58) and using $\omega_d^2 = \omega_n^2(1-\zeta^2)$ yields:

$$\big(-jk\omega - r_1\big)\big(-jk\omega - r_2\big) = \omega_n^2\Big[(1-\Omega_k^2) - j2\zeta\Omega_k\Big] \tag{59}$$

Replacing (59) into (58) gives the compact normalized form:

$$D_{k,1} = -\frac{1}{2j\,\omega_n^2}\left[\frac{1}{(1-\Omega_k^2) - j2\zeta\Omega_k}\right] \tag{60}$$

Upon rationalization, the denominator becomes real, making the final steady-state amplitude and phase shift immediately accessible for analysis as clearly seen in equation (61).

$$D_{k,1} = -\frac{1}{2\omega_n^2}\left[\frac{(2\zeta\Omega_k) - j(1-\Omega_k^2)}{(1-\Omega_k^2)^2 + (2\zeta\Omega_k)^2}\right] \tag{61}$$

Now, for the pole at $s = jk\omega$ we conduct an expansion of equation (54) and rationalization of the denominator, this provides the equation for $D_{k,2}$. Also, we know that $D_{k,1}^* = D_{k,2}$ because sinusoids come in complex conjugates.

$$D_{k,2} = -\frac{1}{2\omega_n^2}\left[\frac{(2\zeta\Omega_k) + j(1-\Omega_k^2)}{(1-\Omega_k^2)^2 + (2\zeta\Omega_k)^2}\right] \tag{62}$$

Taking the inverse Laplace transform to get the time-domain solution of the forcing sine to the ZSR yields:

$$\Delta\delta_{ZS}^{(\sin)}(t) = \gamma\sum_{k=1}^{N} b_k\left[C_{k,r_1}e^{r_1 t} + C_{k,r_2}e^{r_2 t} + D_{k,1}e^{-jk\omega t} + D_{k,2}e^{jk\omega t}\right] \tag{63}$$

The complete zero-state response can be obtained by considering the equation (22): $\Delta\delta_{ZS}(t) =$

$\mathscr{L}^{-1}\{\Delta\delta_{ZS}(s)\}$. Hence, adding (29), (49) and (63).

$$\begin{aligned}\Delta\delta_{ZS}(t) = \frac{\gamma a_0}{2}\left[A_0 + B_0 e^{r_1 t} + C_0 e^{r_2 t}\right] \\ + \gamma \sum_{k=1}^{N} a_k \left(A_{k,r_1} e^{r_1 t} + A_{k,r_2} e^{r_2 t} + B_{k,1} e^{-jk\omega t} + B_{k,2} e^{+jk\omega t}\right) \\ + \gamma \sum_{k=1}^{N} b_k \left(C_{k,r_1} e^{r_1 t} + C_{k,r_2} e^{r_2 t} + D_{k,1} e^{-jk\omega t} + D_{k,2} e^{+jk\omega t}\right)\end{aligned} \tag{64}$$

An important structural property of the modal coefficients is that they all share a common normalization and denominator, namely the factor $2\omega_n^2$ and the frequency-dependent term $(1-\Omega_k^2)^2 + (2\zeta\Omega_k)^2$. This uniform structure is not incidental: it reflects the fact that all forcing components, regardless of their origin (DC, cosine, or sine) are filtered by the same second-order system dynamics. As a consequence, the relative importance of each modal contribution is governed primarily by the numerator terms, which encode phase shifts and relative weighting between in-phase and quadrature components, while the denominator captures resonance amplification and damping attenuation. For high-frequency forcing ($\Omega_k \gg 1$), the denominator grows as Ω_k^4, while the corresponding numerators grow at most as Ω_k^2, causing the response magnitude to decay rapidly with frequency. This decay is a direct consequence of the second-order pole pair $r_{1,2}$ and implies that fast oscillatory inputs are strongly attenuated, meaning that this system behaves like a low-pass filter.

The DC contribution is noteworthy because its residue at the origin determines the static offset, while the residues at the natural poles quantify the unavoidable excitation of the oscillatory mode induced by a constant forcing. This confirms that the equilibrium shift and the transient ringdown are intrinsically coupled through the system dynamics, a behavior compactly represented in (29) via the residues defined in (26), 27 and (28).

For harmonic forcing, the separation between natural-mode terms and forcing-frequency terms provides a clear distinction between transient and steady-state behavior. The residues associated with $\pm jk\omega$ govern the long term oscillatory response, whereas those at r_1 and r_2 determine how strongly each harmonic injects energy into the natural mode during the transient. This structure, evident in (49) and (63), offers a concise analytical framework for identifying forced oscillations and resonance mechanisms in linear power systems.

1.5.3 Power Angle Total Solution

By linearity, the total solution is computed by adding the zero-input and zero-state solutions $\Delta\delta(t) = \Delta\delta_{ZI}(t) + \Delta\delta_{ZS}(t)$. Considering the time-domain solutions presented in equations (19) and (64), it is easy to see that many coefficients can be factorized.

$$
\begin{aligned}
\Delta\delta(t) = \underbrace{\frac{\gamma a_0}{2}A_0}_{\text{constant term}} \quad & + \underbrace{\left[A + \frac{\gamma a_0}{2}B_0 + \gamma\sum_{k=1}^{N}\left(a_k A_{k,r_1} + b_k C_{k,r_1}\right)\right]e^{r_1 t}}_{\text{mode at } r_1} \\
& + \underbrace{\left[B + \frac{\gamma a_0}{2}C_0 + \gamma\sum_{k=1}^{N}\left(a_k A_{k,r_2} + b_k C_{k,r_2}\right)\right]e^{r_2 t}}_{\text{mode at } r_2} \\
& + \underbrace{\gamma\sum_{k=1}^{N}\left(a_k B_{k,1} + b_k D_{k,1}\right)e^{-jk\omega t} + \gamma\sum_{k=1}^{N}\left(a_k B_{k,2} + b_k D_{k,2}\right)e^{jk\omega t}}_{\text{forced oscillatory terms at } \pm k\omega}
\end{aligned}
\tag{65}
$$

Adding a conjugate pair forms a real signal, simply because the imaginary part cancels out, this leaves the real part only. Since residues come in complex conjugates: $A^* = B$, $A^*_{k,r_1} = A_{k,r_2}$, $C^*_{k,r_1} = C_{k,r_2}$, $B^*_{k,1} = B_{k,2}$ and $D^*_{k,1} = D_{k,2}$ with real a_k, b_k. The identity $z(t) + z^*(t) = 2\Re\{z(t)\}$ is used, this identity allows us to rewrite equation (65) as presented in (66).

$$
\begin{aligned}
\Delta\delta(t) = \frac{\gamma a_0}{2}A_0 + 2\Re\Bigg\{ e^{-\zeta\omega_0 t}\Bigg[\underbrace{A + \frac{\gamma a_0}{2}B_0 \quad + \gamma\sum_{k=1}^{N}\left(a_k A_{k,r_1} + b_k C_{k,r_1}\right)}_{C^{\mathrm{TR}}_{r_1}}\Bigg]e^{j\omega_d t} \quad + \dots \\
\dots + \quad \gamma\sum_{k=1}^{N}\underbrace{\left(a_k B_{k,2} + b_k D_{k,2}\right)}_{C^{\mathrm{SS}}_{k}}e^{jk\omega t}\Bigg\}
\end{aligned}
\tag{66}
$$

1.6 Power Angle Envelope Analysis of the Linear FSMIB Model

The time-domain solution of the linear FSMIB model, given in (66), describes the evolution of the power angle deviation $\Delta\delta(t)$ as the superposition of a decaying natural mode and multiple steady-state forcing harmonics. This expression is explicitly real-valued. While this representation is physically meaningful and directly measurable, it does not provide direct insight into the modulation effects produced by the interaction between transient and forced components.

To characterize the time-varying amplitude of the oscillatory response, the concept of an envelope is introduced. The envelope $a_{\Delta\delta}(t)$ is defined as the modulus of the analytic signal $|\hat{\Delta\delta}|$ associated with $\Delta\delta(t)$. The analytic signal is a complex-valued representation obtained by retaining only the positive frequency content of the signal [4]. By construction, the original real signal is recovered as $\Delta\delta(t) = \Re\{\Delta\hat{\delta}(t)\}$.

It is important to note that although (66) is written using complex exponentials, it is not itself an analytic signal due to the explicit real-part operator. That operator enforces conjugate symmetry, combining positive and negative frequency components to produce a real signal. However, because the solution is already expressed as a sum of complex exponentials with only positive frequency terms, the analytic signal can be obtained naturally by removing the real-part operator. In this representation, the imaginary component of the analytic signal corresponds exactly to the Hilbert transform of $\Delta\delta(t)$ and no additional computation is required beyond suppressing the conjugate terms. The power angle analytic signal is shown in equation (67).

$$\Delta\hat{\delta}(t) = \underbrace{2C_{r_1}^{\mathrm{TR}} e^{-(\zeta\omega_0 - j\omega_d)t}}_{Z_T^{\mathrm{TR}}(t)} + \underbrace{\left(\frac{\gamma a_0}{2} A_0 + 2\gamma \sum_{k=1}^{N} C_k^{\mathrm{SS}} e^{jk\omega t}\right)}_{Z_F^{\mathrm{SS}}(t)} \tag{67}$$

Where the coefficient $C_{r_1}^{\mathrm{TR}}$ groups all residue contributions associated with the transient component and C_k^{SS} denotes the steady-state complex coefficient associated with the kth forcing harmonic. The envelope of the power angle response is defined as the modulus of the analytic signal:

$$a_{\Delta\delta}(t) = \left|\Delta\hat{\delta}(t)\right| = \sqrt{\Delta\hat{\delta}(t)\,\Delta\hat{\delta}^*(t)} \tag{68}$$

Substituting (67) into (68) and expanding the product yields:

$$a_{\Delta\delta}(t) = \sqrt{|Z_T^{\mathrm{TR}}(t)|^2 + |Z_F^{\mathrm{SS}}(t)|^2 + 2\Re\{Z_T^{\mathrm{TR}}(t) Z_F^{\mathrm{SS}*}(t)\}} \tag{69}$$

Equation (69) separates the envelope into contributions arising from the transient response, the steady-state forcing and their mutual interaction (cross-term).

To make the structure of the envelope explicit, each term in (69) can be expanded using the definitions in (67). This expansion leads to the closed-form expression in (70).

$$
\begin{aligned}
a_{\Delta\delta}(t) = \Bigg[& \underbrace{\left|2C_{r_1}^{\mathrm{TOT}}\right|^2 e^{-2\zeta\omega_0 t}}_{\text{decaying term}} + \underbrace{\left|\frac{\gamma a_0}{2}A_0\right|^2 + 4\gamma^2 \sum_{k=1}^{N} \left|C_k^{\mathrm{SS}}\right|^2}_{\text{constant terms}} \\
& + \underbrace{4\gamma \Re\left\{ \left(\frac{\gamma a_0}{2}A_0\right) \sum_{k=1}^{N} C_k^{\mathrm{SS}*} e^{-jk\omega t} \right\}}_{\text{forcing frequencies } k\omega} + \underbrace{8\gamma^2 \Re\left\{ \sum_{k=1}^{N} \sum_{\ell>k} C_k^{\mathrm{SS}} C_\ell^{\mathrm{SS}*} e^{j(k-\ell)\omega t} \right\}}_{\text{difference in forcing frequencies } (k-\ell)\omega} \\
& + \underbrace{2e^{-\zeta\omega_0 t} \Re\left\{ 2C_{r_1}^{\mathrm{TOT}} \left(\frac{\gamma a_0}{2}A_0\right)^* e^{j\omega_d t} \right\}}_{\text{decaying oscillation at } \omega_d} + \underbrace{2e^{-\zeta\omega_0 t} \sum_{k=1}^{N} \Re\left\{ 4\gamma C_{r_1}^{\mathrm{TOT}} C_k^{\mathrm{SS}*} e^{j(\omega_d - k\omega)t} \right\} \Bigg]^{1/2}}_{\text{decaying beat frequencies } (\omega_d - k\omega)}
\end{aligned} \tag{70}
$$

This expression shows that the envelope is a non-linear function of both transient and steady-state components. The exponentially decaying term originates from the natural mode and reflects the effect of damping, while the steady-state terms quantify the contribution of the forcing harmonics. The double summation over k and ℓ arises from interactions among different forcing harmonics and produces modulation at difference frequencies $(k-\ell)\omega$. Finally, the terms involving $e^{-\zeta\omega_0 t}e^{j(\omega_d - k\omega)t}$ represent transient–forcing interactions, giving rise to beat phenomena at frequencies $(\omega_d - k\omega)$. It can be said that the envelope captures not only the instantaneous amplitude of the analytic signal associated with the power angle oscillation, but also the modulation effects induced by multi-frequency forcing.

1.7 Analysis of the Forced Nonlinear SMIB System Through the Multiple Scales Method

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