Cylinder Intersection in Raytracing

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April 2023

1 Intersection With Lateral Surface

1.1 Three conditions of intersection

Ray-cylinder intersection at point P satisfies three equations on the lateral surface of a cylinder. **Equation 1** defines the center of intersection C_I on the cylinder axis,

$$C_I = C_B + \alpha \vec{n} \tag{1}$$

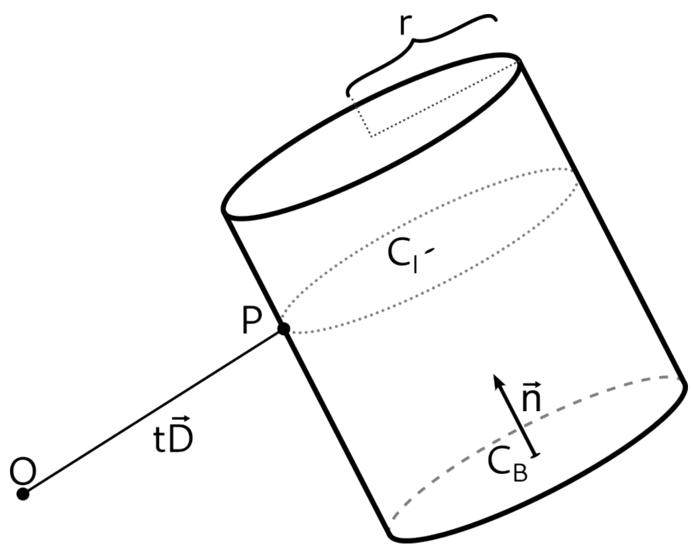
where C_B is the bottom center point of the cylinder and $\vec{n}$ is the unit normal vector which defines the axis. α is the length of the distance between C_I and C_B. It should be noticed that α can be restricted to be smaller than the height of the cylinder and greater than zero to avoid an infinite cylinder.

Similar to plane equation, the vector $(P - C_I)$ is perpendicular to the cylinder axis, therefore,

$$(P - C_I) \bullet \vec{n} = 0 \tag{2}$$

Finally, the distance between the points P, C_I should be equal to radius r,

$$\begin{gathered} \|P - C_I\| = r \\ (P - C_I) \bullet (P - C_I) = r^2 \end{gathered} \tag{3}$$



*The solution for lateral surface intersection is adapted from Chris Dragan's formula.

1.2 Solution of α in terms of point P

In raytracing, we define a ray with origin point O and a unit direction vector $\vec{D}$ multiplied by the length coefficient t. So that, P can be written as a function of t,

$$P(t) = O + t\vec{D}$$

To solve **Equation 1–2** for α, we start by replacing C_I in **Equation 2** with **Equation 1**,

$$(P - C_B - \alpha\vec{n}) \bullet \vec{n} = 0$$
$$(P - C_B) \bullet \vec{n} = \alpha(\vec{n} \bullet \vec{n})$$

Notice that $\vec{n}$ is a unit vector, therefore,

$$(P - C_B) \bullet \vec{n} = \alpha$$
$$(O + t\vec{D} - C_B) \bullet \vec{n} = \alpha$$

For the ease of representation we define,

$$\vec{X} = O - C_B$$

We finally solve the equation for α,

$$\alpha = t\vec{D} \bullet \vec{n} + \vec{X} \bullet \vec{n} \tag{4}$$

This shows that α is dependent on t variable.

1.3 Solution of t

If we write P in **Equation 3** as a function of t, then,

$$(t\vec{D} + \vec{X} - \alpha\vec{n})^2 = r^2$$

Replace α with **Equation 4**, then rearrange the equation to factor out t,

$$\left(t\vec{D} + \vec{X} - \vec{n} \cdot (t\vec{D} \bullet \vec{n} + \vec{X} \bullet \vec{n})\right)^2 = r^2$$
$$\left(t \cdot (\vec{D} - \vec{n} \cdot (\vec{D} \bullet \vec{n})) + \vec{X} - \vec{n} \cdot (\vec{X} \bullet \vec{n})\right)^2 = r^2$$

We define u and v, so that,

$$u = \vec{D} - \vec{n} \cdot (\vec{D} \bullet \vec{n})$$
$$v = \vec{X} - \vec{n} \cdot (\vec{X} \bullet \vec{n})$$
$$(t \cdot u + v)^2 = r^2$$
$$u^2 \cdot t^2 + 2vu \cdot t + v^2 - r^2 = 0$$

Solve the quadratic equation for t where $a = u^2$, $b = 2vu$, and $c = v^2 - r^2$,

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice that $b = 2k$ for some constant $k = vu$ and the equation reduces to,

$$t = \frac{-k \pm \sqrt{k^2 - ac}}{a} \tag{5}$$

Since $\|\vec{n}\| = 1$, u^2, v^2 yield the following equations,

$$\begin{aligned} u^2 &= \vec{D} \bullet \vec{D} - (\vec{D} \bullet \vec{n})^2 \\ v^2 &= \vec{X} \bullet \vec{X} - (\vec{X} \bullet \vec{n})^2 \end{aligned} \tag{6}$$

End the final expansion uv is,

$$uv = \vec{D} \bullet \vec{X} - (\vec{D} \bullet \vec{n}) \cdot (\vec{X} \bullet \vec{n}) \tag{7}$$

2 Intersection With Cylinder Caps

2.1 Solution for the unbounded intersection of each side

Disk intersection can be considered as a plane intersection bounded within the radius of a circle. In case of a cylinder there is a center at the top C_T and a center at the bottom C_B, therefore intersection for the both sides are calculated separately.

On an unbound plane the vector from its reference point (or "center") C to the intersection point P is perpendicular to its normal vector $\vec{n}$,

$$\begin{aligned} (P - C) \bullet \vec{n} &= 0 \\ (t\vec{D} + \vec{X}) \bullet \vec{n} &= 0 \\ t &= \frac{-\vec{X} \bullet \vec{n}}{\vec{D} \bullet \vec{n}} \end{aligned} \tag{8}$$

Once both solutions of t for C_T and C_B are calculated, the greater value can be discarded, since that side of the cylinder will be further away and not be visible.

2.2 Checking if the intersection is within bounds

If P is within the bounds of the circle, then it should satisfy,

$$(P - C) \bullet (P - C) < r^2 \tag{9}$$