Mathematical Modeling and Simulation of Emotional Dynamics using Differential Equations

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Abstract

This technical report explores the mathematical modeling of human emotional states using Ordinary Differential Equations (ODEs). By simulating various scenarios—ranging from simple linear decay to complex coupled interactions—we analyze how mathematical concepts such as stability, equilibrium, and feedback loops map to psychological phenomena like emotional regulation and stress response. The study combines analytical derivations with computational simulations in Python to demonstrate the transient and steady-state behavior of modeled emotional systems.

Contents

1 Introduction

Emotions are complex psychological states that evolve continuously over time. While traditionally studied within the domain of psychology, the temporal dynamics of emotions—such as the decay of anger or the accumulation of stress—can be effectively modeled using dynamical systems theory.

This report presents a series of mathematical models representing emotional behaviors. We treat emotion as a state variable $y(t)$ and investigate its evolution through four distinct analytical stages:

1. Modeling natural emotional resilience using first-order decay principles.
2. Analyzing the equilibrium shifts caused by persistent environmental stimuli.
3. Introducing logistic feedback mechanisms to represent internal emotional regulation limits.
4. Simulating complex interactions and "lag effects" between coupled emotional states.

2 Theoretical Framework

In this section, we establish the mathematical foundation for analyzing emotional dynamics using dynamical systems theory. Rather than implying a direct biological equivalence to physical matter, we employ a *phenomenological approach*. We postulate that emotional states share structural similarities with transient physical systems—such as the natural decay of heat or the stabilization of populations under constraint. By modeling emotion $y(t)$ as a continuous variable governed by differential equations, we can quantify abstract concepts like "resilience" and "emotional capacity." The following subsections derive these models, progressing from idealized linear cases to more realistic coupled interactions.

2.1 Model I: Linear Decay Dynamics

The simplest model for emotional regulation assumes that an intense emotional state fades over time in the absence of external stimuli. This process is governed by the first-order linear differential equation:

$$\frac{dy}{dt} = -ky, \quad k > 0 \tag{1}$$

where k represents the decay rate.

Analytical Solution: The equation is separable. Integrating $\frac{dy}{y} = -kdt$ yields $\ln|y| = -kt + C$. Thus, the general solution is:

$$y(t) = y_0 e^{-kt} \tag{2}$$

Interpretation: This model suggests that without external input, emotions decay exponentially toward zero. However, this model is **insufficient** to represent the full evolution of an emotional state as it only describes simple fading and does not account for external triggers, internal control, or connections with other emotions. External factors can explicitly affect the speed of the emotion (k). For example:

- A calm environment might make k larger (e.g., anger fades faster).
- A stressful environment might make k smaller (e.g., lingering grief).

2.2 Model II: Environmental Forcing & Equilibrium

To account for persistent external factors, we introduce a constant forcing term c:

$$\frac{dy}{dt} = -ky + c, \quad k, c > 0 \tag{3}$$

Equilibrium Analysis: The steady-state solution is found by setting $\frac{dy}{dt} = 0$, which gives $-ky + c = 0$. The equilibrium solution is:

$$y_{eq} = \frac{c}{k} \tag{4}$$

y_{eq} represents a non-zero baseline level. The emotion does not disappear completely; instead, it balances at a level where the fading rate matches the external input.

Interpretation: Constant inputs mean the environment stays the same. Examples include:

- **Visual:** Soft lighting or a view of nature (keeps a baseline of calmness).
- **Auditory:** Background white noise or low-volume instrumental music.

2.3 Model III: Nonlinear Regulation (Logistic Model)

Linear models fail to capture the complexity of internal regulation. We propose a nonlinear model with a logistic feedback term:

$$\frac{dy}{dt} = -ky + ay(1 - y), \quad k, a > 0 \tag{5}$$

Stability Analysis: We solve $-ky + ay(1 - y) = 0 \implies y[-k + a - ay] = 0$. The equilibrium points are:

$$y_1^* = 0, \quad y_2^* = 1 - \frac{k}{a} \quad \text{(assuming } a > k\text{)} \tag{6}$$

Let $f(y) = (a - k)y - ay^2$. The stability is determined by the derivative $f'(y) = (a - k) - 2ay$:

- At $y_1^* = 0$, $f'(0) = a - k$. If $a > k$, this is positive, so $y = 0$ is **unstable**.
- At y_2^*, $f'(y_2^*) = -(a - k)$. If $a > k$, this is negative, so y_2^* is **stable**.

Interpretation: The term $ay(1 - y)$ creates logistic feedback, allowing the emotion to grow from zero. However, due to the decay term $-ky$, the emotion does not reach full capacity (1). Instead, it stabilizes at y^*, modeling how emotional regulation works together with natural decay to establish a balanced state.

2.4 Model IV: Coupled Emotional Interaction

Emotions rarely exist in isolation. We model the interaction between two states, $x(t)$ (e.g., Fatigue) and $y(t)$ (e.g., Anxiety), where Anxiety drives Fatigue:

$$\frac{dx}{dt} = -ax + by \tag{7}$$

$$\frac{dy}{dt} = -cy \tag{8}$$

where $a, b, c > 0$.

Analytical Solution: The second equation yields $y(t) = y_0 e^{-ct}$. Substituting this into the first equation:

$$\frac{dx}{dt} + ax = by_0 e^{-ct} \tag{9}$$

Using the integrating factor e^{at}, the general solution is:

$$x(t) = \frac{by_0}{a-c}e^{-ct} + C_2 e^{-at} \quad (\text{for } a \neq c) \tag{10}$$

Interpretation: The term $+by$ shows that emotion y feeds into emotion x. Even if x would normally fade away, the presence of y keeps it alive. This models connected emotions, such as anxiety causing fatigue.

3 Computational Simulation & Results

The theoretical models discussed above were implemented in Python using the ‘scipy.integrate.odeint‘ solver.

3.1 Implementation Details

The following Python code snippet demonstrates the implementation of the Coupled System model, visualizing the "lag effect" where the secondary emotion persists after the primary one has decayed.

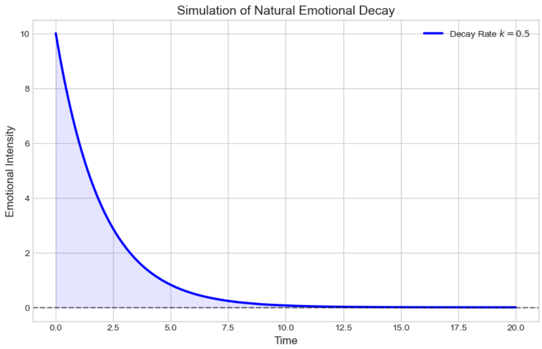
```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

def coupled_system(z, t, a, b, c):
    x, y = z
    dxdt = -a * x + b * y  # x is driven by y
    dydt = -c * y          # y decays naturally
    return [dxdt, dydt]

# Simulation parameters
params = (0.5, 0.5, 0.2) # a, b, c
t = np.linspace(0, 20, 200)
initial_state = [5.0, 5.0]

# Solve ODE
solution = odeint(coupled_system, initial_state, t, args=params)
```

Listing 1: Python implementation of the coupled emotional system

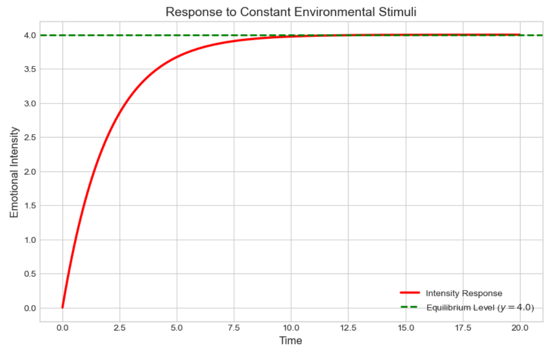
3.2 Simulation Results

The simulation results, presented in Figure 1, validate the analytical predictions derived in the theoretical framework. The visual trajectories confirm four distinct behaviors:

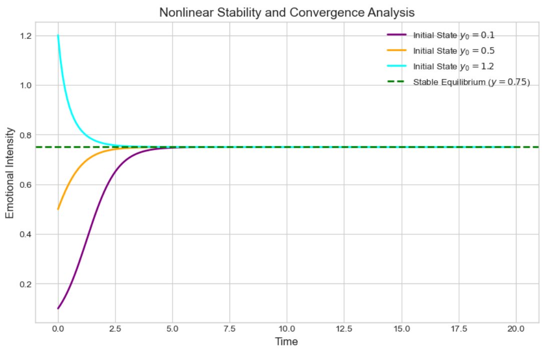
- **Linear Decay (a):** Shows pure exponential fading of emotion in the absence of stimuli.
- **Forced Response (b):** Demonstrates how constant inputs shift the equilibrium away from zero to a new baseline.
- **Nonlinear Regulation (c):** Exhibits logistic stability, where the emotion naturally caps at a specific level due to feedback.
- **Coupled Interaction (d):** Reveals the "lag effect," where a secondary emotion persists even after the source emotion has decayed.



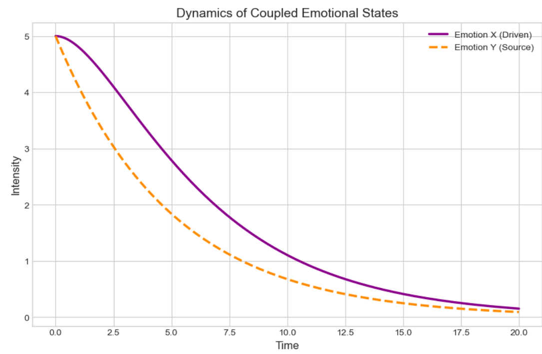
(a) Model I: Linear Decay ($k = 0.5$)



(b) Model II: Forced Response ($c = 2.0$)



(c) Model III: Nonlinear Stability



(d) Model IV: Coupled Systems

Figure 1: Comprehensive simulation results demonstrating four different emotional dynamics modeled in this study.

4 Discussion: Environmental Factors

Mathematical modeling allows us to categorize how different sensory inputs act as "forcing terms" or "parameters" in the differential equations. Table 1 summarizes the proposed relationship between emotional categories and balancing environmental inputs.

Table 1: Mapping of Emotional States to Balancing Inputs

Emotional State	**Proposed Input Type**	**Mechanism of Action**
Happiness	Upbeat, rhythmic sounds	Maintains high energy ($y(t)$)
Sadness	Bright light, warm colors	Counteracts low energy decay
Anger	Slow music, cool colors	Increases decay rate (k)
Fear	Open spaces, clear view	Reduces perceived threat variable
Loneliness	Human voices, social sounds	External coupling input ($+c$)
Guilt/Shame	Dim, soft lighting	Provides safety/privacy
Fatigue	High contrast, fast tempo	Stimulates arousal state

5 Conclusion

This report demonstrated the viability of using differential equations to model psychological phenomena. By transitioning from simple linear decay to coupled and nonlinear systems, we captured essential features of emotional dynamics such as resilience, environmental susceptibility, and complex interaction. While these models are simplified abstractions, they provide a robust framework for understanding the temporal evolution of human emotions, bridging the gap between mathematical systems theory and behavioral science.

References

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