BLG 354E SIGNALS & SYSTEMS FOR COMPUTER ENGINEERING

HOMEWORK 4 REPORT

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Question 1

Multiple convolutions of a signal produce an output that has the shape of a Gaussian curve. Proof:

CLT explains the bell shaped curve

let X_1 be our random variable, and real value of $X \to \hat{x}$

$$\text{Prob}(a \leq X \leq b) = \int_a^b p(x)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} p(x)\,dx = 1$$

Suppose we have 2 independent variables X_1 & X_2 with $P_1(x_1)$ $P_2(x_2)$

How will sum $X_1 + X_2$ be distributed?

For any value of t

$\text{prob}(X_1 + X_2 \leq t)$ is as follows

$$\underbrace{\iint P_1(x)p_2(x)\,dx_1\,dx_2}_{x_1+x_2 \leq t} \quad \text{with change of variable } \tau = x_1, \; T = x_1 + x_2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{t} P_1(\tau)P_2(T-\tau)\,dT\,d\tau = \underbrace{\int_{-\infty}^{t}\int_{-\infty}^{\infty} P_1(\tau)P_2(T-\tau)\,d\tau\,dT}_{\text{We have extracted convolution from probability distribution.}}$$

$$= \int_{-\infty}^{t} P_1(T) * P_2(T)\,dT$$

Now for multi random var case

Let S_n be $X_1 + X_2 \ldots X_N$. Standard dev is $\sqrt{n}$.

$$\frac{S_n}{\sqrt{n}} = \frac{X_1 + \cdots X_N}{\sqrt{n}}$$

$$\text{CLT} \Rightarrow \lim_{n\to\infty} P(u_n \leq u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$p(x)$ is the dist for $X_1, X_2 \ldots X_n$ (They have same distribution)

$(p * p * p * \cdots p) = p^{*n}(x)$

Since they all have mean $= 0$ and var $= 1$. (priori assumption)

Distribution of $\frac{X_1 \ldots X_n}{\sqrt{n}}$ is $\sqrt{n}\,p^{*n}(\sqrt{n}\,x)$

Take FFT to analyze convolution more easily:

$$FT\left(\sqrt{n}\, p^{*n}(\sqrt{n}\, x)\right) = \left(FT(p)\right)^n\left(\frac{s}{\sqrt{n}}\right) = \left(FT\, p\left(\frac{s}{\sqrt{n}}\right)\right)^n$$

$$= \int_{-\infty}^{\infty} e^{-2\pi j \left(\frac{s}{\sqrt{n}}\right)x} p(x)\, dx \quad \text{Using Taylor series} \left(e^x = 1 + x + \frac{x^2}{2!} - - -\right)$$

$$= \int_{-\infty}^{\infty} \left[1 - \frac{2\pi j s}{\sqrt{n}} - \left(\frac{2\pi s}{\sqrt{n}}\right)^2 \frac{1}{2} + \cdots\right] p(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} p(x)\, dx}_{1} - \frac{2\pi j s}{\sqrt{n}} \underbrace{\int_{-\infty}^{\infty} x p(x)\, dx}_{mean = 0} - \frac{2\pi^2 s^2}{n} \underbrace{\int_{-\infty}^{\infty} x^2 p(x)\, dx}_{variance = 1} + \underbrace{\cdots}_{\text{error term } c}$$

$$= 1 - \frac{2\pi^2 s^2}{n} + c \rightarrow \text{noise in multiple convolutions.}$$

$$FT\left(p\left(\frac{s}{\sqrt{n}}\right)\right)^n \approx \left(1 - \frac{2\pi^2 s^2}{n}\right)^n \approx e^{-2\pi^2 s^2}$$

We can obtain CLT by taking IFT of $e^{-2\pi^2 s^2}$

See code for graph plots.

From the plots it can be deduced that longer square waves obtain Gaussian form faster than smaller ones. (Low pass filter with higher bandwidth approaches to Gaussian plot much faster, within less steps of convolution). Since there are more points to evaluate in each multiplication.

Question 2

$A_n = \frac{1}{T}\int_0 f(t)\, e^{-j n w_0 t}\, dt$ (Analysis) $\qquad X(t) = \sum_{-\infty}^{\infty} a_n e^{j n w_0 t} \qquad \frac{2\pi}{6} = \frac{\pi}{3}t$

For first graph

$$a_k = \frac{1}{6}\left(\int_0^3 e^{-jk\frac{\pi}{3}t}\,dt + 3\int_3^4 e^{-jk\frac{\pi}{3}t}\,dt\right) = \frac{1}{6}\left(-\frac{1}{jk\frac{\pi}{3}}\left(e^{-jk\frac{\pi}{3}t}\Big|_0^3\right) - \frac{3}{jk\frac{\pi}{3}} e^{-jk\frac{\pi}{3}t}\Big|_3^4\right)$$

$$= -\frac{1}{2jk\pi}\left(e^{-jk\pi}-1\right) - \frac{3}{2jk\pi}\left(e^{-jk\frac{4\pi}{3}} - e^{-jk\pi}\right) = \frac{1}{2jk\pi}\left(1 - 3e^{-jk\frac{4\pi}{3}} + 2e^{-jk\pi}\right)$$

For second graph

$$a_k = \frac{1}{6}\left(\int_0^1 e^{-jk\frac{\pi}{3}t}\,dt + \int_1^2 (-t+2)e^{-jk\frac{\pi}{3}t}\,dt + \int_4^5 (t-4)e^{-jk\frac{\pi}{3}t}\,dt + \int_5^6 e^{-jk\frac{\pi}{3}t}\,dt\right)$$

$$= \frac{1}{6}\left(-\frac{3}{jk\pi}e^{-jk\frac{\pi}{3}t}\Big|_0^1 - \frac{e^{-\frac{2\pi jk}{3}}\left((3\pi jk-9)e^{\frac{\pi jk}{3}}+9\right)}{\pi^2k^2} - e^{-\frac{5\pi jk}{3}}\frac{\left(9e^{\frac{\pi jk}{3}} - 3\pi jk - 9\right)}{\pi^2k^2}\right.$$

$$\left. - \frac{3}{jk\pi}e^{-jk\frac{\pi}{3}t}\Big|_5^6\right)$$

$$= \frac{1}{6}\left(-\frac{3}{jk\pi}e^{-j\frac{k\pi}{3}} + 1 - \frac{e^{-\frac{2jk\pi}{3}}\left((3\pi jk-9)e^{\frac{\pi jk}{3}}+9\right)}{\pi^2k^2} - e^{-\frac{5\pi jk}{3}}\frac{\left(9e^{\frac{\pi jk}{3}}-3\pi jk-9\right)}{\pi^2k^2} - \frac{3}{jk\pi}\left(e^{-jk2\pi} - e^{-jk\frac{5\pi}{3}}\right)\right)$$

For first graph

$$a_1 = \frac{1}{2j\pi}\left(1 - 3e^{-j\frac{4\pi}{3}} + 2e^{-j\pi}\right) \quad a_2 = \frac{1}{4j\pi}\left(1 - 3e^{-j\frac{8\pi}{3}} + 2e^{-j2\pi}\right)$$

$$a_3 = \frac{1}{6\pi j}\left(1 - 3e^{-j4\pi} + 2e^{-j3\pi}\right) \quad \ldots$$

$$\vdots$$

See code for further