CENG424 - Homework 3

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1. a) For all cats, there is some dog that the cat and the dog are friends. $\equiv$ All cats are friends with some dog.

   b) There exists some cat that are friends with all dogs.

2. a) $\exists x.(\forall y.p(x,y) \rightarrow p(z,z)) \Leftrightarrow (\exists x.p(x,x) \rightarrow \exists y.p(y,y)$

   Assume our Herbrand Universe is $U = \{a, b\}$

   For this logical sentence to be valid, both of the sides should both evaluate to T or F for all interpretations. Let's first observe the right part.

   Right part is always evaluates to true.

   Right part case1: If there exists some p(x,x), $\exists x.p(x,x)$ is true. And also $\exists y.p(y,y)$ becomes also true. $1 \rightarrow 1$ evaluates to 1.

   Right part case 2: If there does not exist some p(x,x), it means that there is no such p(x,x) = 1. Hence, both $\exists x.p(x,x)$ and $\exists y.p(y,y)$ becomes 0. $0 \rightarrow 0$ evaluates to 1.

   Hence, from case1 and case2, right part the double implication $(\exists x.p(x,x) \rightarrow \exists y.p(y,y)$ is 1 for all interpretations.

   For our sentence to be valid, left part of the double implication should also evaluate to 1 for all interpretations.

   For our Herbrand universe, when an existential quantifier is used with a functional constant like p(x), if p(a) $\vee$ p(b) is 1, then $\exists x.p(x)$ is true.

   For our Herbrand universe, when an universal quantifier is used with a functional constant like p(x), if p(a) $\wedge$ p(b) is 1, then $\forall x.p(x)$ is true.

   From this logic, we can rewrite our sentence resides on the right of the double implication as such:

   $\exists x.(\forall y.p(x,y) \rightarrow p(z,z))$

   Define universal quantifier for free variables:
   $\exists x.(\forall y.p(x,y) \rightarrow \forall z.p(z,z))$

$\equiv [(p(a,a) \land p(a,b)) \to (p(a,a) \land p(b,b))] \lor [(p(b,a) \land p(b,b)) \to (p(a,a) \land p(b,b))]$

Let's apply INDO here to obtain the clausal form.
I - $[\neg(p(a,a) \land p(a,b)) \lor (p(a,a) \land p(b,b))] \lor [\neg(p(b,a) \land p(b,b)) \lor (p(a,a) \land p(b,b))]$
N - $[\neg p(a,a) \lor \neg p(a,b) \lor (p(a,a) \land p(b,b))] \lor [\neg p(b,a) \lor \neg p(b,b) \lor (p(a,a) \land p(b,b))]$
D - $[p(a,a) \lor \neg p(b,b) \lor \neg p(b,a) \lor \neg p(a,b) \lor \neg p(a,a)] \land [p(b,b) \lor \neg p(b,b) \lor \neg p(b,a) \lor \neg p(a,b) \lor \neg p(a,a)]$

Both sides of the $\land$ evaluates to 1 from this sentences:
Left side: $p(a,a) \lor \neg p(a,b) \equiv 1$
Right side: $p(b,b) \lor \neg p(b,b) \equiv 1$

Hence, left side of the double implication is also evaluates to 1 for all interpretations. Since, $1 \Leftrightarrow 1 = 1$, this sentence is VALID.

b) We have the Herbrand Universe $= \{a, b\}$. From the same logic obtained above (Q2.1), we can write the given sentence as follows:

$(\forall x.(p(x) \lor q(x))) \to (\exists y.p(y) \to (p(x) \to \forall z.p(z)))$

Defining universal quantifier for free variable, we get:
$(\forall x.(p(x) \lor q(x))) \to (\exists y.p(y) \to (\forall w.p(w) \to \forall z.p(z)))$

Applying the logic:
$((p(a) \lor q(a)) \land (p(b) \lor q(b))) \to [(p(a) \lor p(b)) \to ((p(a) \land p(b)) \to (p(a) \land p(b)))]$

In order the sentence to be valid, the negate of it should be unsatisfiable.

Applying INDO to negated goal transform the sentence into clausal form:

$\neg(((p(a) \lor q(a)) \land (p(b) \lor q(b))) \to [(p(a) \lor p(b)) \to ((p(a) \land p(b)) \to (p(a) \land p(b)))])$

I - $\neg(\neg((p(a) \lor q(a)) \land (p(b) \lor q(b))) \lor [\neg(p(a) \lor p(b)) \lor (\neg(p(a) \land p(b)) \lor (p(a) \land p(b)))])$

N - $(p(a) \lor q(a)) \land (p(b) \lor q(b)) \land (p(a) \lor p(b)) \land (p(a) \land p(b)) \land (\neg p(a) \lor \neg p(b))$
O -
1. $\{p(a), q(a)\}$ *premise*
2. $\{p(b), q(b)\}$ *premise*
3. $\{p(a), p(b)\}$ *premise*
4. $\{p(a)\}$ *premise*
5. $\{p(b)\}$ *premise*
6. $\{\neg p(a), \neg p(b)\}$ *premise*
7. $\{\neg p(a)\}$ 6, 5
8. $\{\}$ 7, 4

Since we have reached an empty clause for the negated goal, negated goal is unsatisfiable. Hence our sentence is VALID.

c) To prove if it is valid or not, let's apply INSEADO to the negated sentence.

sentence: $\exists y.(p(y) \to \exists x.q(x,y)) \to \neg\exists x.q(y,x)$

negated sentence: $\neg[\exists y.(p(y) \to \exists x.q(x,y)) \to \neg\exists x.q(y,x)]$

I - $\neg\exists y.[\neg(\neg p(y) \vee \exists x.q(x,y)) \vee \neg\exists x.q(y,x)]$

N - $\forall y.(\neg p(y) \vee \exists x.q(x,y)) \wedge \exists x.q(y,x)$

S - $\forall y.(\neg p(y) \vee \exists x.q(x,y)) \wedge \exists z.q(y,z)$

E - $\forall y.(\neg p(y) \vee q(h(y),y)) \wedge q(y,a)$

A - $(\neg p(y) \vee q(h(y),y)) \wedge q(y,a)$

D -

O -
1. $\{\neg p(y), q(h(y),y)\}$
2. $\{q(y,a)\}$
3. Failure

Since negation did not result in an empty clause, the premise is not valid.

However, for the Herbrand universe U = $\{a,b,c\}$ assume such an x,y,z assignment:
$(p(a) \to q(b,a)) \to \neg q(a,c)$

For the interpretation I = $\{q(a,c) \leftarrow 0, q(b,a) \leftarrow 1\}$, the sentence is satisfiable. Hence, the sentence is CONTINGENT.

3. | | |
|---|---|
| 1. $\forall x.(p(x) \to q(x))$ | *Premise* |
| 2. $\neg\exists z.r(z)$ | *Premise* |
| 3. $\exists y.p(y) \vee r(a)$ | *Premise* |
| 4. $\neg\exists z.r(z) \to \forall z.\neg r(z)$ | *Premise* |
| 5. $\forall z.\neg r(z)$ | $MP: 4,2$ |
| 6. $\neg r(a)$ | $UI: 5$ |
| 7. $p(b) \vee r(a)$ | $EI: 3$ |
| 8. $(p(b) \vee r(a)) \Leftrightarrow (\neg p(b) \to r(a))$ | OQ |
| 9. $\neg p(b) \to r(a)$ | *replacement theory* |
| 10. $p(b) \to q(b)$ | $UI: 1$ |
| 11. $(\neg p(b) \to r(a)) \to ((\neg p(b) \to \neg r(a)) \to p(b))$ | CR |
| 12. $(\neg p(b) \to \neg r(a)) \to p(b)$ | $MP: 11,9$ |
| 13. $\neg r(a) \to (\neg p(b) \to \neg r(a))$ | II |
| 14. $\neg p(b) \to \neg r(a)$ | $MP: 13,6$ |
| 15. $p(b)$ | $MP: 12,14$ |
| 16. $q(b)$ | $MP: 10,15$ |
| 17. $\exists z.q(z)$ | $EG: 16$ |

4. Our premises are given as follows:
$\forall y.A(a,y)$
$\forall x \forall y.(A(x,y) \to A(B(x),B(y)))$

And our goal is as follows:
$\exists z.(A(a,z) \land A(z,B(B(a))))$

We need to apply INSEADO to premises to obtain clauses:
$\forall y.A(a,y)$
A - $\equiv A(a,y)$
O - $\equiv \{A(a,y)\}$

$\forall x \forall y.(A(x,y) \to A(B(x),B(y)))$
I - $\equiv \forall x \forall y.(\neg A(x,y) \lor A(B(x),B(y)))$
A - $\equiv \neg A(x,y) \lor A(B(x),B(y))$
O - $\equiv \{\neg A(x,y), A(B(x),B(y))\}$

In order our premises to prove the given logic sentence, the set of given premises K $\land$ ($\neg$ goal) should be unsatisfiable. Hence, should result in an empty clause {}

We need to apply INSEADO to the ($\neg$ goal) to obtain the clauses.
$\neg(\exists z.(A(a,z) \land A(z,B(B(a)))))$
N - $\forall z.(\neg A(a,z) \lor \neg A(z,B(B(a))))$
A - $\neg A(a,z) \lor \neg A(z,B(B((a)))$
O - $\{\neg A(a,z), \neg A(z,B(B(a))\}$

1. $\{A(a,y)\}$	*Premise*
2. $\{\neg A(x,y), A(B(x),B(y))\}$	*Premise*
3. $\{\neg A(a,z), \neg A(z,B(B(a))\}$	*negated goal*
4. $\{A(B(a),B(y))\}$	$1,2 : x \leftarrow a$
5. $\{\neg A(a,B(a))\}$	$3,4 : z \leftarrow B(a), y \leftarrow B(a)$
6. {}	$5,1 : y \leftarrow B(a)$

Empty clause is reached. Hence, this sentence is valid.