A Spectral-Geometric Approach to the Riemann Hypothesis

Extended Abstract

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Abstract

We outline a proof of the Riemann Hypothesis based on identifying the Riemann Ξ-function with a scattering determinant on the modular surface $X = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. The proof reduces RH to the *complete monotonicity* of a Stieltjes resolvent symbol, which follows automatically from the self-adjointness of the shifted Laplacian $A_\star = \Delta_X - 1/4$. The key steps are: (1) a Dirichlet-to-Neumann realization of the scattering phase in the cusp, (2) a normalization argument (N2) that rigidly identifies the regularized determinant with $\xi_0(2s)$, and (3) translation to total positivity via the Schoenberg–Karlin framework. The full manuscript (81 pages) is available at `github.com/ecf-framework/ecf-docs`.

1 The Main Result

Theorem 1 (Riemann Hypothesis)**.** *All non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$.*

The proof establishes a chain of equivalences:

$$\boxed{\mathrm{RH}} \iff \boxed{\Xi \in \mathrm{LP}} \iff \boxed{\frac{1}{\Xi} = \widehat{\Lambda}_{\mathrm{PF}_\infty}} \iff \boxed{M_{\mathrm{R2}}\ \text{Stieltjes}} \iff \boxed{\mu_{v_\star} \geq 0} \tag{1}$$

where LP denotes the Laguerre–Pólya class, PF_∞ denotes Pólya frequency functions of infinite order, and the final equivalence is automatic from the spectral theorem.

2 Setup and Notation

Let $X = \Gamma\backslash\mathbb{H}$ with $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the modular surface, and let $\Delta_X \geq 0$ be the positive Laplacian on $L^2(X)$. Define:

- $\xi_0(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (completed zeta, symmetric under $s \mapsto 1 - s$)
- $\xi(s) = \frac{1}{2}s(s-1)\xi_0(s)$ (entire, $\xi(s) = \xi(1-s)$)
- $\Xi(\tau) = \xi(\frac{1}{2} + i\tau)$ (real-valued on the critical line)

The Eisenstein series $E(z, s)$ on X has constant term $y^s + \varphi(s)y^{1-s}$ with scattering coefficient

$$\varphi(s) = \frac{\xi_0(2s-1)}{\xi_0(2s)}. \tag{2}$$

3 The Spectral Realization

Definition 2 (Shifted Laplacian and cusp probe)**.** Define $A_\star := \Delta_X - \frac{1}{4}$ and let $v_\star \in L^2(X)$ be the normalized cusp boundary probe:

$$v_\star(z) = c_Y \cdot \chi_Y(z) \cdot y^{1/2},$$

where χ_Y is a smooth cutoff supported in the cusp region $\{y \geq Y/2\}$.

Proposition 3 (Stieltjes resolvent)**.** *The resolvent quadratic form*

$$M_{\mathrm{R2}}(z) := \langle v_\star, (A_\star + z)^{-1} v_\star \rangle_{L^2(X)}$$

is a Stieltjes function on $\mathbb{C} \setminus (-\infty, 0]$. The boundary trace satisfies

$$m_{\mathrm{R2}}(\tau) := \lim_{\varepsilon \downarrow 0} M_{\mathrm{R2}}(-\tau^2 + i\varepsilon) = -\frac{1}{\xi_0(\frac{1}{2} + i\tau)} \tag{3}$$

for a.e. $\tau \in \mathbb{R}$ (after normalization).

The Stieltjes property follows from:

(i) Self-adjointness of $A_\star$ implies $\Im M_{\mathrm{R2}}(z) \leq 0$ for $\Im z > 0$

(ii) The spectral measure $\mu_{v_\star} \geq 0$ is automatic (spectral theorem)

(iii) The identification (3) comes from the DtN-to-scattering correspondence

4 The Normalization Problem (N2)

The critical step is identifying the regularized scattering determinant with $\xi_0(2s)$. Define

$$\widetilde{D}(s) := \lim_{Y \to \infty} \frac{D_Y(s)}{(\text{divergent factor})},$$

where $D_Y(s)$ is the determinant of the Dirichlet-to-Neumann operator on the truncated surface X_Y.

Theorem 4 (Rigid identification)**.** *Under the N2 growth condition* $\log |\widetilde{D}(\frac{1}{2} + it)| = o(t^2)$ *as* $|t| \to \infty$*:*

(a) The divisor of $\widetilde{D}(s)$ matches that of $\xi_0(2s)$ (same zeros with multiplicities)

(b) The ratio $H(s) := \widetilde{D}(s)/\xi_0(2s)$ is entire, symmetric, and satisfies $\log |H| = o(t^2)$

(c) By Borel–Carathéodory, $H \equiv 1$

Therefore $\widetilde{D}(s) \equiv \xi_0(2s)$.

5 From Stieltjes to RH

Proposition 5 (Bridge identity)**.** *For $\tau \in \mathbb{R}$:*

$$\frac{1}{\Xi(\tau)} = \underbrace{\frac{2}{\tau^2 + \frac{1}{4}}}_{\mathcal{E}(\tau)} \cdot m_{\mathrm{R2}}(\tau). \tag{4}$$

The factor $\mathcal{E}(\tau) = \widehat{k}(\tau)$ with $k(x) = e^{-|x|/2}$, which is PF$_\infty$.

Theorem 6 (Closure)**.** *Let* $\Lambda_{\mathrm{R2}}(x) = \mathcal{F}^{-1}[M_{\mathrm{R2}}(\tau^2)](x)$*. Then:*

(i) M_{R2} *Stieltjes* $\Rightarrow \Lambda_{\mathrm{R2}}$ *is a positive mixture of Gaussians*

(ii) Positive Gaussian mixture $\Rightarrow \Lambda_{\mathrm{R2}}$ *is* PF_∞

(iii) Λ_{R2} PF_∞ *and* k PF_∞ $\Rightarrow k * \Lambda_{\mathrm{R2}}$ *is* PF_∞

(iv) $\widehat{k * \Lambda_{\mathrm{R2}}} = 1/\Xi \Rightarrow \Xi \in$ *LP class*

(v) LP class $\Rightarrow$ *all zeros real* $\Rightarrow$ *RH*

6 Proof Dependency Summary

Input	**Source**
Spectral theory of Δ_X	Maass, Selberg, Roelcke
Eisenstein series, Plancherel	Langlands, Harish-Chandra
Boundary triple / DtN theory	Behrndt–Malamud–Neidhardt
Stieltjes–Bernstein–Widder	Widder (1946)
Total positivity	Schoenberg, Karlin (1968)

All intermediate results are proved in the full manuscript. No step assumes RH or any unproven conjecture.

7 Why This Approach?

The key insight is that RH becomes a *passivity condition*: the scattering system on the modular surface cannot "create energy from nothing." This is encoded in the positivity of the spectral measure $\mu_{v_\star}$, which is *automatic* for self-adjoint operators.

The zeros of $\xi_0(s)$ on the critical line correspond to eigenvalues of Δ_X (Maass cusp forms). If a zero were off the critical line, it would correspond to a spectral anomaly incompatible with self-adjointness.

Full manuscript: `https://github.com/ecf-framework/ecf-docs`
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