A Spectral-Geometric Proof of the Riemann Hypothesis

Technical Summary for Review

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Abstract

This document provides a technical summary of a proposed proof of the Riemann Hypothesis. The proof identifies the Riemann Ξ-function with a normalized scattering determinant on the modular surface and translates the zeros-on-critical-line condition into a total-positivity criterion. The key technical ingredients are: (1) a Dirichlet-to-Neumann boundary triple realization of modular scattering, (2) a normalization theorem (N2) that eliminates regularization ambiguity, (3) the Stieltjes property of the resolvent symbol from spectral positivity, and (4) the Schoenberg–Karlin characterization of the Laguerre–Pólya class. This summary is intended for specialists who may serve as arXiv endorsers; the complete proof spans 81 pages and is available at `github.com/ecf-framework/ecf-docs`.

Contents

1 Introduction and Main Theorem

1.1 Statement

Theorem 1.1 (Riemann Hypothesis)**.** *All non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part* $1/2$*.*

1.2 Proof strategy overview

The proof proceeds through the following equivalence chain:

$$\mathrm{RH} \iff \Xi \in \mathrm{LP} \iff \frac{1}{\Xi(\tau)} = \widehat{\Lambda}(\tau) \text{ with } \Lambda \in \mathrm{PF}_\infty \iff M_{\mathrm{R2}} \in \mathcal{S} \iff \mu_{v_\star} \geq 0 \tag{1}$$

where:

- LP = Laguerre–Pólya class (entire functions with only real zeros, limits of polynomials with real zeros)
- PF_∞ = Pólya frequency functions of infinite order (totally positive kernels)
- $\mathcal{S}$ = Stieltjes functions (positive Nevanlinna functions mapping $\mathbb{C} \setminus (-\infty, 0]$ to $\mathbb{C} \setminus (-\infty, 0]$)
- $\mu_{v_\star}$ = spectral measure of a cyclic vector for a self-adjoint operator

The crucial observation is that the *final* equivalence is automatic: if $A_\star$ is self-adjoint, then its spectral measure is non-negative by the spectral theorem. Thus RH reduces to constructing an appropriate operator realization.

2 Notation and Background

2.1 Zeta functions

Definition 2.1. Define the completed zeta functions:

$$\xi_0(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{2}$$
$$\xi(s) := \tfrac{1}{2}s(s-1)\xi_0(s). \tag{3}$$

Both satisfy the functional equation $f(s) = f(1-s)$. Define the critical-line restriction:

$$\Xi(\tau) := \xi(\tfrac{1}{2} + i\tau) \in \mathbb{R} \quad \text{for } \tau \in \mathbb{R}. \tag{4}$$

Remark 2.2. The function $\Xi(\tau)$ is even, real-entire when extended to $\mathbb{C}$, and has the integral representation

$$\Xi(\tau) = \int_0^\infty \Phi(u)\cos(\tau u)\,du$$

where $\Phi(u) = 2\pi \sum_{n=1}^\infty (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}) \exp(-\pi n^2 e^{2u})$ is the Riemann kernel. RH is equivalent to Ξ having only real zeros.

2.2 The modular surface

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ act on the upper half-plane $\mathbb{H} = \{z = x + iy : y > 0\}$ by Möbius transformations. The quotient $X = \Gamma \backslash \mathbb{H}$ is the modular surface, a non-compact finite-volume hyperbolic orbifold with one cusp at $i\infty$.

The positive Laplacian on $L^2(X, d\mu)$ with $d\mu = y^{-2}dx\,dy$ is

$$\Delta_X = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \geq 0.$$

2.3 Eisenstein series and scattering

The Eisenstein series $E(z,s)$ for $\Re(s) > 1$ is defined by

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\Im \gamma z)^s$$

and extends meromorphically to $s \in \mathbb{C}$. Its constant term in the Fourier expansion is

$$a_0(y,s) = y^s + \varphi(s) y^{1-s} \tag{5}$$

where the **scattering coefficient** is

$$\varphi(s) = \frac{\xi_0(2s-1)}{\xi_0(2s)}. \tag{6}$$

Remark 2.3. Equation (6) is the fundamental link between automorphic spectral theory and the Riemann zeta function. It implies that zeros of $\xi_0(2s)$ in the half-plane $\Re(s) > 1/2$ would appear as poles of $\varphi(s)$, which would be spectral anomalies.

3 The Boundary Triple Framework

3.1 Truncation and the DtN map

Fix a truncation height $Y > 1$ and define the compact core $X_Y = \{z \in X : \Im(z) \leq Y\}$ with horocyclic boundary $\partial X_Y \simeq S^1$.

Definition 3.1 (Dirichlet-to-Neumann map)**.** For $z \in \mathbb{C} \setminus \sigma(\Delta^D_{X_Y})$, define the scalar DtN symbol on the constant boundary mode:

$$m_Y(z) := \langle 1, \Lambda_Y(z) 1 \rangle_{\partial X_Y}$$

where $\Lambda_Y(z) : H^{1/2}(\partial X_Y) \to H^{-1/2}(\partial X_Y)$ is the DtN operator for the Helmholtz equation $(\Delta_{X_Y} - z)u = 0$.

Lemma 3.2 (DtN is Nevanlinna)**.** *The function $m_Y(z)$ is a Nevanlinna (Herglotz) function: it maps the upper half-plane to itself, i.e., $\Im m_Y(z) \geq 0$ for $\Im z > 0$.*

Proof sketch. By Green's identity, for u solving $(\Delta - z)u = 0$ with $u|_{\partial X_Y} = f$:

$$\Im z \cdot \|u\|^2_{L^2(X_Y)} = \Im \langle f, \partial_n u \rangle_{\partial X_Y} = \Im \langle f, \Lambda_Y(z) f \rangle.$$

For $\Im z > 0$, the left side is non-negative, hence $\Im \langle f, \Lambda_Y(z) f \rangle \geq 0$. □

3.2 Boundary triple and Kreĭn formula

The truncated Laplacian Δ_{X_Y} admits a boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ where:

- $\Gamma_0 u = u|_{\partial X_Y}$ (Dirichlet trace, restricted to constant mode)
- $\Gamma_1 u = \partial_n u|_{\partial X_Y}$ (Neumann trace, restricted to constant mode)

Theorem 3.3 (Kreĭn resolvent formula)**.** *Let A_D be the Dirichlet extension and A_Θ the Robin extension with parameter $\Theta \in \mathbb{R}$. Then*

$$(A_\Theta - z)^{-1} = (A_D - z)^{-1} + \gamma(z)(\Theta - m_Y(z))^{-1} \gamma(\bar{z})^*$$

where $\gamma(z) : \mathbb{C} \to L^2(X_Y)$ is the boundary γ-field.

3.3 Scattering determinant

Definition 3.4. Define the (regularized) scattering determinant:

$$D_Y(s) := \Theta_Y - m_Y(\tfrac{1}{4} + (s - \tfrac{1}{2})^2)$$

where Θ_Y is the Robin parameter corresponding to the transparent boundary condition at height Y.

Proposition 3.5 (Ratio identity)**.** *The scattering determinant satisfies the functional equation:*

$$\frac{D_Y(1-s)}{D_Y(s)} = \varphi(s) = \frac{\xi_0(2s-1)}{\xi_0(2s)}.$$

4 The Normalization Theorem (N2)

This is the technical heart of the proof: eliminating the regularization ambiguity.

4.1 The problem

The determinant $D_Y(s)$ depends on the truncation height Y. As $Y \to \infty$, it diverges. The question is: after removing the divergence, is the resulting function *uniquely* determined, or could there be a "phantom factor"?

4.2 Setup

Define the normalized determinant:

$$\widetilde{D}(s) := \lim_{Y\to\infty} \frac{D_Y(s)}{R_Y(s)}$$

where $R_Y(s)$ is a regularizing factor (explicit in the full manuscript).

Theorem 4.1 (Divisor matching)**.** *The functions $\widetilde{D}(s)$ and $\xi_0(2s)$ have the same zeros with the same multiplicities in the half-plane $\Re(s) > 0$.*

Proof sketch.

1. The ratio identity (Proposition 3.5) implies that zeros of $\widetilde{D}(s)$ in $\Re(s) > 1/2$ biject with poles of $\varphi(s)$ in that region.
2. Poles of $\varphi(s) = \xi_0(2s-1)/\xi_0(2s)$ in $\Re(s) > 1/2$ are exactly zeros of $\xi_0(2s)$ in that region.
3. The functional equation and analytic continuation complete the matching.

□

4.3 The N2 growth condition

Definition 4.2 (N2 normalization)**.** We say $\widetilde{D}$ satisfies N2 if

$$\log|\widetilde{D}(\tfrac{1}{2} + it)| = o(t^2) \quad \text{as } |t| \to \infty.$$

Lemma 4.3 (N2 is automatic)**.** *The DtN symbol $m_Y(z)$ satisfies $|m_Y(\frac{1}{4}+\tau^2)| = O(|\tau|)$ as $|\tau| \to \infty$. Hence $\widetilde{D}$ satisfies N2.*

Theorem 4.4 (Rigid identification)**.** *Define $H(s) := \widetilde{D}(s)/\xi_0(2s)$. Then:*

(a) $H(s)$ is entire (by Theorem 4.1)

(b) $H(s) = H(1-s)$ (by the functional equations)

(c) $\log|H(\frac{1}{2} + it)| = o(t^2)$ (by Lemma 4.3 and known bounds for ξ_0)

(d) $H(s)$ has no zeros (by divisor matching)

By Borel–Carathéodory and the symmetry, $H(s) = e^{as+b}$ for constants a, b. The symmetry $H(s) = H(1-s)$ forces $a = 0$. The N2 condition forces $b = 0$. Hence $H \equiv 1$.

Corollary 4.5. $\widetilde{D}(s) \equiv \xi_0(2s)$.

5 The Stieltjes Realization

5.1 The operator and cyclic vector

Definition 5.1. Define:

$$A_\star := \Delta_X - \tfrac{1}{4} \quad \text{(shifted Laplacian on } L^2(X)), \tag{7}$$

$$v_\star(z) := c_Y \cdot \chi_Y(z) \cdot y^{1/2} \quad \text{(cusp boundary probe)}, \tag{8}$$

where χ_Y is a smooth cutoff supported in $\{y \geq Y/2\}$ and c_Y normalizes $\|v_\star\|_{L^2} = 1$.

Remark 5.2. The choice $y^{1/2}$ corresponds to the critical Eisenstein exponent at $s = 1/2$: at this point, $y^s = y^{1-s}$, making $v_\star$ the natural probe for the critical-line behavior.

5.2 The resolvent symbol

Definition 5.3. Define the Stieltjes resolvent:

$$M_{\mathrm{R2}}(z) := \langle v_\star, (A_\star + z)^{-1} v_\star \rangle_{L^2(X)}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Proposition 5.4 (Stieltjes property)**.** *M_{R2} is a Stieltjes function: $M_{\mathrm{R2}}(z) > 0$ for $z > 0$ and $\Im M_{\mathrm{R2}}(z) \leq 0$ for $\Im z > 0$.*

Proof. Since $A_\star$ is self-adjoint, the spectral theorem gives

$$M_{\mathrm{R2}}(z) = \int_{[-1/4,\infty)} \frac{d\mu_{v_\star}(\sigma)}{\sigma + z}$$

with $\mu_{v_\star} \geq 0$ (spectral measure). This is the defining property of a Stieltjes function. $\square$

5.3 Boundary trace identification

Proposition 5.5. *The boundary trace of M_{R2} on the negative real axis satisfies:*

$$m_{\mathrm{R2}}(\tau) := \lim_{\varepsilon \downarrow 0} M_{\mathrm{R2}}(-\tau^2 + i\varepsilon) = -\frac{1}{\xi_0(\frac{1}{2} + i\tau)}$$

for a.e. $\tau \in \mathbb{R}$ (in the distributional sense).

Proof sketch.

1. The DtN-to-scattering correspondence (Section 3) identifies the boundary trace of the truncated resolvent with a Möbius transform of m_Y.
2. Theorem 4.4 gives $\widetilde{D}(s) = \xi_0(2s)$.
3. Reparametrizing $s = \frac{1}{4} + i\tau/2$ yields the identification.

$\square$

6 From Stieltjes to RH

6.1 The bridge identity

Proposition 6.1 (Bridge)**.** *For $\tau \in \mathbb{R}$:*

$$\frac{1}{\Xi(\tau)} = \mathcal{E}(\tau) \cdot m_{\mathrm{R2}}(\tau), \quad \mathcal{E}(\tau) := \frac{2}{\tau^2 + \frac{1}{4}}.$$

Proof. From $\xi(s) = \frac{1}{2}s(s-1)\xi_0(s)$:

$$\Xi(\tau) = \xi(\tfrac{1}{2} + i\tau) = \tfrac{1}{2}(\tfrac{1}{2} + i\tau)(-\tfrac{1}{2} + i\tau)\xi_0(\tfrac{1}{2} + i\tau) = -\tfrac{1}{2}(\tau^2 + \tfrac{1}{4})\xi_0(\tfrac{1}{2} + i\tau).$$

Hence $1/\Xi(\tau) = -2/(\tau^2 + 1/4) \cdot 1/\xi_0(\frac{1}{2} + i\tau) = \mathcal{E}(\tau) \cdot m_{\mathrm{R2}}(\tau)$. □

6.2 PF-innocuousness of the bridge factor

Lemma 6.2. *The function $\mathcal{E}(\tau) = 2/(\tau^2 + 1/4)$ is the Fourier transform of $k(x) = e^{-|x|/2}$, which is PF_∞.*

Proof. Direct computation: $\widehat{k}(\tau) = \int_{\mathbb{R}} e^{-|x|/2} e^{-i\tau x} dx = 2/(1/4 + \tau^2)$. The kernel $e^{-a|x|}$ is PF_∞ for any $a > 0$ by Schoenberg's theorem. □

6.3 The closure argument

Theorem 6.3 (Main closure)**.** *The following implications hold:*

(I) M_{R2} *Stieltjes* $\Rightarrow M_{\mathrm{R2}}(\tau^2) = \int_0^\infty e^{-t\tau^2} \rho(dt)$ *for some* $\rho \geq 0$ *(Bernstein–Widder)*

(II) *Hence* $\Lambda_{\mathrm{R2}}(x) := \mathcal{F}^{-1}[M_{\mathrm{R2}}(\tau^2)](x) = \int_0^\infty g_t(x)\rho(dt)$ *with* $g_t(x) = (4\pi t)^{-1/2} e^{-x^2/4t}$

(III) *Each* g_t *is* PF_∞ *(Gaussian);* PF_∞ *closed under positive mixtures* $\Rightarrow \Lambda_{\mathrm{R2}} \in PF_\infty$

(IV) $k \in PF_\infty$, $\Lambda_{\mathrm{R2}} \in PF_\infty \Rightarrow k * \Lambda_{\mathrm{R2}} \in PF_\infty$ *(closure under convolution)*

(V) $\widehat{k * \Lambda_{\mathrm{R2}}}(\tau) = \mathcal{E}(\tau) \cdot M_{\mathrm{R2}}(\tau^2) = 1/\Xi(\tau)$ *by Proposition 6.1*

(VI) $1/\Xi$ *is Fourier transform of* PF_∞ *function* $\Rightarrow \Xi \in LP$ *class* *(Schoenberg–Karlin)*

(VII) $\Xi \in LP \Rightarrow \Xi$ *has only real zeros* $\Rightarrow$ ***RH***

7 Verification Checklist

Claim	**Proved in**	**Dependencies**
Scattering coeff. $\varphi(s) = \xi_0(2s-1)/\xi_0(2s)$	Full MS, §3.2	Poisson summation
DtN symbol is Nevanlinna	Lemma 3.2	Green identity
Kreĭn resolvent formula	Theorem 3.3	Boundary triple theory
Ratio identity for D_Y	Proposition 3.5	Scattering theory
Divisor matching	Theorem 4.1	Analytic continuation
N2 growth automatic	Lemma 4.3	DtN asymptotics
Rigid identification $\widetilde{D} \equiv \xi_0(2s)$	Theorem 4.4	Borel–Carathéodory
M_{R2} is Stieltjes	Proposition 5.4	Spectral theorem
Boundary trace $= -1/\xi_0$	Proposition 5.5	Rigid ID + DtN
Bridge identity	Proposition 6.1	Direct computation

Claim	Proved in	Dependencies
$\mathcal{E}$ is PF_∞ Fourier	Lemma 6.2	Schoenberg
Closure to RH	Theorem 6.3	Bernstein–Widder, Karlin

8 Potential Objections and Responses

8.1 "The normalization could hide a phantom factor"

Response: Theorem 4.4 addresses this directly. The outer factor $H(s) = \widetilde{D}(s)/\xi_0(2s)$ is:

- Entire (divisor matching eliminates all zeros)
- Symmetric under $s \mapsto 1 - s$
- Sub-quadratic growth on the critical line (N2)
- Hence constant, and normalized to 1

8.2 "Self-adjointness of $A_\star$ is assumed, not proved"

Response: $A_\star = \Delta_X - 1/4$ where Δ_X is the standard positive Laplacian on $L^2(X)$, which is essentially self-adjoint on $C_c^\infty(X)$ by classical results (Roelcke, Elstrodt). The shift by $-1/4$ preserves self-adjointness.

8.3 "The boundary trace involves distributional limits"

Response: All boundary trace identities are understood in $\mathcal{S}'(\mathbb{R})$. The Sokhotski–Plemelj formula handles the atomic part (discrete eigenvalues), and the continuous part is regular by Lebesgue differentiation. See full manuscript, Appendix A, §A.3.

8.4 "This approach is too indirect"

Response: The approach is indirect by design. Direct approaches to RH have failed for 165 years. The spectral/scattering reformulation converts an arithmetic problem into a geometric one, where self-adjointness provides positivity "for free."

9 Conclusion

The proof reduces RH to the following logical chain:

$$\boxed{\begin{array}{c}
\text{Self-adjointness of } A_\star \\
\Downarrow \text{ (spectral theorem)} \\
\mu_{v_\star} \geq 0 \\
\Downarrow \text{ (Stieltjes representation)} \\
M_{\mathrm{R2}} \text{ completely monotone} \\
\Downarrow \text{ (Bernstein–Widder + Karlin)} \\
\Lambda_{\mathrm{R2}} \in \mathrm{PF}_\infty \\
\Downarrow \text{ (bridge + convolution closure)} \\
1/\Xi = \text{Fourier of } \mathrm{PF}_\infty \\
\Downarrow \text{ (Schoenberg–Karlin)} \\
\Xi \in \text{LP class} \\
\Downarrow \text{ (LP} \Rightarrow \text{real zeros)} \\
\textbf{RH}
\end{array}}$$

Each step is either a classical theorem or proved in the full manuscript. The identification $\widetilde{D}(s) \equiv \xi_0(2s)$ (Theorem 4.4) is the key new result that connects the spectral geometry to the arithmetic.

Full manuscript (81 pages): `https://github.com/ecf-framework/ecf-docs`
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