Galileo: A Pseudospectral Collocation Framework for Legged Robots

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***Abstract*— Athletic maneuvers for legged robots present a difficult challenge due to the complex dynamics and contact constraints. This paper introduces a versatile trajectory optimization framework for continuous problems over multiple phases. We introduce a new transcription scheme that enables pseudospectral collocation to optimize directly on Lie Groups, such as SE(3) and quaternions without special normalization constraints. The key insight is the change of variables—we choose to optimize over the history of the tangent vectors rather than the states themselves. Our approach uses a modified Legendre-Gauss-Radau (LGR) method to produce dynamic motions for various legged robots. We implement our approach in a Model Predictive Control (MPC) context and track the MPC output using a Quadratic Program (QP) based whole-body controller. Results on the Go1 Unitree and WPI's HURON humanoid confirm the feasibility of the planned trajectories.**

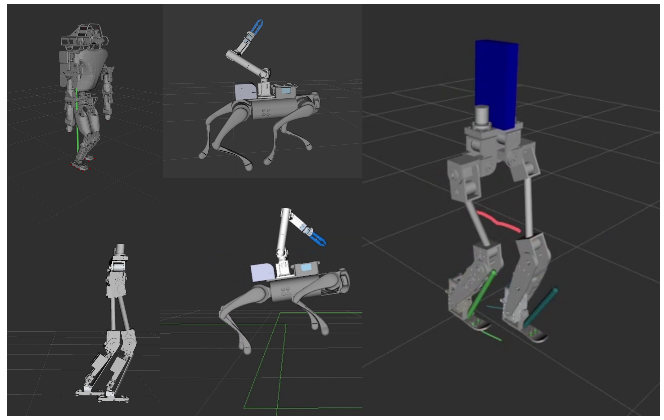

Fig. 1: Galileo MPC solving legged robot motions at 50Hz, simulated in Gazebo. Video results shown in [13].

I. Introduction

Trajectory synthesis presents a fundamental challenge in robotics, particularly in the domain of legged robots. The main objective is to determine control inputs that guide a robot from an initial to a desired state while navigating a constraint-rich state space. These restrictions motivate the need to automatically generate valid trajectories for dynamic motions with an approach that can be generalized to various types of robots.

A. Literature Review

We have categorized prior studies into three subsections: dynamic modeling, contact modeling, and trajectory optimization for legged robots.

1) Dynamic Modeling: To formulate the optimal control problem, it is important to select an appropriate dynamic model to represent the system, as it will influence the complexity and fidelity of the solution. [1] proposed an online Zero Moment Point (ZMP)-based optimization scheme that generates trajectories between contact polygons, but is incapable of generating full flight phases. As a result, the robot may experience difficulties in traversing uneven terrain or navigating obstacles. In [2], [3], the authors used a linearized Single Rigid Body (SRB) model to enable full flight phases in a compact quadratic program but did not take into account the roll and pitch angles of the base which can lead to loss of stability and inaccuracies in foot placement. In [4]–[6], the full nonlinear SRB was employed, but the former two opt for Euler angle representations, which introduces singularities in the orientation Jacobian. [6] utilized the full nonlinear SO(3) representation in their SRB model, but required a second trajectory optimization layer with full rigid-body dynamics to compensate for inaccuracies in the SRB dynamics, leading to longer solve times. Each of these works using the SRB model neglected leg inertia, which reduced their ability to perform dynamic motions.

On the other hand, higher fidelity models such as centroidal momentum dynamics [7] were used in approaches such as [8]–[11]. The parameterization of the joint variables allows for a crisp description of kinematic constraints while still keeping the problem size small compared to the full rigid-body dynamics [12]. The centroidal model also accounts for inertia in the legs, overcoming the limitations of the SRB model. However, all of these approaches rely on an Euler angle representation, preventing them from fully exploiting truly acrobatic motions due to the singularity in the orientation Jacobian.

2) Contact Modeling: One of the most important aspects of legged robots is planning the discrete contact sequence. Mixed integer approaches such as [14], [15] plan the contact sequence by segmenting the obstacle-free space into convex regions, which can then be optimized using MICP. Mixed integer formulations are attractive because they directly tackle the binary nature of contact decisions. Unfortunately, mixed-integer programs are computationally heavy and typically require binary constraint relaxations for convergence, making them largely unsuitable for real-time applications—despite their impressive offline performance in works such as [16].

In [17], the authors introduced an approach based on the complementary constraints of unilateral contact. These complementary constraints are defined by the condition

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between the contact force and the distance to contact (i.e., contact force can only exist if the distance to contact is zero, and, likewise, if the distance to contact is nonzero, the contact force must be zero). This technique was further refined in [18], [19]. However, these approaches do not satisfy the Linear Independence Constraint Qualification (LICQ), which most off-the-shelf nonlinear program solvers assume. [20] takes advantage of the intuition that every leg will alternate between stance and swing phases, and as such, the contact sequence optimization can be turned into an entirely continuous problem by merely optimizing the timings of each leg's phase independently. This yields an elegant formulation that is mathematically identical to the LCP approaches but without the need for a specialized solver. However, this approach is difficult to implement and relies on a continuous approximation of contact surfaces, leading to poor convergence for discontinuous surfaces such as stairs.

Recently, holistic approaches [21], [22] that solve the discrete contact sequence and continuous trajectory optimization problem simultaneously have aroused attention. Most notably, [23] proposed an approach that combines trajectory optimization with an informed graph search coupled with sampling-based planning. Our paper introduces an alternative switched system solver that is suitable for the bilevel optimization introduced in [23].

3) Trajectory Optimization: Literature in trajectory optimization for legged robots can generally be grouped into two schools of thought: Shooting methods, and collocation methods.

Collocation methods solve the trajectory optimization problem via function approximation [24]. In this paper, we will focus on pseudospectral collocation [25]–[27], which is traditionally formed around the Lagrange basis polynomials. At its core, pseudospectral collocation achieves convergence by simultaneously optimizing the state and the controls at the Lagrange polynomial roots, which are used as the collocation points. Pseudospectral collocation works by matching the state dynamics at each of the collocation points with a defect constraint and approximating the integral of the cost function using Gaussian quadrature. This defect matching yields an implicit integrator, which is known to have improved numerical stability over explicit methods [28]. Unfortunately, the pseudospectral collocation methods presented in these works do not immediately lend themselves to interpolation over manifolds.

We introduce a simple yet holistic solution to this problem by optimizing over the tangent space. Optimizing over the tangent space rather than the state manifold is not new—stemming primarily from approaches seeking to parameterize the $SO(3)$ rotation matrix such as [3], [6], [29]—however, these approaches are only performed using the forward Euler transcription method, which is known to have substantial drift over practical time horizons [30]. Furthermore, the choice of dynamic model in these works severely limits their ability to fully exploit the full orientation state space, due to the neglect of leg inertias. We believe that it is a combination of optimization methods and choice of system model that enables truly dynamic behaviors.

Our contributions include: 1) Galileo: A lightweight, extensible, and open-source optimization library for switched systems, based on LGR pseudospectral collocation. The library is open-source and freely available at [13].

1) A new transcription method to enable pseudospectral collocation over differentiable manifolds, without the need for additional normalization constraints.
2) A Galileo MPC interface for legged robots, including hardware and simulation examples for Unitree Go1 quadruped, Atlas, and WPI's HURON.

II. GALILEO

The purpose of Galileo is to transcribe continuous-time, multi-phase trajectory optimization problems into ones solvable by nonlinear program (NLP) solvers, such as IPOPT [31] or SNOPT [32].

A. Multi-phase Optimization Problem Formulation

We seek to solve the following class of trajectory optimization problems:

$$\underset{\mathbf{x},\mathbf{u}}{\text{minimize}} \quad \sum_{p=1}^{P}\left[\Phi_p(\mathbf{x}(t_{f,p})) + \int_{t_{0,p}}^{t_{f,p}} \mathcal{L}_p(\mathbf{x}(t),\mathbf{u}(t))dt\right] \tag{1}$$

$$\begin{aligned}
\text{subject to}\quad & \dot{\mathbf{x}}(t) = \mathbf{f}_p(\mathbf{x}(t),\mathbf{u}(t)) \qquad \forall t \in [t_{0,p}, t_{f,p}] && (2)\\
& \mathbf{c}_p(\mathbf{x}(t),\mathbf{u}(t)) \leq \mathbf{0} \qquad \forall t \in [t_{0,p}, t_{f,p}] && (3)\\
& \mathbf{b}_p(\mathbf{x}(t_{0,p}),\mathbf{x}(t_{f,p})) = \mathbf{0} && (4)\\
& \mathbf{x}(t_{0,p+1}) = \mathbf{S}_p(\mathbf{x}(t_{f,p})) && (5)\\
& \forall p \in [1, ..., P]
\end{aligned}$$

This is known as multi-phase Bolza form [33]. Where $t_{0,p}$ and $t_{f,p}$ represent the boundary times of phase p. Here, (1) represents the costs, (2) the system flow map, (3) the inequality constraints associated with phase p, and (4) the boundary constraints. $\mathcal{L}_p$ and Φ_p are the Lagrange and Mayer terms for phase p, respectively, and are used to apply incremental and terminal costs. (5) is the jump map—a constraint enacted on the transitions between phases.

Overall, the optimization seeks a trajectory $\mathbf{x}$ and control inputs $\mathbf{u}$ that minimize the cost while satisfying the active constraints in each phase. Efficiently solving this class of trajectory optimization problems is the focus of this paper.

B. Galileo: Optimizing with Deviations

Practical applications in robotics often require subsets of the decision variables to lie on differentiable manifolds. Quaternions, for instance, are required to stay on a hypersphere in $\mathbb{R}^4$, which is difficult to enforce without employing explicit normalization constraints at each iteration.

Our method directly optimizes over these differentiable manifolds—without the use of additional equality constraints—through a change in decision variables. Instead of including explicit states (orientations, in our example) in our decision variables, we use the state deviants, which are tangents on the state manifold. These state deviants will henceforth be referred to with $\tilde{\mathbf{x}}$.

Thus, our new decision variables are now $\tilde{\mathbf{x}}$ and $\mathbf{u}$. Modifying the Gaussian quadrature used in [24], we transcribe the continuous-time objective (1) with the following objective function

$$J = \sum_{p=0}^{P}\Bigg[\Phi_p(\mathbf{x}_{N_p}) + \sum_{k=0}^{N_p}\Bigg[\sum_{j=0}^{d_{\tilde{x}}}\Bigg[B_{\tilde{x},j}\mathcal{L}_p(\mathbf{F}_{\text{exp}}(\mathbf{x}_0, \tilde{\mathbf{x}}_{k,j}), \mathbf{u}(\tau_{\tilde{x},j}))\Bigg]\Bigg]\Bigg] \tag{6}$$

where $\mathbf{F}_{\text{exp}}$ is the differentiable manifold's exponential map, and $\tilde{\mathbf{x}}_{k,j}$ refers to the state deviant at the k-th knot segment and j-th collocation point, determined by the Legendre-Gauss-Radau (LGR) points in our work. Here, $B_{\tilde{x},j}$ is the j-th element of the quadrature coefficients for the Lagrange polynomial of degree $d_{\tilde{x}}$. $\mathbf{u}(\tau_{\tilde{x},j})$ refers to the control input at the state deviant collocation time $\tau_{\tilde{x},j}$ (mapped to the range $[-1, 1]$) on the control input's Lagrange polynomial. For numerical accuracy, this control input value is determined using Barycentric Lagrange interpolation.

In order to ensure the collocated trajectory satisfies the system dynamics, we transcribe (2) with the collocation constraint

$$h \cdot \mathbf{f}_p(\mathbf{F}_{\text{exp}}(\mathbf{x}_0, \tilde{\mathbf{x}}_{k,j}), \mathbf{u}(\tau_{\tilde{x},j})) - \mathbf{D}_{\tilde{x}}\tilde{\mathbf{x}} = 0 \tag{7}$$

where $\mathbf{D}_{\tilde{x}}$ is the differentiation matrix corresponding to the Lagrange polynomial of degree $d_{\tilde{x}}$ and h is the timestep. Transcription of the inequality (3) and boundary constraints (4) follows similar logic.

Recall that LGR points do not have a collocation point at the end of the $[-1, 1]$ interval. Thus, we use a continuity constraint that ensures that, within each phase, the expression for the state deviant at the end of each knot segment is equal to the state deviant value at the knot point at the start of the next knot segment. This can be written with the equality constraint

$$\tilde{\mathbf{x}}_{k+1,0} = \sum_{j=0}^{d_{\tilde{x}}} \mathbf{C}_{\tilde{x}}\tilde{\mathbf{x}}_{k,j} \tag{8}$$

where $\mathbf{C}_{\tilde{x}}$ is the vector of continuity coefficients, with elements corresponding to the Lagrange basis polynomials of degree $d_{\tilde{x}}$ evaluated at $t = 1$.

The number of decision variables for our transcription is $\sum_{p=0}^{P}\left(N_p(n_{\tilde{x}}d_{\tilde{x}} + n_u d_u) + (N_p + 1)(n_{\tilde{x}} + n_u)\right)$ and the number of equality constraints is $\sum_{p=0}^{P}\left(n_x + N_p(n_x d_x + n_{\tilde{x}} + n_u)\right)$. This is in contrast to normalization-based methods which add an additional $\sum_{p=0}^{P}\left(N_p(n_x d_x)\right)$ equality constraints to ensure that the state lies on the manifold.

III. Galileo for Legged Robots

We will now show how our transcription scheme can be used to control legged robots.

A. Centroidal Momentum Dynamics

Let the state be given by $\mathbf{x} = [\mathbf{k}_{\text{com}}, \mathbf{q}_b, \mathbf{q}_j]^\intercal$ where $\mathbf{k}_{\text{com}}$ is the centroidal momentum about the center of mass, $\mathbf{q}_b$ is the base position and orientation w.r.t the fixed inertial frame, and $\mathbf{q}_j$ is the joint configuration. We choose to represent the base orientation as a quaternion to overcome the singularity presented by Euler angles, and the overparameterization of decision variables caused by rotation matrices. For the remainder of this paper, we shall refer to the quaternion component of $\mathbf{q}_b$ with $\mathbf{q}_{bo}$.

For the control input, we define $\mathbf{u} = [\mathbf{f}_{\text{wrench,i}}, \mathbf{v}_j]^\intercal$ where $\mathbf{f}_{\text{wrench,i}}$ is the stacked contact force $\mathbf{f}_{e_i}$ and contact torque $\boldsymbol{\tau}_{e_i}$ for end effectors i with contact patches, or simply the contact force at the i-th end effector if it is a point contact. $\mathbf{v}_j$ is the joint velocity variable.

The centroidal momentum dynamics [7] can then be written as

$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} \sum_{i=0}^{n_e} \mathbf{f}_{e_i} + m\mathbf{g} \\ \sum_{i=0}^{n_e}(\mathbf{r}_{\text{com},e_i}(\mathbf{q}) \times \mathbf{f}_{e_i}) + \boldsymbol{\tau}_{e_i} \\ \mathbf{A}_b^{-1}(\mathbf{q})\,(\mathbf{h}_{\text{com}} - \mathbf{A}_j(\mathbf{q})\dot{\mathbf{q}}_j) \\ \mathbf{v}_j \end{bmatrix} \tag{9}$$

where $\mathbf{q} = [\mathbf{q}_b, \mathbf{q}_j]^\intercal$, and $\mathbf{A}(\mathbf{q})$ is the centroidal momentum matrix taken at the center of mass and aligned with the fixed inertial frame. It should be noted that $\mathbf{r}_{\text{com},e_i}$ is the i-th end effector's foot position w.r.t the current center of mass position.

We use this definition for the dynamics of the legged robot due to the physical accuracy of the resulting dynamics when compared to simpler template models such as the Single Rigid Body (SRB) model, and the reduced computation cost compared to the whole-body dynamics.

B. Quaternion Parameterization for Legged Robot Pose

For the legged robot problem, we use position vectors and orientation quaternions to describe the pose. Thus, $\mathbf{q}_{bo}$ is restricted to lie on $\mathcal{H}$, which is a hypersphere in $\mathcal{R}^4$.

Given our previously defined state definition (III-A), we define the state deviants with $\tilde{\mathbf{x}} = \left[\tilde{\mathbf{k}}_{\text{com}}, \tilde{\mathbf{q}}_b, \tilde{\mathbf{q}}_j\right]^\intercal$ where $\tilde{\mathbf{q}}_b$ can be intuitively described as the concatenated base linear and angular velocity unit vectors required to transform the body from its initial pose to its current pose. This transformation can be described with the pose exponential maps.

The quaternion exponential function corresponds to the map $\exp_{q_{bo}} : \mathcal{R}^3 \mapsto \mathcal{H}$. This map is given by

$$\exp_{q_{bo}}(\boldsymbol{\theta}) = \cos(||\boldsymbol{\theta}||) + \frac{\sin(||\boldsymbol{\theta}||)}{||\boldsymbol{\theta}||}\boldsymbol{\theta} \cdot \mathbf{i} \tag{10}$$

where $\boldsymbol{\theta} \in \mathcal{R}^3$, and $\mathbf{i}$ is the imaginary unit vector in $\mathcal{H}$.

The new quaternion after the transformation $\boldsymbol{\omega}$ is then given by $\mathbf{q}_{bo}(t + \delta t) = \mathbf{q}_r(t) \otimes \exp_{q_{bo}}(\mathbf{q}_{bo,\omega}(t))$

where $\mathbf{q}_{bo,\omega}$ is the unit quaternion given by applying an angular velocity to the identity rotation over a timestep δt. This transformation is given by $\mathbf{q}_{bo,\omega} = \boldsymbol{\omega} \cdot \delta t/2 \cdot \mathbf{i}$.

To get the integrated position, we make use of the Rodrigues rotation formula $\mathbf{p}(t+\delta t) = \mathbf{p}(t) + \mathbf{q}_{bo}(t)\mathbf{R}\mathbf{v}(t)\cdot\delta t$ where $\mathbf{R}$ is the Rodrigues rotation matrix obtained from $\boldsymbol{\omega}(t)$. Note that the second term makes use of the quaternion vector rotation formula.

After some derivations, we have

$$\mathbf{F}_{\text{exp}}(\mathbf{x}(t), \tilde{\mathbf{x}}(t), \delta t) = \begin{bmatrix} \mathbf{k}_{\text{com}}(t) + \tilde{\mathbf{k}}_{\text{com}}(t)\cdot\delta t \\ \mathbf{p}(t) + \mathbf{q}_{bo}(t)\mathbf{R}\mathbf{v}(t)\cdot\delta t \\ \mathbf{q}_{bo}(t) \otimes \exp_{q_{bo}}(\mathbf{q}_{bo,\omega}(t)) \\ \mathbf{q}_j + \tilde{\mathbf{q}}_j\cdot\delta t \end{bmatrix} \quad (11)$$

for legged robots using the centroidal momentum model[1].

IV. Results

To showcase the feasibility of our method, we developed Galileo [13], a powerful C++ MPC framework for legged robots, and tested our approach in simulation (Gazebo) for the Unitree Go1 quadruped. Our work results in 50Hz MPC trajectories, which rivals state-of-the-art such as [8] and [34]. For the whole-body controller and perception, we use the framework introduced in [**?**] to track our MPC trajectories and generate steppable region constraints, which allows us to perform stairclimbing behaviors as shown in Fig. 1 (video in [13]).

V. Conclusions and Future Work

We presented Galileo [13], a new C++ library for robot trajectory optimization, using a modified pseudospectral collocation method to optimize over state manifolds such as SE(3). In future work, we would like to pursue three distinct avenues: Automating the generation of initial guesses to improve convergence speed, using variation-based linearization on the history of the state deviants, and segmenting the nonconvex dynamics into convex regions, which can be optimized over in a Graph of Convex Sets (GCS) framework.

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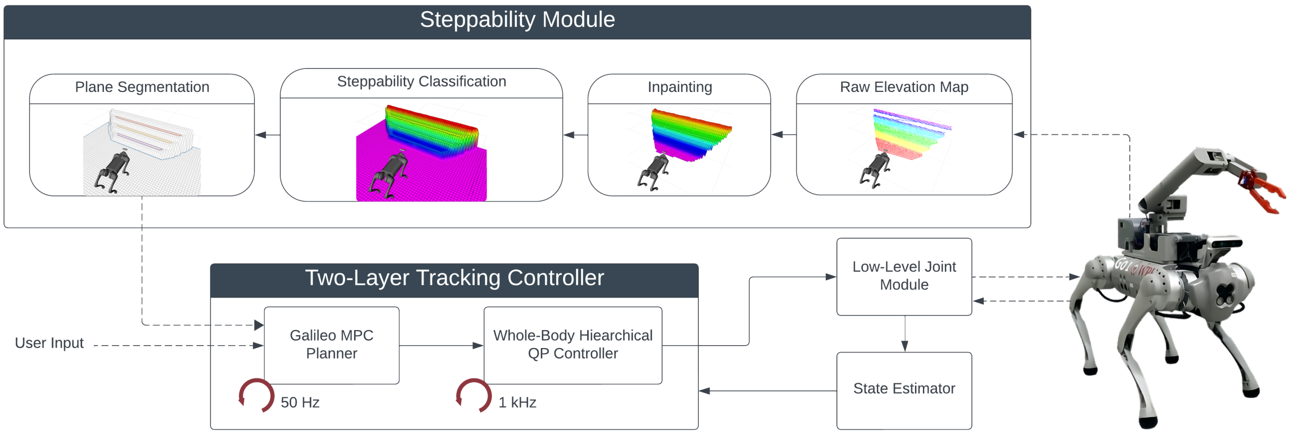

Fig. 2: Placeholder text

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