Name: ______________________________

1. (20 points) Use the known values of the function $f(x) = \sin(x)$ (with $y = f(x)$) at $x = 0, \pi/6, \pi/4, \pi/3$ and $\pi/2$ in order to derive an interpolating polynomial $p(x)$ using a **monomial basis**, i.e., $\phi_j(x) = x^j$. What is the **degree** of your polynomial? What is the **interpolation error magnitude** $|p(1.2) - \sin(1.2)|$? Make a plot of your data points and the underlying interpolating polynomial for $x \in [0, \pi/2]$ **on the same graph**.

   Attach any codes/scripts producing your figure and the figure itself.

2. (20 points) Assume the data pairs $\{(x_i, y_i)\}_{i=0}^n$ together with the functions:

$$\begin{aligned} \rho_j &= \prod_{i \neq j} (x_j - x_i), \quad j = 0, 1, \ldots, n, \\ \psi(x) &= \prod_{i=0}^{n} (x - x_i). \end{aligned}$$

   (a) (12 points) Show that:

$$\rho_j = \psi'(x_j).$$

   (b) (8 points) While using Lagrange interpolation, show that the interpolating polynomial of degree at most n can be written as

$$p_n(x) = \psi(x) \sum_{j=0}^{n} \frac{y_j}{(x - x_j)\, \psi'(x_j)}.$$

3. (10 points) Consider the data set $\{x_i\}_{i=0}^n$ containing $(n+1)$ **distinct points** and the corresponding **Lagrange basis functions** $\{L_i(x)\}_{i=0}^n$. Then, prove that

$$\sum_{j=0}^{n} L_j(x) = 1.$$

   Hint: Consider interpolating the function $f(x) = 1$ at the points given!

4. (30 points) Let $f(x) = 1/x$ and data points $x_0 = 2$, $x_1 = 3$ and $x_2 = 4$. Note that you can use the abscissae to find the corresponding ordinates.
   (a) (8 points) Find **by hand** the **Lagrange form**, the **standard form**, and the **Newton form** of the interpolating polynomial $p_2(x)$ of $f(x)$ at the given points. **State which is which!** Then, expand out the Newton and Lagrange form to verify that they agree with the standard form of p_2 that you obtained [this is true due to the **uniqueness of polynomial interpolation**!]. Also, verify that $p_2(x_i) = f(x_i)$ for $i = 0, 1, 2$.
   (b) (10 points) Use the **Polynomial Interpolation Error** theorem to find an upper bound for the error
$$||f - p_2||_\infty = \max_{2 \le x \le 4} |f(x) - p_2(x)|.$$
   (c) (12 points) Find the **exact value** of $||f - p_2||_\infty$ to at least 5 decimal places of accuracy. Of course, the answer should be less than or equal to the upper bound you found in part (b).

5. (20 points) For some function f, the divided difference table is given:

i	x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	1	$f[x_0]$	–	–	–
1	5	$f[x_1]$	$f[x_0, x_1]$	–	–
2	6	4	0	$-1/4$	–
3	4	2	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Fill in the **unknown entries** in the table.

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