Name: ____________________

1. (45 points) Evaluate by **hand** the **definite integrals** below. Subsequently, **approximate** them by using the **composite Trapezoidal** rule $T_n(f)$ for $n = 2, 4, 8, 16, 32$ and 64, where $h = (b - a)/n$.

   (a) (15 points) $I = \displaystyle\int_0^1 (3x + 1)\, dx$

   (b) (15 points) $I = \displaystyle\int_0^1 xe^{-x^2}\, dx$

   (c) (15 points) $I = \displaystyle\int_0^{2\pi} (\cos(x) + 1)\, dx$

   To do so, write a MATLAB function (or in **any** other programming language) stored as `trap.m` whose first line should read

   ```
   function [ y ] = trap( f, a, b, n )
   ```

   Include a copy of your code. For each of the cases above make a table with columns:

   - column 1: n
   - column 2: h
   - column 3: T_n (this is the approximation, i.e., the output `y`)
   - column 4: |error| (this is the **absolute** error)
   - column 5: $|\text{error}|/h^2$

   Are the numbers in the last column **converging**, and if so, **what does this mean**? Specifically, comment on the behavior of the error for (a) and (b). If your code is correct, you will notice that for (c) the last column is **not** converging, and that the approximation is **very accurate**. Can you explain **briefly** why?

2. (25 points) Consider the basic **Simpson's rule**:

$$S_2(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right], \quad \text{with} \quad h = \frac{b-a}{2}. \tag{1}$$

This approximation satisfies:

$$E(f) = -\frac{f^{(4)}(\eta)}{90} h^5, \tag{2}$$

for some $\eta \in [a, b]$ which implies that Simpson's rule is **exact** if $f(x)$ is a polynomial of degree <= 3, i.e., $p_n(x)$ for $0 \le n \le 3$.

(a) (15 points) **Derive** Simpson's rule, i.e., Eq. (1).

(b) (10 points) Consider $f(x) = x^3$. Then, find I_{Simp} (**by hand** !) which approximates $I_f = \int_a^b f(x)\, dx$. Is the answer exact? **Justify your answer**.

3. (45 points) Repeat the numerical evaluation of the integrals in parts (a)-(c) of Question 1 using the **composite Simpson's** rule $S_n(f)$ for $n = 2, 4, 8, 16, 32$ and 64, where $h = (b - a)/n$. To do so, write a MATLAB function (or in **any** other programming language) stored as `simp.m` whose first line should read

```
function [ y ] = simp( f, a, b, n )
```

Similary as in Question 1, and for each case, make a table with the same format as in Question 1 except of column 5 in which you must compute $|\text{error}|/h^4$. Include a copy of your code and compare your results with the **composite Trapezoidal** rule.

4. (15 points) Besides the Trapezoidal and Simpson's rules, there exists another approximation to $\int_a^b f(x)\, dx$ by replacing $f(x)$ with the **constant** $f((a + b)/2)$ on $[a, b]$.

(a) (5 points) Show that the aforementioned process leads to the numerical integration formula

$$M_1(f) = (b - a)\, f\left(\frac{a + b}{2}\right).$$

This is called the **basic Midpoint rule**.

(b) (10 points) Derive the **composite version** of the Midpoint rule

$$M_n(f) = h\left[f(x_1) + f(x_2) + \ldots + f(x_n)\right],$$

where

$$h = (b - a)/n, \quad x_j = a + (j - 1/2)\, h, \quad j = 1, \ldots, n.$$

5. (20 points) Consider the integral of part (b) of Question 1:

$$\int_0^1 x e^{-x^2}\, dx.$$

and the **error formulas** for the Trapezoidal and Simpson's rules:

$$E_n^T(f) \equiv \int_a^b f(x)\,dx - T_n(f) = -\frac{h^2(b-a)}{12} f''(c_n),$$
$$E_n^S(f) \equiv \int_a^b f(x)\,dx - S_n(f) = -\frac{h^4(b-a)}{180} f^{(4)}(c_n),$$

respectively, where $c_n \in [a, b]$. Then, find the value of n (number of sub-intervals) to be chosen in order to ensure

$$|E_n^T(f)| \leq 10^{-4}, \quad |E_n^S(f)| \leq 10^{-4}.$$

Compare your theoretical results with the numerical findings of Question 1.

Hint: You should find the extrema of $f''(x)$ and $f^{(4)}(x)$ and determine their maximum values. To do so, graphing $f''(x)$ and $f^{(4)}(x)$ might help and you could use Newton's method to find their extrema!

6. (20 points) Use the **method of undetermined coefficients** to derive the approximation for the first derivative of a function $f(x)$ at x_0:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2).$$

Study the error in the above formula and determine it in terms of $f(x)$ and its derivatives. Then, if $f(x) = e^x$ and $x_0 = 1/2$ write a MATLAB script which does the following: approximates the first derivative of $f(x)$ given using the above formula and shows in a **log-log** plot the **absolute error** by halving h at each step, i.e., use $h = 0.1$, $h = 0.05$, $h = 0.025$, $h = 0.0125$, and so on. In addition, the script should plot the theoretical prediction for the error on top of the previous graph in order to compare the findings. Finally, show that your approximation has $O(h^2)$ truncation error, i.e., the order of accuracy is 2. When your approximation starts getting worse and **why does this happen**?

Hints: Note that if error $\approx Ch^q$ where q is the order of accuracy, then $\log(\text{error}) \approx \log C + q \log h$. Therefore in the log-log plot, you must determine the slope of the straight line obtained therein which eventually is the value of q. Recall that for a straight line of the form of $y = b + ax$, two points can be selected on the x-axis, e.g., x_1 and x_2 and we can find the corresponding ordinates $y_1 = y(x_1)$ and $y_2 = y(x_2)$. Then, we obtain the slope via

$$a = \frac{y_2 - y_1}{x_2 - x_1}.$$

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