Math 451 **Homework Assignment #8 (Due on 03/13)**

Name: ____

1. (15 points) In class, we discussed iterative schemes based on a specific **splitting** of an $n \times n$ matrix A to solve the linear system $A\mathbf{x} = \mathbf{b}$.

   (a) (5 points) If $A = M - N$, then show that the following schemes are equivalent:

$$\begin{aligned} M\mathbf{x}_{k+1} &= N\mathbf{x}_k + \mathbf{b}; \\ \mathbf{x}_{k+1} &= (\mathbb{I} - M^{-1}A)\mathbf{x}_k + M^{-1}\mathbf{b}, \quad \text{where } \mathbb{I} \text{ is the } n \times n \text{ identity matrix;} \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + M^{-1}\mathbf{r}_k, \quad \text{where} \quad \mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k. \end{aligned}$$

   (b) (10 points) A totally **equivalent splitting** of A that we discussed is of the form: $A = L + U + D$. Note that, L is **strictly** lower triangular, U **strictly** upper triangular and D diagonal. This way, the Jacobi method reads

$$\boxed{\mathbf{x}_{k+1} = R_J\mathbf{x}_k + D^{-1}\mathbf{b},}$$

   with R_J the Jacobi iteration matrix $R_J = -D^{-1}(L + U)$.
   If A is **strictly diagonally dominant**, show that the Jacobi iteration matrix satisfies

$$\|R_J\|_\infty < 1.$$

   Note that if this condition holds, then the Jacobi method converges for **any** initial vector $\mathbf{x}_0$.

   Hints:

   - An $n \times n$ matrix A is strictly diagonally dominant if $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ holds.
   - Note that if A is an $n \times n$ matrix, then $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$.

2. (a) (10 points) Find **by hand** the **eigenvalues**, **eigenvectors** and **spectral radius** of the following matrices:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

You may use MATLAB's `eig` command to **verify** your answers. Furthermore, you can find the spectral radius of a matrix easily using MATLAB by typing `max(abs(eig))`.

(b) (15 points) Find **all** the values of a and b for which the matrix

$$A = \begin{bmatrix} a & 1 & 1+b \\ 1 & a & 1 \\ 1-b^2 & 1 & a \end{bmatrix}$$

is **symmetric positive definite**.

3. (30 points) The linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

has the unique solution $\mathbf{x} = [1\,1]^T$.

(a) (10 points) Determine **by hand** the $R_J = -D^{-1}(L+U)$ and $R_{GS} = -(L+D)^{-1}U$, that is, the Jacobi and Gauss-Seidel iteration matrices, respectively (of course you may use MATLAB to **verify** your answers).

(b) (5 points) Find the ∞-norm and spectral radius of R_J and R_{GS}.

(c) (15 points) Perform **5 iterations** of both Jacobi and Gauss-Seidel methods using $\mathbf{x}_0 = [0\,0]^T$. For each present your results in a table with the following format:

- column 1: k (iteration step)
- column 2: $x_1^{(k)}$ (1st component of the computed solution vector at step k)
- column 3: $x_2^{(k)}$ (2nd component of the computed solution vector at step k)
- column 4: $\|e^{(k)}\|_\infty$ (error norm at step k)
- column 5: $\|e^{(k)}\|_\infty / \|e^{(k-1)}\|_\infty$ (ratio of successive error norms at step k).

Which method is converging faster? Attach any of your codes and **justify your answer**.

4. (20 points) Employ the Successive Over-Relaxation (SOR) method to solve the linear system

$$\begin{aligned} 2x_1 - x_2 &= 5, \\ -x_1 + 2x_2 - x_3 &= -2, \\ -x_2 + 2x_3 &= 2, \end{aligned}$$

with $\omega = 1.3$ and initial vector $\mathbf{x}_0 = [0\,0\,0]^T$. Stop the iterations when $\|\mathbf{r}_k\|_2 \leq \mathrm{tol}\,\|\mathbf{b}\|_2$ holds with $\mathrm{tol} = 10^{-10}$. Provide your MATLAB code and output which includes the solution.

5. (20 points) Assume that $\omega \in [0.5, 1.8]$ and notice that the case with $\omega < 1$ corresponds to the Successive **Under**-Relaxation and $\omega > 1$ to SOR. Also, when $\omega = 1$, this is the original Gauss-Seidel method.

   Make a graph of the spectral radius of the iteration matrix:

$$R_{SOR} = (\omega L + D)^{-1} \left[(1-\omega) D - \omega U\right],$$

   for the matrix A given in Question 4 as a function of ω. What is the **optimal value** of ω here, i.e., ω_{opt}? **Verify your answer** by running your code developed in Question 4 for $\omega = \omega_{\text{opt}}$. Include its output together with the figure for $\rho(R_{SOR})$ as a function of ω and the code producing it.

6. (20 points) Use your codes developed for the **Jacobi**, **Gauss-Seidel**, and **SOR** (with $\omega = 1.1$) iterative methods to solve the following linear system of equations:

$$\begin{bmatrix} 7 & 1 & -1 & 2 \\ 1 & 8 & 0 & -2 \\ -1 & 0 & 4 & -1 \\ 2 & -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \\ -3 \end{bmatrix}.$$

   Stop the iterations when $\|\mathbf{r}_k\|_2 \leq \text{tol}\,\|\mathbf{b}\|_2$ holds with tol $= 10^{-10}$. As per the initial guess (for **all** methods), use the **zero** vector, i.e., $\mathbf{x}_0 = [0\,0\,0\,0]^T$. Make a graph in a **semilog scale** showcasing the $\|\mathbf{r}_k\|_2$ against the number of iterations k in each case and compare your findings. Include **all** your codes, MATLAB output and solution.

7. (20 points) Implement the Conjugate Gradient (CG) method in MATLAB (or in any other scientific programming language). To do so, write an m-file `my_cg.m`, the first line of which should be:

```
function xk = my_cg( A, b, x0, tol, nmax )
```

   Test your code with the linear system given by

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix},$$

   using initial vector $\mathbf{x}_0 = [0\,0\,0]^T$. Stop the iterations when $\langle \mathbf{r}_k, \mathbf{r}_k \rangle \leq \text{tol}^2 \langle \mathbf{b}, \mathbf{b} \rangle$ holds and tol $= 10^{-10}$. Include **all** your codes and MATLAB output.

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