Name: ______________________________

1. (20 points) Use the known values of the function $f(x) = \sin(x)$ (with $y = f(x)$) at $x = 0, \pi/6, \pi/4, \pi/3$ and $\pi/2$ in order to derive an interpolating polynomial $p(x)$ using a **monomial basis**, i.e., $\phi_j(x) = x^j$. What is the **degree** of your polynomial? What is the **interpolation error magnitude** $|p(1.2) - \sin(1.2)|$? Make a plot of your data points and the underlying interpolating polynomial for $x \in [0, \pi/2]$ **on the same graph**.

   Attach any codes/scripts producing your figure and the figure itself.

Solution: Let us create a table containing our data points $\{(x_i, y_i)\}_{i=0}^4$:

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
y	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

Since the number of data points is 5, we look for an interpolating polynomial of degree 4:

$$p_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4, \tag{1}$$

where the unknown coefficients $\{c_i\}_{i=0}^4$ can be obtained via the **interpolation conditions**. Recall that the latter forms a linear system for the coefficients which can be solved easily in MATLAB by using a very simple script, reminiscent of the one discussed in class (and employing the "backslash operator!):

```
clear all; close all; clc; format long;

% Setup and interpolation conditions:
x = [0;pi/6;pi/4;pi/3;pi/2];
y = sin(x);
coeff = [1,x(1),x(1)^2,x(1)^3,x(1)^4;...
         1,x(2),x(2)^2,x(2)^3,x(2)^4;...
         1,x(3),x(3)^2,x(3)^3,x(3)^4;...
         1,x(4),x(4)^2,x(4)^3,x(4)^4;...
```

```
          1,x(5),x(5)^2,x(5)^3,x(5)^4;];

% Use backslash to solve the system:
c = coeff \ y;

% Then, construct the interpolating polynomial in question:
p4 = @(x) c(1) + c(2) * x + c(3) * x.^2 ...
        + c(4) * x.^3 + c(5) * x.^4;

% Finally, plot the outcome:
figure;
set(gca,'FontSize',24);
scatter(x,y,'*k','LineWidth',2);
hold on;
xx = linspace(x(1),x(end),100);
plot(xx,p4(xx),'-r','LineWidth',2);
xlabel('$x$','Interpreter','latex');
ylabel('$y$','Interpreter','latex');
```

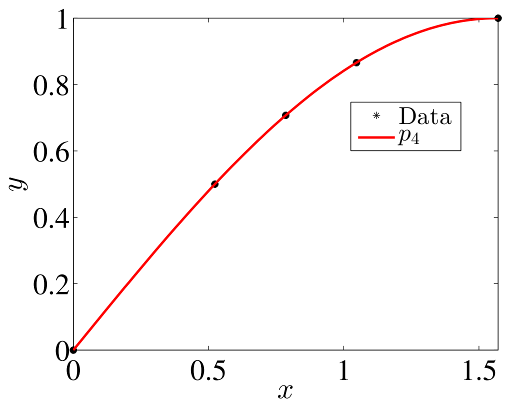

Figure 1: The interpolating polynomial $p_4(x)$ together with the data points (see, the legend).

The plot in question is shown in Fig. 1. Finally, the interpolation error magnitude is:

```
>> abs(p4(1.2)-sin(1.2))

ans =

     1.024356771470725e-04
```

2. (20 points) Assume the data pairs $\{(x_i, y_i)\}_{i=0}^{n}$ together with the functions:

$$\begin{aligned} \rho_j &= \prod_{i \neq j} (x_j - x_i), \quad j = 0, 1, \dots, n, \\ \psi(x) &= \prod_{i=0}^{n} (x - x_i). \end{aligned}$$

   (a) (12 points) Show that:
$$\rho_j = \psi'(x_j).$$

   (b) (8 points) While using Lagrange interpolation, show that the interpolating polynomial of degree at most n can be written as
$$p_n(x) = \psi(x) \sum_{j=0}^{n} \frac{y_j}{(x - x_j)\,\psi'(x_j)}.$$

Solution:

(a) Let us calculate ψ' directly:

$$\begin{aligned} \psi'(x) &= (x - x_1)(x - x_2) + \cdots + (x - x_n) \\ &+ (x - x_0)(x - x_2) + \cdots + (x - x_n) \\ &+ \dots \\ &+ (x - x_0)(x - x_1) + \cdots + (x - x_{n-2})(x - x_{n-1}) \Rightarrow \\ \psi'(x) &= \prod_{i=1}^{n} (x - x_i) + \prod_{\substack{i=0 \\ i \neq 1}}^{n} (x - x_i) + \cdots + \prod_{i=0}^{n-1} (x - x_i), \end{aligned}$$

or, in compact form

$$\boxed{\psi'(x) = \sum_{k=0}^{n} \left[\prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i) \right]}. \tag{2}$$

However, at $x = x_j$ with $j = 0, 1, \dots, n$, only the $k = j$ term survives in Eq. (2). Since, $i \neq k$ and $k = j$, then $i \neq j$ and Eq. (2) yields to

$$\psi'(x_j) = \prod_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i) \Rightarrow \boxed{\psi'(x_j) = \rho_j}.$$

(b) Based on Lagrange interpolation, we have that

$$p(x) = \sum_{j=0}^{n} y_j L_j = \sum_{j=0}^{n} \left[\prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)} \right] y_j = \sum_{j=0}^{n} \left[\frac{\prod_{\substack{i=0 \\ i \neq j}}^{n} (x - x_i)}{\rho_j \, (x - x_j)} \right] y_j,$$

and using the definition of $\psi(x)$ and its derivative (in terms of ρ_j) we obtain:

$$\boxed{p(x) = \psi(x) \sum_{j=0}^{n} \frac{y_j}{\psi'(x_j)\,(x - x_j)}}.$$

3. (10 points) Consider the data set $\{x_i\}_{i=0}^{n}$ containing $(n+1)$ **distinct points** and the corresponding **Lagrange basis functions** $\{L_i(x)\}_{i=0}^{n}$. Then, prove that

$$\sum_{j=0}^{n} L_j(x) = 1.$$

Hint: Consider interpolating the function $f(x) = 1$ at the points given!

Solution: Since the hint suggests $f(x) = 1$, the Lagrange form of the polynomial $p_n(x)$ of degree $\leq n$ that interpolates $f(x)$ at the points given reads

$$p_n(x) = \sum_{j=0}^{n} \cancelto{1}{f(x_j)} L_j(x) = \sum_{j=0}^{n} L_j(x).$$

Now consider the polynomial $p(x) = p_n(x) - 1$, which has degree $\leq n$. However, $p(x_j) = p_n(x_j) - 1 = 0$ since $p_n(x)$ interpolates the function $f(x) = 1$ at the points x_j with $j = 0, 1, \ldots n$. The latter implies that $p(x)$ has at least $n+1$ roots. Since $deg(p(x)) \leq n$, by the **Fundamental Theorem of Algebra** it can have at most n roots unless it is the zero polynomial. Thus, we have $p(x) \equiv 0$, or:

$$\boxed{p_n(x) = \sum_{j=0}^{n} L_j(x) = 1, \quad \forall x}. \quad \blacksquare$$

4. (30 points) Let $f(x) = 1/x$ and data points $x_0 = 2$, $x_1 = 3$ and $x_2 = 4$. Note that you can use the abscissae to find the corresponding ordinates.

(a) (8 points) Find **by hand** the **Lagrange form**, the **standard form**, and the **Newton form** of the interpolating polynomial $p_2(x)$ of $f(x)$ at the given points. **State which is which!** Then, expand out the Newton and Lagrange form to verify that they agree with the standard form of p_2 that you obtained [this is true due to the **uniqueness of polynomial interpolation**!]. Also, verify that $p_2(x_i) = f(x_i)$ for $i = 0, 1, 2$.

(b) (10 points) Use the **Polynomial Interpolation Error** theorem to find an upper bound for the error

$$||f - p_2||_\infty = \max_{2\le x\le 4} |f(x) - p_2(x)|.$$

(c) (12 points) Find the **exact value** of $||f - p_2||_\infty$ to at least 5 decimal places of accuracy. Of course, the answer should be less than or equal to the upper bound you found in part (b).

Solution:

(a) At first, we have the following table:

x	2	3	4
y	1/2	1/3	1/4

Then, the Lagrange form is:

$$\begin{aligned} p_2(x) &= \sum_{j=0}^{2} y_j L_j(x) \\ &= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= \frac{1}{4}(x-3)(x-4) - \frac{1}{3}(x-2)(x-4) + \frac{1}{8}(x-2)(x-3), \end{aligned}$$

or, expanding out terms we arrive at

$$\boxed{p_2(x) = \frac{13}{12} - \frac{3}{8}x + \frac{1}{24}x^2}. \tag{3}$$

A quick check reveals that $p_2(x_i) = f(x_i)$ holds for $i = 0, 1, 2$.

Next, by utilizing the formulas for the divided differences in our case we end up with the corresponding divided difference table:

i	x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$
0	2	1/2	–	–
1	3	1/3	−1/6	–
2	4	1/4	−1/12	1/24

Thus, extracting its diagonal entries yields the coefficients for the Newton interpolating polynomial given by

$$p_2 = \frac{1}{2} - \frac{1}{6}(x-2) + \frac{1}{24}(x-2)(x-3), \tag{4}$$

which is precisely (after some algebra) the same polynomial as the one given by Eq. (3).

(b) Next, the error expression for polynomial interpolation reads

$$|f(x) - p_2(x)| = \frac{|f^{(3)}(\xi)|}{3!}\left|\prod_{i=0}^{2}(x-x_i)\right|, \quad \xi \in [2,4].$$

Let us bound each term in the rhs separately:

- $|f^{(3)}(x)| = |6/x^4|$ and is **maximized** in the interval $[2,4]$ when $x = 2$.
- $\left|\prod_{i=0}^{2}(x-x_i)\right| = \left|(x-2)(x-3)(x-4)\right|$, and let us introduce $g(x)$ being the function inside the absolute value: $g(x) = -24 + 26x - 9x^2 + x^3$. The latter has **extrema**, i.e., solutions to $g'(x) = 26 - 18x + 3x^2 = 0$ at $x^* = 3 \pm \sqrt{3}/3$. Plugging either of the extrema we get $|g(x^*)| = |0.3849001794597484 \approx 0.4|$ since we have the absolute value.

Based on the above, we are ready to apply the **Polynomial Interpolation Error** theorem to bound the error. This way, we have that

$$||f - p_2||_\infty = \max_{2\le x\le 4}|f(x) - p_2(x)| \le \frac{1}{6}\max_{2\le \xi\le 4}|f^{(3)}(\xi)|\,\frac{2}{5} = \frac{1}{40}.$$

Thus, an upper bound on the error is $1/40 = 0.025$.

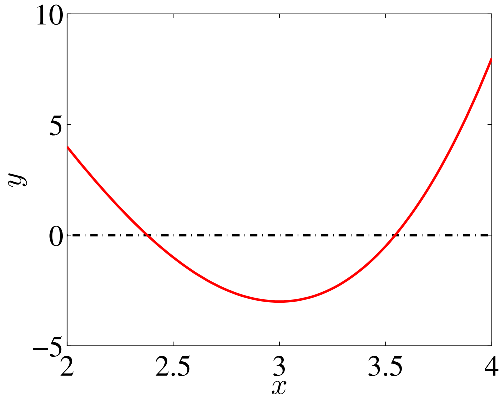

Figure 2: Graph of the numerator revealing the existence of **two** roots.

(c) In the last part, we introduce the function $w(x) = f(x) - p_2(x) \Rightarrow w(x) = -\frac{13}{12} + \frac{1}{x} + \frac{3}{8}x - \frac{1}{24}x^2$ and our task is to maximize $|w(x)|$ in the interval $[2,4]$.

That means, we have to find the **extrema** of $w(x)$ first, which in turn, corresponds to the **zeros** of

$$w'(x) = \frac{3}{8} - \frac{1}{x^2} - \frac{x}{12} = -\frac{2x^3 - 9x^2 + 24}{24x^2},$$

or equivalently, the zeros of the numerator (since the denominator is not equal to zero for the interval consider herein). This problem is precisely a **root-finding** one and we can use, say, **Newton's method** to find the fixed points! For our convenience, the graph of the numerator is shown in Fig. 2 which reveals the existence of two roots. Using Newton's method with initial guesses: $x_0^{(1)} = 2.2$ and $x_0^{(2)} = 3.6$ we obtain the corresponding fixed points: $x_1^* = 2.378075758565231$ and $x_2^* = 3.545260255335104$. Finally, the respective values of $w(x)$ are $w(x_1^*) = 6.682053593859871 \times 10^{-3}$ and $w(x_2^*) = 4.503073410654035 \times 10^{-3}$, thus:

$$\max_{2\le x\le 4} |f(x) - p_2(x)| = \max_{2\le x\le 4} |w(x)| = 6.682053593859871 \times 10^{-3}.$$

It should be pointed out (in line with theory) that this is **less** than the upper bound of $1/40 = 0.025$ found in part (b).

5. (20 points) For some function f, the divided difference table is given:

i	x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	1	$f[x_0]$	–	–	–
1	5	$f[x_1]$	$f[x_0,x_1]$	–	–
2	6	4	0	$-1/4$	–
3	4	2	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]$

Fill in the **unknown entries** in the table.

Solution: Initially, we observe that $f[x_1,x_2] = 0$ which implies

$$0 = \frac{f[x_2] - f[x_1]}{x_2 - x_1} \Rightarrow f[x_1] = f[x_2] \Rightarrow \boxed{f[x_1] = 4}. \tag{5}$$

Furthermore, the $f[x_2,x_3]$ is given by

$$f[x_2,x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} \Rightarrow \boxed{f[x_2,x_3] = 1}. \tag{6}$$

Next, we use the fact that $f[x_0, x_1, x_2] = -1/4$; however, the formula for $f[x_0, x_1, x_2]$ reads

$$f[x_0, x_1, x_2] = \frac{\cancelto{0}{f[x_1, x_2]} - f[x_0, x_1]}{x_2 - x_0} \Rightarrow \frac{1}{4} = \frac{f[x_0, x_1]}{5} \Rightarrow \boxed{f[x_0, x_1] = \frac{5}{4}}. \qquad (7)$$

Note that from Eqs. (7) and (5), the $f[x_0]$ can be obtained as follows:

$$f[x_0, x_1] = \frac{5}{4} \Rightarrow \frac{5}{4} = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \Rightarrow \boxed{f[x_0] = -1}. \qquad (8)$$

We can now easily find $f[x_1, x_2, x_3]$ via

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - \cancelto{0}{f[x_1, x_2]}}{x_3 - x_1} \Rightarrow \boxed{f[x_1, x_2, x_3] = -1}. \qquad (9)$$

Finally, and as per the $f[x_0, x_1, x_2, x_3]$, we have

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \Rightarrow \boxed{f[x_0, x_1, x_2, x_3] = -\frac{1}{4}}, \qquad (10)$$

and from Eqs. (5)-(10), the divided difference table becomes

i	x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	1	-1	$-$	$-$	$-$
1	5	4	$5/4$	$-$	$-$
2	6	4	0	$-1/4$	$-$
3	4	2	1	-1	$-1/4$

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