Name: ____________________

1. (30 points) Consider the data points $\{(0,1),(1,1),(2,5)\}$. Then:

   (a) (10 points) Find the **piecewise linear** interpolating function for the data.

   (b) (10 points) Find the **quadratic** interpolating polynomial.

   (c) (10 points) Find the **natural cubic spline** that interpolates the data.

Solution:

(a) We denote the points given as $\{(x_0,y_0),(x_1,y_1),(x_2,y_2)\}$. Then, we construct the piecewise linear interpolating functions $p_1^{(1)}(x)$ and $p_1^{(2)}(x)$ for the data sets $\{(x_0,y_0),(x_1,y_1)\}$ and $\{(x_1,y_1),(x_2,y_2)\}$, respectively. This way, we have

$$p_1^{(1)}(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \Rightarrow p_1^{(1)}(x) = 1 + \frac{1-1}{1}(x-0) \Rightarrow \boxed{p_1^{(1)}(x) = 1},$$

and

$$p_1^{(2)}(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Rightarrow p_1^{(2)}(x) = 1 + \frac{5-1}{2-1}(x-1) \Rightarrow \boxed{p_1^{(2)}(x) = 4x - 3},$$

which suggest

$$\boxed{s(x) = \begin{cases} 1, & x \in [0,1], \\ 4x - 3, & x \in [1,2]. \end{cases}}$$

(b) As per the quadratic interpolating polynomial, we use Lagrange interpolation such that

$$
\begin{aligned}
p_2(x) &= \sum_{j=0}^{2} y_j L_j(x) \\
&= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= \frac{(x-1)(x-2)}{(-1)(-2)} + \frac{x(x-2)}{(1-2)} + 5\frac{x(x-1)}{2} \Rightarrow \\
p_2(x) &= \frac{1}{2}(x-1)(x-2) - x(x-2) + \frac{5}{2}x(x-1) \Rightarrow \\
&\boxed{p_2(x) = 2x^2 - 2x + 1}.
\end{aligned}
$$

(c) For the natural cubic spline, we first re-define the points given as $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$. Then, we make use of the formula discussed in class

$$
\begin{aligned}
s(x) = {} & \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x) y_{j-1} + (x - x_{j-1}) y_j}{x_j - x_{j-1}} \\
& - \frac{x_j - x_{j-1}}{6}[(x_j - x) M_{j-1} + (x - x_{j-1}) M_j],
\end{aligned}
$$

for $x_{j-1} \leq x \leq x_j$, and

$$
\begin{aligned}
& \frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1} \\
& = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j+1}}, \quad j = 2, 3, \ldots n-1,
\end{aligned}
$$

supplemented with $M_1 = M_n = 0$. In our case, $M_1 = M_3 = 0$! Indeed, for $j = 2$ we have

$$
\begin{aligned}
\frac{x_2 - x_1}{6} M_1 + \frac{x_3 - x_1}{3} M_2 + \frac{x_3 - x_2}{6} M_3 &= \frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_3} \Rightarrow \\
\frac{2}{3} M_2 &= \frac{4}{1} \Rightarrow \\
&\boxed{M_2 = 6}.
\end{aligned}
$$

This way, the natural cubic spline that interpolates the data in $[0, 1]$ $(j = 2)$

is given by

$$
\begin{aligned}
s^{(1)}(x) &= \frac{(x_2-x)^3M_1+(x-x_1)^3M_2}{6(x_2-x_1)} + \frac{(x_2-x)y_1+(x-x_1)y_2}{x_2-x_1} \\
&\quad - \frac{x_2-x_1}{6}[(x_2-x)M_1+(x-x_1)M_2] \Rightarrow \\
s^{(1)}(x) &= x^3 + \frac{(1-x)+x}{1} - x \Rightarrow \\
&\boxed{s^{(1)}(x) = x^3 - x + 1}.
\end{aligned}
$$

Similarly, the natural cubic spline that interpolates the data in $[1,2]$ ($j=3$) is given by

$$
\begin{aligned}
s^{(2)}(x) &= \frac{(x_3-x)^3M_2+(x-x_2)^3M_3}{6(x_3-x_2)} + \frac{(x_3-x)y_2+(x-x_2)y_3}{x_3-x_2} \\
&\quad - \frac{x_3-x_2}{6}[(x_3-x)M_2+(x-x_2)M_3] \Rightarrow \\
s^{(2)}(x) &= \frac{6(2-x)^3}{6} + \frac{2-x+5(x-1)}{1} - \frac{1}{6}(2-x)6 \Rightarrow \\
s^{(2)}(x) &= (2-x)^3 + 4x - 3 - 2 + x \Rightarrow \\
&\boxed{s^{(2)}(x) = (2-x)^3 + 5(x-1)}.
\end{aligned}
$$

The above suggest the natural cubic spline

$$
\boxed{s(x) = \begin{cases} x^3 - x + 1, & x \in [0,1], \\ (2-x)^3 + 5(x-1), & x \in [1,2]. \end{cases}}
$$

2. (20 points) Is the following function a cubic spline on the interval $0 \le x \le 2$?

$$
s(x) = \begin{cases} (x-1)^3, & x \in [0,1], \\ 2(x-1)^3, & x \in [1,2]. \end{cases}
$$

Solution: Let $f_1(x) = (x-1)^3$ and $f_2(x) = 2(x-1)^3$. If those functions satisfy the following conditions

$$
f_1(1) = f_2(1), \quad f_1'(1) = f_2'(1), \quad f_1''(1) = f_2''(1),
$$

then the function given is a cubic spline.

Indeed, we can easily check first that $f_1(1) = f_2(1) \equiv 0$. Then, $f_1'(x) = 3(x-1)^2$ and $f_2'(x) = 6(x-1)^2$ such that $f_1'(1) = f_2'(1) \equiv 0$. Finally, $f_1''(x) = 6(x-1)$ and $f_2''(x) = 12(x-1)$, and this way, $f_1''(1) = f_2''(1) \equiv 0$. Thus, we conclude that the function $s(x)$ is indeed a cubic spline!

3. (30 points) Consider the function $f(x) = \sin(x)$ of Question 1 of HW4.

   (a) (15 points) Interpolate the function $f(x)$ at 5 Chebyshev points over the interval $[0, \pi/2]$ and compare your results with those of Question 1 of HW4, i.e., evaluate the interpolation error magnitude $|p(1.2) - \sin(1.2)|$. Also, plot your data points and the underlying interpolating polynomial for $x \in [0, \pi/2]$ in the same figure.

   (b) (15 points) Repeat the interpolation, although use 5 Chebyshev points over the interval $[0, \pi]$ this time. Plot $f(x)$ at the Chebyshev points as well as the interpolant for $x \in [0, \pi]$. What are your conclusions?

Hints:

- Recall the definition of Chebyshev points! Also there exists a transformation, mapping $x \in [-1, 1]$ onto $\widetilde{x} \in [a, b]$ according to

$$\widetilde{x} = \frac{a+b}{2} + \frac{b-a}{2} x.$$

- Use the m-files **divdif.m** and **evalnewt.m** to construct the interpolant.

Solution:

(a) The script constructing the interpolant in this case is given as follows:

```
clearvars; close all; clc; format long;

% Setup and interval [a,b]:
n = 4;       % This is the n in the formula (5.22)!
a = 0;
b = pi/2;

% Chebyshev points and affine transformation:
  ii = 0:1:n;
  xi = cos( (2 * ii + 1 )*pi/ (2*(n+1)));
 xor = a + 0.5 * ( b - a ) * ( xi + 1);
 yor = sin(xor);              % Obtain the ordinates.
coef = divdif(xor,yor);      % Obtain the coefficients.
```

```
% Evaluate the interpolant obtained:
x = linspace(0,pi/2,101);
y = evalnewt(x,xor,coef);

% Plot the result:
figure;
set(gca,'FontSize',24,'Fontname','Times');
scatter(xor,yor,'*k','LineWidth',2);
hold on;
plot(x,y,'-r','LineWidth',2);
xlabel('$x$','Interpreter','latex');
ylabel('$y$','Interpreter','latex');
```

We compare our results in the left panel of Fig. 1. Furthermore, the interpolation error magnitude in this case is:

```
>> abs(evalnewt(1.2,xor,coef)-sin(1.2))

ans =

     3.537202267278605e-05
```

which is smaller than the one obtained in Question 1.

(b) Simply, we modify `b=pi;` and the vector `x = linspace(0,pi,101);` in the above script where the comparison is highlighted in the right panel of Fig. 1. Finally the interpolation error magnitude is:

```
>> abs(evalnewt(1.2,xor,coef)-sin(1.2))

ans =

     2.699341157743618e-04
```

which has been increased slightly. Recall our discussion about errors in polynomial interpolation. In this example, the width of the interval has been increased, although the number of Chebyshev points was kept fixed ($n = 5$). Thus, we may need to add more points in the latter case to maintain desired accuracy.

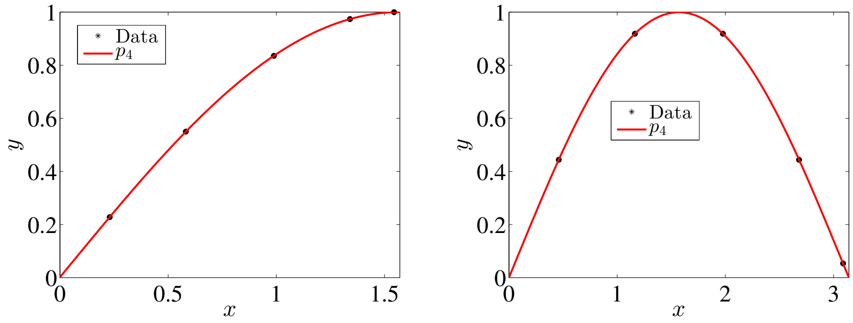

Figure 1: The interpolating polynomial $p_4(x)$ using 5 Chebyshev points together with the data points of Question 6 (see, the legend as well) for the respective cases: $[0, \pi/2]$ (left panel) and $[0, \pi]$ (right panel).

4. (20 points) Find the **linear** least squares approximation to $f(x) = e^x$ on $[-1, 1]$.

Solution:

Since $n = 1$, from the least squares approximation we have

$$l_1(x) = \beta_0 P_0(x) + \beta_1 P_1(x),$$

where

$$\beta_j = \frac{(f, P_j)}{(P_j, P_j)}, \quad j = 1, 2,$$

and P_j stand for the Legendre polynomials, i.e., $P_0(x) = 1$ and $P_1(x) = x$. This way, we have

$$(f, P_0) = \int_{-1}^{1} e^x \, dx \Rightarrow (f, P_0) = e - e^{-1},$$
$$(f, P_1) = \int_{-1}^{1} xe^x \, dx \Rightarrow (f, P_1) = \frac{2}{e}.$$

Note also that $(P_0, P_0) = 2$ and $(P_1, P_1) = 2/3$ such that $\beta_0 = (e - e^{-1})/2$ and $\beta_1 = 3/e$, respectively. Therefore, the linear least squares approximation to e^x is

$$l_1(x) = \frac{e - e^{-1}}{2} + \frac{3}{e}x \Rightarrow \boxed{l_1(x) = 1.1752011936438014 + 1.103638323514327x}.$$

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