



Name: \_\_\_\_\_

1. (15 points) Let  $f(x) = x^2 - 4x + 3$ . Then:
- (a) (5 points) Find  $p_1(x)$ ,  $p_2(x)$  and  $p_3(x)$  around  $x_0 = 0$ . How  $P_3(x)$  is related to  $f(x)$ ?
  - (b) (5 points) Same as part (a) but consider  $x_0 = 1$ .
  - (c) (5 points) In general, given a polynomial  $f(x)$  with degree  $m$ , what can you say about  $f(x) - p_k(x)$  for  $k \geq m$ ?

**Solution:** Note that  $f'(x) = 2x - 4$ ,  $f''(x) = 2$  and  $f'''(x) = 0$ .

- (a) To this end, we employ Taylor's Theorem:

$$p_1(x) = f(0) + f'(0)x \Rightarrow p_1(x) = 3 - 4x,$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \Rightarrow p_2(x) = 3 - 4x + x^2,$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \Rightarrow p_3(x) = 3 - 4x + x^2,$$

since  $f'''(x) = 0$ . Thus,  $p_3(x) = f(x)$  and we must have  $R_3(x) = 0$ .

- (b) Similarly, around  $x_0 = 1$  we have:

$$p_1(x) = f(1) + f'(1)(x - 1) \Rightarrow p_1(x) = 2 - 2x,$$

$$p_2(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2 \Rightarrow p_2(x) = 3 - 4x + x^2,$$

$$p_3(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2 + \frac{f'''(1)}{6}x^3 \\ \Rightarrow p_3(x) = 3 - 4x + x^2,$$

and again as in part (a),  $p_3(x) = f(x)$  because  $f'''(x) = 0$  where  $R_3(x) = 0$ .

- (c) In that case, we will have that  $f(x) - p_k(x) = 0$  since  $f(x)$  is a polynomial of degree at most  $m$ , thus all the  $k + 1$  derivatives are zero, i.e.,  $f^{(k+1)}(x) = 0$  when  $k \geq m$ . In addition, the error term  $R_k(x)$  is **identically zero**.

2. (25 points) Given  $f(x) = \cos x$ , find both  $p_2(x)$  and  $p_3(x)$  about  $x_0 = 0$ , and use them to approximate  $\cos(0.1)$ . Show that in each case the remainder term provides an upper bound for the true (absolute) error.

**Solution:** At first, note that  $f'(x) = -\sin x$ ,  $f''(x) = -\cos x$ ,  $f'''(x) = \sin x$  as well as  $f^{(4)}(x) = \cos x$ . Next, we calculate the corresponding Taylor polynomials about  $x_0 = 0$ :

$$\begin{aligned} p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \Rightarrow p_2(x) = 1 - \frac{1}{2}x^2, \\ p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \\ &\Rightarrow p_3(x) = 1 - \frac{1}{2}x^2. \end{aligned}$$

Therefore,  $p_2(x) = p_3(x)$  and  $p_2(0.1) = 0.995$ . Since both Taylor polynomials are the same, the error in both cases is

$$|\cos(0.1) - 0.995| \approx 4.16528 \times 10^{-6}.$$

From the above Taylor polynomials, we have the respective remainder terms with  $\xi \in (0, 0.1)$  for the errors

$$\begin{aligned} |R_2(x)| &= \left| \frac{f^{(3)}(\xi)}{6} (0.1)^3 \right| = \left| \frac{\sin \xi}{6000} \right| \leq \frac{\sin(0.1)}{6000} \approx 1.66389 \times 10^{-5}, \\ |R_3(x)| &= \left| \frac{f^{(4)}(\xi)}{24} (0.1)^4 \right| = \left| \frac{\cos \xi}{240000} \right| \leq \frac{\cos(0)}{240000} \approx 4.16667 \times 10^{-6}. \end{aligned}$$

Thus, an upper bound derived using the error term for  $R_k$  of the respective cases is indeed **larger** than the actual error.

3. (30 points) If  $f(x) = e^x$ , then
- (10 points) derive the Maclaurin series of the function  $f(x) = e^x$ , i.e., the Taylor series about  $x_0 = 0$  (write **separately**  $p_k(x)$  and  $R_k(x)$ ),
  - (20 points) find a minimum value of  $k$  necessary for  $p_k(x)$  to approximate  $f(x)$  to within  $10^{-6}$  on the interval  $[0, 0.5]$  (here, you must use the remainder term).

**Solution:**

- (a) At first, it should be noted in passing that  $f^{(k)}(x) = e^x$ ,  $\forall k \geq 0$ . This way, from Taylor's Theorem we have that

$$\begin{aligned} f(x) &= p_k(x) + R_k(x) = \\ &= \sum_{n=0}^k \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(k+1)}(\xi)}{(k+1)!} x^{k+1} \\ &= \sum_{n=0}^k \frac{x^n}{n!} + \frac{f^{(k+1)}(\xi)}{(k+1)!} x^{k+1} \\ &= \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!}}_{p_k(x)} + \underbrace{\frac{e^\xi}{(k+1)!} x^{k+1}}_{R_k(x)}, \end{aligned}$$

for some  $\xi$ .

- (b) Based on the remainder from part (a), we want to find the integer value of  $k$  where  $\xi \in (0, 0.5)$  such that

$$\begin{aligned} \max_{x \in [0, 0.5]} |R_k(x)| &= \max_{x \in [0, 0.5]} \frac{e^\xi}{(k+1)!} x^{k+1} \leq \frac{e^{1/2}}{(k+1)!} \frac{1}{2^{k+1}} \leq 10^{-6} \\ \Rightarrow 2^{k+1}(k+1)! &\geq e^{1/2} \times 10^6 \approx 1.64872 \times 10^6. \end{aligned}$$

Typical values give  $2^7 \times 7! = 645120$  and  $2^8 \times 8! = 10321920$ , thus  $k+1 = 8 \Rightarrow k = 7$ , or we must have  $k \geq 7$ .

4. (20 points) Let  $f(x) = \sqrt[3]{x}$ . Does  $f(x)$  have a Taylor polynomial of degree 1 based on expanding about  $x = 0$  and  $x = 1$ ? Justify your answers. Include a copy of the graph of  $f(x)$  and its associated polynomials (when applicable) on the same figure, as well as your MATLAB script producing the figure.

**Solution:** From Taylor's approximation formula and for a polynomial of degree 1, we have

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

From  $f(x)$  given, we have that  $f'(x) = x^{-2/3}/3$ . However, the polynomial of degree 1, i.e.,  $p_1(x)$  of  $f(x)$  around  $x_0 = 0$  does **not** exist due to the fact that  $f(x)$  is **not** differentiable at  $x = 0$ . On the other hand, the  $p_1(x)$  around  $x = 1$  exists (with  $f'(1) = 1/3$ ) and is given by

$$p_1(x) = 1 + \frac{1}{3}(x - 1) \Rightarrow p_1(x) = \frac{1}{3}(x + 2).$$

Based on the above results, we conclude that  $f(x)$  does **not** have a Taylor polynomial approximation of degree 1 around  $x_0 = 0$  whereas  $f(x)$  does around  $x_0 = 1$ . Finally, the MATLAB code and its output for making the graph in question are given next

```

1 clearvars; close all; clc; format long;
2
3 % Declare f(x) and p_{1}(x):
4 func = @(x) x.^(1/3);
5 p1 = @(x) (x+2)/3;
6
7 % Introduce the interval [0.5,1.5]:
8 x = linspace(0.5,1.5,501);
9
10 % Make the figure:
11 figure;
12 plot(x, func(x), '-k', 'linewidth', 3);
13 hold on;
14 plot(x, p1(x), '-r', 'linewidth', 3);
15 xlabel('$x$', 'interpreter', 'latex');
16 hg = legend('$\sqrt[3]{x}$', '$p_{1}(x)$', ...
17             'interpreter', 'latex', 'fontname', 'times');
18 set(gca, 'fontsize', 24, 'fontname', 'times');

```

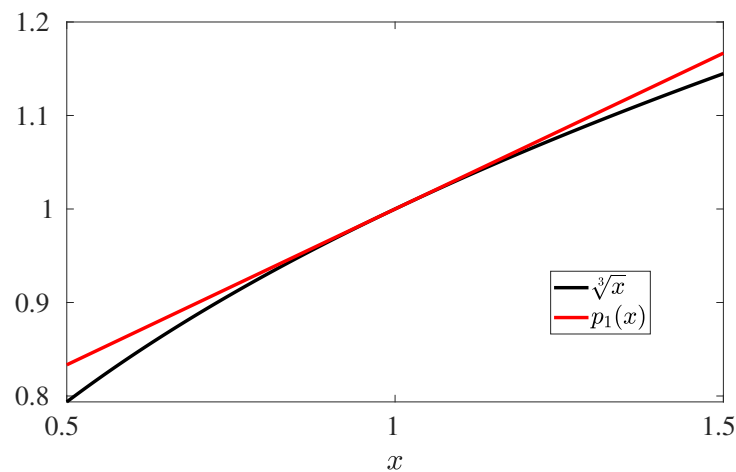


Figure 1: The graph of  $f(x)$  (solid black line) and  $p_1(x)$  (solid red line) corresponding to the polynomial around  $x_0 = 1$  (the one around  $x_0$  does **not** exist).

5. (10 points) Consider the polynomial

$$p(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \frac{x^{12}}{12!} - \frac{x^{15}}{15!}.$$

Evaluate  $p(x)$  as efficiently as possible. How many multiplications are necessary? Assume all coefficients have been computed and stored for later use.

**Solution:** In class, we discussed about nested multiplication applied to

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

which involves  $n$  additions and  $n$  multiplications. In our example, we first introduce  $z = x^3$  and write the polynomial given as

$$p(z) = 1 - \frac{z}{3!} + \frac{z^2}{6!} - \frac{z^3}{9!} + \frac{z^4}{12!} - \frac{z^5}{15!}.$$

This way, we can write it using nested multiplication as

$$p(z) = 1 + z \left( -\frac{1}{3!} + z \left( \frac{1}{6!} + z \left( -\frac{1}{9!} + z \left( \frac{1}{12!} + z \left( -\frac{1}{15!} \right) \right) \right) \right) \right),$$

or

$$p(x) = 1 + x^3 \left( -\frac{1}{3!} + x^3 \left( \frac{1}{6!} + x^3 \left( -\frac{1}{9!} + x^3 \left( \frac{1}{12!} + x^3 \left( -\frac{1}{15!} \right) \right) \right) \right) \right),$$

involving 5 multiplications. It should be noted that to evaluate  $x^3$ , we could do so by considering recursive multiplication, i.e.,  $x^2 = x * x$  and  $x^3 = x^2 * x$ . Thus, together with the nested multiplication involved in the polynomial evaluation, we have **7 multiplications** in total. Finally, we can introduce the following sequence of operations (along the lines of the Theorem discussed in class):

$$\begin{aligned} b_5 &= -\frac{1}{15!}, & b_4 &= \frac{1}{12!} - x^3 b_5 \\ b_3 &= -\frac{1}{9!} + x^3 b_4, & b_2 &= \frac{1}{6!} + x^3 b_3 \\ b_1 &= -\frac{1}{3!} + x^3 b_2, & b_0 &= 1 + x^3 b_1. \end{aligned}$$

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