Name: ______________________________

1. (15 points) Let $f(x) = x^2 - 4x + 3$. Then:
   (a) (5 points) Find $p_1(x)$, $p_2(x)$ and $p_3(x)$ around $x_0 = 0$. How $P_3(x)$ is related to $f(x)$?
   (b) (5 points) Same as part (a) but consider $x_0 = 1$.
   (c) (5 points) In general, given a polynomial $f(x)$ with degree m, what can you say about $f(x) - p_k(x)$ for $k \geq m$?

Solution: Note that $f'(x) = 2x - 4$, $f''(x) = 2$ and $f'''(x) = 0$.

(a) To this end, we employ Taylor's Theorem:

$$p_1(x) = f(0) + f'(0)x \Rightarrow p_1(x) = 3 - 4x,$$
$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \Rightarrow p_2(x) = 3 - 4x + x^2,$$
$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \Rightarrow p_3(x) = 3 - 4x + x^2,$$

since $f'''(x) = 0$. Thus, $p_3(x) = f(x)$ and we must have $R_3(x) = 0$.

(b) Similarly, around $x_0 = 1$ we have:

$$p_1(x) = f(1) + f'(1)(x-1) \Rightarrow p_1(x) = 2 - 2x,$$
$$p_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \Rightarrow p_2(x) = 3 - 4x + x^2,$$
$$p_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}x^3$$
$$\Rightarrow p_3(x) = 3 - 4x + x^2,$$

and again as in part (a), $p_3(x) = f(x)$ because $f'''(x) = 0$ where $R_3(x) = 0$.

(c) In that case, we will have that $f(x) - p_k(x) = 0$ since $f(x)$ is a polynomial of degree at most m, thus all the $k+1$ derivatives are zero, i.e., $f^{(k+1)}(x) = 0$ when $k \geq m$. In addition, the error term $R_k(x)$ is **identically zero**.

2. (25 points) Given $f(x) = \cos x$, find both $p_2(x)$ and $p_3(x)$ about $x_0 = 0$, and use them to approximate $\cos(0.1)$. Show that in each case the remainder term provides an upper bound for the true (absolute) error.

Solution: At first, note that $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$ as well as $f''''(x) = \cos x$. Next, we calculate the corresponding Taylor polynomials about $x_0 = 0$:

$$
\begin{aligned}
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \Rightarrow p_2(x) = 1 - \frac{1}{2}x^2, \\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}(x-1)^3 \\
\Rightarrow p_3(x) &= 1 - \frac{1}{2}x^2.
\end{aligned}
$$

Therefore, $p_2(x) = p_3(x)$ and $p_2(0.1) = 0.995$. Since both Taylor polynomials are the same, the error in both cases is

$$|\cos(0.1) - 0.995| \approx 4.16528 \times 10^{-6}.$$

From the above Taylor polynomials, we have the respective remainder terms with $\xi \in (0, 0.1)$ for the errors

$$
\begin{aligned}
|R_2(x)| &= \left| \frac{f^{(3)}(\xi)}{6}(0.1)^3 \right| = \left| \frac{\sin \xi}{6000} \right| \leq \frac{\sin(0.1)}{6000} \approx 1.66389 \times 10^{-5}, \\
|R_3(x)| &= \left| \frac{f^{(4)}(\xi)}{24}(0.1)^4 \right| = \left| \frac{\cos \xi}{240000} \right| \leq \frac{\cos(0)}{240000} \approx 4.16667 \times 10^{-6}.
\end{aligned}
$$

Thus, an upper bound derived using the error term for R_k of the respective cases is indeed **larger** than the actual error.

3. (30 points) If $f(x) = e^x$, then
   (a) (10 points) derive the Maclaurin series of the function $f(x) = e^x$, i.e., the Taylor series about $x_0 = 0$ (write **separately** $p_k(x)$ and $R_k(x)$),
   (b) (20 points) find a minimum value of k necessary for $p_k(x)$ to approximate $f(x)$ to within 10^{-6} on the interval $[0, 0.5]$ (here, you must use the remainder term).

Solution:

(a) At first, it should be noted in passing that $f^{(k)}(x) = e^x$, $\forall k \geq 0$. This way, from Taylor's Theorem we have that

$$
\begin{aligned}
f(x) = p_k(x) + R_k(x) = \\
&= \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(k+1)}(\xi)}{(k+1)!} x^{k+1} \\
&= \sum_{n=0}^{k} \frac{x^n}{n!} + \frac{f^{(k+1)}(\xi)}{(k+1)!} x^{k+1} \\
&= \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!}}_{p_k(x)} + \underbrace{\frac{e^\xi}{(k+1)!} x^{k+1}}_{R_k(x)},
\end{aligned}
$$

for some ξ.

(b) Based on the remainder from part (a), we want to find the integer value of k where $\xi \in (0, 0.5)$ such that

$$
\begin{aligned}
&\max_{x\in[0,0.5]} |R_k(x)| = \max_{x\in[0,0.5]} \frac{e^\xi}{(k+1)!} x^{k+1} \leq \frac{e^{1/2}}{(k+1)!} \frac{1}{2^{k+1}} \leq 10^{-6} \\
&\Rightarrow 2^{k+1}(k+1)! \geq e^{1/2} \times 10^6 \approx 1.64872 \times 10^6.
\end{aligned}
$$

Typical values give $2^7 \times 7! = 645120$ and $2^8 \times 8! = 10321920$, thus $k+1 = 8 \Rightarrow k = 7$, or we must have $k \geq 7$.

4. (20 points) Let $f(x) = \sqrt[3]{x}$. Does $f(x)$ have a Taylor polynomial of degree 1 based on expanding about $x = 0$ and $x = 1$? Justify your answers. Include a copy of the graph of $f(x)$ and its associated polynomials (when applicable) on the same figure, as well as your MATLAB script producing the figure.

Solution: From Taylor's approximation formula and for a polynomial of degree 1, we have

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

From $f(x)$ given, we have that $f'(x) = x^{-2/3}/3$. However, the polynomial of degree 1, i.e., $p_1(x)$ of $f(x)$ around $x_0 = 0$ does **not** exist due to the fact that $f(x)$ is **not** differentiable at $x = 0$. On the other hand, the $p_1(x)$ around $x = 1$ exists (with $f'(1) = 1/3$) and is given by

$$p_1(x) = 1 + \frac{1}{3}(x - 1) \Rightarrow p_1(x) = \frac{1}{3}(x + 2).$$

Based on the above results, we conclude that $f(x)$ does **not** have a Taylor polynomial approximation of degree 1 around $x_0 = 0$ whereas $f(x)$ does around $x_0 = 1$. Finally, the MATLAB code and its output for making the graph in question are given next

```
clearvars; close all; clc; format long;

% Declare f(x) and p_{1}(x):
func = @(x) x.^(1/3);
  p1 = @(x) (x+2)/3;

% Introduce the interval [0.5,1.5]:
   x = linspace(0.5,1.5,501);

% Make the figure:
figure;
plot(x,func(x),'-k','linewidth',3);
hold on;
plot(x,p1(x),'-r','linewidth',3);
xlabel('$x$','interpreter','latex');
hg = legend('$\sqrt[3]{x}$','$p_{1}(x)$', ...
       'interpreter','latex','fontname','times');
set(gca,'fontsize',24,'fontname','times');
```

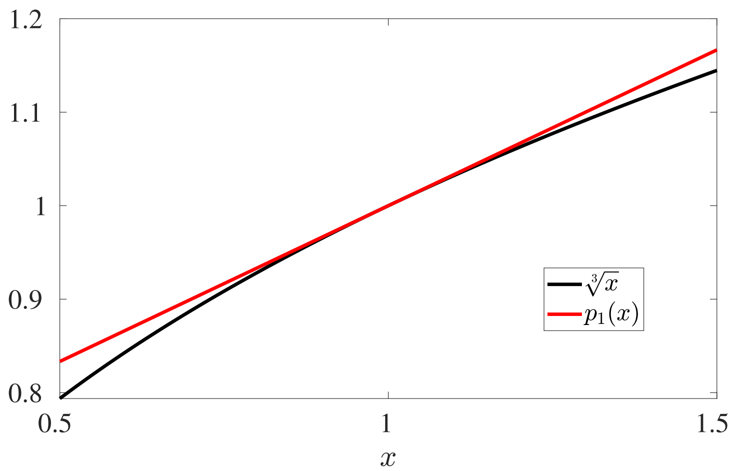

Figure 1: The graph of $f(x)$ (solid black line) and $p_1(x)$ (solid red line) corresponding to the polynomial around $x_0 = 1$ (the one around x_0 does **not** exist).

5. (10 points) Consider the polynomial

$$p(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \frac{x^{12}}{12!} - \frac{x^{15}}{15!}.$$

Evaluate $p(x)$ as efficiently as possible. How many multiplications are necessary? Assume all coefficients have been computed and stored for later use.

Solution: In class, we discussed about nested multiplication applied to

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

which involves n additions and n multiplications. In our example, we first introduce $z = x^3$ and write the polynomial given as

$$p(z) = 1 - \frac{z}{3!} + \frac{z^2}{6!} - \frac{z^3}{9!} + \frac{z^4}{12!} - \frac{z^5}{15!}.$$

This way, we can write it using nested multiplication as

$$p(z) = 1 + z\left(-\frac{1}{3!} + z\left(\frac{1}{6!} + z\left(-\frac{1}{9!} + z\left(\frac{1}{12!} + z\left(-\frac{1}{15!}\right)\right)\right)\right)\right),$$

or

$$p(x) = 1 + x^3\left(-\frac{1}{3!} + x^3\left(\frac{1}{6!} + x^3\left(-\frac{1}{9!} + x^3\left(\frac{1}{12!} + x^3\left(-\frac{1}{15!}\right)\right)\right)\right)\right),$$

involving 5 multiplications. It should be noted that to evaluate x^3, we could do so by considering recursive multiplication, i.e., $x^2 = x * x$ and $x^3 = x^2 * x$. Thus, together with the nested multiplication involved in the polynomial evaluation, we have 7 **multiplications** in total. Finally, we can introduce the following sequence of operations (along the lines of the Theorem discussed in class):

$$\begin{aligned} b_5 &= -\frac{1}{15!}, & b_4 &= \frac{1}{12!} - x^3 b_5 \\ b_3 &= -\frac{1}{9!} + x^3 b_4, & b_2 &= \frac{1}{6!} + x^3 b^3 \\ b_1 &= -\frac{1}{3!} + x^3 b_2, & b_0 &= 1 + x^3 b_1. \end{aligned}$$

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