Name: ______________________________

1. (15 points) Let $f(x)$ be written as

$$f(x) = (x - x^*)^m q(x), \quad q(x^*) \neq 0,$$

where x^* is a **root of multiplicity** $m > 1$. In such a case scenario, Newton's method may converge when $m > 1$ but **not** quadratically. Note also that Newton's method can be written as $x_{n+1} = g(x_n)$ where

$$g(x) = x - \frac{f(x)}{f'(x)},$$

which is called the **iteration function**.

(a) (5 points) Write out the **iteration function** $g(x)$ for Newton's method in this case (it will involve $q(x)$ and $q'(x)$).

(b) (10 points) Show that $g'(x^*) = 1 - 1/m \neq 0$, and **explain why** this implies only **linear** convergence of Newton's method in the case of multiple roots.

2. (25 points) Show that in the case of a root of multiplicity m, the **modified Newton's method** given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)},$$

is quadratically convergent.

3. (15 points) Suppose

$$x + \ln x = 0, \quad x > 0.$$

Implement the **secant method** in MATLAB (or in **any** programming language) and **find the root** of the above equation. Use $x_0 = 0.5$, $x_1 = 0.6$ and $|x_{k+1} - x_k| < 10^{-10}$ as a convergence criterion. In addition, use your function from Question 3 employing Newton's method and repeat the calculation with same initial guess x_0 and convergence criterion as before. Attach your code for the secant method and provide MATLAB outputs for both cases. **Which method converges faster?** Briefly explain.

4. (15 points) Assume the following **fixed point iterations** ($x_{k+1} = g(x_k)$):
   (a) (5 points) $x_{k+1} = -16 + 6x_k + \frac{12}{x_k}$ with $x^* = 2$
   (b) (5 points) $x_{k+1} = \frac{2}{3}x_k + \frac{1}{x_k^2}$ with $x^* = 3^{1/3}$
   (c) (5 points) $x_{k+1} = \frac{12}{1+x_k}$ with $x^* = 3$

   Note that x^* above corresponds to the respective **fixed point**.

   Then, which of the above iterations **will converge** to the fixed point x^* indicated above, provided that $x_0 \approx x^*$, i.e., the initial iterate x_0 is sufficiently close to x^*? If it does converge, then find the order of convergence.

5. (30 points) Write a MATLAB script which can identify the **three roots** of

$$e^x - 2x^2 = 0,$$

   using **fixed-point** iterations with $|x_{k+1} - x_k| < 10^{-10}$ as a convergence criterion. Note that plotting will help here. Furthermore, **explain** your choices for the $g(x)$ utilized in order to ensure convergence and attach your MATLAB script.

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