

# MATH 143 (CALCULUS III) THEORY REVIEW NOTES

<b>Instructor:</b>	Dr. Stathis Charalampidis	<b>Office:</b>	25-319 (Faculty Offices East)
<b>Email:</b>	echarala@calpoly.edu	<b>Phone:</b>	(805) 756-2465

**Definition [Parametric curves and equations]** A **parametric curve**  $C$  is a two-dimensional curve whose  $x$  and  $y$  points are defined by a **pair** of functions

$$x = f(t), \quad y = g(t), \quad t \in I$$

called **parametric equations** with  $t$  usually standing for time. If  $t \in [a, b]$ , then the points  $(f(a), g(a))$  and  $(f(b), g(b))$  correspond to the **initial** and **terminal points** of the curve, respectively.

**Definition [Tangents, Areas, and Lengths]** Let  $x = f(t)$  and  $y = g(t)$  be a parametrization of a curve  $C$  described by  $y = F(x)$ . Then:

- The **slope of the tangent line** to the curve is  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ .

- Similarly,  $\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy/dt}{dx/dt} \right)$ .

- The area under  $y = F(x)$  is given by

$$A = \int_a^b g(t) f'(t) dt, \quad x(a) = \alpha, \quad x(b) = \beta.$$

- The **length** (or **arclength**) of the curve  $C$  is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

**Definition [Polar coordinates, slope, polar area and length formulas]**

- Given  $P(r, \theta)$ , we can find  $P(x, y)$  via  $x = r \cos \theta$  and  $y = r \sin \theta$ .
- Given  $P(x, y)$ , we can find  $P(r, \theta)$  via  $r = \sqrt{x^2 + y^2}$  and  $\theta = \arctan(y/x)$ .
- Let  $r = f(\theta)$  be a polar equation with parametric equations:

$$x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta.$$

Then the **slope** at  $(r, \theta)$  is given by

$$\left. \frac{dy}{dx} \right|_{(r, \theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

- The **area** of  $r = f(\theta)$  between  $\theta = a$  and  $\theta = b$  is given by

$$A = \frac{1}{2} \int_a^b [f(\theta)]^2 d\theta.$$

- The **length** of a polar curve is given by

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

- **Remarks for graphing polar curves:**

1. The graph with  $r = a$  is a circle with radius  $|a|$  centered at  $(0, 0)$ .
2. The graph with polar equation  $\theta = \theta_0$  is a line through  $(0, 0)$  making an angle  $\theta_0$  with the  $x$ -axis.
3. The equation for a **circle**:  $r = a \sin \theta$  or  $r = a \cos \theta$ .
4. The equations for **flowers**:  $r = a \cos(n\theta)$  or  $r = a \sin(n\theta)$ . Note that if  $n$  is odd, we have  $n$  petals; if  $n$  is even we have  $2n$  petals.
5. The equations for the **cardioid**:  $r = a + b \cos \theta$  or  $r = a + b \sin \theta$ .

**Definition [Sequence]** A **sequence** is a list of ordered numbers:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

**Definition [Limit Test for convergence/divergence of sequences]** A sequence  $\{a_n\}$  has the **limit**  $L$ :

$$\lim_{n \rightarrow \infty} a_n = L \Rightarrow a_n \rightarrow L, \quad \text{as } n \rightarrow \infty,$$

if we can make the terms  $a_n$  as close to  $L$  as  $n \gg 1$ . If the limit exists, we say that the sequence **converges** (or is **convergent**). Otherwise the sequence **diverges**.

**Theorem [Limit Rules]** If  $\{a_n\}$  and  $\{b_n\}$  are **convergent** sequences, and  $c = \text{const.}$ , then

- $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n \quad \left( \lim_{n \rightarrow \infty} c = c \right)$
- $\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$  if  $\lim_{n \rightarrow \infty} b_n \neq 0$
- $\lim_{n \rightarrow \infty} a_n^p = \left[ \lim_{n \rightarrow \infty} a_n \right]^p$  if  $p > 0$  and  $a_n > 0$ .

**Theorem [Sandwich Theorem]** Let  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$ . If

$$a_n \leq b_n \leq c_n \text{ for } \forall n \geq n_0 \text{ and } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L, \text{ then } \lim_{n \rightarrow \infty} b_n = L.$$

**Theorem** If  $\lim_{n \rightarrow \infty} a_n = L$  and the function  $f$  is continuous at  $L$ , then  $\lim_{n \rightarrow \infty} f(a_n) = f(L)$ .

**Theorem** Let  $f(x)$  be defined  $\forall x \geq n_0$  and  $\{a_n\}$  such that  $a_n = f(n)$  for  $n \geq n_0$ . Then

$$\lim_{x \rightarrow \infty} f(x) = L \Rightarrow \lim_{n \rightarrow \infty} a_n = L.$$

**Theorem** The above sequences converge to the limits indicated below:

- $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$
- $\lim_{n \rightarrow \infty} x^{1/n} = 1, \forall x > 0.$
- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x, \forall x.$
- $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$
- $\lim_{n \rightarrow \infty} x^n = 0$  if  $|x| \leq 1.$
- $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x.$

**Theorem [Monotonicity of sequences]**

- A sequence  $\{a_n\}$  is called **increasing** (non-decreasing) if  $a_n < a_{n+1}$  holds  $\forall n \geq 1$ .
- A sequence  $\{a_n\}$  is called **decreasing** (non-increasing) if  $a_n > a_{n+1}$  holds  $\forall n \geq 1$ .
- A sequence  $\{a_n\}$  is called **monotonic** if it is either increasing or decreasing.

**Theorem [Monotonic Sequence Theorem]** Every bounded and monotonic sequence is convergent.

**Definition [Infinite series, partial sums and convergence]**

- Given  $\{a_n\}$ , an expression of the form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is an **infinite series**.

- The  $n$ th **partial sum** is the sum of the first  $n$  terms:

$$s_n \doteq a_1 + a_2 + a_3 + \dots + a_n$$

In addition,  $\{s_n\}$  is the **sequence of partial sums**.

- If  $s_n \rightarrow s$ , the series **converges** and that its sum is  $s$ :

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n = s.$$

- On the contrary, if  $\{s_n\}$  does not converge, the series **diverges**.

**Definition** [Geometric series]

- Geometric series is a series of the form of

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \doteq \sum_{n=1}^{\infty} ar^{n-1} \left( \doteq \sum_{n=0}^{\infty} ar^n \right),$$

where  $a \in \mathbb{R}$  is called the **leading term** and  $r \in \mathbb{R}$  is the **ratio**.

- A geometric series **converges** if  $|r| < 1$ . Its sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad (|r| < 1).$$

- If  $|r| \geq 1$  the geometric series **diverges**.
- Its  $n$ th partial sum,  $s_n$ , is given by

$$s_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$$

**Definition** [Telescoping series] A **telescoping series** is any series where nearly every term cancels with a preceding or following term.

**Theorem** [The  $n$ th-term test] If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$  (or  $a_n \rightarrow 0$ ). On the contrary,  $\sum_{n=1}^{\infty} a_n$  diverges if  $\lim_{n \rightarrow \infty} a_n$  fails to exist or if  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

**Theorem** [Combining series] If  $\sum a_n$  and  $\sum b_n$  are convergent series, then so are the series  $\sum c a_n$  (where  $c = \text{const.}$ ), and  $\sum (a_n \pm b_n)$ , and

- $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$ .
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$ .

**Theorem** [The integral test]

Suppose  $f(x)$  is a continuous, positive function where  $f'(x) < 0$  (i.e., decreasing) on  $[N, \infty)$  with  $N > 0$ , and let  $a_n = f(n)$ . Then the series  $\sum_{n=N}^{\infty} a_n$  and the integral  $\int_N^{\infty} f(x) dx$  both converge **or** both diverge.

**Theorem** [The  $p$ -test for series] The  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ :

- Converges** if  $p > 1$ .
- Diverges** if  $p \leq 1$ .

**Theorem** [Remainder Estimate for the Integral Test] Suppose  $f(k) = a_k$  where  $f$  is a continuous, positive, and decreasing function for  $x \geq n$ , and  $\sum a_n$  is convergent to  $s$ . If  $R_n = s - s_n$ , then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$

The value of the sum  $s$  satisfies:

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx.$$

**Theorem** [The Direct Comparison Test (DCT)] Suppose that  $\sum a_n$  and  $\sum b_n$  are series with  $a_n, b_n > 0$  (i.e., non-negative terms) and let  $N$  being some fixed integer. Then, the following two cases are discerned:

- If  $\sum b_n$  is convergent and  $a_n \leq b_n, \forall n \geq N$ , then  $\sum a_n$  is also convergent.
- If  $\sum b_n$  is divergent and  $a_n \geq b_n, \forall n \geq N$ , then  $\sum a_n$  is also divergent.

**Theorem** [The Limit Comparison Test (LCT)] Suppose  $a_n, b_n > 0, \forall n \geq N$  for some fixed integer  $N$ .

- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ , then  $\sum a_n$  and  $\sum b_n$  both converge or both diverge.

- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  **converges**, then  $\sum a_n$  **converges**.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  **diverges**, then  $\sum a_n$  **diverges**.

**Theorem** [Alternating series test (AST)] If the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \quad b_n > 0 \quad \forall n,$$

satisfies

- $b_{n+1} \leq b_n, \forall n \geq N$  (for some  $N$ )
- $b_n \rightarrow 0$  as  $n \rightarrow \infty$

then the series is convergent.

**Theorem** [Alternating series estimation theorem] If the alternating series converges, then for  $n \geq N$

$$s_n = b_1 - b_2 + \cdots + (-1)^{n+1} b_n,$$

approximates the sum  $s$  of the series with an error whose absolute value is less than  $b_{n+1}$ , the absolute value of the first unused term (i.e.,  $|a_{n+1}| = b_n$ ). Furthermore, the sum  $s$  lies between any two successive partial sums  $s_n$  and  $s_{n+1}$  and the remainder,  $s - s_n$  has the same sign as the first unused term. In other words, we write

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

**Definition** [Absolute and Conditional convergence]

- $\sum a_n$  is **absolutely convergent** if  $\sum |a_n|$  is convergent.
- $\sum a_n$  is **conditionally convergent** if it is convergent but **not absolutely convergent**.
- If  $\sum a_n$  is **absolutely convergent**, then  $\sum a_n$  is **convergent**.

**Theorem** [The Ratio Test] Let  $\sum a_n$  be a series and suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then,

- The series is **absolutely convergent** if  $L < 1$  (and therefore convergent!).
- The series is **divergent** if  $L > 1$  (or infinite).
- The test fails (or is inconclusive) if  $L = 1$  (no conclusion can be drawn about the convergence or divergence).

**Theorem** [The Root Test] Let  $\sum a_n$  be a series and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L.$$

Then,

- The series is **absolutely convergent** if  $L < 1$  (and therefore convergent!).
- The series is **divergent** if  $L > 1$  (or infinite).
- The test fails (or is inconclusive) if  $L = 1$ .

**Definition** [Power series and their convergence] A **power series in  $x$**  or a **power series about 0** or **centered at 0** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

**Remarks:**

- The series converges **only** when  $x = a$  corresponding to  $R = 0$ .
- Use the **ratio test** to find the **radius  $R$**  and **interval  $I$**  of convergence. Note that  $I$  is an **open interval**.
- Check the endpoints** separately to find the **complete** interval of convergence.
- $\exists$  **three** possibilities:  $R = 0$ ,  $R = \infty$ , or  $R = c$  where  $c$  is a finite number.
- In the third case, there are **four** possibilities:  
 $(a - R, a + R), [a - R, a + R], [a - R, a + R), [a - R, a + R]$ .

**Definition** [Taylor and Mclaurin series] The power series representation of  $f(x)$  is called **Taylor series** and is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k, \quad f^{(k)}(a) \doteq f^{(k)}(x) \Big|_{x=a}.$$

If  $a = 0$ , then the Taylor series is called **Mclaurin series**.

**Definition** [Taylor's formula/theorem] If  $f^{(k)} \in C[a, b] \forall k = 1, n$  and  $f^{(n+1)} \in C(a, b)$  then

$$f(x) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}}_{R_n(x)},$$

where  $T_n(x)$  is the **Taylor polynomial of order  $n$**  and  $R_n$  is the **remainder**. The value  $c$  is between  $a$  and  $b$ .

**Definition** [Remainder estimation theorem/Taylor's inequality] If  $\exists M > 0$  such that  $|f^{(n+1)}(x)| \leq M$  holds for  $|x-a| \leq d$ , then the remainder term  $R_n(x)$  of the Taylor series satisfies

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}, \quad |x-a| \leq d.$$

**Definition** [Binomial series] If  $k$  is any real number and  $|x| < 1$ , then the series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n,$$

where

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!},$$

are called the **binomial series** and  $\binom{k}{n}$  the **binomial coefficients**.

**Definition** [Frequently used Mclaurin series]

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  with  $|x| < 1$  or  $I = (-1, 1)$ .

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  with  $I = \mathbb{R}$ .

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$  with  $I = \mathbb{R}$ .

- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$  with  $I = \mathbb{R}$ .

- $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$  with  $I = (-1, 1]$ .

- $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$  with  $I = [-1, 1]$ .

**Definition** [Vector] Given points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$  in  $\mathbb{R}^3$ , then  $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$  denotes the *arrow* or *vector* based at  $A$  with terminal point  $B$ .

**Definition** [Equation of the sphere] The equation of the sphere centered at  $(h, k, l)$  with radius  $r$  is given by

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

**Theorem** [Length and distance] Let  $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$  be vector and let  $A(x_2, y_2, z_2)$ ,  $B(x_3, y_3, z_3)$  be points. Then:

1. The length of  $\mathbf{a}$  is  $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{x_1^2 + y_1^2 + z_1^2}$ .

2. The distance  $d(A, B)$  from the point  $A$  to the point  $B$  is:

$$d(A, B) = |\overrightarrow{AB}| = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2}.$$

**Theorem** [Dot product and angle of vectors] For nonzero vectors  $\mathbf{a}, \mathbf{b}$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta,$$

where  $\theta \in [0, \pi]$  is the angle between the vectors. Furthermore, if  $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$  and  $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$ , then

$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

**Definition** [Direction angles and direction cosines] The **direction angles** of a vector  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  are the angles  $\alpha, \beta$  and  $\gamma$  that  $\vec{a}$  makes with the positive  $x$ -,  $y$ - and  $z$ - axes:

$$\cos \alpha = \frac{a_1}{|\vec{a}|}, \quad \cos \beta = \frac{a_2}{|\vec{a}|}, \quad \cos \gamma = \frac{a_3}{|\vec{a}|}.$$

**Definition** [Vector and scalar projections]

- The **vector projection** of  $\mathbf{b}$  onto (in the direction of)  $\mathbf{a}$  is

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} = \text{comp}_{\mathbf{a}} \mathbf{b} \frac{\mathbf{a}}{|\mathbf{a}|}.$$

- The **scalar projection** of  $\mathbf{b}$  onto (in the direction of)  $\mathbf{a}$  is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}| \cos \theta.$$

**Definition** [Cross product] The **cross product**  $\mathbf{a} \times \mathbf{b}$  of vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  can be calculated by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Note that  $\mathbf{a} \times \mathbf{b}$  is  $\perp$  to both  $\mathbf{a}$  and  $\mathbf{b}$  by construction.

**Theorem** The length of  $\mathbf{a} \times \mathbf{b}$  is given by:  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta)$ , where  $\theta \in [0, \pi]$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . Also  $|\mathbf{a} \times \mathbf{b}|$  is **area of the parallelogram** with sides  $\mathbf{a}$  and  $\mathbf{b}$ . Note that it follows that area of the triangle with vertices  $\langle 0, 0, 0 \rangle$  and the position vectors  $\mathbf{a}$  and  $\mathbf{b}$  is:  $|\mathbf{a} \times \mathbf{b}|/2$ .

**Theorem** Let two vectors  $\mathbf{a}$  and  $\mathbf{b}$ . Then:

- $\mathbf{a} \perp \mathbf{b}$  if  $\mathbf{a} \cdot \mathbf{b} = 0$  (number).
- $\mathbf{a} // \mathbf{b}$  if  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$  (vector!).

**Definition** [Scalar triple product] The **scalar triple product** of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ ,  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  and  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$  can be calculated by the determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

**Theorem** The **volume of the parallelepiped** determined by the vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$  is the **absolute value**  $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ .

**Definition** Given a point  $P(x_0, y_0, z_0)$  and a vector  $\mathbf{v} = \langle a, b, c \rangle$ , the **vector equation** of the line  $L$  passing through  $P_0$  in the direction of  $\mathbf{v}$  is:

$$\mathbf{r}(t) = \mathbf{OP} + t\mathbf{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.$$

The resulting equations:

$$x = x_0 + at,$$

$$y = y_0 + bt,$$

$$z = z_0 + ct,$$

are called the **parametric equations** for  $L$ .

**Definition** If  $t$  is eliminated from the parametric equations of the line  $L$ , we obtain the **symmetric equations** for  $L$  given by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

**Definition** The distance  $D$  from a point  $S$  to a line that passes through  $P$  parallel to a vector  $\mathbf{v}$  is given by

$$D = |\mathbf{PS}| \sin \theta = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

**Definition**

- The **plane** passing through the point  $P(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is given by the following **vector equation**:

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,$$

where  $(x, y, z)$  denotes a **general point** on the plane.

- Equivalently, we have:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d \quad \text{with} \quad d = -(ax_0 + by_0 + cz_0).$$

- The **angle** between two intersecting planes with normal vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2$  is the angle between their normal vectors given by

$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right).$$

- The distance  $D$  from a point  $P_1(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  with normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is given by

$$D = |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

**Definition** [Vector functions and properties] Consider  $\mathbf{r}(t)$  with  $x = f(t)$ ,  $y = g(t)$ , and  $z = h(t)$  and  $t \in [a, b]$ . Then:

- $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \langle \lim_{t \rightarrow t_0} f(t), \lim_{t \rightarrow t_0} g(t), \lim_{t \rightarrow t_0} h(t) \rangle$ .
- If  $\mathbf{r}$  is continuous at  $t = t_0$ , then  $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$ .
- $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$  is continuous if  $\mathbf{r}(t)$  is continuous  $\forall t \in I$ .
- The **velocity field** is  $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ .
- The **acceleration field** is  $\mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle = \mathbf{v}'(t)$ .
- The **indefinite integral** of  $\mathbf{r}(t)$  wrt  $t$  is

$$\int \mathbf{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle + \mathbf{c},$$

where  $\mathbf{c}$  is an **arbitrary constant vector**.

- The **definite integral** of  $\mathbf{r}(t)$  from  $a$  to  $b$  is

$$\int_a^b \mathbf{r}(t) dt = \langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \rangle,$$

- The **length**  $L$  of  $\mathbf{r}(t)$  is the integral of the speed:

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

**Definition** [Curvature and TNB frame] Let a parametrization of  $C$  in  $\mathbb{R}^3$  be  $\mathbf{r} = \langle f(t), g(t), h(t) \rangle$ . Then:

- The **unit tangent vector** is  $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ .
- The **unit normal vector** is  $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ .
- The **unit binormal vector** is  $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ .
- The **curvature of the curve** is

$$\kappa \doteq \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3},$$

- The **speed** is given by  $\frac{ds}{dt} = |\mathbf{r}'(t)|$ , where  $s$  stands for the **ar-length function**:

$$s(t) = \int_a^t |\mathbf{r}'(u)| du.$$

**Definition** The **acceleration vector** can be decomposed as

$$\mathbf{a} \doteq a_T \mathbf{T} + a_N \mathbf{N},$$

whence

- $a_T \doteq \frac{d}{dt} |\mathbf{r}'|$  is the **tangential** component of  $\mathbf{a}$ , and
- $a_N = \kappa |\mathbf{r}'|^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \sqrt{|\mathbf{a}|^2 - a_T^2}$  is the **normal** component of  $\mathbf{a}$ .