Name: ______________________________

1. (10 points) Let $a_{ij} = \max\{i, j\}$ with $1 \leq i, j \leq n$ be the coffiecients of the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned} \tag{1}$$

   Furthermore, suppose $b_i \equiv 1$ for $i = 1, 2, \ldots, n$. In class, we discussed about the MATLAB implemention of the Gaussian elimination algorithm. The associated MATLAB function is `GEpivot.m`. Use this MATLAB function to solve the above linear system for $n = 2, 5, 10$ and $n = 20$. Attach your MATLAB **driver** and include MATLAB output.

2. (10 points) Repeat Question 1, but use the modifed coefficient matrix given by

$$a_{ij} = \min\{i, j\}.$$

3. (30 points) Write a MATLAB function that solves a general linear system

$$A\mathbf{x} = \mathbf{b},$$

   by using **forward** and **backward** substitutions. Store your function as `my_lin_solver.m` whose first line should read

```
function [ x ] = my_lin_solver( A, b )
```

   Inside this function, you must use the *LU* decomposition provided by the MATLAB function `lu_doolittle` that was given in class. Of course, you could either use the MATLAB's built-in function `lu` for this purpose as well!

Then, test your code with the 3×3 system:

$$\begin{aligned} 3x_1 + x_2 + 4x_3 &= 6, \\ x_2 - 2x_3 &= -3, \\ x_1 + 2x_2 - x_3 &= -2. \end{aligned}$$

The exact solution is $\mathbf{x} = [1, -1, 1]^T$. Then, use your function in order to solve the 4×4 system:

$$\begin{aligned} x_1 + x_2 + x_4 &= 2, \\ 2x_1 + x_2 - x_3 + x_4 &= 1, \\ 4x_1 - x_2 - 2x_3 + 2x_4 &= 0, \\ 3x_1 - x_2 - x_3 + x_4 &= -3. \end{aligned}$$

Compute the l_2-norm of the residual $\|\mathbf{b} - A\hat{\mathbf{x}}\|_2$ where $\hat{\mathbf{x}}$ is the solution computed for the 4×4 system. Attach **all** your codes and provide MATLAB output.

4. (20 points) Write a MATLAB function called `tridiag.m` in order to solve the linear system $A\mathbf{x} = \mathbf{f}$ where A is an $n \times n$ **tridiagonal matrix** of the form of

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & b_n & a_n \end{bmatrix}.$$

Its first line should read

```
function [ x ] = tridiag( a, b, c, f )
```

The **inputs** are the n-dimensional vectors: **a**, **b**, **c** and **f**, and the **output** is the **solution vector x**. Test your code with the 5×5 system with $a_i = 2$, $b_i = -1$, $c_i = -1$, and rhs vector $\mathbf{f} = [1, 0, 0, 0, 1]^T$. The exact solution is $\mathbf{x} = [1, 1, 1, 1, 1]^T$. Attach **all** your codes and provide MATLAB output.

5. (30 points) Consider the second-order, non-homogeneous ordinary differential equation (ODE):

$$u'' - u = x, \tag{2}$$

where $u = u(x)$ satisfies the boundary conditions: $u(0) = u(1) = 0$. Problems of this sort [cf. (2)] together with boundary conditions on the unknown function $u(x)$ are called **boundary value problems** (BVPs), while the ODE given is known as the *Helmholtz equation*.

In class, we derived the so-called second-order **centered**, **finite difference approximation** of the second derivative:

$$u''(x_0) \approx \frac{u(x_0+h) - 2u(x_0) + u(x_0-h)}{h^2} + O(h^2).$$

Use this approximation and the MATLAB function `tridiag.m` in order to solve the BVP of Eq. (2) with $n = 24$ points in $[0, 1]$ (see *Hints*, for details). Furthermore, if the exact solution to Eq. (2) is given by

$$u_{\text{exact}}(x) = \frac{e}{e^2-1}\left(e^x - e^{-x}\right) - x,$$

plot the **numerical** and **exact** solutions on the **same** figure using a different marker (say, open circles and solid line, respectively) and include a legend. Finally, calculate the l_2-norm of the absolute error: $\|u_{\text{exact}} - u_{\text{numerical}}\|_2$. Include your code, any figure and MATLAB output.

Hints:

(a) Divide the interval $[0, 1]$ into $n + 1$ equal subintervals and set $x_i = ih$, $i = 0, 1, \ldots, n+1$ such that $(n+1)h = 1$ holds. This way, we create an one-dimensional computational grid (or mesh).

(b) Then, we look for an approximate solution $u(x_i) \doteq u_i$ with $i = 1, \ldots, n$ using the boundary conditions $u_0 = u_{n+1} \equiv 0$.

(c) To do so, the BVP at the discrete level is written as a **difference equation**:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - u_i = x_i, \Rightarrow$$
$$u_{i+1} - \left(2 + h^2\right) u_i + u_{i-1} = h^2 x_i, \quad i = 1, \ldots, n.$$

Note that the latter equation is just a linear system of the form of $A\mathbf{x} = \mathbf{f}$ with A being a **tridiagonal matrix**.

6. (15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}, \quad |c| \neq 1,$$

and find its condition number $\text{cond}(A)$ **by hand**. When does A become **ill-conditioned**? If we are supposed to solve $A\mathbf{x} = \mathbf{b}$, what does the ill-conditioning of A say about the linear system? How is $\text{cond}(A)$ related to $\det(A)$?

7. (15 points) Consider the following matrix, rhs vector and two approximate solutions

$$A = \begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0.9911 \\ -0.4870 \end{bmatrix},$$

respectively.

(a) (2 points) Show **by hand** that $\mathbf{x} = [2, -2]^T$ is the exact solution of $A\mathbf{x} = \mathbf{b}$.

(b) (3 points) Compute the error and residual vectors in MATLAB for $\mathbf{x}_1$ and $\mathbf{x}_2$.

(c) (5 points) Use MATLAB to find $\|A\|_\infty$, $\|A^{-1}\|_\infty$ and the condition number $\mathrm{cond}(A)$ in the ∞ norm. Note that in MATLAB the inverse of a matrix A is `inv(A)` while the condition number is available as a built-in command (try `help cond` for more details).

(d) (5 points) We proved that the relative error in the solution is bounded by

$$\frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \leq \mathrm{cond}(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|},$$

where $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ and $\hat{\mathbf{r}}$ are the error and residual vectors, respectively with $\mathbf{r} = \mathbf{b} - A\hat{\mathbf{x}}$. Verify this result for the two approximate solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ given by using the ∞ norm.

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