

MATH 143 (CALCULUS III) THEORY REVIEW NOTES

Instructor:	Dr. Stathis Charalampidis	Office:	25-319 (Faculty Offices East)
Email:	echarala@calpoly.edu	Phone:	(805) 756-2465

Definition [Parametric curves and equations] A **parametric curve** C is a two-dimensional curve whose x and y points are defined by a **pair** of functions

$$x = f(t), \quad y = g(t), \quad t \in I$$

called **parametric equations** with t usually standing for time. If $t \in [a, b]$, then the points $(f(a), g(a))$ and $(f(b), g(b))$ correspond to the **initial** and **terminal points** of the curve, respectively.

Definition [Tangents, Areas, and Lengths] Let $x = f(t)$ and $y = g(t)$ be a parametrization of a curve C described by $y = F(x)$. Then:

- The **slope of the tangent line** to the curve is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

- Similarly, $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy/dt}{dx/dt} \right)$.

- The area under $y = F(x)$ is given by

$$A = \int_a^b g(t) f'(t) dt, \quad x(a) = \alpha, \quad x(b) = \beta.$$

- The **length** (or **arclength**) of the curve C is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Definition [Polar coordinates, slope, polar area and length formulas]

- Given $P(r, \theta)$, we can find $P(x, y)$ via $x = r \cos \theta$ and $y = r \sin \theta$.
- Given $P(x, y)$, we can find $P(r, \theta)$ via $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$.
- Let $r = f(\theta)$ be a polar equation with parametric equations:

$$x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta.$$

Then the **slope** at (r, θ) is given by

$$\left. \frac{dy}{dx} \right|_{(r, \theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

- The **area** of $r = f(\theta)$ between $\theta = a$ and $\theta = b$ is given by

$$A = \frac{1}{2} \int_a^b [f(\theta)]^2 d\theta.$$

- The **length** of a polar curve is given by

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

- **Remarks for graphing polar curves:**

1. The graph with $r = a$ is a circle with radius $|a|$ centered at $(0, 0)$.
2. The graph with polar equation $\theta = \theta_0$ is a line through $(0, 0)$ making an angle θ_0 with the x -axis.
3. The equation for a **circle**: $r = a \sin \theta$ or $r = a \cos \theta$.
4. The equations for **flowers**: $r = a \cos(n\theta)$ or $r = a \sin(n\theta)$. Note that if n is odd, we have n petals; if n is even we have $2n$ petals.
5. The equations for the **cardioid**: $r = a + b \cos \theta$ or $r = a + b \sin \theta$.

Definition [Sequence] A **sequence** is a list of ordered numbers:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

Definition [Limit Test for convergence/divergence of sequences] A sequence $\{a_n\}$ has the **limit** L :

$$\lim_{n \rightarrow \infty} a_n = L \Rightarrow a_n \rightarrow L, \quad \text{as } n \rightarrow \infty,$$

if we can make the terms a_n as close to L as $n \gg 1$. If the limit exists, we say that the sequence **converges** (or is **convergent**). Otherwise the sequence **diverges**.

Theorem [Limit Rules] If $\{a_n\}$ and $\{b_n\}$ are **convergent** sequences, and $c = \text{const.}$, then

- $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n \quad \left(\lim_{n \rightarrow \infty} c = c \right)$
- $\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } \lim_{n \rightarrow \infty} b_n \neq 0$
- $\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0.$

Theorem [Sandwich Theorem] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$. If

$$a_n \leq b_n \leq c_n \text{ for } \forall n \geq n_0 \text{ and } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L, \text{ then } \lim_{n \rightarrow \infty} b_n = L.$$

Theorem If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then $\lim_{n \rightarrow \infty} f(a_n) = f(L)$.

Theorem Let $f(x)$ be defined $\forall x \geq n_0$ and $\{a_n\}$ such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \rightarrow \infty} f(x) = L \Rightarrow \lim_{n \rightarrow \infty} a_n = L.$$

Theorem The above sequences converge to the limits indicated below:

- $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$
- $\lim_{n \rightarrow \infty} x^{1/n} = 1, \forall x > 0.$
- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x, \forall x.$
- $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$
- $\lim_{n \rightarrow \infty} x^n = 0$ if $|x| \leq 1.$
- $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x.$

Theorem [Monotonicity of sequences]

- A sequence $\{a_n\}$ is called **increasing** (non-decreasing) if $a_n < a_{n+1}$ holds $\forall n \geq 1$.
- A sequence $\{a_n\}$ is called **decreasing** (non-increasing) if $a_n > a_{n+1}$ holds $\forall n \geq 1$.
- A sequence $\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Theorem [Monotonic Sequence Theorem] Every bounded and monotonic sequence is convergent.

Definition [Infinite series, partial sums and convergence]

- Given $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is an **infinite series**.

- The n th **partial sum** is the sum of the first n terms:

$$s_n \doteq a_1 + a_2 + a_3 + \dots + a_n$$

In addition, $\{s_n\}$ is the **sequence of partial sums**.

- If $s_n \rightarrow s$, the series **converges** and that its sum is s :

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n = s.$$

- On the contrary, if $\{s_n\}$ does not converge, the series **diverges**.

Definition [Geometric series]

- Geometric series is a series of the form of

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \doteq \sum_{n=1}^{\infty} ar^{n-1} \left(\doteq \sum_{n=0}^{\infty} ar^n \right),$$

where $a \in \mathbb{R}$ is called the **leading term** and $r \in \mathbb{R}$ is the **ratio**.

- A geometric series **converges** if $|r| < 1$. Its sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad (|r| < 1).$$

- If $|r| \geq 1$ the geometric series **diverges**.
- Its n th partial sum, s_n , is given by

$$s_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$$

Definition [Telescoping series] A **telescoping series** is any series where nearly every term cancels with a preceding or following term.

Theorem [The n th-term test] If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$ (or $a_n \rightarrow 0$). On the contrary, $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} a_n$ fails to exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$.

Theorem [Combining series] If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum c a_n$ (where $c = \text{const.}$), and $\sum (a_n \pm b_n)$, and

- $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$.
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Theorem [The integral test]

Suppose $f(x)$ is a continuous, positive function where $f'(x) < 0$ (i.e., decreasing) on $[N, \infty)$ with $N > 0$, and let $a_n = f(n)$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ both converge **or** both diverge.

Theorem [The p -test for series] The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$:

- Converges** if $p > 1$.
- Diverges** if $p \leq 1$.

Theorem [Remainder Estimate for the Integral Test] Suppose $f(k) = a_k$ where f is a continuous, positive, and decreasing function for $x \geq n$, and $\sum a_n$ is convergent to s . If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$

The value of the sum s satisfies:

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx.$$

Theorem [The Direct Comparison Test (DCT)] Suppose that $\sum a_n$ and $\sum b_n$ are series with $a_n, b_n > 0$ (i.e., non-negative terms) and let N being some fixed integer. Then, the following two cases are discerned:

- If $\sum b_n$ is convergent and $a_n \leq b_n, \forall n \geq N$, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n, \forall n \geq N$, then $\sum a_n$ is also divergent.

Theorem [The Limit Comparison Test (LCT)] Suppose $a_n, b_n > 0, \forall n \geq N$ for some fixed integer N .

- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ **converges**, then $\sum a_n$ **converges**.
- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ **diverges**, then $\sum a_n$ **diverges**.

Theorem [Alternating series test (AST)] If the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \quad b_n > 0 \quad \forall n,$$

satisfies

- $b_{n+1} \leq b_n, \forall n \geq N$ (for some N)
- $b_n \rightarrow 0$ as $n \rightarrow \infty$

then the series is convergent.

Theorem [Alternating series estimation theorem] If the alternating series converges, then for $n \geq N$

$$s_n = b_1 - b_2 + \cdots + (-1)^{n+1} b_n,$$

approximates the sum s of the series with an error whose absolute value is less than b_{n+1} , the absolute value of the first unused term (i.e., $|a_{n+1}| = b_n$). Furthermore, the sum s lies between any two successive partial sums s_n and s_{n+1} and the remainder, $s - s_n$ has the same sign as the first unused term. In other words, we write

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Definition [Absolute and Conditional convergence]

- $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent.
- $\sum a_n$ is **conditionally convergent** if it is convergent but **not absolutely convergent**.
- If $\sum a_n$ is **absolutely convergent**, then $\sum a_n$ is **convergent**.

Theorem [The Ratio Test] Let $\sum a_n$ be a series and suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then,

- The series is **absolutely convergent** if $L < 1$ (and therefore convergent!).
- The series is **divergent** if $L > 1$ (or infinite).
- The test fails (or is inconclusive) if $L = 1$ (no conclusion can be drawn about the convergence or divergence).

Theorem [The Root Test] Let $\sum a_n$ be a series and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L.$$

Then,

- The series is **absolutely convergent** if $L < 1$ (and therefore convergent!).
- The series is **divergent** if $L > 1$ (or infinite).
- The test fails (or is inconclusive) if $L = 1$.

Definition [Power series and their convergence] A **power series in x or a power series about 0 or centered at 0** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

Remarks:

- The series converges **only** when $x = a$ corresponding to $R = 0$.
- Use the **ratio test** to find the **radius R** and **interval I** of convergence. Note that I is an **open interval**.
- Check the endpoints** separately to find the **complete interval** of convergence.
- \exists **three** possibilities: $R = 0$, $R = \infty$, or $R = c$ where c is a finite number.
- In the third case, there are **four** possibilities:
 $(a - R, a + R), [a - R, a + R], [a - R, a + R), [a - R, a + R]$.

Definition [Taylor and Mclaurin series] The power series representation of $f(x)$ is called **Taylor series** and is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k, \quad f^{(k)}(a) \doteq f^{(k)}(x) \Big|_{x=a}.$$

If $a = 0$, then the Taylor series is called **Mclaurin series**.

Definition [Taylor's formula/theorem] If $f^{(k)} \in C[a, b] \forall k = 1, n$ and $f^{(n+1)} \in C(a, b)$ then

$$f(x) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}}_{R_n(x)},$$

where $T_n(x)$ is the **Taylor polynomial of order n** and R_n is the **remainder**. The value c is between a and b .

Definition [Remainder estimation theorem/Taylor's inequality] If $\exists M > 0$ such that $|f^{(n+1)}(x)| \leq M$ holds for $|x-a| \leq d$, then the remainder term $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}, \quad |x-a| \leq d.$$

Definition [Binomial series] If k is any real number and $|x| < 1$, then the series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n,$$

where

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!},$$

are called the **binomial series** and $\binom{k}{n}$ the **binomial coefficients**.

Definition [Frequently used Mclaurin series]

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with $|x| < 1$ or $I = (-1, 1)$.

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with $I = \mathbb{R}$.

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ with $I = \mathbb{R}$.

- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ with $I = \mathbb{R}$.

- $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ with $I = (-1, 1]$.

- $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ with $I = [-1, 1]$.

Definition [Vector] Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in \mathbb{R}^3 , then $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ denotes the *arrow* or *vector* based at A with terminal point B .

Definition [Equation of the sphere] The equation of the sphere centered at (h, k, l) with radius r is given by

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

Theorem [Length and distance] Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ be vector and let $A(x_2, y_2, z_2)$, $B(x_3, y_3, z_3)$ be points. Then:

1. The length of \mathbf{a} is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

2. The distance $d(A, B)$ from the point A to the point B is:

$$d(A, B) = |\overrightarrow{AB}| = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2}.$$

Theorem [Dot product and angle of vectors] For nonzero vectors \mathbf{a}, \mathbf{b}

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta,$$

where $\theta \in [0, \pi]$ is the angle between the vectors. Furthermore, if $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Definition [Direction angles and direction cosines] The **direction angles** of a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ are the angles α, β and γ that \vec{a} makes with the positive x -, y - and z - axes:

$$\cos \alpha = \frac{a_1}{|\vec{a}|}, \quad \cos \beta = \frac{a_2}{|\vec{a}|}, \quad \cos \gamma = \frac{a_3}{|\vec{a}|}.$$

Definition [Vector and scalar projections]

- The **vector projection** of \mathbf{b} onto (in the direction of) \mathbf{a} is

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} = \text{comp}_{\mathbf{a}} \mathbf{b} \frac{\mathbf{a}}{|\mathbf{a}|}.$$

- The **scalar projection** of \mathbf{b} onto (in the direction of) \mathbf{a} is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}| \cos \theta.$$

Definition [Cross product] The **cross product** $\mathbf{a} \times \mathbf{b}$ of vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ can be calculated by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Note that $\mathbf{a} \times \mathbf{b}$ is \perp to both \mathbf{a} and \mathbf{b} by construction.

Theorem The length of $\mathbf{a} \times \mathbf{b}$ is given by: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta)$, where $\theta \in [0, \pi]$ is the angle between \mathbf{a} and \mathbf{b} . Also $|\mathbf{a} \times \mathbf{b}|$ is **area of the parallelogram** with sides \mathbf{a} and \mathbf{b} . Note that it follows that area of the triangle with vertices $\langle 0, 0, 0 \rangle$ and the position vectors \mathbf{a} and \mathbf{b} is: $|\mathbf{a} \times \mathbf{b}|/2$.

Theorem Let two vectors \mathbf{a} and \mathbf{b} . Then:

- $\mathbf{a} \perp \mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} = 0$ (number).
- $\mathbf{a} // \mathbf{b}$ if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (vector!).

Definition [Scalar triple product] The **scalar triple product** of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ can be calculated by the determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem The **volume of the parallelepiped** determined by the vectors \mathbf{a}, \mathbf{b} and \mathbf{c} is the **absolute value** $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Definition Given a point $P(x_0, y_0, z_0)$ and a vector $\mathbf{v} = \langle a, b, c \rangle$, the **vector equation** of the line L passing through P_0 in the direction of \mathbf{v} is:

$$\mathbf{r}(t) = \mathbf{OP} + t\mathbf{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.$$

The resulting equations:

$$x = x_0 + at,$$

$$y = y_0 + bt,$$

$$z = z_0 + ct,$$

are called the **parametric equations** for L .

Definition If t is eliminated from the parametric equations of the line L , we obtain the **symmetric equations** for L given by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Definition The distance D from a point S to a line that passes through P parallel to a vector \mathbf{v} is given by

$$D = |\mathbf{PS}| \sin \theta = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

Definition

- The **plane** passing through the point $P(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by the following **vector equation**:

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,$$

where (x, y, z) denotes a **general point** on the plane.

- Equivalently, we have:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d \quad \text{with} \quad d = -(ax_0 + by_0 + cz_0).$$

- The **angle** between two intersecting planes with normal vectors \mathbf{n}_1 and \mathbf{n}_2 is the angle between their normal vectors given by

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right).$$

- The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by

$$D = |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Definition [Vector functions and properties] Consider $\mathbf{r}(t)$ with $x = f(t)$, $y = g(t)$, and $z = h(t)$ and $t \in [a, b]$. Then:

- $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \langle \lim_{t \rightarrow t_0} f(t), \lim_{t \rightarrow t_0} g(t), \lim_{t \rightarrow t_0} h(t) \rangle$.
- If \mathbf{r} is continuous at $t = t_0$, then $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.
- $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous if $\mathbf{r}(t)$ is continuous $\forall t \in I$.
- The **velocity field** is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **acceleration field** is $\mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle = \mathbf{v}'(t)$.
- The **indefinite integral** of $\mathbf{r}(t)$ wrt t is

$$\int \mathbf{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle + \mathbf{c},$$

where \mathbf{c} is an **arbitrary constant vector**.

- The **definite integral** of $\mathbf{r}(t)$ from a to b is

$$\int_a^b \mathbf{r}(t) dt = \langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \rangle,$$

- The **length** L of $\mathbf{r}(t)$ is the integral of the speed:

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

Definition [Curvature and TNB frame] Let a parametrization of C in \mathbb{R}^3 be $\mathbf{r} = \langle f(t), g(t), h(t) \rangle$. Then:

- The **unit tangent vector** is $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$.
- The **unit normal vector** is $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$.
- The **unit binormal vector** is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.
- The **curvature of the curve** is

$$\kappa \doteq \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3},$$

- The **speed** is given by $\frac{ds}{dt} = |\mathbf{r}'(t)|$, where s stands for the **ar-length function**:

$$s(t) = \int_a^t |\mathbf{r}'(u)| du.$$

Definition The **acceleration vector** can be decomposed as

$$\mathbf{a} \doteq a_T \mathbf{T} + a_N \mathbf{N},$$

whence

- $a_T \doteq \frac{d}{dt} |\mathbf{r}'|$ is the **tangential** component of \mathbf{a} , and
- $a_N = \kappa |\mathbf{r}'|^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \sqrt{|\mathbf{a}|^2 - a_T^2}$ is the **normal** component of \mathbf{a} .