Statistics for Business and Economics

6th Edition

Chapter 5

Discrete Random Variables and Probability Distributions

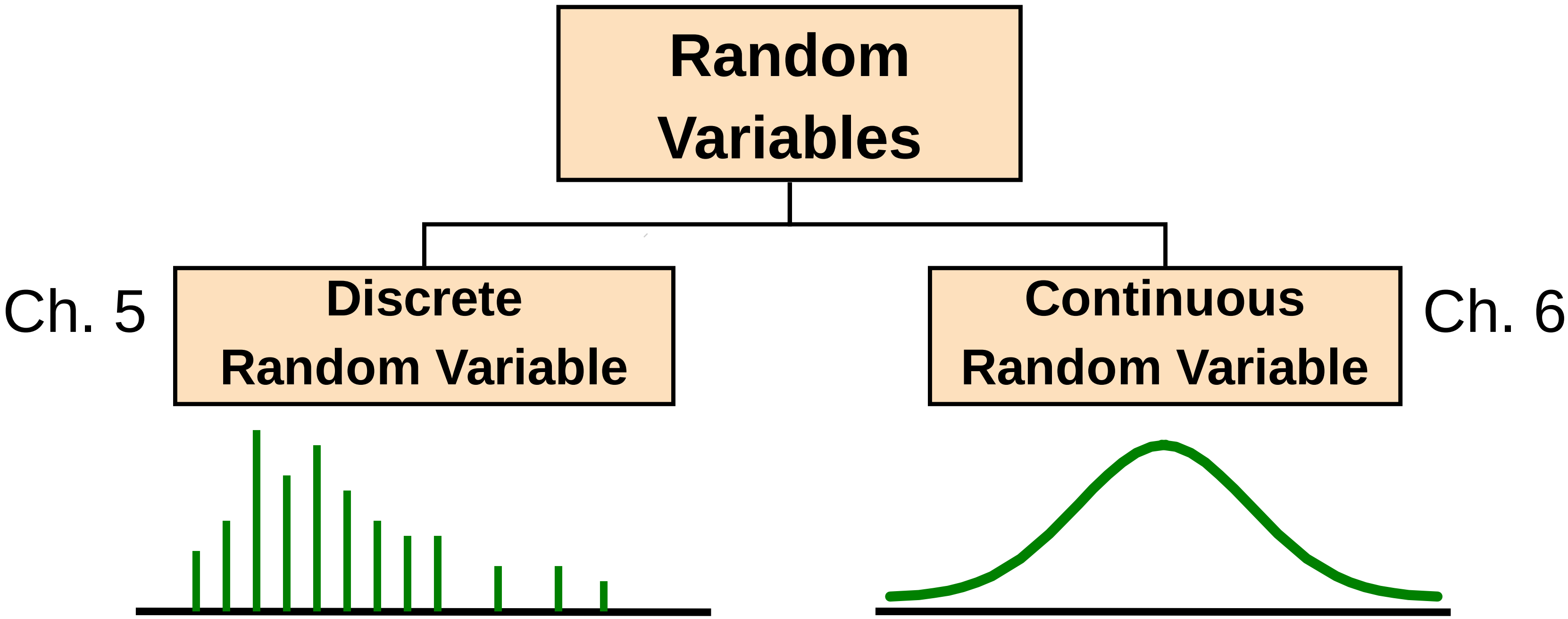
Chapter Goals

After completing this chapter, you should be able to:

- Interpret the mean and standard deviation for a discrete random variable
- Use the binomial probability distribution to find probabilities
- Describe when to apply the binomial distribution
- Use the hypergeometric and Poisson discrete probability distributions to find probabilities
- Explain covariance and correlation for jointly distributed discrete random variables

Introduction to Probability Distributions

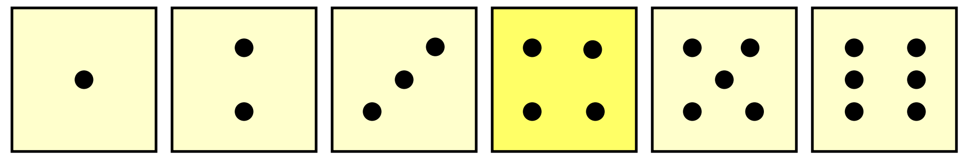
- **Random Variable**
  - Represents a possible numerical value from a random experiment



Discrete Random Variables

- Can only take on a countable number of values

Examples:



- **Roll a die twice**
  Let X be the number of times 4 comes up
  (then X could be 0, 1, or 2 times)

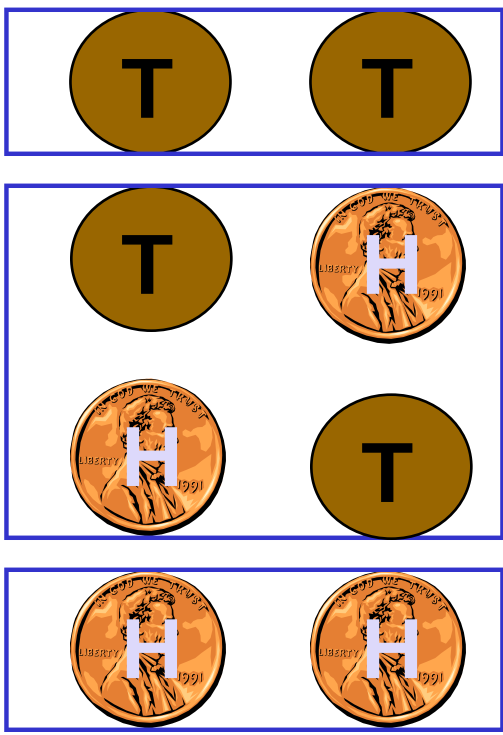
- **Toss a coin 5 times.**
  Let X be the number of heads
  (then X = 0, 1, 2, 3, 4, or 5)

Discrete Probability Distribution

Experiment: Toss 2 Coins. Let X = # heads.

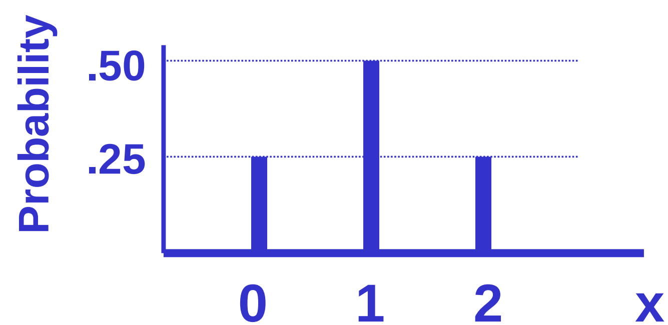
Show P(x) , i.e., P(X = x) , for all values of x:

4 possible outcomes



Probability Distribution

x Value	Probability
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25



Probability Distribution Required Properties

- $P(x) \geq 0$ for any value of x
- The individual probabilities sum to 1;

$$\sum_{x} P(x) = 1$$

(The notation indicates summation over all possible x values)

Cumulative Probability Function

- The cumulative probability function, denoted $F(x_0)$, shows the probability that X is less than or equal to x_0

$$\boxed{F(x_0) = P(X \le x_0)}$$

- In other words,

$$\boxed{F(x_0) = \sum_{x \le x_0} P(x)}$$

Expected Value

- Expected Value (or mean) of a discrete distribution (Weighted Average)

$$\mu = E(x) = \sum_x xP(x)$$

x	P(x)
0	.25
1	.50
2	.25

- Example: Toss 2 coins,
  x = # of heads,
  compute expected value of x:

E(x) = (0 x .25) + (1 x .50) + (2 x .25)
= 1.0

Variance and Standard Deviation

- Variance of a discrete random variable X

$$\sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 P(x)$$

- Standard Deviation of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_x (x-\mu)^2 P(x)}$$

Standard Deviation Example

- Example: Toss 2 coins, X = # heads, compute standard deviation (recall E(x) = 1)

$$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)}$$

$$\sigma = \sqrt{(0-1)^2(.25) + (1-1)^2(.50) + (2-1)^2(.25)} = \sqrt{.50} = .707$$

Possible number of heads = 0, 1, or 2

Functions of Random Variables

- If P(x) is the probability function of a discrete random variable X , and g(X) is some function of X , then the expected value of function g is

$$\boxed{E[g(X)] = \sum_{x} g(x)P(x)}$$

Linear Functions of Random Variables

- Let a and b be any constants.

- a) $\boxed{E(a) = a \quad \text{and} \quad Var(a) = 0}$

  i.e., if a random variable always takes the value a, it will have mean a and variance 0

- b) $\boxed{E(bX) = b\mu_X \quad \text{and} \quad Var(bX) = b^2\sigma_X^2}$

  i.e., the expected value of $b \cdot X$ is $b \cdot E(x)$

Linear Functions of Random Variables

(continued)

- Let random variable X have mean μ_x and variance σ^2_x
- Let a and b be any constants.
- Let Y = a + bX
- Then the mean and variance of Y are

$$\boxed{\mu_Y = E(a + bX) = a + b\mu_X}$$

$$\boxed{\sigma^2_Y = Var(a + bX) = b^2\sigma^2_X}$$

- so that the standard deviation of Y is

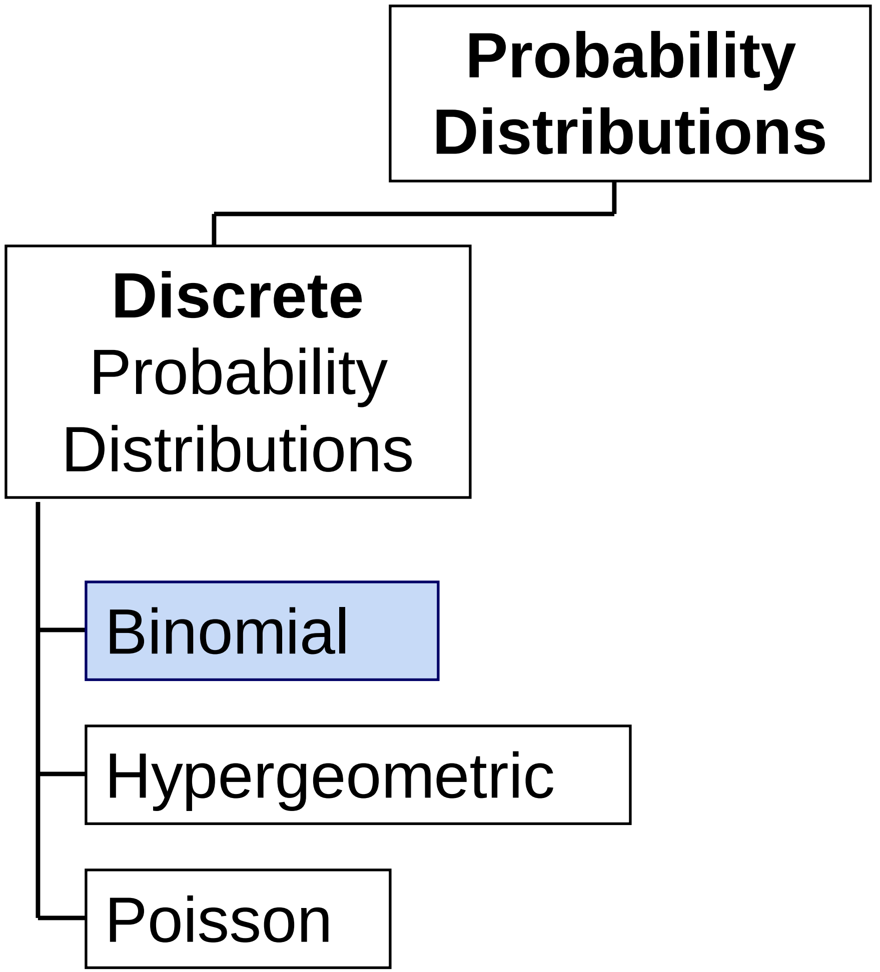
$$\boxed{\sigma_Y = |b|\sigma_X}$$

Probability Distributions

Probability Distributions

- Ch. 5 — **Discrete** Probability Distributions
  - Binomial
  - Hypergeometric
  - Poisson
- Ch. 6 — **Continuous** Probability Distributions
  - Uniform
  - Normal
  - Exponential

The Binomial Distribution



Bernoulli Distribution

- Consider only two outcomes: "success" or "failure"
- Let P denote the probability of success
- Let $1 - P$ be the probability of failure
- Define random variable X:

  $x = 1$ if success, $x = 0$ if failure
- Then the Bernoulli probability function is

$$\boxed{P(0) = (1 - P) \quad \text{and} \quad P(1) = P}$$

Bernoulli Distribution Mean and Variance

- The mean is $\boxed{\mu = P}$

$$\mu = E(X) = \sum_{X} xP(x) = (0)(1-P) + (1)P = P$$

- The variance is $\boxed{\sigma^2 = P(1 - P)}$

$$\sigma^2 = E[(X-\mu)^2] = \sum_{X} (x-\mu)^2 P(x)$$

$$= (0-P)^2(1-P) + (1-P)^2 P = P(1-P)$$

Sequences of x Successes in n Trials

- The number of sequences with x successes in n independent trials is:

$$\boxed{C_x^n = \frac{n!}{x!(n-x)!}}$$

Where $n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 1$ and $0! = 1$

- These sequences are mutually exclusive, since no two can occur at the same time

Binomial Probability Distribution

- A fixed number of observations, n
  - e.g., 15 tosses of a coin; ten light bulbs taken from a warehouse
- Two mutually exclusive and collectively exhaustive categories
  - e.g., head or tail in each toss of a coin; defective or not defective light bulb
  - Generally called "success" and "failure"
  - Probability of success is P , probability of failure is 1 – P
- Constant probability for each observation
  - e.g., Probability of getting a tail is the same each time we toss the coin
- Observations are independent
  - The outcome of one observation does not affect the outcome of the other

Possible Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it

Binomial Distribution Formula

$$P(x) = \frac{n!}{x!(n-x)!} P^{x}(1-P)^{n-x}$$

P(x) = probability of **x** successes in **n** trials, with probability of success **P** on each trial

x = number of 'successes' in sample, (x = 0, 1, 2, ..., n)

n = sample size (number of trials or observations)

P = probability of "success"

Example: Flip a coin four times, let x = # heads:

n = 4

P = 0.5

1 - P = (1 - 0.5) = 0.5

x = 0, 1, 2, 3, 4

Example: Calculating a Binomial Probability

What is the probability of one success in five observations if the probability of success is 0.1?

$x = 1$, $n = 5$, and $P = 0.1$

$$P(x=1) = \frac{n!}{x!(n-x)!}P^X(1-P)^{n-X}$$

$$= \frac{5!}{1!(5-1)!}(0.1)^1(1-0.1)^{5-1}$$

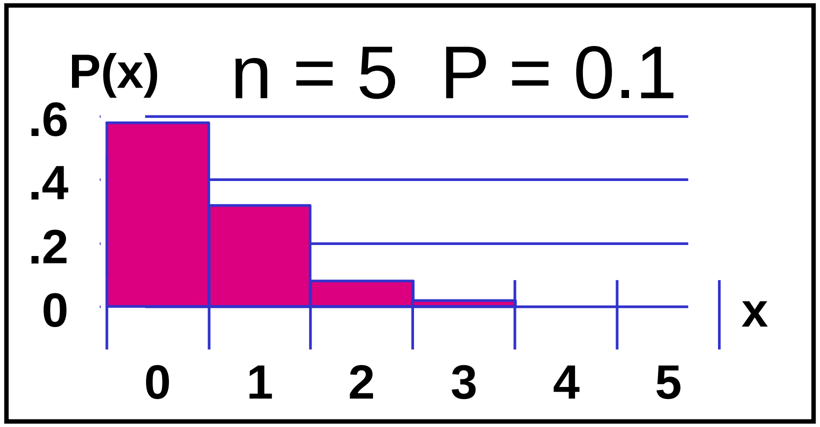
$$= (5)(0.1)(0.9)^4$$

$$= .32805$$

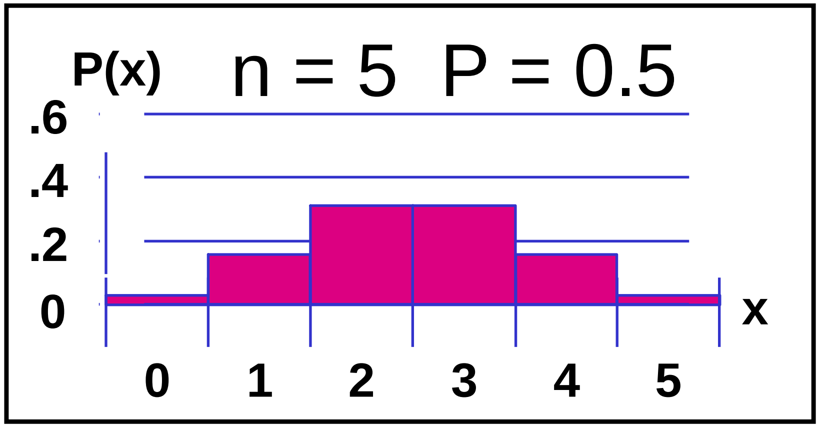
Binomial Distribution

- The shape of the binomial distribution depends on the values of P and n

- Here, n = 5 and P = 0.1



- Here, n = 5 and P = 0.5



Binomial Distribution Mean and Variance

- Mean

$$\mu = E(x) = nP$$

- Variance and Standard Deviation

$$\sigma^2 = nP(1-P)$$

$$\sigma = \sqrt{nP(1-P)}$$

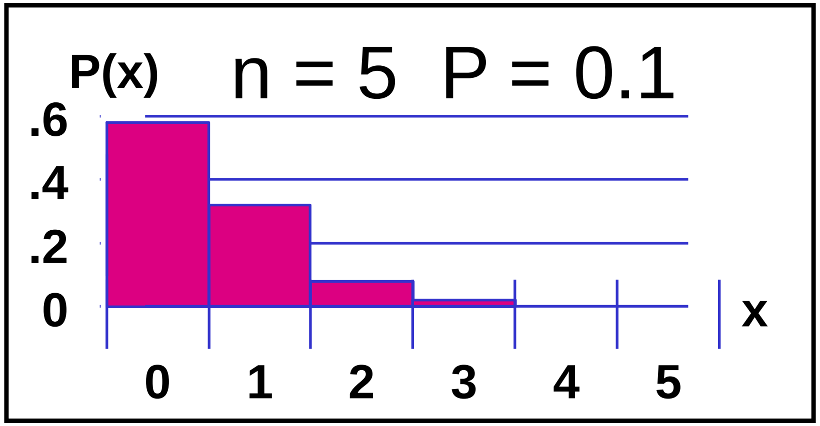
Where n = sample size
P = probability of success
(1 – P) = probability of failure

Binomial Characteristics

Examples

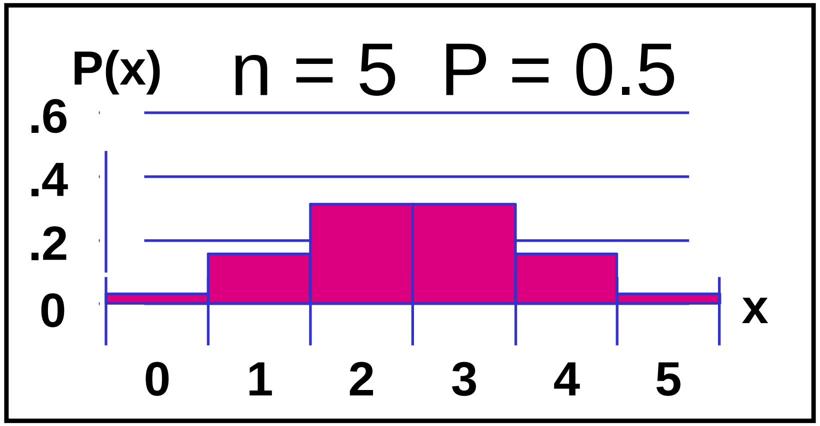
$$\mu = nP = (5)(0.1) = 0.5$$

$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.1)(1-0.1)}$$
$$= 0.6708$$



$$\mu = nP = (5)(0.5) = 2.5$$

$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.5)(1-0.5)}$$
$$= 1.118$$



Using Binomial Tables

N	x	…	p=.20	p=.25	p=.30	p=.35	p=.40	p=.45	p=.50
10	0	…	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
	1	…	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098
	2	…	0.3020	0.2816	0.2335	0.1757	0.1209	0.0763	0.0439
	3	…	0.2013	0.2503	0.2668	**0.2522**	0.2150	0.1665	0.1172
	4	…	0.0881	0.1460	0.2001	0.2377	0.2508	0.2384	0.2051
	5	…	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461
	6	…	0.0055	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051
	7	…	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172
	8	…	0.0001	0.0004	0.0014	0.0043	0.0106	**0.0229**	0.0439
	9	…	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098
	10	…	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010

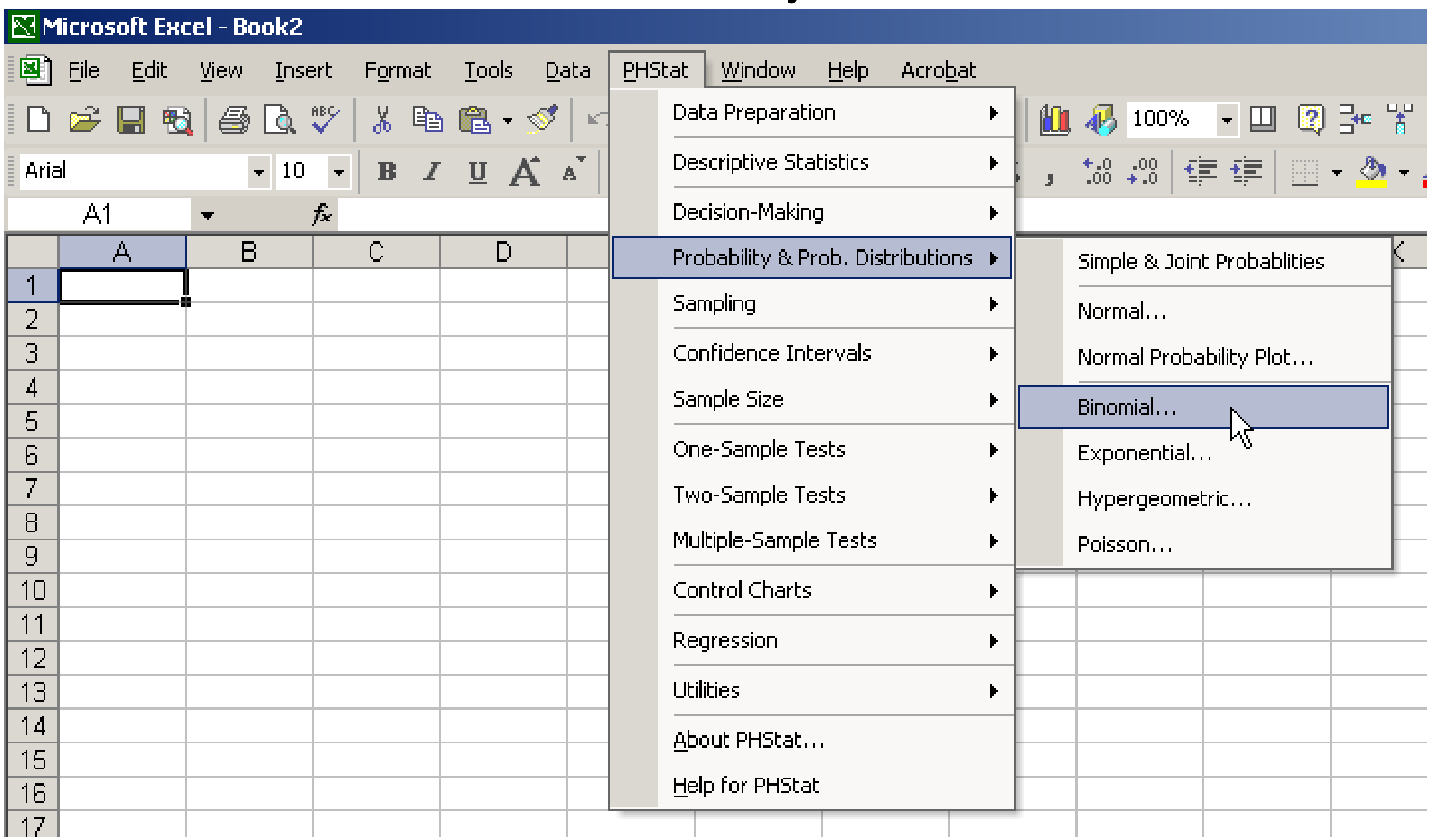
Examples:

n = 10, x = 3, P = 0.35: $P(x = 3|n = 10, p = 0.35) = .2522$

n = 10, x = 8, P = 0.45: $P(x = 8|n = 10, p = 0.45) = .0229$

Using PHStat

- Select PHStat / Probability & Prob. Distributions / Binomial…



Using PHStat

(continued)

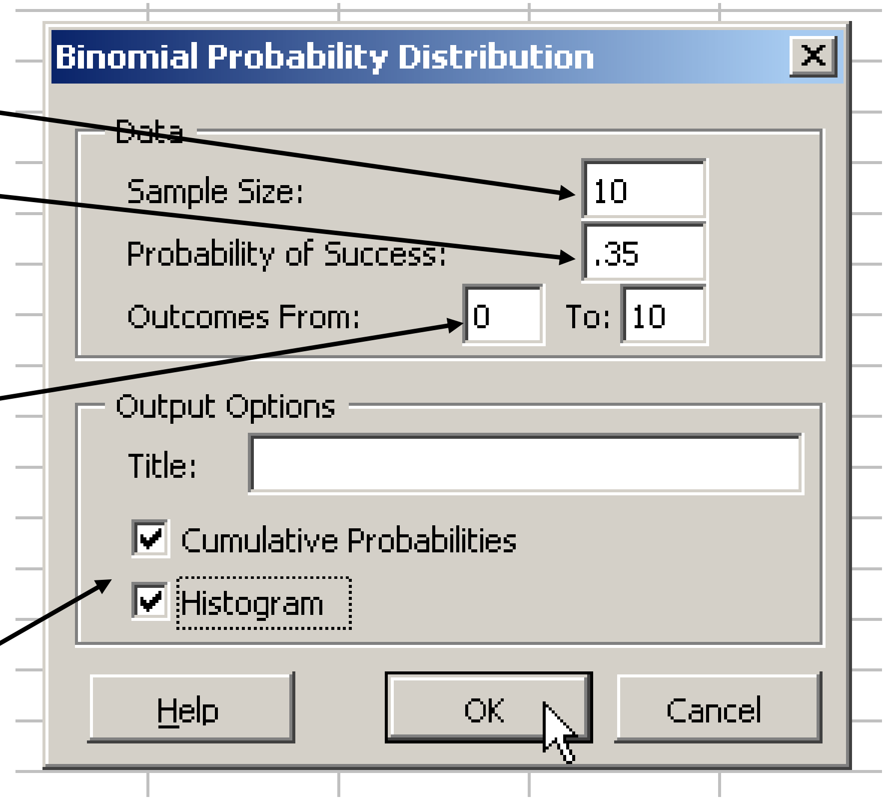
- Enter desired values in dialog box

Here: n = 10

p = .35

Output for x = 0 to x = 10 will be generated by PHStat

Optional check boxes for additional output



PHStat Output

	A	B	C	D	E	F	G	H
1	Binomial Probabilities							
2								
3	Data							
4	Sample size	10						
5	Probability of success	0.35						
6								
7	Statistics							
8	Mean	3.5						
9	Variance	2.275						
10	Standard deviation	1.50831						
11								
12	Binomial Probabilities Table							
13		X	P(X)	P(<=X)	P(<X)	P(>X)	P(>=X)	
14		0	0.013463	0.013463	0	0.986537	1	
15		1	0.072492	0.085954	0.013463	0.914046	0.986537	
16		2	0.175653	0.261607	0.085954	0.738393	0.914046	
17		3	0.25222	0.513827	0.261607	0.486173	0.738393	
18		4	0.237668	0.751496	0.513827	0.248504	0.486173	
19		5	0.15357	0.905066	0.751496	0.094934	0.248504	
20		6	0.06891	0.973976	0.905066	0.026024	0.094934	
21		7	0.021203	0.995179	0.973976	0.004821	0.026024	
22		8	0.004281	0.99946	0.995179	0.00054	0.004821	
23		9	0.000512	0.999972	0.99946	2.76E-05	0.00054	
24		10	2.76E-05	1	0.999972	0	2.76E-05	
25								
26								
27								
28								

P(x = 3 | n = 10, P = .35) = .2522

P(x > 5 | n = 10, P = .35) = .0949

The Hypergeometric Distribution

Probability Distributions

- **Discrete** Probability Distributions
  - Binomial
  - Hypergeometric
  - Poisson

The Hypergeometric Distribution

- "n" trials in a sample taken from a finite population of size N
- Sample taken without replacement
- Outcomes of trials are dependent
- Concerned with finding the probability of "X" successes in the sample where there are "S" successes in the population

Hypergeometric Distribution Formula

$$P(x) = \frac{C_x^S C_{n-x}^{N-S}}{C_n^N} = \frac{\dfrac{S!}{x!(S-x)!} \times \dfrac{(N-S)!}{(n-x)!(N-S-n+x)!}}{\dfrac{N!}{n!(N-n)!}}$$

Where

N = population size

S = number of successes in the population

N – S = number of failures in the population

n = sample size

x = number of successes in the sample

n – x = number of failures in the sample

Using the Hypergeometric Distribution

- Example: 3 different computers are checked from 10 in the department. 4 of the 10 computers have illegal software loaded. What is the probability that 2 of the 3 selected computers have illegal software loaded?

N = 10	n = 3
S = 4	x = 2

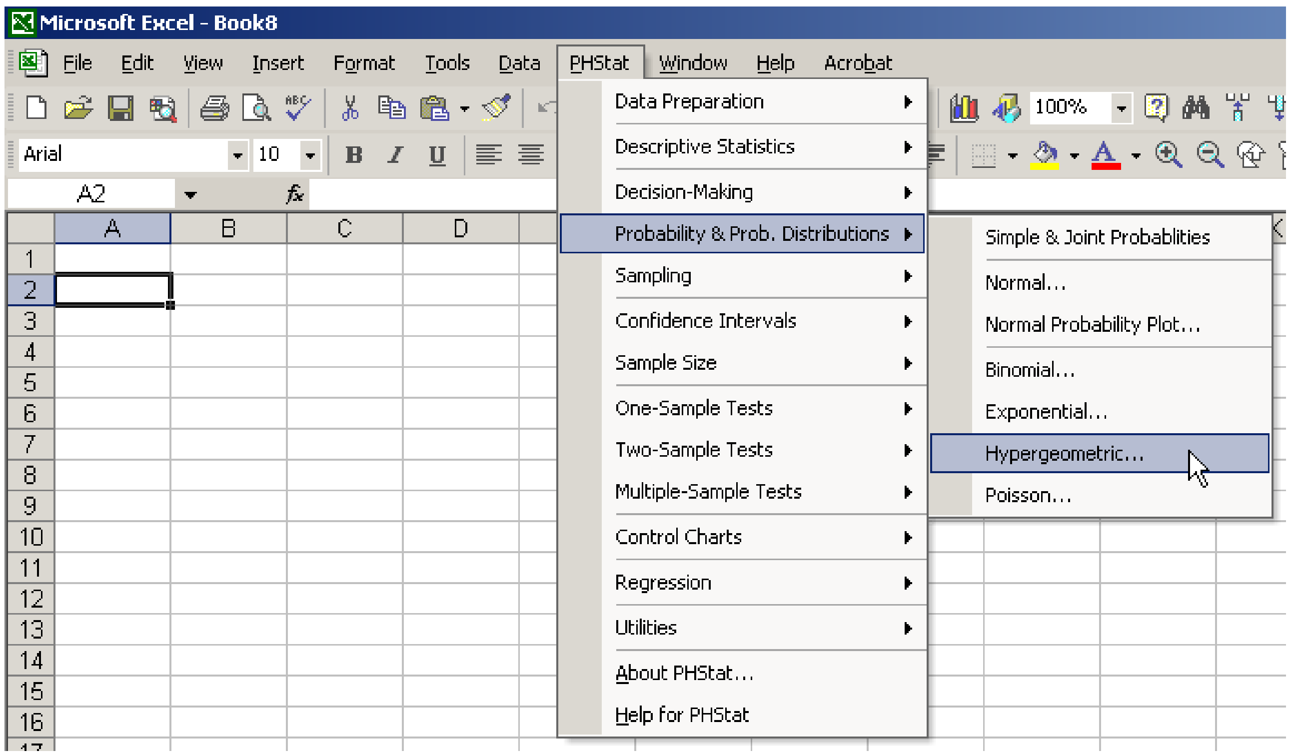
$$P(x=2) = \frac{C_x^S C_{n-x}^{N-S}}{C_n^N} = \frac{C_2^4 C_1^6}{C_3^{10}} = \frac{(6)(6)}{120} = 0.3$$

The probability that 2 of the 3 selected computers have illegal software loaded is 0.30, or 30%.

Hypergeometric Distribution in PHStat

- Select:

PHStat / Probability & Prob. Distributions / Hypergeometric …

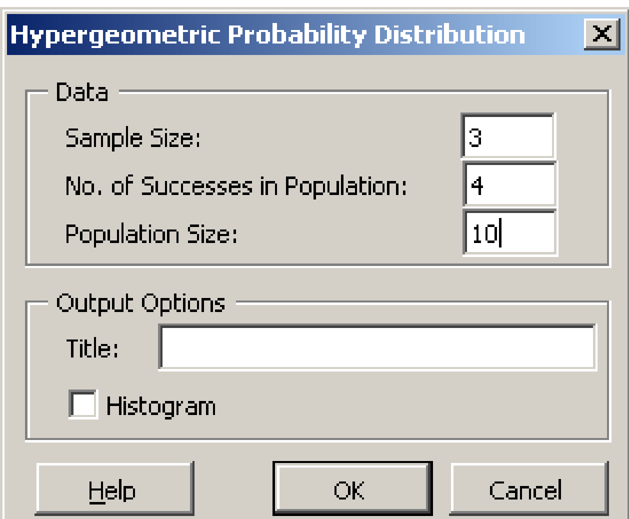


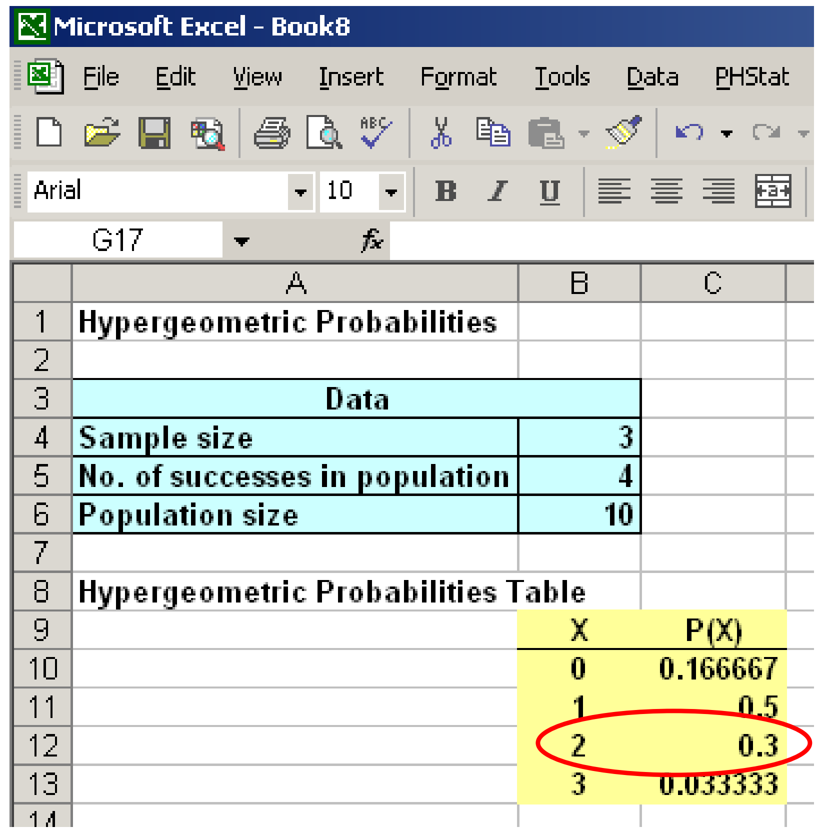
Hypergeometric Distribution in PHStat

(continued)

- Complete dialog box entries and get output …

N = 10	n = 3
S = 4	x = 2





P(X = 2) = 0.3

The Poisson Distribution

- **Probability Distributions**
  - **Discrete** Probability Distributions
    - Binomial
    - Hypergeometric
    - Poisson

The Poisson Distribution

- Apply the Poisson Distribution when:
  - You wish to count the number of times an event occurs in a given continuous interval
  - The probability that an event occurs in one subinterval is very small and is the same for all subintervals
  - The number of events that occur in one subinterval is independent of the number of events that occur in the other subintervals
  - There can be no more than one occurrence in each subinterval
  - The average number of events per unit is λ (lambda)

Poisson Distribution Formula

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where:

x = number of successes per unit

λ = expected number of successes per unit

e = base of the natural logarithm system (2.71828...)

Poisson Distribution Characteristics

- Mean

$$\mu = E(x) = \lambda$$

- Variance and Standard Deviation

$$\sigma^2 = E[(X-\mu)^2] = \lambda$$

$$\sigma = \sqrt{\lambda}$$

where λ = expected number of successes per unit

Using Poisson Tables

X	λ								
	0.10	0.20	0.30	0.40	**0.50**	0.60	0.70	0.80	0.90
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	**0.0758**	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example: Find $P(X = 2)$ if $\lambda = .50$

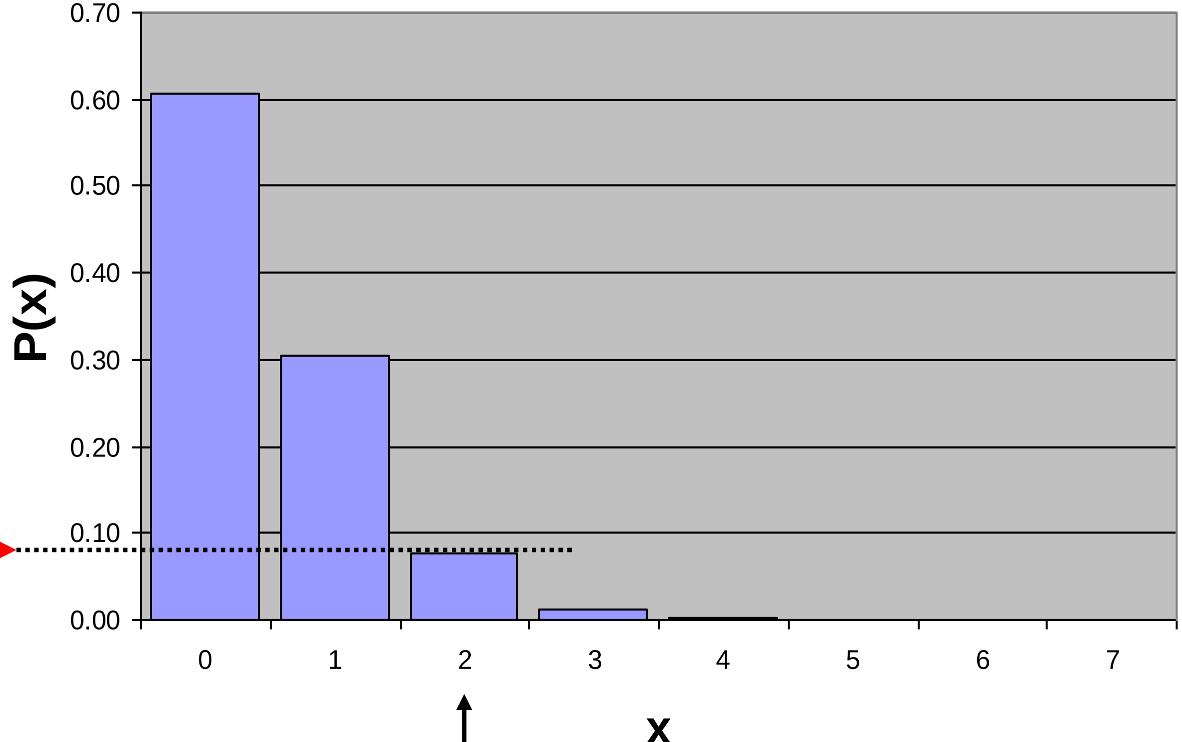
$$P(X=2)=\frac{e^{-\lambda}\lambda^{X}}{X!}=\frac{e^{-0.50}(0.50)^{2}}{2!}=.0758$$

Graph of Poisson Probabilities

Graphically:

λ = .50

X	**λ = 0.50**
0	0.6065
1	0.3033
2	0.0758
3	0.0126
4	0.0016
5	0.0002
6	0.0000
7	0.0000

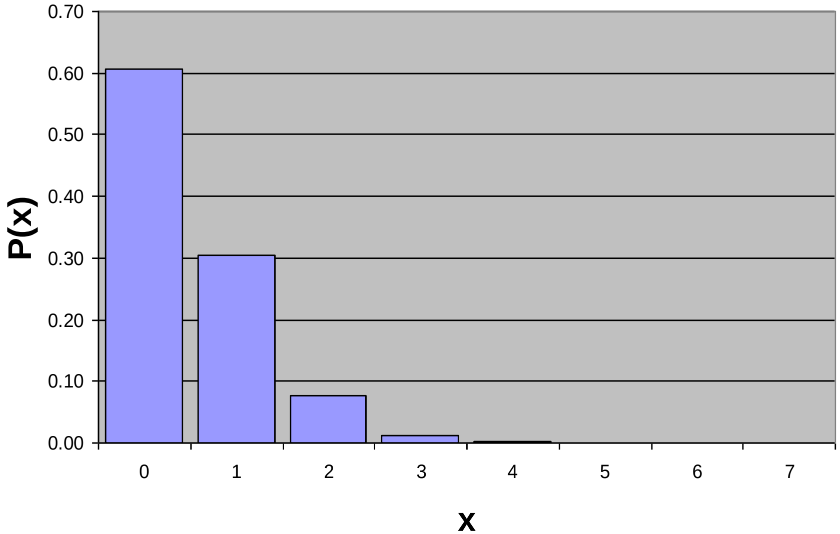


P(X = 2) = .0758

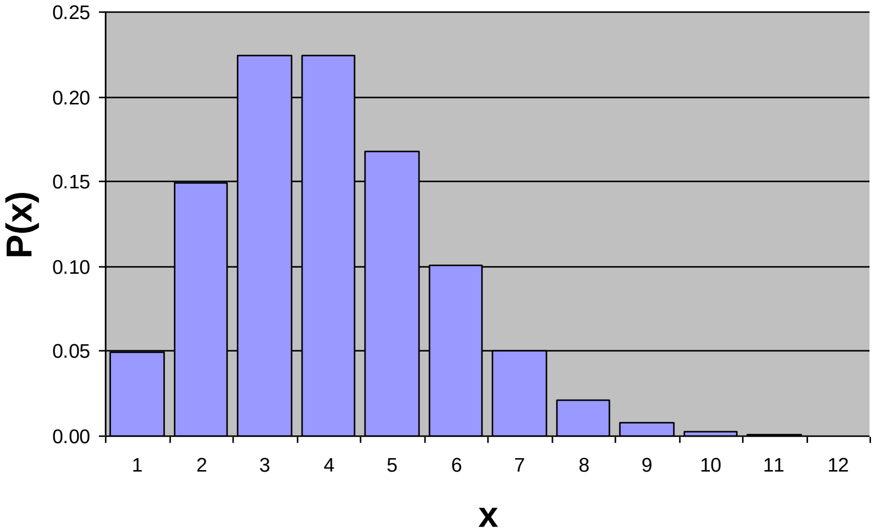
Poisson Distribution Shape

- The shape of the Poisson Distribution depends on the parameter λ :

λ = 0.50



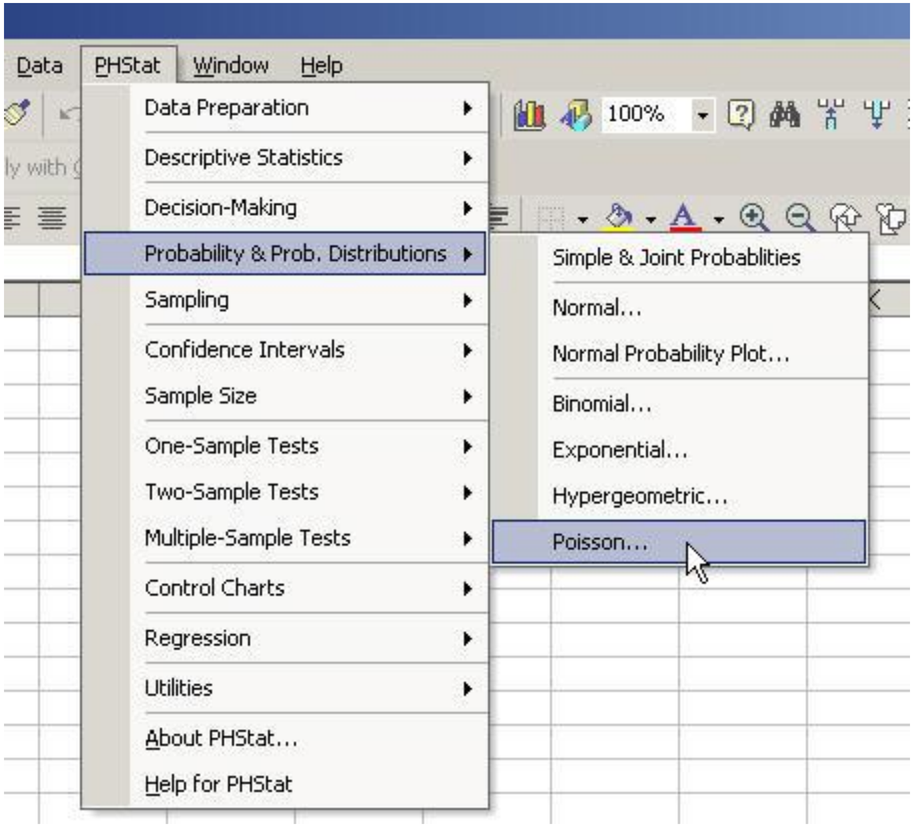
λ = 3.00



Poisson Distribution in PHStat

- Select:

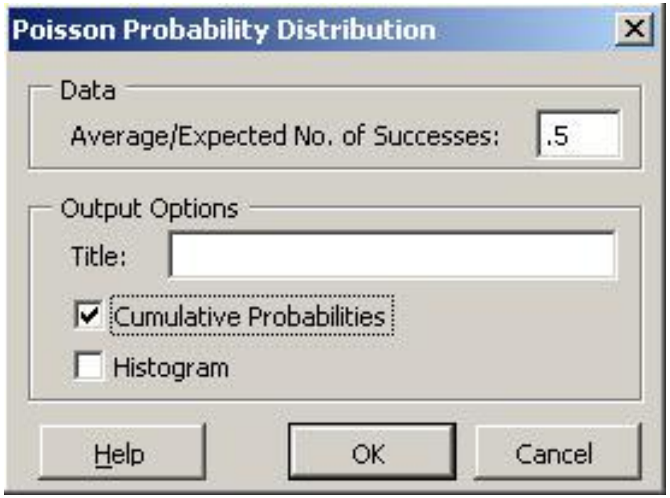
  PHStat / Probability & Prob. Distributions / Poisson…



Poisson Distribution in PHStat

(continued)

- Complete dialog box entries and get output …



	A	B	C	D	E	F	G
1	Poisson Probabilities for Customer Arrivals						
2							
3	Data						
4	Average/Expected number of successes:				0.5		
5							
6	Poisson Probabilities Table						
7		X	P(X)	P(<=X)	P(<X)	P(>X)	P(>=X)
8		0	0.606531	0.606531	0.000000	0.393469	1.000000
9		1	0.303265	0.909796	0.606531	0.090204	0.393469
10		2	0.075816	0.985612	0.909796	0.014388	0.090204
11		3	0.012636	0.998248	0.985612	0.001752	0.014388
12		4	0.001580	0.999828	0.998248	0.000172	0.001752
13		5	0.000158	0.999986	0.999828	0.000014	0.000172
14		6	0.000013	0.999999	0.999986	0.000001	0.000014
15		7	0.000001	1.000000	0.999999	0.000000	0.000001
16		8	0.000000	1.000000	1.000000	0.000000	0.000000

$P(X = 2) = 0.0758$

Joint Probability Functions

- A joint probability function is used to express the probability that X takes the specific value x and simultaneously Y takes the value y, as a function of x and y

$$P(x, y) = P(X = x \cap Y = y)$$

- The marginal probabilities are

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

Conditional Probability Functions

- The conditional probability function of the random variable Y expresses the probability that Y takes the value y when the value x is specified for X.

$$P(y \mid x) = \frac{P(x,y)}{P(x)}$$

- Similarly, the conditional probability function of X, given Y = y is:

$$P(x \mid y) = \frac{P(x,y)}{P(y)}$$

Independence

- The jointly distributed random variables X and Y are said to be independent if and only if their joint probability function is the product of their marginal probability functions:

$$\boxed{P(x,y) = P(x)P(y)}$$

for all possible pairs of values x and y

- A set of k random variables are independent if and only if

$$\boxed{P(x_1, x_2, \cdots, x_k) = P(x_1)P(x_2)\cdots P(x_k)}$$

Covariance

- Let X and Y be discrete random variables with means μ_X and μ_Y
- The expected value of $(X - \mu_X)(Y - \mu_Y)$ is called the covariance between X and Y
- For discrete random variables

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_x)(y-\mu_y)P(x,y)$$

- An equivalent expression is

$$\mathrm{Cov}(X,Y) = E(XY) - \mu_x\mu_y = \sum_x \sum_y xyP(x,y) - \mu_x\mu_y$$

Covariance and Independence

- The covariance measures the strength of the linear relationship between two variables

- If two random variables are statistically independent, the covariance between them is 0
  - The converse is not necessarily true

Correlation

- The correlation between X and Y is:

$$\boxed{\rho = \text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}}$$

- $\rho = 0 \Rightarrow$ no linear relationship between X and Y
- $\rho > 0 \Rightarrow$ positive linear relationship between X and Y
  - when X is high (low) then Y is likely to be high (low)
  - $\rho = +1 \Rightarrow$ perfect positive linear dependency
- $\rho < 0 \Rightarrow$ negative linear relationship between X and Y
  - when X is high (low) then Y is likely to be low (high)
  - $\rho = -1 \Rightarrow$ perfect negative linear dependency

Portfolio Analysis

- Let random variable X be the price for stock A
- Let random variable Y be the price for stock B
- The market value, W, for the portfolio is given by the linear function

$$\boxed{W = aX + bY}$$

(a is the number of shares of stock A,
b is the number of shares of stock B)

Portfolio Analysis

(continued)

- The mean value for W is

$$\boxed{\begin{aligned}\mu_W &= E[W] = E[aX + bY] \\ &= a\mu_X + b\mu_Y\end{aligned}}$$

- The variance for W is

$$\boxed{\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X, Y)}$$

or using the correlation formula

$$\boxed{\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Corr}(X, Y)\sigma_X\sigma_Y}$$

Example: Investment Returns

Return per $1,000 for two types of investments

		Investment	
$P(x_i y_i)$	**Economic condition**	**Passive Fund X**	**Aggressive Fund Y**
.2	Recession	- $ 25	- $200
.5	Stable Economy	+ 50	+ 60
.3	Expanding Economy	+ 100	+ 350

$$E(x) = \mu_x = (-25)(.2) + (50)(.5) + (100)(.3) = 50$$

$$E(y) = \mu_y = (-200)(.2) + (60)(.5) + (350)(.3) = 95$$

Computing the Standard Deviation for Investment Returns

$P(x_i y_i)$	Economic condition	Investment: Passive Fund X	Investment: Aggressive Fund Y
0.2	Recession	- $ 25	- $200
0.5	Stable Economy	+ 50	+ 60
0.3	Expanding Economy	+ 100	+ 350

$$\sigma_X = \sqrt{(-25-50)^2(0.2)+(50-50)^2(0.5)+(100-50)^2(0.3)}$$
$$= 43.30$$

$$\sigma_y = \sqrt{(-200-95)^2(0.2)+(60-95)^2(0.5)+(350-95)^2(0.3)}$$
$$= 193.71$$

Covariance for Investment Returns

$P(x_iy_i)$	Economic condition	Investment	
		Passive Fund X	**Aggressive Fund Y**
.2	Recession	- $ 25	- $200
.5	Stable Economy	+ 50	+ 60
.3	Expanding Economy	+ 100	+ 350

$$\begin{aligned}\text{Cov}(X, Y) &= (-25-50)(-200-95)(.2)+(50-50)(60-95)(.5)\\ &\quad +(100-50)(350-95)(.3)\\ &= 8250\end{aligned}$$

Portfolio Example

Investment X:	$\mu_x = 50$	$\sigma_x = 43.30$
Investment Y:	$\mu_y = 95$	$\sigma_y = 193.21$

$$\sigma_{xy} = 8250$$

Suppose 40% of the portfolio (P) is in Investment X and 60% is in Investment Y:

$$E(P) = .4(50) + (.6)(95) = 77$$

$$\sigma_P = \sqrt{(.4)^2(43.30)^2 + (.6)^2(193.21)^2 + 2(.4)(.6)(8250)}$$

$$= 133.04$$

The portfolio return and portfolio variability are between the values for investments X and Y considered individually

Interpreting the Results for Investment Returns

- The aggressive fund has a higher expected return, but much more risk

$$\mu_y = 95 \;>\; \mu_x = 50$$

but

$$\sigma_y = 193.21 \;>\; \sigma_x = 43.30$$

- The Covariance of 8250 indicates that the two investments are positively related and will vary in the same direction

Chapter Summary

- Defined discrete random variables and probability distributions
- Discussed the Binomial distribution
- Discussed the Hypergeometric distribution
- Reviewed the Poisson distribution
- Defined covariance and the correlation between two random variables
- Examined application to portfolio investment