Statistics for Business and Economics

6th Edition

Chapter 6

Continuous Random Variables and Probability Distributions

Chapter Goals

After completing this chapter, you should be able to:

- Explain the difference between a discrete and a continuous random variable
- Describe the characteristics of the uniform and normal distributions
- Translate normal distribution problems into standardized normal distribution problems
- Find probabilities using a normal distribution table

Chapter Goals

(continued)

After completing this chapter, you should be able to:

- Evaluate the normality assumption
- Use the normal approximation to the binomial distribution
- Recognize when to apply the exponential distribution
- Explain jointly distributed variables and linear combinations of random variables

Probability Distributions

Probability Distributions

- Ch. 5 — **Discrete** Probability Distributions
  - Binomial
  - Hypergeometric
  - Poisson
- **Continuous** Probability Distributions — Ch. 6
  - Uniform
  - Normal
  - Exponential

Continuous Probability Distributions

- A continuous random variable is a variable that can assume any value in an interval
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches
- These can potentially take on any value, depending only on the ability to measure accurately.

Cumulative Distribution Function

- The cumulative distribution function, F(x), for a continuous random variable X expresses the probability that X does not exceed the value of x

$$F(x) = P(X \le x)$$

- Let a and b be two possible values of X, with a < b. The probability that X lies between a and b is

$$P(a < X < b) = F(b) - F(a)$$

Probability Density Function

The probability density function, $f(x)$, of random variable X has the following properties:

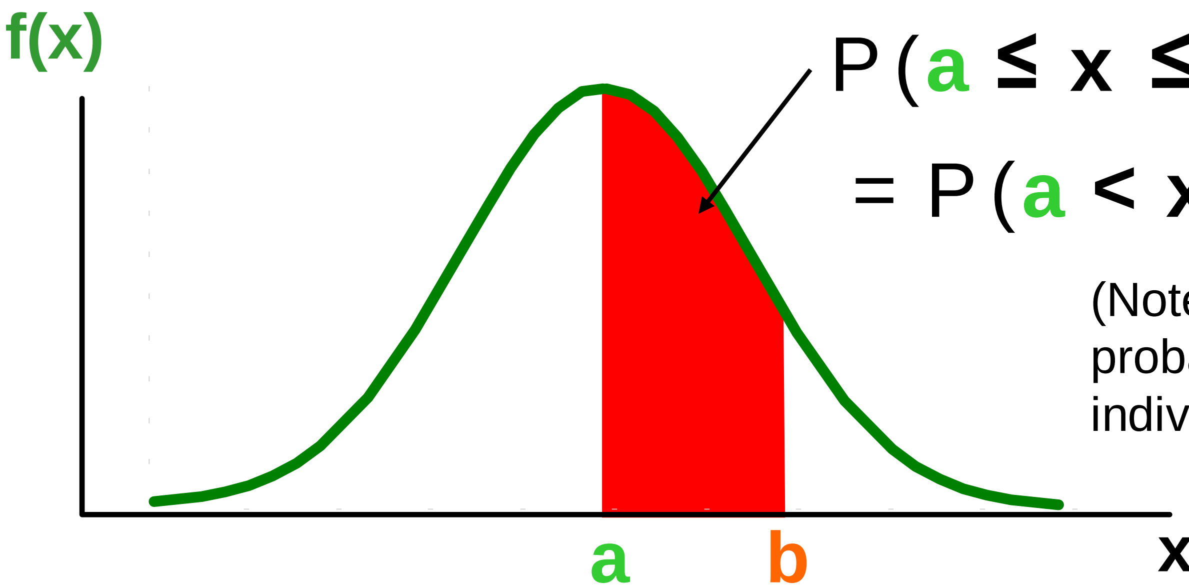
1. $f(x) > 0$ for all values of x
2. The area under the probability density function $f(x)$ over all values of the random variable X is equal to 1.0
3. The probability that X lies between two values is the area under the density function graph between the two values
4. The cumulative density function $F(x_0)$ is the area under the probability density function $f(x)$ from the minimum x value up to x_0

$$f(x_0) = \int_{x_m}^{x_0} f(x)dx$$

where x_m is the minimum value of the random variable x

Probability as an Area

Shaded area under the curve is the probability that X is between a and b



$$P(a \leq x \leq b)$$

$$= P(a < x < b)$$

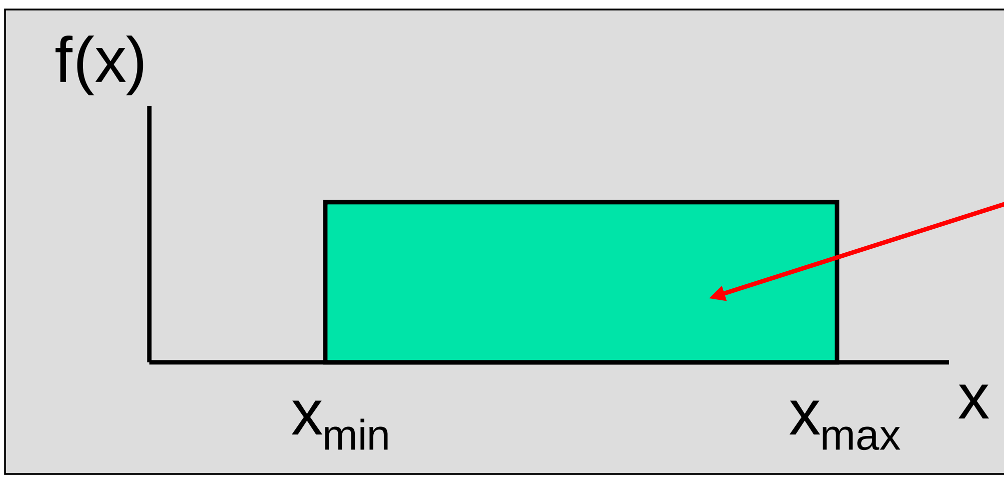
(Note that the probability of any individual value is zero)

The Uniform Distribution

- **Probability Distributions**
  - **Continuous** Probability Distributions
    - Uniform
    - Normal
    - Exponential

The Uniform Distribution

- The uniform distribution is a probability distribution that has equal probabilities for all possible outcomes of the random variable



Total area under the uniform probability density function is 1.0

The Uniform Distribution

(continued)

The Continuous Uniform Distribution:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where

$f(x)$ = value of the density function at any x value

a = minimum value of x

b = maximum value of x

Properties of the Uniform Distribution

- The mean of a uniform distribution is

$$\boxed{\mu = \frac{a+b}{2}}$$

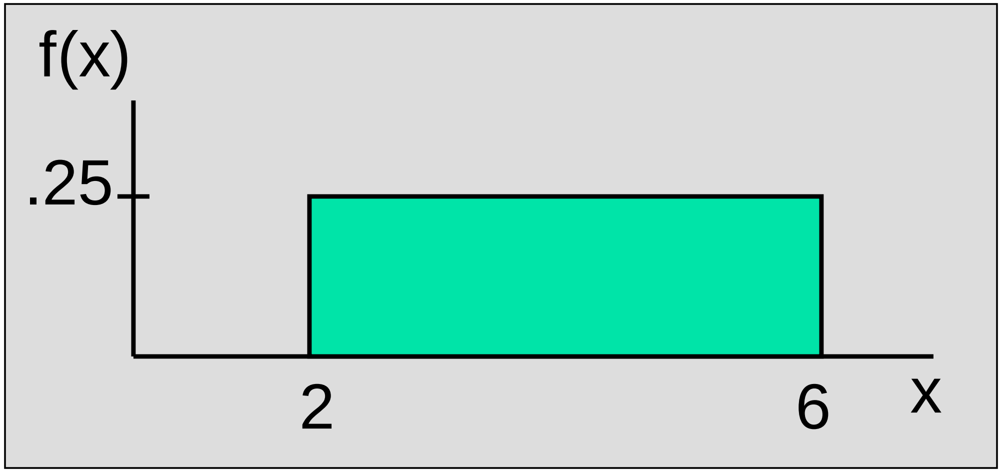
- The variance is

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

Uniform Distribution Example

Example: Uniform probability distribution over the range $2 \le x \le 6$:

$$f(x) = \frac{1}{6 - 2} = .25 \quad \text{for } 2 \le x \le 6$$



$$\mu = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(6-2)^2}{12} = 1.333$$

Expectations for Continuous Random Variables

- The mean of X, denoted μ_X , is defined as the expected value of X

$$\boxed{\mu_X = E(X)}$$

- The variance of X, denoted σ_X^2 , is defined as the expectation of the squared deviation, $(X - \mu_X)^2$, of a random variable from its mean

$$\boxed{\sigma_X^2 = E[(X - \mu_X)^2]}$$

Linear Functions of Variables

- Let $\boxed{W = a + bX}$, where X has mean μ_X and variance σ_X^2, and a and b are constants

- Then the mean of W is

$$\boxed{\mu_W = E(a+bX) = a+b\mu_X}$$

- the variance is

$$\boxed{\sigma_W^2 = \mathrm{Var}(a+bX) = b^2\sigma_X^2}$$

- the standard deviation of W is

$$\boxed{\sigma_W = |b|\sigma_X}$$

Linear Functions of Variables

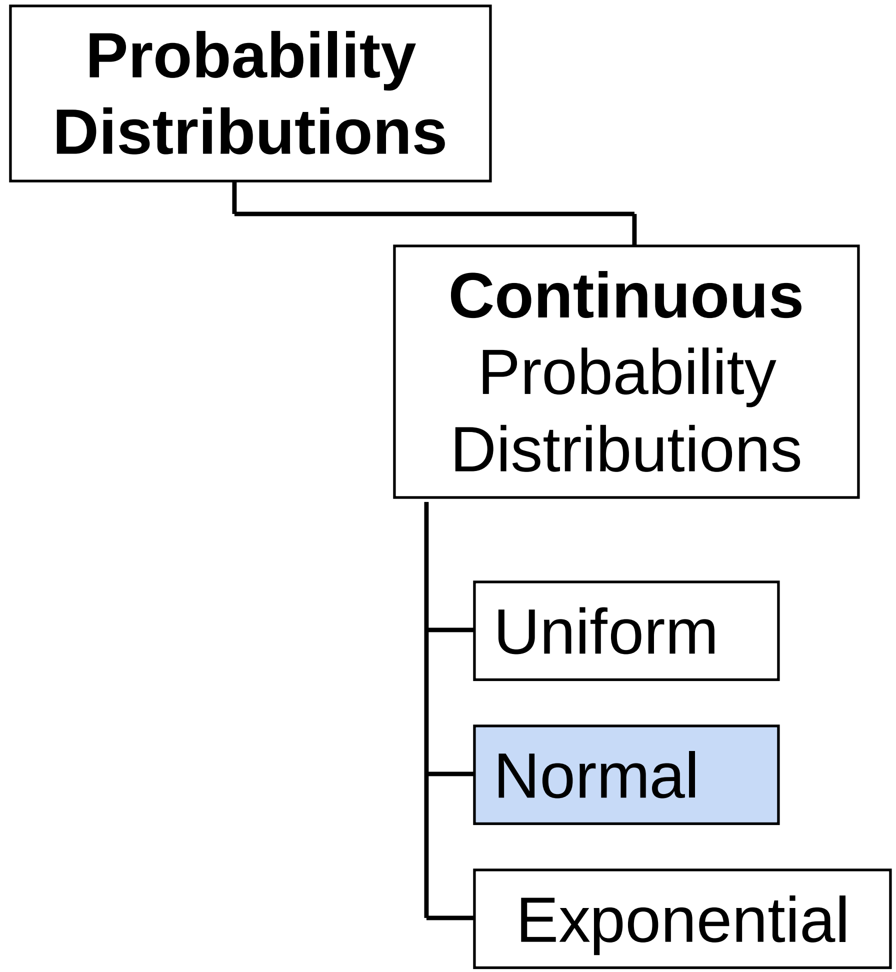
(continued)

- An important special case of the previous results is the standardized random variable

$$Z = \frac{X - \mu_X}{\sigma_X}$$

- which has a mean 0 and variance 1

The Normal Distribution



The Normal Distribution

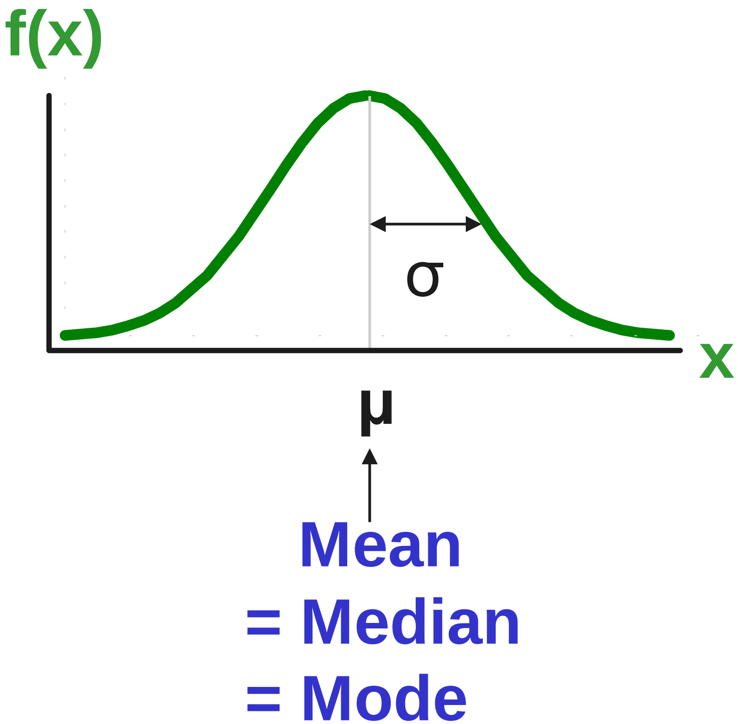
(continued)

- Bell Shaped
- Symmetrical
- Mean, Median and Mode are Equal

Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range:
$+\infty$ to $-\infty$

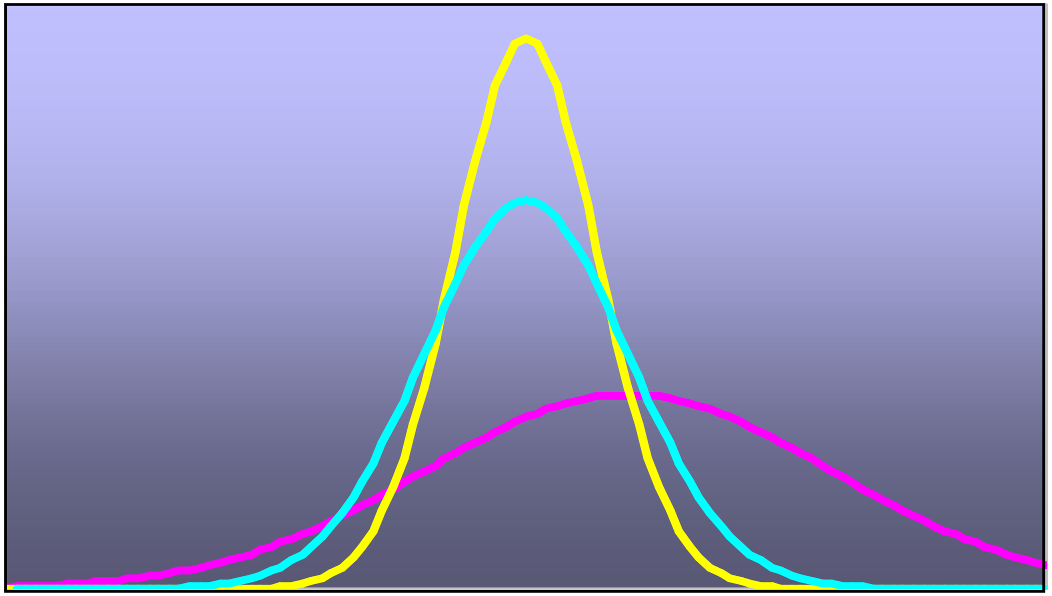


The Normal Distribution

(continued)

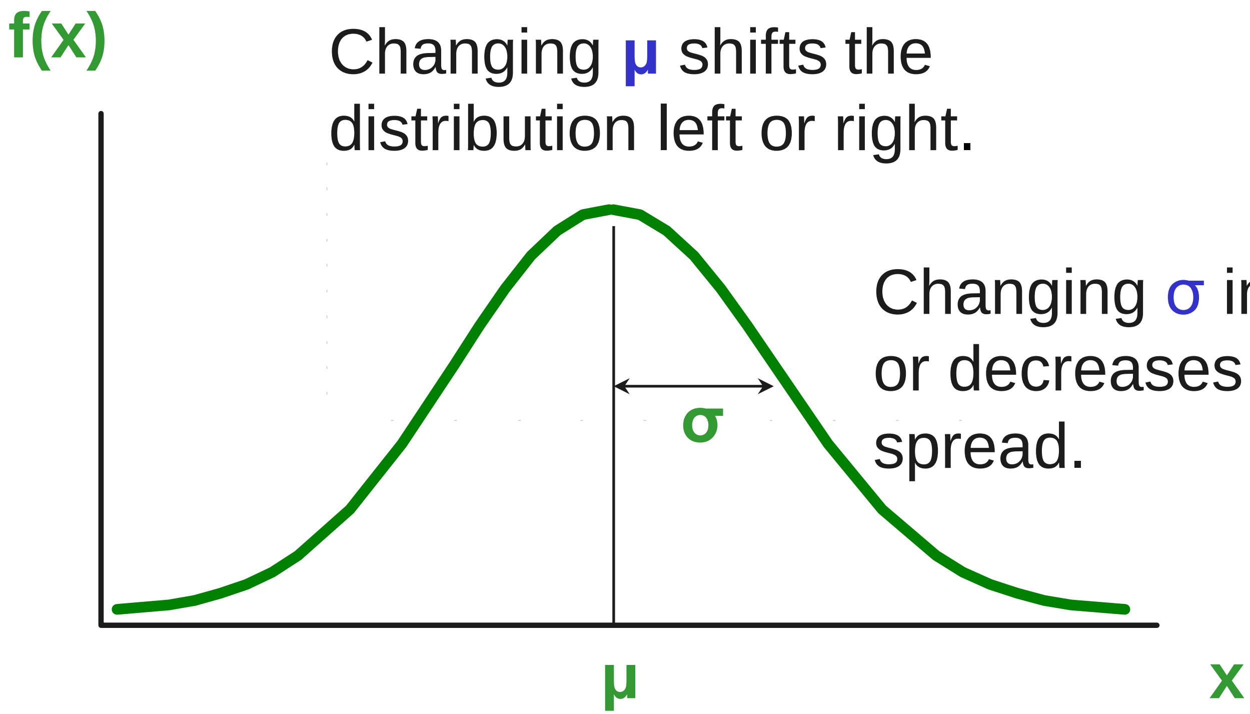
- The normal distribution closely approximates the probability distributions of a wide range of random variables
- Distributions of sample means approach a normal distribution given a “large” sample size
- Computations of probabilities are direct and elegant
- The normal probability distribution has led to good business decisions for a number of applications

Many Normal Distributions



By varying the parameters μ and σ, we obtain different normal distributions

The Normal Distribution Shape



Changing **μ** shifts the distribution left or right.

Changing **σ** increases or decreases the spread.

Given the mean μ and variance σ we define the normal distribution using the notation

$$\boxed{X \sim N(\mu, \sigma^2)}$$

The Normal Probability Density Function

- The formula for the normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

μ = the population mean

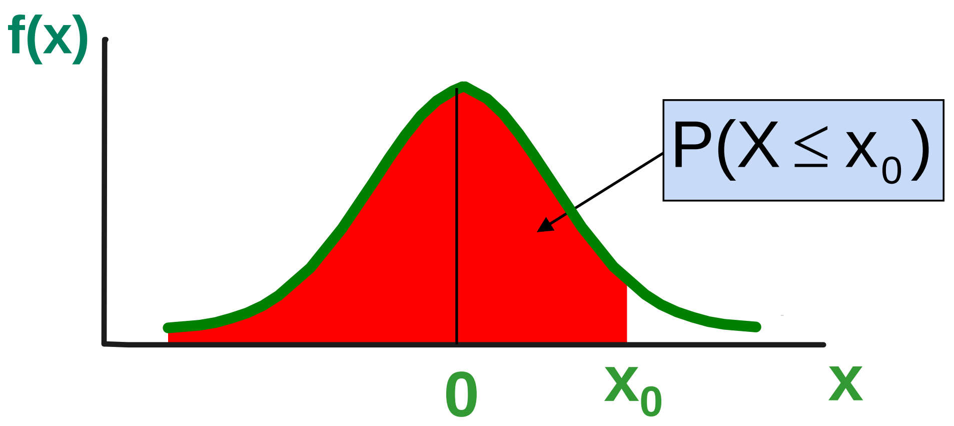
σ = the population standard deviation

x = any value of the continuous variable, $-\infty < x < \infty$

Cumulative Normal Distribution

- For a normal random variable X with mean μ and variance σ^2 , i.e., $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is

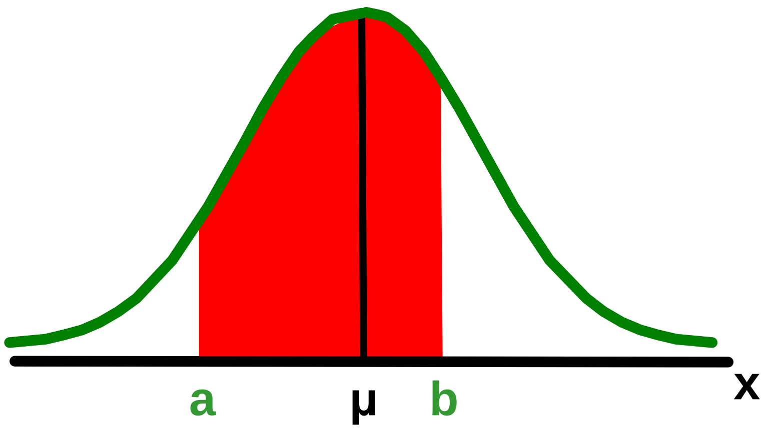
$$F(x_0) = P(X \leq x_0)$$



Finding Normal Probabilities

The probability for a range of values is measured by the area under the curve

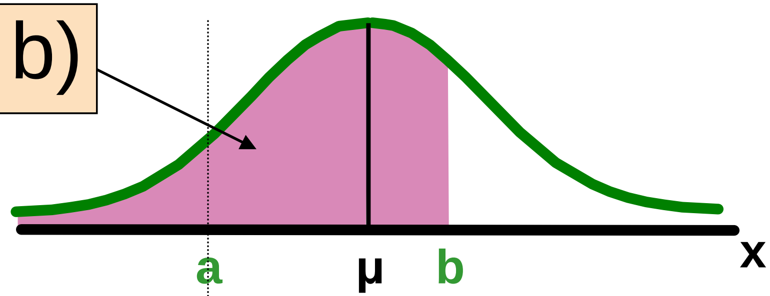
$$\boxed{P(a < X < b) = F(b) - F(a)}$$



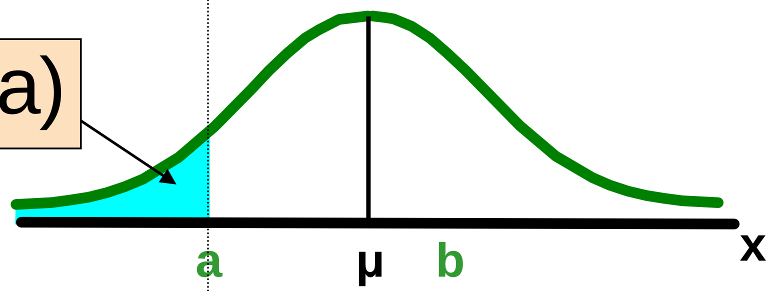
Finding Normal Probabilities

(continued)

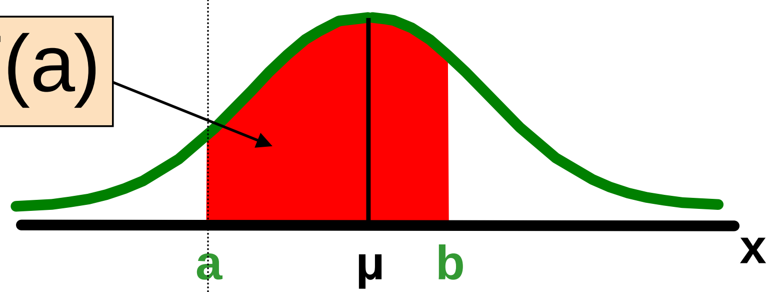
$$F(b) = P(X < b)$$



$$F(a) = P(X < a)$$



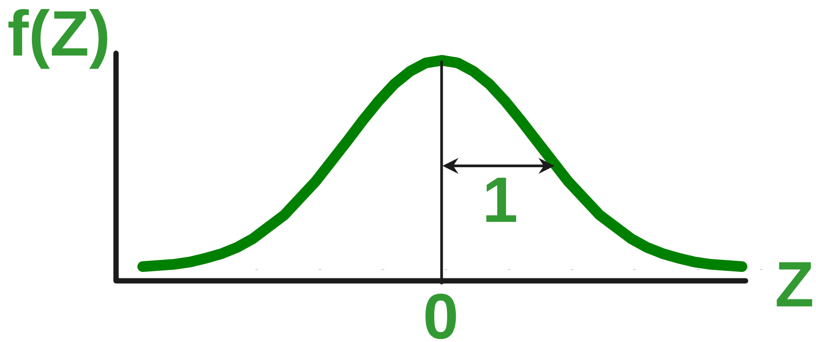
$$P(a < X < b) = F(b) - F(a)$$



The Standardized Normal

- Any normal distribution (with any mean and variance combination) can be transformed into the standardized normal distribution (Z), with mean 0 and variance 1

$$\boxed{Z \sim N(0,1)}$$



- Need to transform X units into Z units by subtracting the mean of X and dividing by its standard deviation

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

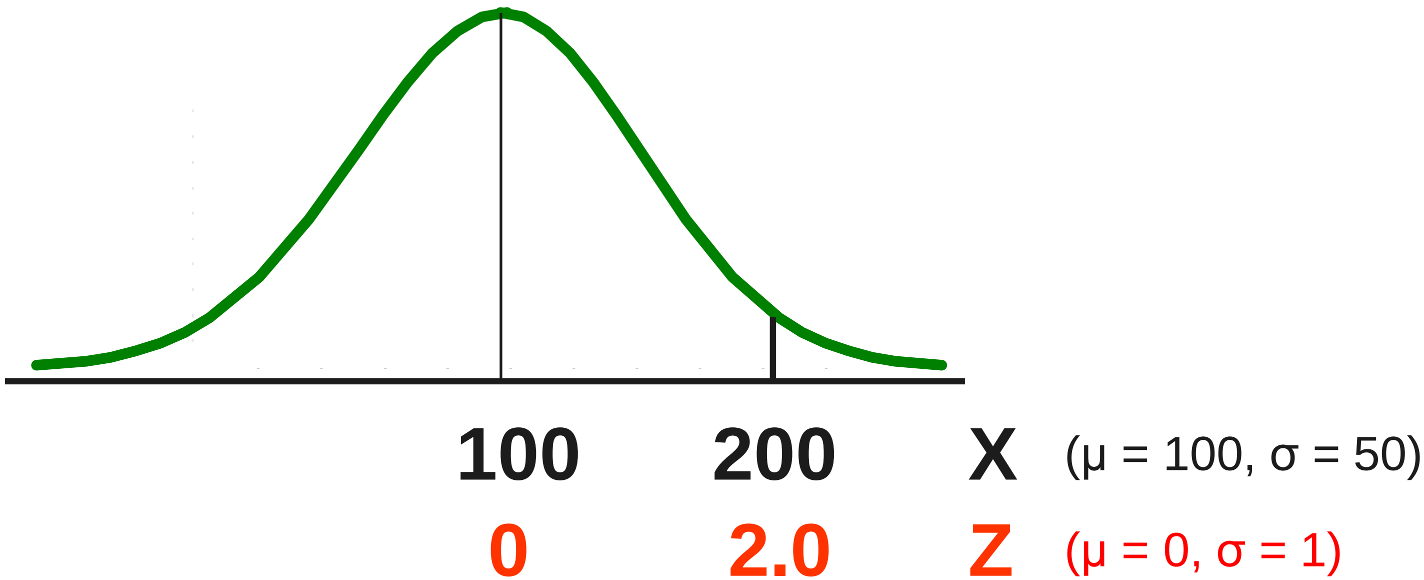
Example

- If X is distributed normally with mean of 100 and standard deviation of 50, the Z value for X = 200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2.0$$

- This says that X = 200 is two standard deviations (2 increments of 50 units) above the mean of 100.

Comparing X and Z units

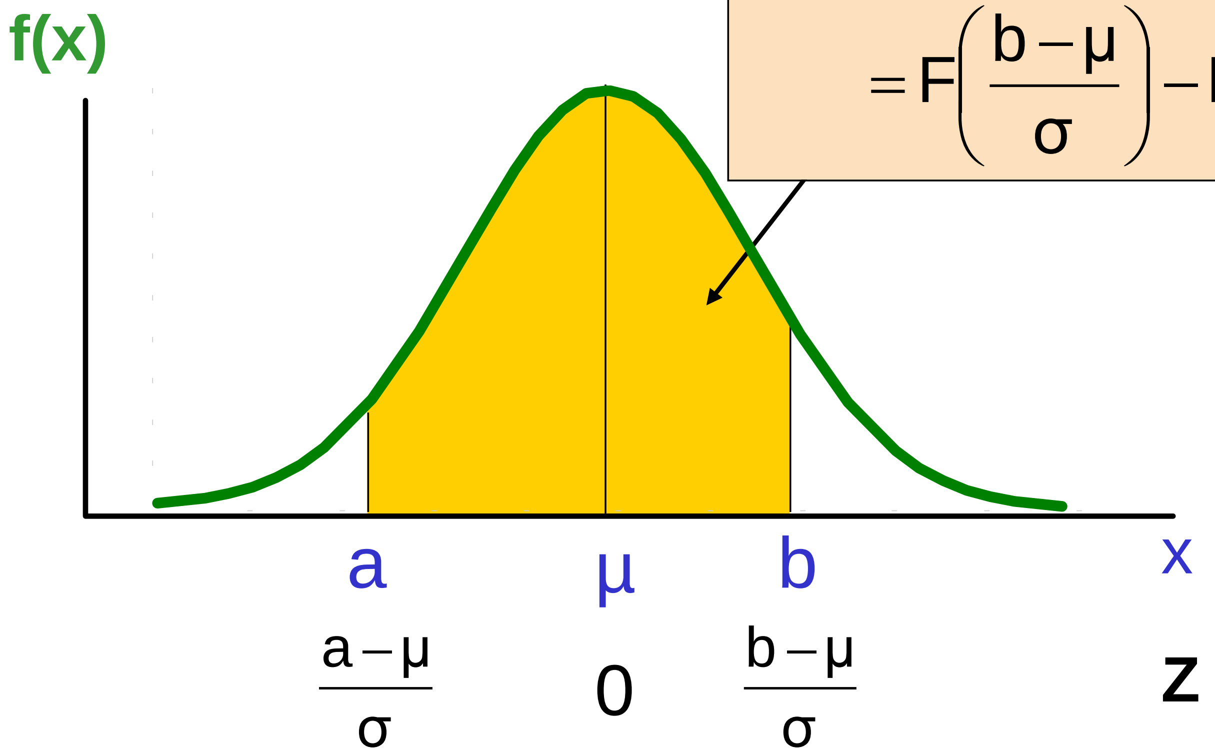


Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

Finding Normal Probabilities

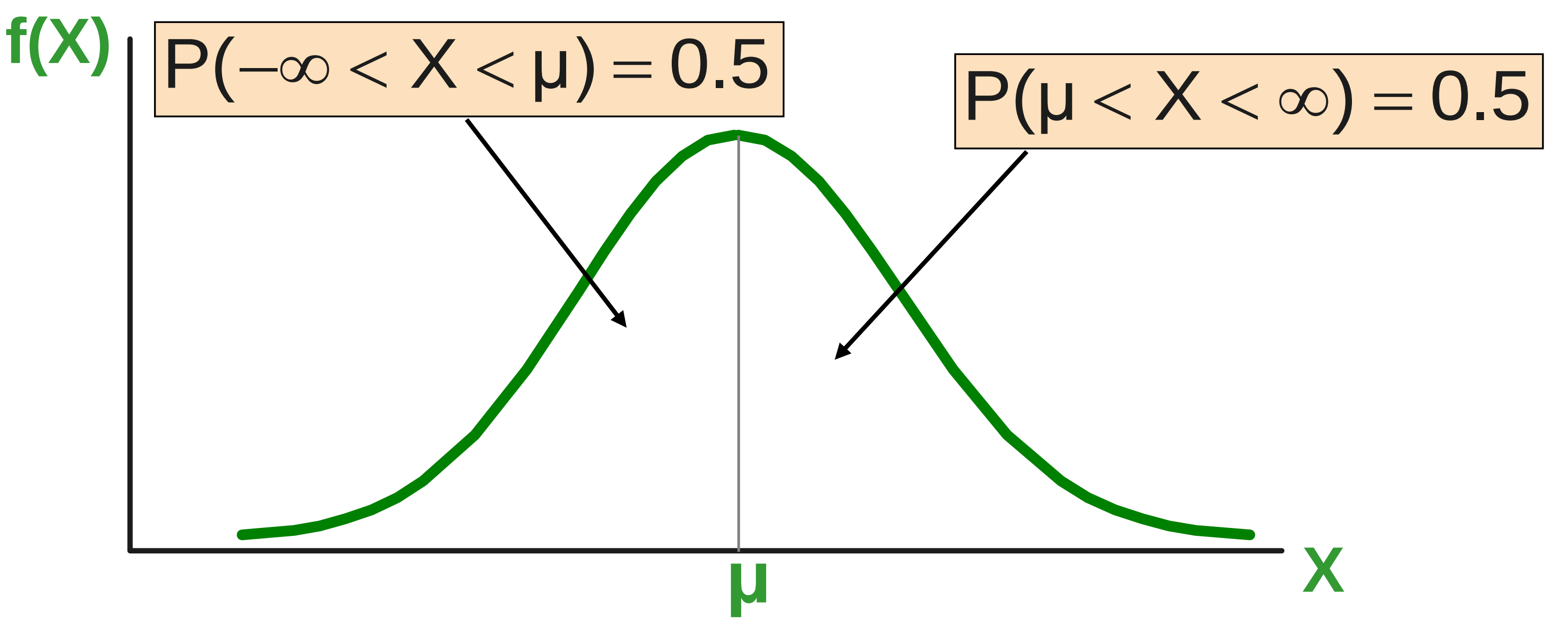
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$



Probability as Area Under the Curve

The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



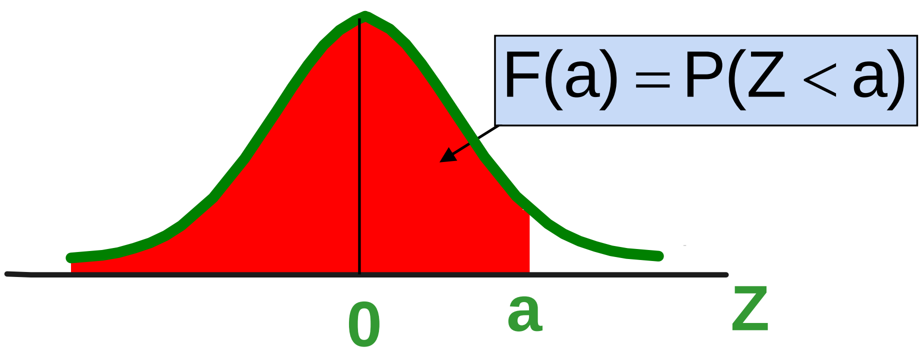
$$P(-\infty < X < \infty) = 1.0$$

Appendix Table 1

- The Standardized Normal table in the textbook (Appendix Table 1) shows values of the cumulative normal distribution function

- For a given Z-value a , the table shows F(a)

  (the area under the curve from negative infinity to a)

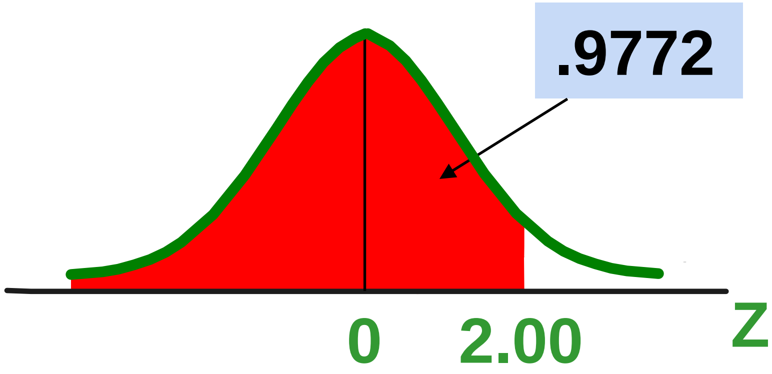


The Standardized Normal Table

- Appendix Table 1 gives the probability F(a) for any value a

Example:

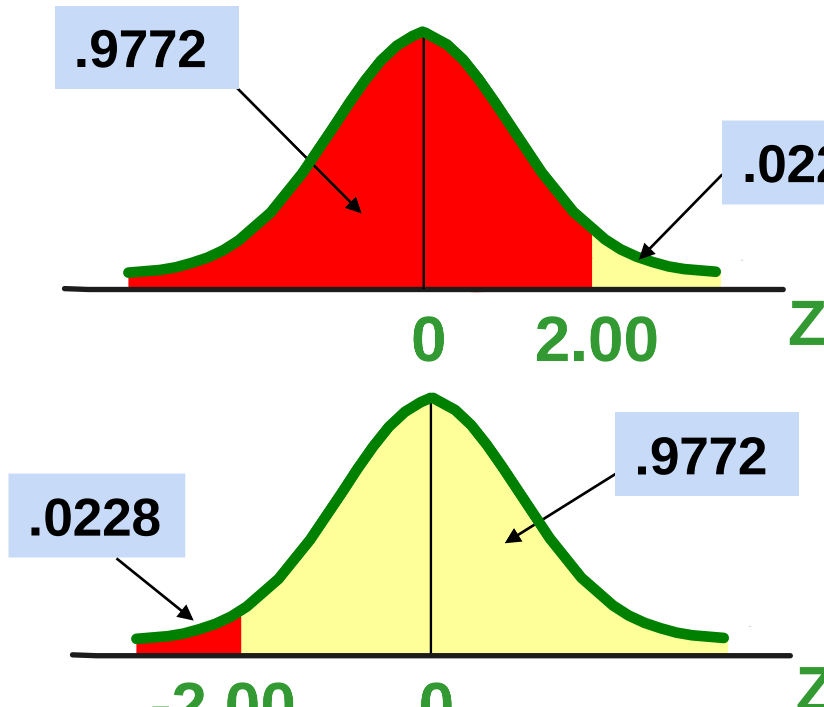
P(Z < 2.00) = .9772



The Standardized Normal Table

(continued)

- For negative Z-values, use the fact that the distribution is symmetric to find the needed probability:



Example:

P(Z < -2.00) = 1 – 0.9772

= 0.0228

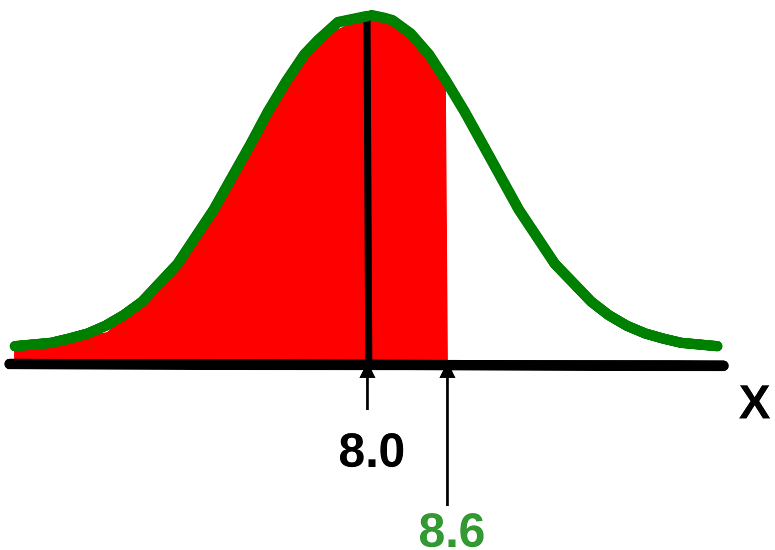
General Procedure for Finding Probabilities

To find $P(a < X < b)$ when X is distributed normally:

- Draw the normal curve for the problem in terms of X
- Translate X-values to Z-values
- Use the Cumulative Normal Table

Finding Normal Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0
- Find $P(X < 8.6)$

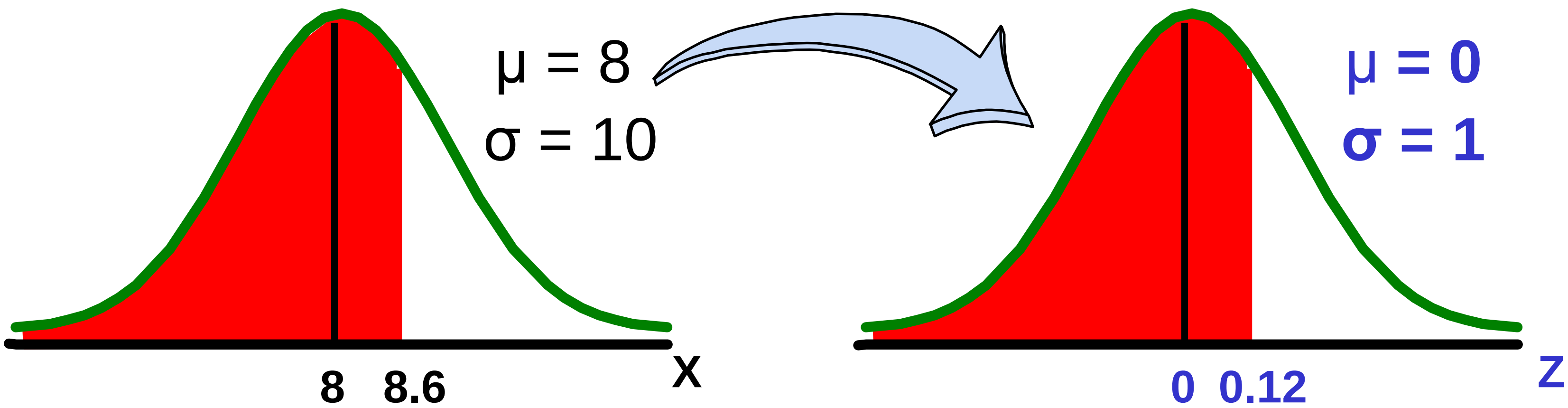


Finding Normal Probabilities

(continued)

- Suppose X is normal with mean 8.0 and standard deviation 5.0. Find P(X < 8.6)

$$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12$$



P(X < 8.6) P(Z < 0.12)

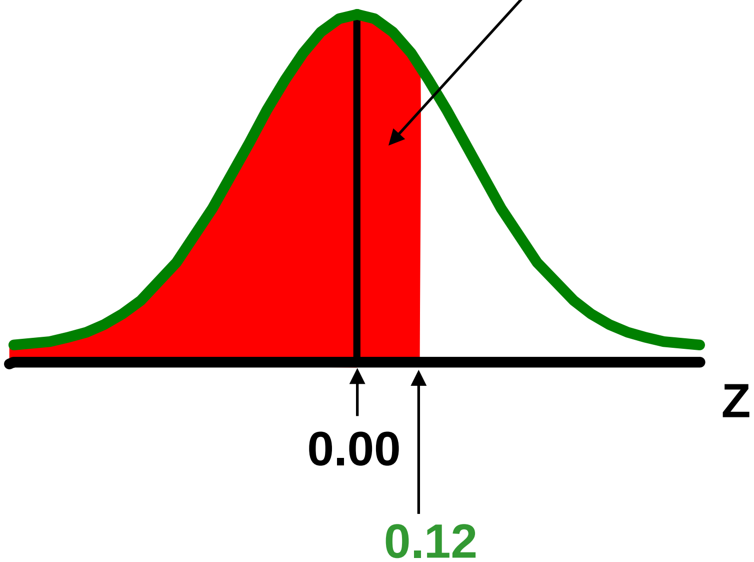
Solution: Finding P(Z < 0.12)

Standardized Normal Probability Table (Portion)

z	F(z)
.10	.5398
.11	.5438
.12	.5478
.13	.5517

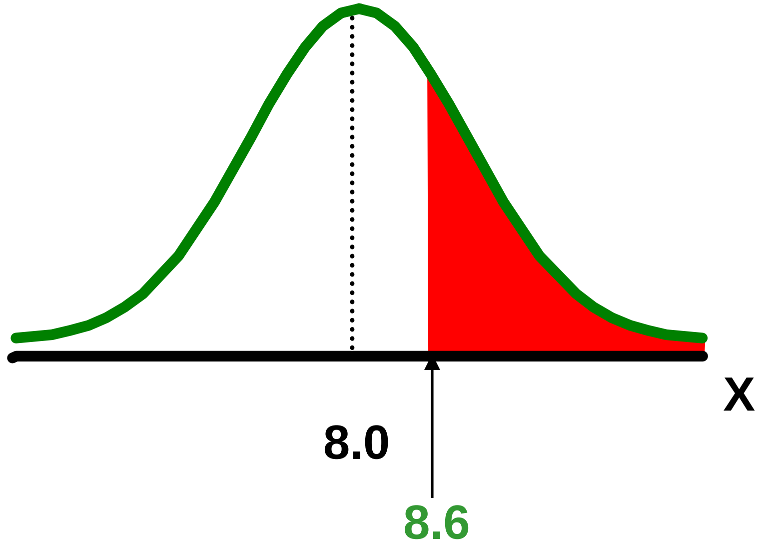
P(X < 8.6)
= P(Z < 0.12)

F(0.12) = 0.5478



Upper Tail Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(X > 8.6)



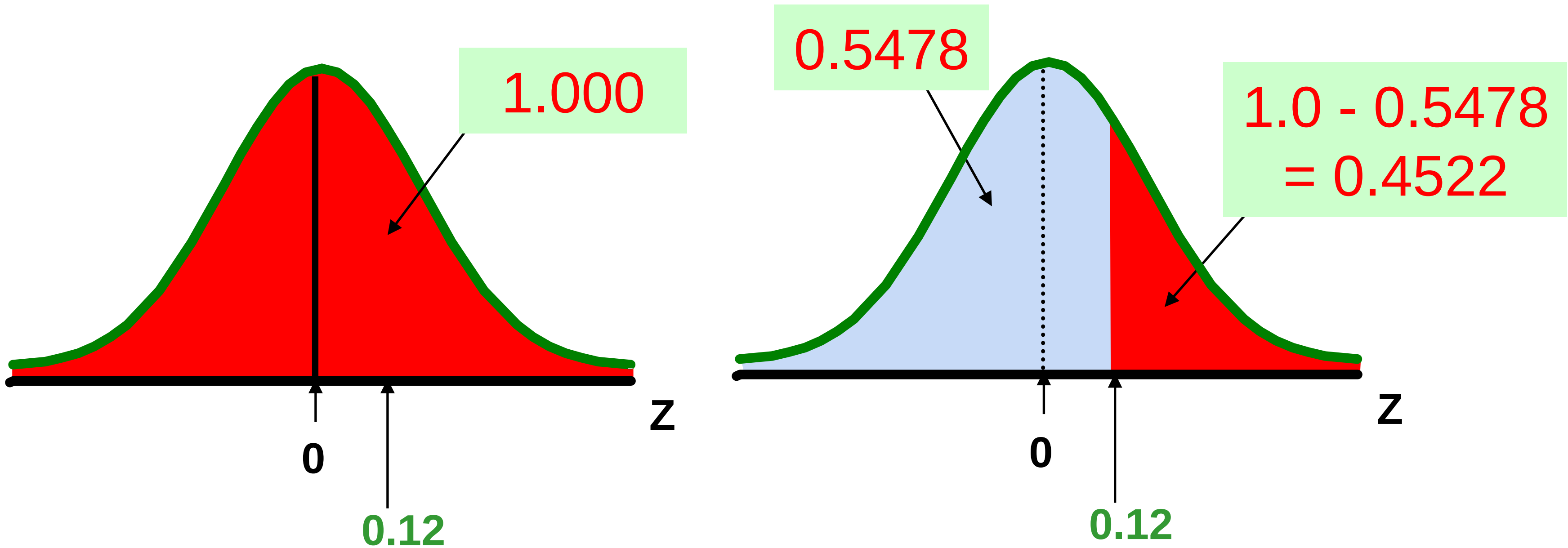
Upper Tail Probabilities

(continued)

- Now Find P(X > 8.6)…

$$P(X > 8.6) = P(Z > 0.12) = 1.0 - P(Z \le 0.12)$$

$$= 1.0 - 0.5478 = \boxed{0.4522}$$



Finding the X value for a Known Probability

- Steps to find the X value for a known probability:
  1. Find the Z value for the known probability
  2. Convert to X units using the formula:

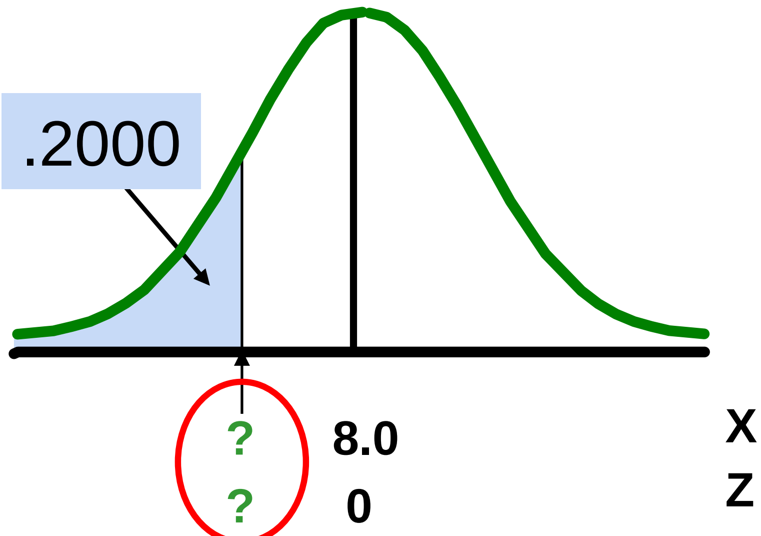
$$\boxed{X = \mu + Z\sigma}$$

Finding the X value for a Known Probability

(continued)

Example:

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now find the X value so that only 20% of all values are below this X



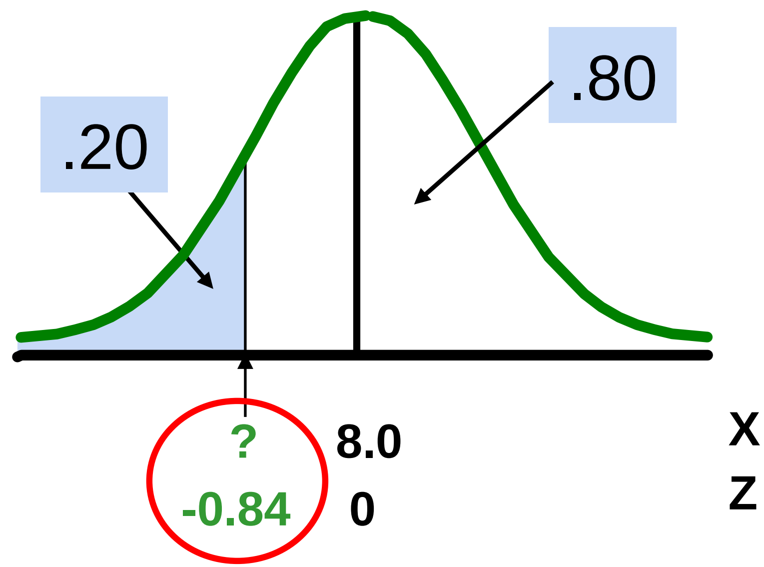
Find the Z value for 20% in the Lower Tail

1. Find the Z value for the known probability

Standardized Normal Probability Table (Portion)

z	F(z)
.82	.7939
.83	.7967
.84	.7995
.85	.8023

- 20% area in the lower tail is consistent with a Z value of -0.84



Finding the X value

2. Convert to X units using the formula:

$$
\begin{aligned}
X &= \mu + Z\sigma \\
&= 8.0 + (-0.84)5.0 \\
&= 3.80
\end{aligned}
$$

So 20% of the values from a distribution with mean 8.0 and standard deviation 5.0 are less than 3.80

Assessing Normality

- Not all continuous random variables are normally distributed
- It is important to evaluate how well the data is approximated by a normal distribution

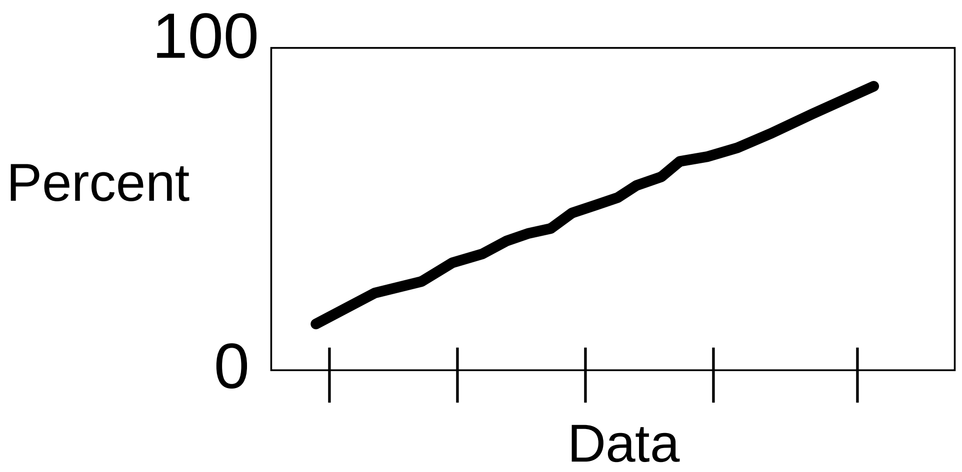
The Normal Probability Plot

- Normal probability plot
  - Arrange data from low to high values
  - Find cumulative normal probabilities for all values
  - Examine a plot of the observed values vs. cumulative probabilities (with the cumulative normal probability on the vertical axis and the observed data values on the horizontal axis)
  - Evaluate the plot for evidence of linearity

The Normal Probability Plot

(continued)

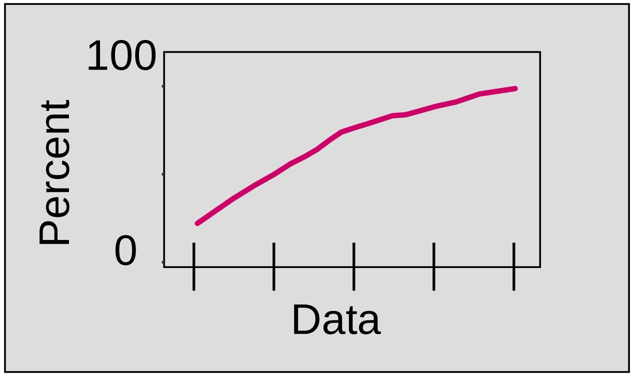
A normal probability plot for data from a normal distribution will be approximately linear:



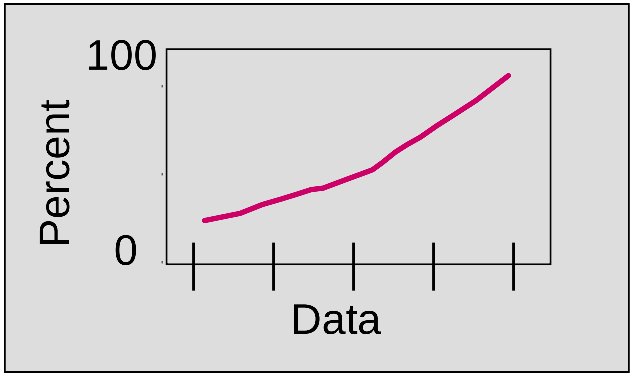
The Normal Probability Plot

(continued)

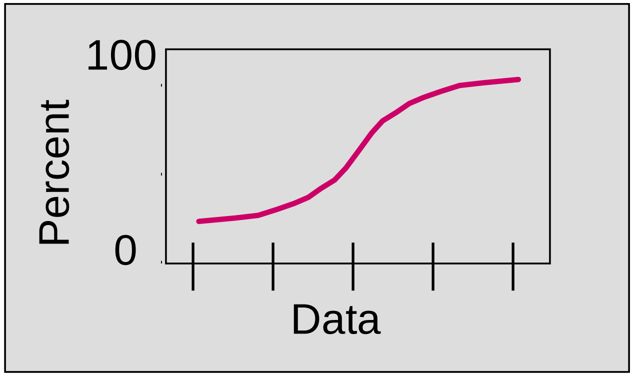
Left-Skewed



Right-Skewed



Uniform



Nonlinear plots indicate a deviation from normality

Normal Distribution Approximation for Binomial Distribution

- Recall the binomial distribution:
  - n independent trials
  - probability of success on any given trial = P
- Random variable X:
  - X_i =1 if the i^{th} trial is "success"
  - X_i =0 if the i^{th} trial is "failure"

$$\boxed{E(X) = \mu = nP}$$

$$\boxed{Var(X) = \sigma^2 = nP(1-P)}$$

Normal Distribution Approximation for Binomial Distribution

(continued)

- The shape of the binomial distribution is approximately normal if n is large
- The normal is a good approximation to the binomial when $nP(1 - P) > 9$
- Standardize to Z from a binomial distribution:

$$Z = \frac{X - E(X)}{\sqrt{Var(X)}} = \frac{X - np}{\sqrt{nP(1-P)}}$$

Normal Distribution Approximation for Binomial Distribution

(continued)

- Let X be the number of successes from n independent trials, each with probability of success P.

- If $nP(1 - P) > 9$,

$$P(a < X < b) = P\left(\frac{a - nP}{\sqrt{nP(1-P)}} \le Z \le \frac{b - nP}{\sqrt{nP(1-P)}}\right)$$

Binomial Approximation Example

- 40% of all voters support ballot proposition A. What is the probability that between 76 and 80 voters indicate support in a sample of n = 200 ?
  - $E(X) = \mu = nP = 200(0.40) = 80$
  - $Var(X) = \sigma^2 = nP(1 – P) = 200(0.40)(1 – 0.40) = 48$

    (note: $nP(1 – P) = 48 > 9$)

$$
\begin{aligned}
P(76 < X < 80) &= P\left(\frac{76-80}{\sqrt{200(0.4)(1-0.4)}} \le Z \le \frac{80-80}{\sqrt{200(0.4)(1-0.4)}} \right) \\
&= P(-0.58 < Z < 0) \\
&= F(0) - F(-0.58) \\
&= 0.5000 - 0.2810 = 0.2190
\end{aligned}
$$

The Exponential Distribution

- **Probability Distributions**
  - **Continuous** Probability Distributions
    - Normal
    - Uniform
    - Exponential

The Exponential Distribution

- Used to model the length of time between two occurrences of an event (the time between arrivals)

  - Examples:
    - Time between trucks arriving at an unloading dock
    - Time between transactions at an ATM Machine
    - Time between phone calls to the main operator

The Exponential Distribution

(continued)

- The exponential random variable T (t>0) has a probability density function

$$f(t) = \lambda e^{-\lambda t} \quad \text{for } t > 0$$

- Where
  - λ is the mean number of occurrences per unit time
  - t is the number of time units until the next occurrence
  - e = 2.71828
- T is said to follow an exponential probability distribution

The Exponential Distribution

- Defined by a single parameter, its mean λ (lambda)

- The cumulative distribution function (the probability that an arrival time is less than some specified time t) is

$$\boxed{F(t) = 1 - e^{-\lambda t}}$$

where e = mathematical constant approximated by 2.71828

λ = the population mean number of arrivals per unit

t = any value of the continuous variable where t > 0

Exponential Distribution Example

Example: Customers arrive at the service counter at the rate of 15 per hour. What is the probability that the arrival time between consecutive customers is less than three minutes?

- The mean number of arrivals per hour is 15, so $\lambda = 15$
- Three minutes is .05 hours
- P(arrival time < .05) $= 1 - e^{-\lambda X} = 1 - e^{-(15)(.05)} = \boxed{0.5276}$
- So there is a 52.76% probability that the arrival time between successive customers is less than three minutes

Joint Cumulative Distribution Functions

- Let $X_1, X_2, \ldots X_k$ be continuous random variables
- Their joint cumulative distribution function,

  $F(x_1, x_2, \ldots x_k)$

  defines the probability that simultaneously X_1 is less than x_1, X_2 is less than x_2, and so on; that is

$$\boxed{F(x_1, x_2, \ldots, x_k) = P(X_1 < x_1 \cap X_2 < x_2 \cap \cdots X_k < x_k)}$$

Joint Cumulative Distribution Functions

(continued)

- The cumulative distribution functions

  $F(x_1), F(x_2), \ldots, F(x_k)$

  of the individual random variables are called their marginal distribution functions

- The random variables are independent if and only if

$$\boxed{F(x_1, x_2, \ldots, x_k) = F(x_1)F(x_2)\cdots F(x_k)}$$

Covariance

- Let X and Y be continuous random variables, with means μ_x and μ_y
- The expected value of $(X - \mu_x)(Y - \mu_y)$ is called the covariance between X and Y

$$\mathrm{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

- An alternative but equivalent expression is

$$\mathrm{Cov}(X,Y) = E(XY) - \mu_x\mu_y$$

- If the random variables X and Y are independent, then the covariance between them is 0. However, the converse is not true.

Correlation

- Let X and Y be jointly distributed random variables.

- The correlation between X and Y is

$$\rho = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Sums of Random Variables

Let $X_1, X_2, \ldots X_k$ be k random variables with means $\mu_1, \mu_2, \ldots \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Then:

- The mean of their sum is the sum of their means

$$\boxed{E(X_1 + X_2 + \cdots + X_k) = \mu_1 + \mu_2 + \cdots + \mu_k}$$

Sums of Random Variables

(continued)

Let $X_1, X_2, \ldots X_k$ be k random variables with means $\mu_1, \mu_2, \ldots \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Then:

- If the covariance between every pair of these random variables is 0, then the variance of their sum is the sum of their variances

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_k) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2$$

- However, if the covariances between pairs of random variables are not 0, the variance of their sum is

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_k) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2 + 2\sum_{i=1}^{K-1}\sum_{j=i+1}^{K}\mathrm{Cov}(X_i, X_j)$$

Differences Between Two Random Variables

For two random variables, X and Y

- The mean of their difference is the difference of their means; that is

$$\boxed{E(X-Y)=\mu_X-\mu_Y}$$

- If the covariance between X and Y is 0, then the variance of their difference is

$$\boxed{Var(X-Y)=\sigma_X^2+\sigma_Y^2}$$

- If the covariance between X and Y is not 0, then the variance of their difference is

$$\boxed{Var(X-Y)=\sigma_X^2+\sigma_Y^2-2Cov(X,Y)}$$

Linear Combinations of Random Variables

- A linear combination of two random variables, X and Y, (where a and b are constants) is

$$\boxed{W = aX + bY}$$

- The mean of W is

$$\boxed{\mu_W = E[W] = E[aX + bY] = a\mu_X + b\mu_Y}$$

Linear Combinations of Random Variables

(continued)

- The variance of W is

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X,Y)$$

- Or using the correlation,

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Corr}(X,Y)\sigma_X\sigma_Y$$

- If both X and Y are joint normally distributed random variables then the linear combination, W, is also normally distributed

Example

- Two tasks must be performed by the same worker.
  - X = minutes to complete task 1; $\mu_x = 20$, $\sigma_x = 5$
  - Y = minutes to complete task 2; $\mu_y = 20$, $\sigma_y = 5$
  - X and Y are normally distributed and independent

- What is the mean and standard deviation of the time to complete both tasks?

Example

(continued)

- X = minutes to complete task 1; $\mu_x = 20$, $\sigma_x = 5$
- Y = minutes to complete task 2; $\mu_y = 30$, $\sigma_y = 8$

- What are the mean and standard deviation for the time to complete both tasks?

$$W = X + Y$$

$$\mu_W = \mu_X + \mu_Y = 20 + 30 = 50$$

- Since X and Y are independent, Cov(X,Y) = 0, so

$$\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 + 2\mathrm{Cov}(X,Y) = (5)^2 + (8)^2 = 89$$

- The standard deviation is

$$\sigma_W = \sqrt{89} = 9.434$$

Chapter Summary

- Defined continuous random variables
- Presented key continuous probability distributions and their properties
  - uniform, normal, exponential
- Found probabilities using formulas and tables
- Interpreted normal probability plots
- Examined when to apply different distributions
- Applied the normal approximation to the binomial distribution
- Reviewed properties of jointly distributed continuous random variables