Statistics for Business and Economics

6th Edition

Chapter 4

Probability

Chapter Goals

After completing this chapter, you should be able to:

- Explain basic probability concepts and definitions
- Use a Venn diagram or tree diagram to illustrate simple probabilities
- Apply common rules of probability
- Compute conditional probabilities
- Determine whether events are statistically independent
- Use Bayes' Theorem for conditional probabilities

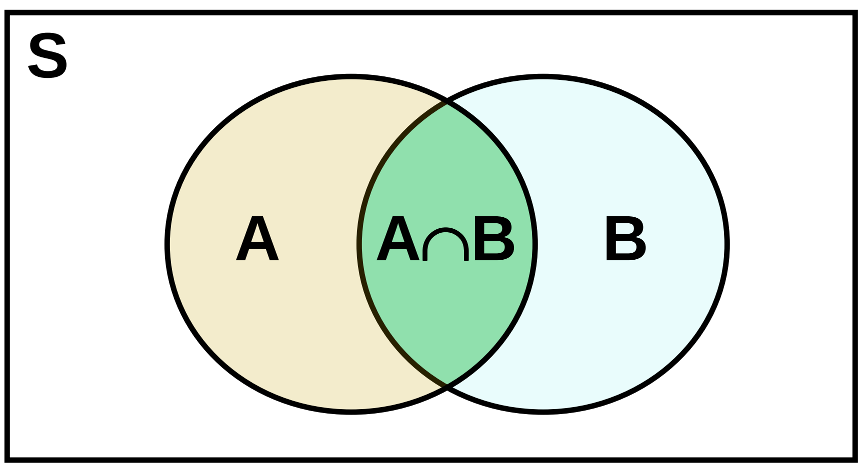
Important Terms

- Random Experiment – a process leading to an uncertain outcome
- Basic Outcome – a possible outcome of a random experiment
- Sample Space – the collection of all possible outcomes of a random experiment
- Event – any subset of basic outcomes from the sample space

Important Terms

(continued)

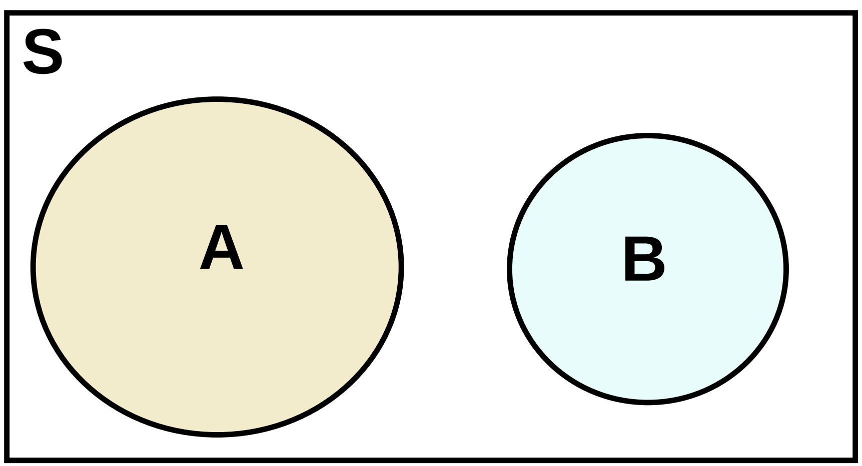
- Intersection of Events – If A and B are two events in a sample space S, then the intersection, A ∩ B, is the set of all outcomes in S that belong to both A and B



Important Terms

(continued)

- A and B are Mutually Exclusive Events if they have no basic outcomes in common
  - i.e., the set $A \cap B$ is empty

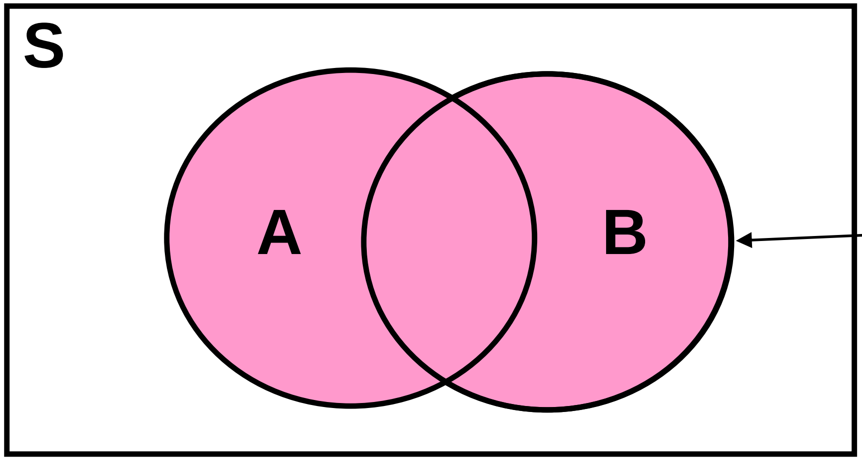


Important Terms

(continued)

- Union of Events – If A and B are two events in a sample space S, then the union, A ∪ B, is the set of all outcomes in S that belong to either

  A or B



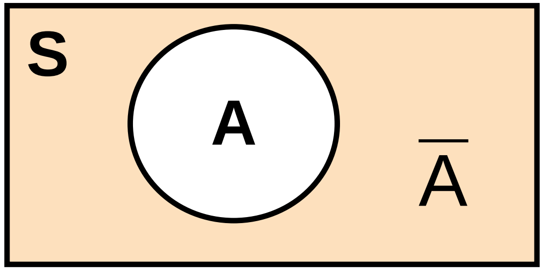
The entire shaded area represents **A** ∪ **B**

Important Terms

(continued)

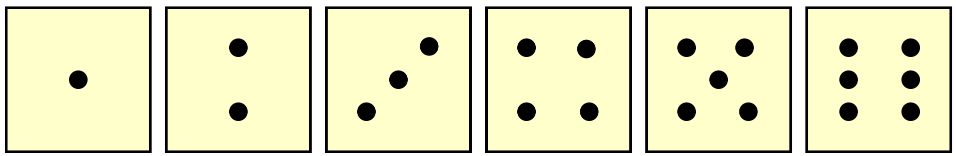
- Events $E_1, E_2, \ldots E_k$ are Collectively Exhaustive events if $E_1 \cup E_2 \cup \ldots \cup E_k = S$
  - i.e., the events completely cover the sample space

- The Complement of an event A is the set of all basic outcomes in the sample space that do not belong to A. The complement is denoted $\overline{A}$



Examples

Let the Sample Space be the collection of all possible outcomes of rolling one die:



S = [1, 2, 3, 4, 5, 6]

Let A be the event “Number rolled is even”

Let B be the event “Number rolled is at least 4”

Then

A = [2, 4, 6] and B = [4, 5, 6]

Examples

(continued)

S = [1, 2, 3, 4, 5, 6]	A = [2, 4, 6]	B = [4, 5, 6]

Complements:

$$\overline{A} = [1, 3, 5] \qquad \overline{B} = [1, 2, 3]$$

Intersections:

$$A \cap B = [4, 6] \qquad \overline{A} \cap B = [5]$$

Unions:

$$A \cup B = [2, 4, 5, 6]$$

$$A \cup \overline{A} = [1, 2, 3, 4, 5, 6] = S$$

Examples

(continued)

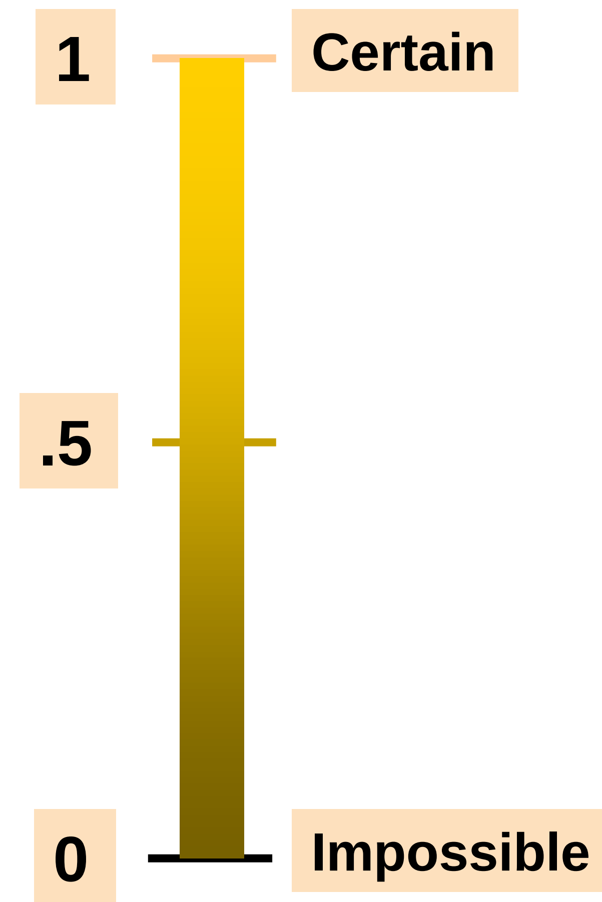
S = [1, 2, 3, 4, 5, 6]	A = [2, 4, 6]	B = [4, 5, 6]

- Mutually exclusive:
  - A and B are not mutually exclusive
    - The outcomes 4 and 6 are common to both
- Collectively exhaustive:
  - A and B are not collectively exhaustive
    - $A \cup B$ does not contain 1 or 3

Probability

- Probability – the chance that an uncertain event will occur (always between 0 and 1)

$$0 \le P(A) \le 1 \quad \text{For any event A}$$



Assessing Probability

- There are three approaches to assessing the probability of an uncertain event:

1. classical probability

$$\text{probability of event A} = \frac{N_A}{N} = \frac{\text{number of outcomes that satisfy the event}}{\text{total number of outcomes in the sample space}}$$

  - Assumes all outcomes in the sample space are equally likely to occur

Counting the Possible Outcomes

- Use the Combinations formula to determine the number of combinations of n things taken k at a time

$$C_k^n = \frac{n!}{k!(n-k)!}$$

- where
  - $n! = n(n-1)(n-2)\dots(1)$
  - $0! = 1$ by definition

Assessing Probability

Three approaches (continued)

2. relative frequency probability

$$\text{probability of event A} = \frac{n_A}{n} = \frac{\text{number of events in the population that satisfy event A}}{\text{total number of events in the population}}$$

- the limit of the proportion of times that an event A occurs in a large number of trials, *n*

3. subjective probability

an individual opinion or belief about the probability of occurrence

Probability Postulates

1. If A is any event in the sample space S, then

$$\boxed{0 \leq P(A) \leq 1}$$

2. Let A be an event in S, and let O_i denote the basic outcomes. Then

$$\boxed{P(A) = \sum_{A} P(O_i)}$$

(the notation means that the summation is over all the basic outcomes in A)

3. $\boxed{P(S) = 1}$

Probability Rules

- The Complement rule:

$$\boxed{P(\bar{A}) = 1 - P(A)} \qquad \text{i.e., } P(A) + P(\bar{A}) = 1$$

- The Addition rule:
  - The probability of the union of two events is

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

A Probability Table

Probabilities and joint probabilities for two events A and B are summarized in this table:

	B	$\overline{B}$	
A	$P(A \cap B)$	$P(A \cap \overline{B})$	$P(A)$
$\overline{A}$	$P(\overline{A} \cap B)$	$P(\overline{A} \cap \overline{B})$	$P(\overline{A})$
	$P(B)$	$P(\overline{B})$	$P(S) = 1.0$

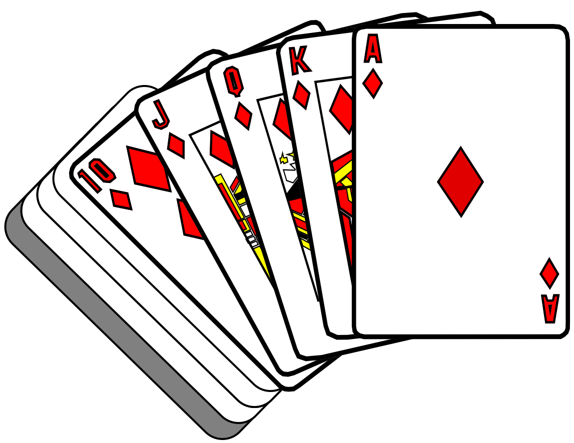
Addition Rule Example

Consider a standard deck of 52 cards, with four suits:

♥ ♣ ♦ ♠

Let event A = card is an Ace

Let event B = card is from a red suit



Addition Rule Example

(continued)

P(Red ∪ Ace) = P(Red) + P(Ace) - P(Red ∩ Ace)

= 26/52 + 4/52 - 2/52 = 28/52

Don't count the two red aces twice!

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Conditional Probability

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

The conditional probability of A given that B has occurred

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

The conditional probability of B given that A has occurred

Conditional Probability Example

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

- What is the probability that a car has a CD player, given that it has AC ?

i.e., we want to find $P(CD \mid AC)$

Conditional Probability Example

(continued)

- Of the cars on a used car lot, **70%** have air conditioning (AC) and **40%** have a CD player (CD).
  20% of the cars have both.

	CD	No CD	Total
AC	**.2**	**.5**	**.7**
No AC	**.2**	**.1**	**.3**
Total	**.4**	**.6**	**1.0**

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Conditional Probability Example

(continued)

- Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is 28.57%.

	CD	No CD	Total
AC	**.2**	**.5**	**.7**
No AC	**.2**	**.1**	**.3**
Total	**.4**	**.6**	**1.0**

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Multiplication Rule

- Multiplication rule for two events A and B:

$$\boxed{P(A \cap B) = P(A|B)P(B)}$$

- also

$$\boxed{P(A \cap B) = P(B|A)P(A)}$$

Multiplication Rule Example

$$P(\text{Red} \cap \text{Ace}) = P(\text{Red} \mid \text{Ace})P(\text{Ace})$$

$$= \left(\frac{2}{4}\right)\left(\frac{4}{52}\right) = \frac{2}{52}$$

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Statistical Independence

- Two events are statistically independent if and only if:

$$P(A \cap B) = P(A)P(B)$$

  - Events A and B are independent when the probability of one event is not affected by the other event

- If A and B are independent, then

$$P(A \mid B) = P(A) \qquad \text{if } P(B)>0$$

$$P(B \mid A) = P(B) \qquad \text{if } P(A)>0$$

Statistical Independence Example

- Of the cars on a used car lot, **70%** have air conditioning (AC) and **40%** have a CD player (CD).
  20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

- Are the events AC and CD statistically independent?

Statistical Independence Example

(continued)

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$P(AC \cap CD) = 0.2$

$$\left.\begin{array}{l} P(AC) = 0.7 \\ P(CD) = 0.4 \end{array}\right\} P(AC)P(CD) = (0.7)(0.4) = 0.28$$

$$P(AC \cap CD) = 0.2 \quad \neq \quad P(AC)P(CD) = 0.28$$

So the two events are not statistically independent

Bivariate Probabilities

Outcomes for bivariate events:

	B_1	B_2	. . .	B_k
A_1	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	. . .	$P(A_1 \cap B_k)$
A_2	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	. . .	$P(A_2 \cap B_k)$
. . .	. . .	. . .	. . .	. . .
A_h	$P(A_h \cap B_1)$	$P(A_h \cap B_2)$	. . .	$P(A_h \cap B_k)$

Joint and Marginal Probabilities

- The probability of a joint event, A ∩ B:

$$P(A \cap B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a marginal probability:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Marginal Probability Example

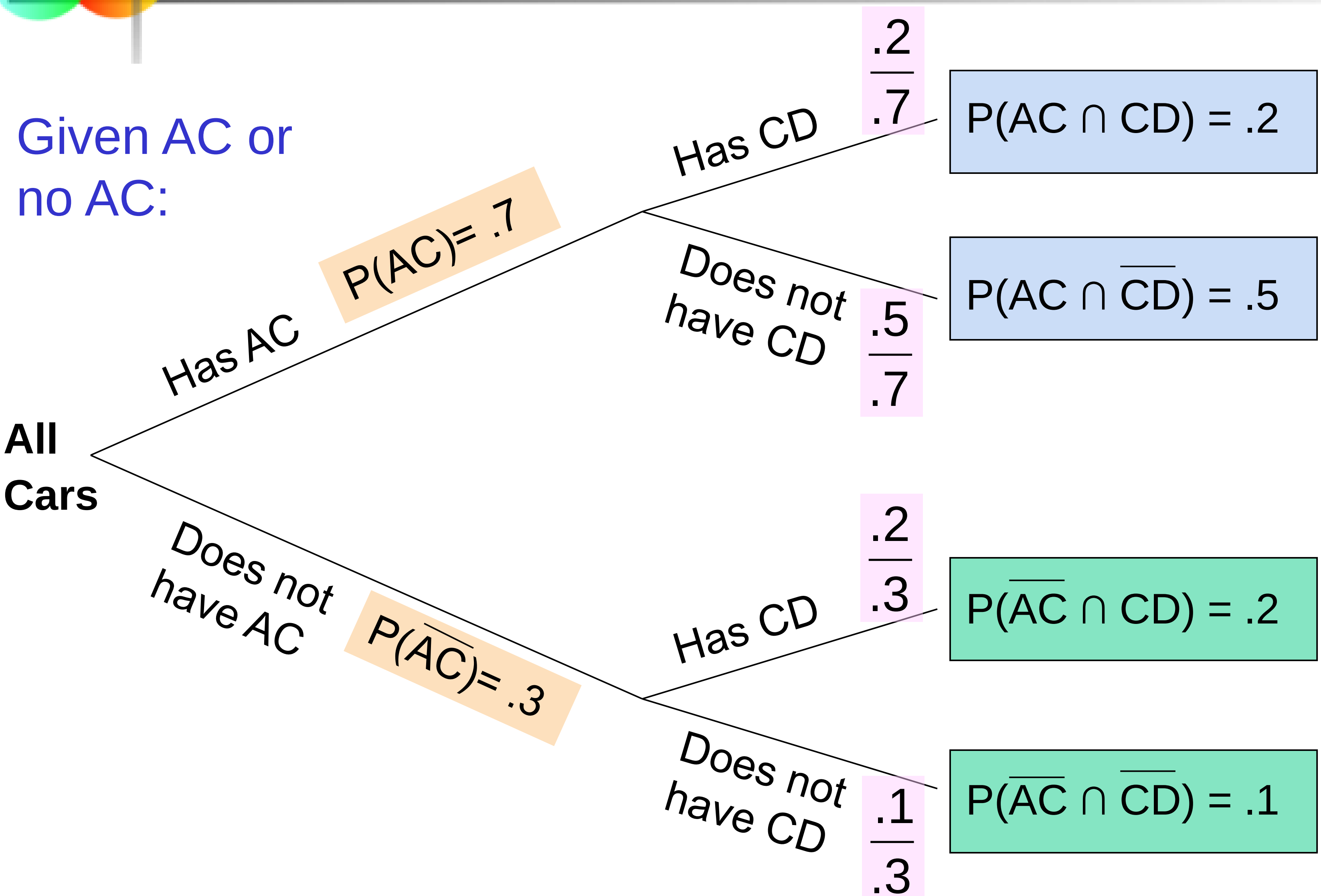
P(Ace)

$$= P(\text{Ace} \cap \text{Red}) + P(\text{Ace} \cap \text{Black}) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Using a Tree Diagram

Given AC or no AC:



Odds

- The odds in favor of a particular event are given by the ratio of the probability of the event divided by the probability of its complement
- The odds in favor of A are

$$\boxed{\text{odds} = \frac{P(A)}{1 - P(A)} = \frac{P(A)}{P(\overline{A})}}$$

Odds: Example

- Calculate the probability of winning if the odds of winning are 3 to 1:

$$\text{odds} = \frac{3}{1} = \frac{P(A)}{1\text{-}P(A)}$$

  - Now multiply both sides by 1 – P(A) and solve for P(A):

$$3 \text{ x } (1\text{-}\ P(A)) = P(A)$$

$$3 - 3P(A) = P(A)$$

$$3 = 4P(A)$$

$$\boxed{P(A) = 0.75}$$

Overinvolvement Ratio

- The probability of event A_1 conditional on event B_1 divided by the probability of A_1 conditional on activity B_2 is defined as the overinvolvement ratio:

$$\boxed{\frac{P(A_1 \mid B_1)}{P(A_1 \mid B_2)}}$$

- An overinvolvement ratio greater than 1 implies that event A_1 increases the conditional odds ration in favor of B_1:

$$\boxed{\frac{P(B_1 \mid A_1)}{P(B_2 \mid A_1)} > \frac{P(B_1)}{P(B_2)}}$$

Bayes' Theorem

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A)}$$

$$= \frac{P(A \mid E_i)P(E_i)}{P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \ldots + P(A \mid E_k)P(E_k)}$$

- where:

  E_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(E_i)$

Bayes' Theorem Example

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Bayes' Theorem Example

(continued)

- Let S = successful well

  U = unsuccessful well

- P(S) = .4 , P(U) = .6 (prior probabilities)
- Define the detailed test event as D
- Conditional probabilities:

  P(D|S) = .6 P(D|U) = .2

- Goal is to find P(S|D)

Bayes' Theorem Example

(continued)

Apply Bayes' Theorem:

$$P(S \mid D) = \frac{P(D \mid S)P(S)}{P(D \mid S)P(S) + P(D \mid U)P(U)}$$

$$= \frac{(.6)(.4)}{(.6)(.4) + (.2)(.6)}$$

$$= \frac{.24}{.24 + .12} = .667$$

So the revised probability of success (from the original estimate of .4), given that this well has been scheduled for a detailed test, is .667

Chapter Summary

- Defined basic probability concepts
  - Sample spaces and events, intersection and union of events, mutually exclusive and collectively exhaustive events, complements
- Examined basic probability rules
  - Complement rule, addition rule, multiplication rule
- Defined conditional, joint, and marginal probabilities
- Reviewed odds and the overinvolvement ratio
- Defined statistical independence
- Discussed Bayes' theorem