MATH 3406: A Second Course in Linear Algebra Lecture Notes

Ethan Chen
echen347.github.io

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Contents

1 Intro

These are publicly available notes for MATH 3406 at Georgia Tech written by me (Ethan Chen). All errors are my own. You can contact me at echen347.github.io or at ec@gatech.edu.

2 Notation

These are assumed unless otherwise specified. The specific section's definition of these variables take priority over these definitions. If something seems unclear please contact me.

2.1 U, V, W

Denotes a vector space.

2.2 u, v, w

Denotes a vector in its corresponding vector space.

2.3 $\mathbb{F}$

Denotes the field V is over, usually $\mathbb{R}$ or $\mathbb{C}$.

2.4 x_i

Used to refer to a list; i ranges from 1 or 0 to some arbitrary natural number.

2.5 S, T

Denotes a linear map, usually from V to W.

2.6 A, B

Denotes a n-by-m matrix.

2.7 ϕ

Denotes a linear functional from V to $\mathbb{F}$.

3 Vector Spaces

Corresponding to Chapter 1, sections B and C of Axler.

3.1 Properties of a Vector Space

V is a vector space iff $\forall \lambda, \lambda_1, \lambda_2 \in \mathbb{F}$ and $\forall u, v, w \in V$:

1. $u + v \in V$
2. $\lambda v \in V$
3. $u + v = v + u$
4. $(u + v) + w = u + (v + w)$
5. $\exists 0$ s.t. $v + 0 = v$
6. $\exists(-v)$ s.t. $v + (-v) = 0$
7. $\exists 1$ s.t. $1v = v$
8. $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$, $\lambda(u + v) = \lambda u + \lambda v$

Sometimes written as V.S. in shorthand.

3.2 Subspace

U is a subspace of V iff $\forall \lambda \in \mathbb{F}, \forall u, w \in U$:

1. $0 \in U$
2. $u + w \in U$
3. $\lambda u \in U$

Note that it is *usually* more efficient to show something is a subspace of another V.S. than to show it is a V.S. directly.

3.3 Sums

If U_i are subsets of V then $\sum U_i = \{\sum u_i | u_i \in U_i\}$. Note this is similar to the union of sets in set theory.

3.4 Direct Sums

Denoted $U_1 \oplus \cdots \oplus U_m$, a direct sum is said to be when each element of $\sum U_i$ can be uniquely written as a sum of u_i.

3.4.1 Condition for a Direct Sum

$\sum U_i$ is a direct sum iff $\sum u_i = 0$ only when $\forall u_i, u_i = 0$.

3.4.2 Condition for a Direct Sum

$U + W$ is a direct sum iff $U \cap W = \{0\}$.

4 Span

Corresponding to Chapter 2, Section A, first half of Axler.

4.1 Span

The *span* of a set of vectors v_i is $\{\sum a_i v_i | a_i \in \mathbb{F}\}$, denoted $\text{span}(v_i)$. Sometimes defined as the set of all linear combinations of v_i; a *linear combination* of a set of vectors v_i is simply $\sum a_i v_i$ for some $a_i \in \mathbb{F}$.

4.1.1 Span and Vector Spaces

We say v_i *spans* a V.S. V if V is the smallest V.S. that contains every vector in $\text{span}(v_i)$.

4.2 Finite-Dimensional Vector Space

V is *finite-dimensional* if $\exists v_i$ that spans V. **Note:** by definition, a list has finite length.

4.3 Polynomials

The definition of a polynomial is assumed, and is denoted $p(z)$. However, note that some polynomials may be over a different field. $p(z) = (2i+7)z^3 - (3i-11)z^2 + 12$ is a polynomial over $\mathbb{C}$, for example.

4.3.1 $\mathcal{P}(\mathbb{F})$

The set of all polynomials with coefficients in $\mathbb{F}$.

4.3.2 Degree of a Polynomial

The *degree* of a polynomial is the highest degree m s.t. $p(z)$ can be expressed as

$$p(z) = \sum_{i=0}^{m} a_i p^i, a_i \in \mathbb{F}.$$

Then we say deg $p = m$. If a polynomial is identically 0, then its degree is $-\infty$.

4.3.3 $\mathcal{P}_m(\mathbb{F})$

The set of all polynomials of degree m, coefficients $\in \mathbb{F}$.

4.4 Infinite-Dimensional Vector Space

A V.S. that is not finite-dimensional.

5 Linear (In)Dependence

Corresponding to Chapter 2, Section A, second half of Axler.

5.1 Linear Independence

v_i is *linearly independent* if there exists a unique solution to $\sum a_i v_i = 0$ for $a_i \in F$. The solution is then all $a_i = 0$. Note the empty list $()$ is also linearly independent.

5.2 Linear Dependence

v_i is *linearly dependent* if it is not linearly independent. Thus there exists a_i not all 0 such that $\sum a_i v_i = 0$.

5.2.1 Linear Dependence Lemma

Suppose $v_i, i \in [m]$ is linearly dependent. Then $\exists j \in [m]$ s.t.

1. $v_j \in \text{span}(v_1, \ldots v_{j-1})$
2. $\text{span}(v_i) = \text{span}(v_i - v_j)$. Note that $v_i - v_j$ denotes the original list of v_i with v_j removed.

Note that this implies that in a finite-dimensional V.S., the length of every linearly independent list of vectors is $\leq$ the length of every spanning list of vectors.

5.3 Finite-Dimensional Subspaces

Every subspace of a finite-dimensional V.S. is finite-dimensional.

6 Bases

Corresponding to Chapter 2, Section B of Axler.

6.1 Basis

A list of vectors in V that is linearly independent and spans V.

6.2 Criterion for Basis

v_i is a basis for V iff $\forall v \in V$, $v = \sum a_i v_i$.

6.3 Spanning Lists and Bases

Every spanning list is a superlist of a basis.

6.4 Basis of Finite-Dimensional Vector Spaces

$\exists$ a basis for every finite-dimensional V.S.

6.5 Linearly Independent Lists and Bases

Every linearly independent list in a finite-dimensional V.S. is a sublist of a basis.

6.6 Existence of Subspaces in Direct Sums

If V is finite-dimensional, and $U \subseteq V$, then $\exists W \subseteq V$ s.t. $V = U \oplus W$.

7 Dimension

Corresponding to Chapter 2, Section C of Axler.

7.1 Dimension

The length a basis of the V.S.; denoted $\dim V$.

7.2 Dimension of a Subspace

Given finite-dimensional V, $U \subseteq V$, $\dim U \leq \dim V$.

7.3 Linearly Independent Lists and Bases (and Dimension)

Every linearly independent list in V with length $\dim V$ is a basis of V.

7.4 Spanning Lists and Bases (and Dimension)

Every spanning list in V with length $\dim V$ is a basis of V.

7.5 Dimension of a Sum

Given $U, W \subseteq V$, then $\dim (U + W) = \dim U + \dim V - \dim (U \cap W)$. Note for direct sums $\dim (U + W) = \dim U + \dim V$, since $(U \cap W) = \{0\}$, and hence $\dim (U \cap W) = 0$

8 Vector Space of Linear Maps

Corresponding to Chapter 3, Section A of Axler.

8.1 Linear Map

The function $T : V \to W$ s.t. $\forall \lambda \in \mathbb{F}, \forall u, v \in V$:

1. $T(u + v) = Tu + Tv$
2. $T(\lambda v) = \lambda(Tv)$

Note that $T(v) = Tv$, and usually parenthesis are removed.

8.1.1 Zero Map

The *zero map*, or 0, is defined as $\forall v \in V, 0v = 0$.

8.1.2 Identity Map

The *identity map*, or I, is defined as $\forall v \in V, Iv = v$.

8.2 $\mathcal{L}(V, W)$

The set of all linear maps from V to W.

8.3 Linear Maps and Bases

If v_i is a basis of V and w_i is a basis of W, then $\exists T \in \mathcal{L}(V, W)$ s.t. $\forall j, Tv_j = w_j$.

8.4 Addition, Scalar Multiplication on $\mathcal{L}(V, W)$

For $S, T \in \mathcal{L}(V, W)$, $v \in V$, $\lambda \in \mathbb{F}$, we define $(S+T)(v) = Sv + Tv$, and $(\lambda T)(v) = \lambda(Tv)$. Note that this implies $\mathcal{L}(V, W)$ is a V.S.

8.5 Product of Linear Maps

Given $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$, $u \in U$, define $ST \in \mathcal{L}(U, W)$ s.t. $(ST)(u) = S(Tu)$.

8.6 Algebraic Properties of Linear Maps

The following are some notable properties of linear maps. Given $T, T_i \in \mathcal{L}(U, V)$, $S, S_i \in \mathcal{L}(V, W)$:

1. $(T_1T_2)T_3 = T_1(T_2T_3)$
2. $TI = IT = T$
3. $(S_1 + S_2)T = S_1T + S_2T$, $S(T_1 + T_2) = ST_1 + ST_2$
4. $T(0) = 0$

9 Null Spaces and Ranges

Corresponding to Chapter 3, Section B of Axler.

9.1 Null Space

Denoted null T, defined as $\{v \in V | Tv = 0\}$. This is a subspace of V.

9.2 Injective

T is *injective* if $Tu = Tv \Rightarrow u = v$. This is equivalent to null $T = \{0\}$

9.2.1 Dimension and Injectivity

If $T \in \mathcal{L}(V, W)$ where dim $V >$ dim W, then T is not injective.

9.3 Range

Denoted range T, defined as $\{Tv | v \in V\}$. This is a subspace of V.

9.4 Surjective

T is *surjective* if range $T = W$.

9.4.1 Dimension and Surjectivity

If $T \in \mathcal{L}(V, W)$ where dim $V <$ dim W, then T is not surjective.

9.5 The Fundamental Theorem of Linear Maps

dim $V =$ dim null $T +$ dim range T

9.6 (In)Homogeneous Systems of Linear Equations

Not covered in Hannah Turner's Section of MATH 3406. Please contact me if you have questions regarding this section of Axler, preferably when I don't have any exams coming up.

10 Matrices

Corresponding to Chapter 3, Section C of Axler.

10.1 Matrix

The definition of a matrix is assumed, however it is useful to have a reminder that an m-by-n matrix with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}$$

Note that $A_{j,k}$ refers to the entry in row j, column k in A.

10.2 Matrix of a Linear Map

Denoted $\mathcal{M}(T)$; $v_i, i \in [1, n]$ is a basis for V, and $w_i, i \in [1, m]$ is a basis for W. Then the *matrix of* T wrt v_i, w_i is a matrix s.t. $T_{v_k} = \sum A_{i,k} w_i$. If bases are unclear, use $\mathcal{M}(T, (v_i), (w_i))$.

10.3 Matrix Addition

Matrices of the same size can be added as such: $(A+B)_{j,k} = A_{j,k} + B_{j,k}, \forall j, k$. Note that $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

10.4 Scalar Multiplication of a Matrix

$\lambda \in \mathbb{F}$: $\lambda A = B, B_{j,k} = \lambda A_{j,k}$. Note that $\lambda \mathcal{M}(T) = \mathcal{M}(\lambda T)$.

10.5 $\mathbb{F}^{m,n}$

The set of all m-by-n matrices with entries in $\mathbb{F}$. Note that dim $\mathbb{F}^{m,n} = mn$.

10.6 Matrix Multiplication

$A \in \mathbb{F}^{m,n}, B \in \mathbb{F}^{n,p}$. $AB \in \mathbb{F}^{m,p}, (AB)_{j,k} = \sum A_{j,i} B_{i,k}$. Note that if $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W) \Rightarrow \mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

10.7 $A_{j,\cdot}, A_{k,\cdot}$

Denotes a 1-by-n matrix consisting of row j of A, or a m-by-1 matrix consisting of a column k of A.

10.7.1 Entries and Columns in a Matrix Product

$A \in \mathbb{F}^{m,n}, B \in \mathbb{F}^{n,p}$. $(AB)_{j,k} = A_{j,\cdot} B_{k,\cdot}$, and $(AB)_{\cdot,k} = AB_{k,\cdot}$

10.7.2 Linear Combination of Columns

c is a n-by-1 matrix. Then $Ac = \sum c_i A_{\cdot,i}$.

11 Invertibility and Isomorphisms

Corresponding to Chapter 3, Section D of Axler.

11.1 Invertible, Inverse

$T \in \mathcal{L}(V, W)$ is *invertible* if $\exists S \in \mathcal{L}(W, V)$ s.t. ST is the identity map on V and TS is the identity map on W. S is said to be the *inverse* of T. Note that any invertible linear map has a unique inverse, and is denoted T^{-1}. T is invertible iff Y is injective and surjective.

11.2 Isomorphism, Isomorphic

An invertible linear map; two V.S. are *isomorphic* if $\exists$ an *isomorphism* from one V.S. to the other.

11.2.1 $\mathcal{L}(V, W)$, $\mathbb{F}^{m,n}$

$\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

11.3 $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

Only applies to finite-dimensional V.S.

11.4 Matrix of a Vector

Denoted $\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$; For a basis v_i of V, $v = \sum c_i v_i$. Note $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(v_k)$.

Further, $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$.

11.5 Operator

T, s.t. $T \in \mathcal{L}(V)$. $\mathcal{L}(V) = \mathcal{L}(V, V)$

11.6 Invertible, Injective, Surjective

For finite-dimensional V, $T \in \mathcal{L}(V)$, either all 3 conditions are true, or none.

12 Invariant Subspaces

Corresponding to Chapter 5, Section A of Axler.

12.1 Notation

$T \in \mathcal{L}(V)$, unless otherwise stated.

12.2 Invariant Subspace

Subspace U is *invariant* if $u \in U \Rightarrow Tu \in U$.

12.3 Eigenvalue

$\lambda \in \mathbb{F}$ is an *eigenvalue* if $\exists v \in V$ s.t. $v \neq 0, Tv = \lambda v$.

12.3.1 Conditions to be an Eigenvalue

$T - \lambda I$ is not injective, surjective, or invertible, where one condition implies the other two.

12.4 Eigenvector

λ is an eigenvalue of T. $v \in V$ is an *eigenvector* of T corresponding to λ if $v \neq 0, Tv = \lambda v$.

12.5 Linear Independence of Eigenvectors

λ_i are distinct eigenvalues of T, and v_i the corresponding eigenvectors. Then v_i is a linearly independent set.

12.6 Number of Eigenvalues

Given finite-dimensional V there are at most dim V distinct eigenvalues for any $T \in \mathcal{L}(V)$.

12.7 $T|_U, T/U$

For an invariant subspace U:

1. The *restriction operator* $T|_U \in \mathcal{L}(U)$ is given by $T|_U(u) = Tu$.
2. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is given by $(T/U)(v + U) = Tv + U$.

The quotient operator was not covered in class, and is henceforth not used.

13 Eigenvectors and Upper-Triangular Matrices

Corresponding to Chapter 5, Section B of Axler.

13.1 Notation

$T \in \mathcal{L}(V)$, unless otherwise stated.

13.2 T^m

T applied m times; $\underbrace{T \cdots T}_{m \text{ times}}$. Note $T^0 = I$. If T is invertible, then $T^{-m} = (T^{-}1)^m$.

13.3 $p(T)$

Given a polynomial $p(z) = \sum a_i z^i$, $p(T) = a_0 I + \sum a_i T^i$.

13.3.1 Product of Polynomials

$p, q \in \mathcal{P}(\mathbb{F})$, $(pq)(z) = p(z)q(z)$.

13.4 Multiplicative Properties

$p, q \in \mathcal{P}(\mathbb{F})$, then $(pq)(T) = p(T)q(T)$, and $p(T)q(T) = q(T)p(T)$.

13.5 Existence of Eigenvalues in Complex Vector Spaces

$\forall$ finite-dimensional V, $\forall T \in V$, $\exists$ an eigenvalue.

13.6 Matrix of an Operator

The matrix of an operator is defined the same way as the matrix of a linear transform from V to V with the same basis.

13.6.1 Diagonal of a Matrix

$A_{i,i}$ in a square matrix.

13.6.2 Upper-Triangular Matrix

A matrix with all entries below the diagonal equal to 0.

13.7 Conditions for Upper-Triangularity

v_i is a basis of V. Then the following are equivalent:

1. $\mathcal{M}(T)$ is upper triangular.
2. $Tv_j \in \text{span}(v_i), i \in [1, j], \forall j \in [1, n]$
3. $\text{span}(v_i), i \in [1, j], \forall j \in [1, n]$ is invariant under T.

13.8 Existence of Upper-Triangular Matrix over $\mathbb{C}$

$\forall$ finite-dimensional V, $\forall T \in V$, $\exists$ a basis of V s.t. T has an upper-triangular matrix in respect to the basis.

13.9 Invertibility in Upper-Triangular Matrix

T has an upper-triangular matrix; T is invertible iff $\forall i$, $A_{i,i} \neq 0$.

13.10 Eigenvalues in Upper-Triangular Matrix

The eigenvalues of T lie on the diagonal of the upper-triangular matrix of T.

14 Eigenspaces and Diagonal Matrices

Corresponding to Chapter 5, Section C of Axler.

14.1 Notation

$T \in \mathcal{L}(V)$, unless otherwise stated.

14.2 Diagonal Matrix

A matrix with all non-diagonal entries 0.

14.3 Eigenspace

$E(\lambda, T) = \text{null}(T - \lambda I)$ is an *eigenspace* of T corresponding to an eigenvalue λ.

14.3.1 Sum of Eigenspaces

$\sum E(\lambda_i, T)$ where λ_i are distinct eigenvalues of finite-dimensional T is a direct sum, and further $\sum \dim E(\lambda_i, T) \leq \dim V$.

14.4 Diagonalizable

T has a diagonal matrix with respect to some basis of V.

14.4.1 Conditions Equivalent to Diagonalizability

For λ_i distinct eigenvalues of finite-dimensional T:

1. V has a basis consisting of eigenvectors of T.
2. $\exists$ 1-dimensional invariant subspaces U_i of V s.t. $V = U_1 \oplus \ldots \oplus U_n$.
3. $V = E(\lambda_1, T) \oplus \ldots \oplus E(\lambda_m, T)$.
4. $\dim V = \sum \dim E(\lambda_i, T)$.

14.5 Enough Eigenvalues imply Diagonalizability

If T has $\dim V$ distinct eigenvalues then T is diagonalizable.

15 Inner Products and Norms

Corresponding to Chapter 6, Section A of Axler.

15.1 Notation

V denotes an inner product space after 14.3.1.

15.2 Dot Product

Assumed. However, note that in $\mathbb{C}^n$, the euclidean dot product is $\langle u_i, v_i \rangle = \sum u_i \bar{v}_i$.

15.3 Inner Product

A function on V that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbb{F}$ s.t.

1. $\langle v, v \rangle \geq 0, \forall v \in v$.
2. $\langle v, v \rangle = 0$ iff $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \forall u, v, w \in V$.
4. $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle, \forall \lambda \in \mathbb{F}, \forall u, v \in V$.
5. $\langle u, v \rangle = \overline{\langle v, u \rangle}, \forall u, v \in V$.

15.3.1 Inner Product Space

A V.S. V along with an inner product on V.

15.3.2 Properties of an Inner Product

1. $\forall u \in V$, the function $f : v \to \langle v, u \rangle$ is a linear map from V to $\mathbb{F}$.
2. $\langle 0, u \rangle = 0, \forall u \in V$.
3. $\langle u, 0 \rangle = 0, \forall u \in V$.
4. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle, \forall u, v, w \in V$.
5. $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle, \forall \lambda \in \mathbb{F}, \forall u, v \in V$.

15.4 Norm

$||v|| = \sqrt{\langle v, v \rangle}$

15.4.1 Basic Properties of the Norm

1. $||v|| = 0$ iff $v = 0$.
2. $||\lambda v|| = |\lambda| \, ||v||, \forall \lambda \in \mathbb{F}$.

15.5 Orthogonal

$u, v \in V$ are *orthogonal* if $\langle u, v \rangle = 0$.

15.5.1 Orthogonality and 0

1. 0 is orthogonal to all $v \in V$.
2. 0 is the only vector in V orthogonal to itself.

15.6 Pythagorean Theorem

u, v are orthogonal, then $||u+v||^2 = ||u||^2 + ||v||^2$.

15.7 Orthogonal Decomposition

$u, v \in V$ where $v \neq 0$. Then set $c = \dfrac{\langle u, v \rangle}{||v||^2}$, and $w = u - \dfrac{\langle u, v \rangle}{||v||^2} v$. Then $\langle u, v \rangle = 0$ and $u = cv + w$.

15.8 Cauchy-Schwarz Inequality

$|\langle u, v \rangle| \leq ||u|| \, ||v||$, where equality is achieved iff $u = \lambda v, \lambda \in \mathbb{F}$.

15.9 Triangle Inequality

$||u+v|| \leq ||u|| + ||v||$, where equality is achieved iff $u = \lambda v, \lambda > 0 \in \mathbb{F}$.

15.10 Parallelogram Equality

$||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$.

16 Orthonormal Bases

Corresponding to Chapter 6, Section B of Axler.

16.1 Notation

V denotes an inner product space.

16.2 Orthonormal

v_i is *orthonormal* if each $||v_i|| = 1$ and is orthogonal to all other vectors in the list.

16.3 Norm of an Orthonormal Linear Combination

e_i is an orthonormal list of V, then $||\sum a_i e_i||^2 = \sum |a_i|^2, \forall a_i \in \mathbb{F}$.

16.4 Linear Independence of Orthonormal Lists

Every orthonormal list of vectors is linear independent.

16.5 Orthonormal Basis

An orthonormal list of vectors that are also a basis.

16.5.1 Length of Orthonormal List and Bases

Every orthonormal list v_i with length dim V is an orthonormal basis of V.

16.6 Vector in terms of Orthonormal Basis

e_i is an orthonormal basis, then $v = \sum \langle v, e_i \rangle e_i$ and $||v||^2 = \sum |\langle v, e_i \rangle|^2$.

16.7 Gram-Schmidt Procedure

Given linearly independent v_i, we have $e_1 = \frac{v_1}{||v_1||}$, and for j, we have

$$e_j = \frac{v_j - \left(\sum_{i=1}^{j-1} \langle v_j, e_i \rangle e_i \right)}{\left|\left| v_j - \left(\sum_{i=1}^{j-1} \langle v_j, e_i \rangle e_i \right) \right|\right|}$$

Then e_i is orthonormal, and span$(v_i) =$ span(e_i).

16.8 Existence of Orthonormal Basis

$\forall$ finite-dimensional inner product space, $\exists$ an orthonormal basis.

16.8.1 Orthonormal List and Orthonormal Bases

Every orthonormal v_i in finite-dimensional V can be extended to an orthonormal basis of V.

16.9 Upper-triangular Matrices and Orthonormal Bases

T has an upper-triangular matrix with respect to some basis $\Rightarrow \exists$ an upper-triangular matrix with respect to some orthonormal basis.

16.10 Schur's Theorem

For finite-dimensional complex V, $\exists$ upper-triangular $\mathcal{M}(T)$ with respect to some orthonormal basis of V.

16.11 Linear Functional

$\phi \in \mathcal{L}(V, \mathbb{F})$.

16.12 Riesz Representation Theorem

$\exists u \in V$ s.t. $\phi(v) = \langle v, u \rangle \forall v \in V$.

17 Orthogonal Complements and Minimization Problems

Corresponding to Chapter 6, Section C of Axler.

17.1 Notation

V denotes an inner product space.

17.2 Orthogonal Complement, $U^\perp$

Given $U \subseteq V$, $U^\perp = \{v \in V : \langle v, u\rangle = 0, \forall u \in U\}$.

17.2.1 Basic Properties of Orthogonal Complement

1. $U \subseteq V \Rightarrow U^\perp$ is a subspace of V.
2. $\{0\}^\perp = V$.
3. $V^\perp = \{0\}$.
4. $U \subseteq V \Rightarrow U \cap U^\perp \subset \{0\}$.
5. $U, W \subseteq V$ and $U \subset W$, then $W^\perp \subset U^\perp$.

17.3 Direct Sum of Subspace and Orthogonal Complement

$V = U \oplus U^\perp$ for finite-dimensional subspace U.

17.4 Dimension of the Orthogonal Complement

dim $U^\perp =$ dim $V -$ dim U for finite-dimensional V and subspace U of V.

17.5 Orthogonal Complement of Orthogonal Complement

$U = (U^\perp)^\perp$ for finite-dimensional subspace U.

17.6 Orthogonal Projection, P_U

$P_U \in \mathcal{L}(V)$ s.t. for $v \in V$, write $v = u + v$ where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$, where U is finite-dimensional.

17.6.1 Properties of the Orthogonal Projection

1. $P_U \in \mathcal{L}(V)$.
2. $P_U u = u, \forall u \in U$.
3. $P_U w = 0, \forall w \in U^\perp$.
4. range $P_U = U$.
5. null $P_U = U^\perp$.

6. $v - P_U v \in U^\perp$.
7. $P_U^2 = P_U$.
8. $||P_U v|| \leq ||v||$.
9. $\forall$ orthonormal basis e_i of U, $P_U v = \sum \langle v, e_i \rangle e_i$

17.7 Minimizing the Distance to a Subspace

Given finite-dimensional subspace U, $v \in V$, $U \in U$, $||v - P_U v|| \leq ||v - u||$, where equality is achieved iff $u = P_U v$.

18 Self-Adjoint and Normal Operators

Corresponding to Chapter 7, Section A of Axler.

18.1 Notation

U, V, W denote inner product spaces.

18.2 Adjoint, T^*

$T^* : W \to V$ s.t. $\langle Tv, w \rangle = \langle v, T^* w \rangle, \forall v \in V, \forall w \in W$.

18.2.1 Adjoint is a Linear Map

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

18.2.2 Properties of the Adjoint

1. $(S + T)^* = S^* + T^*, \forall S, T \in \mathcal{L}(V, W)$.
2. $(\lambda T)^* = \bar{\lambda} T^*, \forall \lambda \in \mathbb{F}, \forall T$.
3. $(T^*)^* = T, \forall T$.
4. $I^* = I$.
5. $(ST)^* = T^* S^*$, where $T \in \mathcal{L}(V, W)$, and $S \in \mathcal{L}(W, U)$.

18.2.3 Null Space and Range of T^*

1. null $T^* = (\text{range } T)^\perp$.
2. range $T^* = (\text{null } T)^\perp$.
3. null $T = (\text{range } T^*)^\perp$.
4. range $T = (\text{null } T^*)^\perp$.

18.3 Conjugate Transpose

The *conjugate transpose* of an m-by-n matrix is the n-by-m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

18.4 The Matrix of T^*

Suppose e_i is an orthonormal basis of V and f_i is an orthonormal basis of W. Then $\mathcal{M}(T^*, f_i, e_i)$ is the conjugate transpose of $\mathcal{M}(T, e_i, f_i)$.

18.5 Self-Adjoint

$T = T^*$, or $\langle Tv, w \rangle = \langle v, Tv \rangle, \forall v, w \in V$.

18.5.1 Eigenvalues of Self-Adjoint Operators

All eigenvalues of self-adjoint operators are real.

18.5.2 Orthogonality of Tv

Over $\mathbb{C}$, if $\langle Tv, v \rangle = 0, \forall v \in V$, then $T = 0$.

18.5.3 Self-Adjoint Operators and $\langle Tv, v \rangle$

Over $\mathbb{C}$, T is self-adjoint iff $\langle Tv, v \rangle \in \mathbb{R}, \forall v \in V$.

18.5.4 Self-Adjoint Operators and $\langle Tv, v \rangle = 0$

If T is self-adjoint s.t. $\langle Tv, v \rangle = 0, \forall v \in V$, then $T = 0$.

18.6 Normal

$TT^* = T^*T$. Note every self-adjoint operator is normal, but not all normal operators are self-adjoint.

18.6.1 Condition for Normality

T is normal iff $||Tv|| = ||T^*v||, \forall v \in V$.

18.6.2 Orthogonal Eigenvectors for Normal Operators

Given normal T, then the eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

19 The Spectral Theorem

Corresponding to Chapter 7, Section B of Axler.

19.1 Notation

U, V, W denote inner product spaces.

19.2 The Complex Spectral Theorem

For $\mathbb{F} = \mathbb{C}$:

1. T is normal.
2. V has an orthonormal basis consisting of eigenvectors of T.
3. T has a diagonal matrix with respect to some orthonormal basis of V.

19.3 Invertible Quadratic Expressions

Given self-adjoint T, and $b, c \in \mathbb{R}$ s.t. $b^2 < 4c$. The $T^2 + bT + cI$ is invertible.

19.4 Eigenvalues of Self-Adjoint Operators

Given $V \neq \{0\}$, then T has an eigenvalue.

19.5 Self-Adjoint Operators and Invariant Subspaces

T is self-adjoint and U is an invariant subspace of V. Then

1. $U^\perp$ is invariant under T.
2. $T|_U \in \mathcal{L}(U)$ is self-adjoint.
3. $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

19.6 The Real Spectral Theorem

For $\mathbb{F} = \mathbb{R}$:

1. T is self-adjoint.
2. V has an orthonormal basis consisting of eigenvectors of T.
3. T has a diagonal matrix with respect to some orthonormal basis of V.