Johnson Noise

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In the early 1900's, establishing a clear connection between the microscopic atomic theory and the measurable macroscopic properties of the real world was a key dilemma in physics. Although the ideal gas law and its associated gas constant $R_g = kN$ were known, Avogadro's number N and the Boltzmann constant k were not known individually. In 1928 Johnson B. Johnson measured a thermally varying electrical noise and published his findings [1] alongside a theoretical derivation of the same phenomena provided by his colleague Harry Nyquist [2]. This resistance and temperature dependent Johnson noise in a circuit provides a system analogous to a gas under pressure, and its measurement allows for the extraction of the Boltzmann constant k as well as the centigrade temperature of absolute zero T_0. In this experiment we make a measurement of Johnson noise and extract the value of the Boltzmann constant to be $k = (1.34 \pm 0.020e^{-23})\frac{J}{K}$ and the centigrade temperature of absolute zero to be $T_0 = (-236.0 \pm 32.0)\ C$.

I. JOHNSON NOISE THEORY

In his original paper on Johnson noise [2], Nyquist presents a simple system consisting of transmission line shorted with two resistors of equal resistance R at either end. At a temperature T, thermal agitation of charge carriers within this system produces electrical modes of oscillation in a range of frequencies. These thermally dependent oscillations produce the fluctuating voltage noise that was measured by Johnson [1]. By assuming the individual oscillation mode frequencies to be closely spaced, they can be treated as a continuous spectrum. The modes that fall within each small frequency band df of the spectrum contribute $dV = 2R(dI)df$ and $dP = \frac{dV^2}{4R}df$ to the total root mean square voltage and average power within the system, where k refers to the Boltzmann constant. By the equipartition theorem, each frequency band df contributes an average power $dP = kTdf$ to the total average power. These equations can be combined to derive the following expression for the mean square voltage contribution of a frequency band df:

$$dV^2 = 4kRfTdf \tag{1}$$

This equation draws an analogy between the mean square voltage of an electrical system and the pressure of a gas in a container. The equipartition theorem renders both systems proportional to kT, and relates each to the number of degrees of freedom. The number of degrees of freedom for a compressed gas is given in terms of Avogadro's number N, while the number of degrees of freedom for an electrical system is equal to the number of oscillating modes within a frequency range. At the turn of the 20th century, an electrical system with a derivable number of degrees of freedom provided a scenario in which to measure the Boltzmann constant and extract a relationship between the microscopic granularity of nature and macroscopic observation.

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II. EXPERIMENTAL SETUP

The experimental apparatus consists of a resistor, an amplifier, a band-pass filter and an oscilloscope, as shown in Fig. 1.

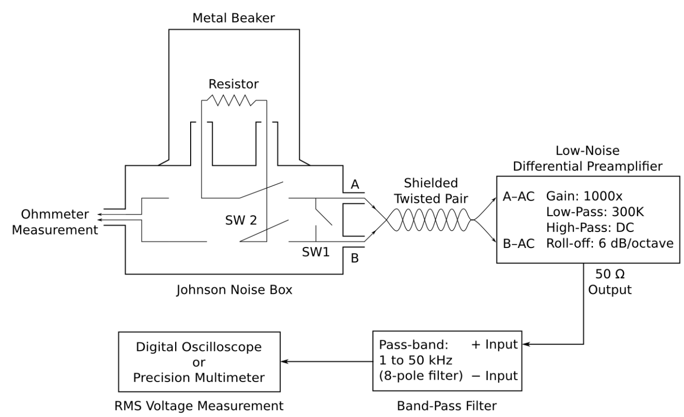

FIG. 1. The experimental setup used to measure Johnson noise voltage over a resistor. Shorting switches SW1 and SW2 allow for isolation of the noise over the resistor placed on the Johnson Noise Box from the noise over the amplifier. Adapted from [3].

II.1. Johnson Noise Box

In order to select a specific resistance over which to measure voltage noise, a specially designed circuit box is used. Alligator clips protruding from the box's surface allow specific resistors to be swapped in and out of the system. High precision metal-oxide film resistors are used, and a metal beaker is placed over the resistor to mitigate external noise. The box contains ports used to measure the resistance of the resistor placed in the clips using a multimeter [4]. A switch mechanism is incorporated in order to short the resistor in and out of the mechanism.

II.2. Amplifier

In order to amplify the noise signal to a mV range over which accurate measurements can be made, an SRS differential amplifier provides a $\times 1000$ gain to the input noise voltage. Current is fed from the Johnson box into the amplifier through a pair of cables twisted together to mitigate cross-talk interference, and shielded from external interference with aluminum foil.

II.3. Band-pass Filter

To achieve a frequency spectrum between 1 kHz and 50 kHz, a Krohn-Hite FMB300 filter module [5] is positioned after the amplifier in the signal. As we will discuss, limiting the measured voltage noise to a certain frequency range allows for calculation of a finite system gain, and subsequent extraction of the pre-processed noise over the resistor on the Johnson box.

II.4. Measurement Readout

Measurements are read from a standard digital oscilloscope. Due to the fluctuating nature of Johnson noise, raw data is taken in the form of root mean square voltages. This measurement relates a fluctuating AC voltage signal to the DC signal that transfers equal power in a set time period.

III. PROCEDURE

The basic experimental procedure consisted of placing a resistor between the clips on the Johnson box, measuring the system resistance for each resistor by means of a multimeter, and recording root mean square voltage values from the oscilloscope. Because the amplifier in the signal chain contributes a large portion of its own electrical noise to the final measurement at the oscilloscope, voltage values V_R and V_S were taken with the resistor shorted in and out of the circuit, respectively. These measurements were performed 10 times each for each resistor or temperature varied point in order to mitigate statistical uncertainty. Due to the independent nature of the resistor and amplifier voltage noise contributions, the resistor noise contribution V_m was then extracted from the total oscilloscope readout by subtracting the sources in quadrature: $V_m^2 = V_R^2 - V_S^2$.

III.1. Gain Calibration

The experimental setup used has a multiplicative effect on the original voltage Johnson noise generated over the resistor on the Johnson box. In order to extract the pre-processed noise voltage from the post-processed noise voltage read at the oscilloscope, it is necessary to quantify the effect of the experimental apparatus. This effect can be modeled as a multiplication of the pre-processed noise voltage with a frequency dependent gain factor, yielding a relationship between the measured resistor noise and the input resistor noise: $dV_m^2 = g(f)dV_n^2$. By substituting in Eq. 1 and integrating over a finite frequency spectrum between 0 kHz to 90 kHz, the following relationship is obtained:

$$V_m^2 = 4kRTG \tag{2}$$

where

$$G = \int_{0\ kHz}^{90\ kHz} \frac{g^2(f)}{1 + (2\pi f RC)^2} df \tag{3}$$

The $(1 + (2\pi f RC)^2)^{-1}$ factor is introduced to account for the effect of capacitance in between connecting cables in the apparatus. In order to measure the gain factor value $g(f)$ for our specific apparatus, the Johnson box was replaced by a function generator. The generator was set to emit a sinusoid of known voltage amplitude V_{in} for a variety of frequencies in the 0 kHz to 90 kHz range. The output voltage on the oscilloscope V_{out} was measured at each frequency, and the gain factor was extracted by computing $g(f) = \frac{V_{out}}{V_{in}}$. A plot of the capacitance adjusted gain and raw gain for a 477.0 $kOhm$ resistor is shown in Fig. 2.

Capacitance of the system was measured at the start of each session, by means of an LCR meter connected across the Johnson box. G was extracted for each resistor value using two numerical integration methods - trapezoidal integration as well as Simpson's rule - over the 0 kHz to 90 kHz range.

IV. RESULTS AND ANALYSIS

Armed with the ability to extract the pre-processed voltage noise from the measured voltages, we conducted two sets of measurements. The first set of voltage measurements were taken at constant room temperature 23.0 C, with 12 different resistors in a range between 10 $kOhm$ and 1000 $kOhm$. The Boltzmann constant was extracted from these values. The second set of measurements was taken with a single resistor value of 99.8 $kOhm$ at varying temperatures. Resistor temperature was varied over 9 different values by inverting the Johnson noise box into a liquid nitrogen bath at -195.8 C, and a temperature varying oven chamber set

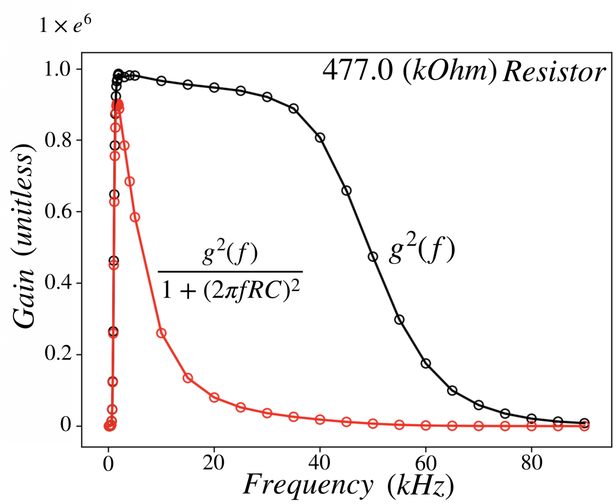

FIG. 2. The raw gain curve obtained is shown in black. The capacitance adjusted gain curve for a 477.0 $kOhm$ resistor is shown in red. Numerical integration over the adjusted gain curve can be used to extract the gain factor G for each resistor.

to values between 23.0 C and 140.0 C. From these data points, values for both the Boltzmann constant and the centigrade temperature of absolute zero were extracted.

IV.1. Extracting k

Using the values of V_m^2 obtained by varying resistance R at constant room temperature $T = 23.0\ C$, Eq. 2 was used to extract a value of k for each resistor. A plot relating each computed value of k to resistance R is shown in Fig. 3.

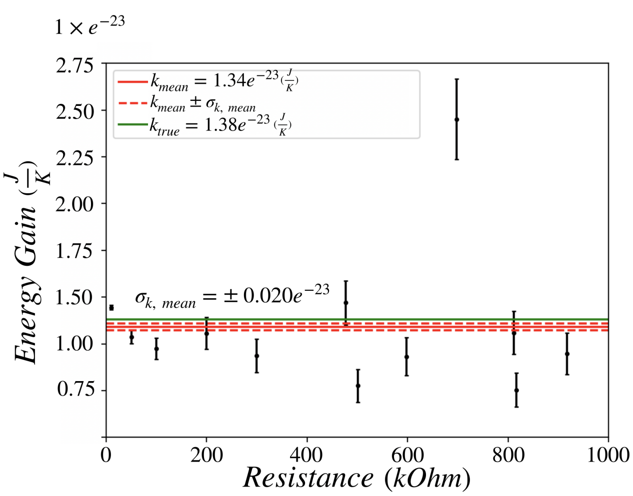

FIG. 3. k values extracted for 9 resistors of varying value. The true value of k is plotted in green, while the average of the data and corresponding error bands are plotted in red.

The systematic uncertainty on k stems from sources of error in R, G, and V_m^2. Uncertainty in resistance σ_R is based on the precision of the multimeter measurement for each resistor, and was obtained from the specification sheet [4]. This uncertainty ranges from $\sigma_R = \pm 0.2\%$ to $\sigma_R = \pm 0.6\%$, depending on the resistor. Uncertainty in the mean squared voltage measurement $V_m^2 = V_R^2 - V_S^2$ can be derived by adding the independent variances of the squared set of shorted voltage measurements and unshorted voltage measurements as such: $\sigma_{V^2} = \sqrt{var(V_R^2) + var(V_S^2)}$. Depending on the resistor, this uncertainty ranged from $\sigma_{V^2} = \pm 2.5e^{-7}\ volts$ to $\sigma_{V^2} = \pm 9.6e^{-7}\ volts$. The last error contribution stems from the uncertainty in the gain integral σ_G for each resistor. This uncertainty can be broken down into two components - error from the method of numerical integration used to compute G, and error from the systematic uncertainty in the capacitance of the apparatus. The uncertainty in G as a result of integration method is obtained by using two integration methods - trapezoid rule and Simpson's rule. By computing $\frac{G_{trapezoid} - G_{simpson's}}{G_{trapezoid}}$, a range of uncertainty from $\sigma_{integration} = \pm 0.04\%$ to $\sigma_{integration} = \pm 0.9\%$ is obtained. The largest source of systematic uncertainty stems from the uncertainty in capacitance between connecting cables in the apparatus, and the resulting affect on the uncertainty in the gain integral G. In order to quantify this uncertainty, we took two sets of measurements with differing cable positions. For each resistance, the ratio of gain integrals between each setup was calculated and used to solve for the corresponding fluctuation in capacitance. The uncertainty contribution due to capacitance is by far the greatest source of uncertainty in this system, ranging from $\sigma_C = \pm 5\%$ to $\sigma_C = \pm 10\%$. As resistances get higher, the measurements become increasingly systematically dominated as the error in capacitance gains greater effect as seen in Eq. 3. Propagating the above sources of error in quadrature through the equation used to obtain k, uncertainties on k are found to be in the $\sigma_k = \pm 0.03e^{-23}\frac{J}{K}$ to $\sigma_k = \pm 0.2e^{-23}\frac{J}{K}$ range.

Using the uncertainties on each point, it is possible to compute the uncertainty on their average using a Monte Carlo simulation. For each iteration in a 10,000 iteration cycle, a set of k values is raffled from a distribution centered at 0 with width $2\sigma_k$. The average of each raffled set of k values is compared to the measured k average, and the standard deviation of these differences is taken as the uncertainty in the average value of k. Doing so yields an uncertainty on the average k value of $\pm 0.02e^{-23}\frac{J}{K}$.

The average Boltzmann constant value as extracted from this measurement is given by $k = (1.34 \pm 0.020)e^{-23}\frac{J}{K}$. Compared to the accepted value of the Boltzmann constant [6] $k_{true} = 1.38e^{-23}\frac{J}{K}$, the data seems to provide a good measurement of k. However, it is important to note that the near-infinite precision uncertainty associated with the 10.02 $kOhm$ measurement as well as the abnormally high value obtained for the 698.0 $kOhm$ resistor dominate the average of the measurements. Averaging without these two points yields a more reasonable Boltzmann constant measurement of $k = (1.21 \pm 0.018)e^{-23}\frac{J}{K}$.

IV.2. Extracting T_0

A relationship between Johnson noise and temperature can be extracted by plotting $\frac{V_m^2}{4RG}$ against T as shown in Fig. 4.

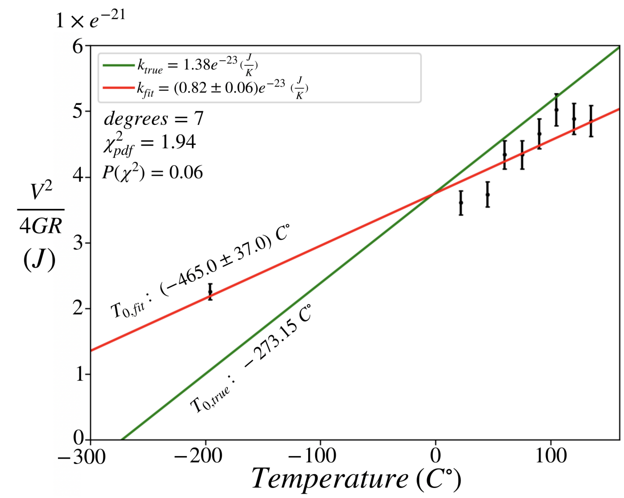

FIG. 4. The centigrade temperature of absolute zero is extracted by finding the x-intercept of the linear fit in red. Although the linear fit is fairly reasonable, the slope and intercept differ significantly from the accepted relationship plotted in green.

The uncertainty in absolute zero stem from the same sources as the uncertainty in the k measurement discussed previously. The only two variations are an added uncertainty in temperature T, and the addition of uncertainty in G stemming from uncertainty in R. Uncertainty in R is propagated in quadrature, and yields approximately the same range of uncertainty in G as calculated above. Uncertainty in T was quantified by measuring the temperature fluctuation within the oven over the 45 *sec* period needed to collect V_R and V_S data at each temperature. This horizontal uncertainty was then converted to a vertical uncertainty by multiplying it by the Boltzmann constant. The uncertainty on each point translates into an uncertainty in the extracted value of absolute zero given by the error of the linear fit: $\sigma_{T_0} = \pm 37.0\ C$. Inspection of the linear fit leads to suspicion concerning the validity of the liquid nitrogen measurement. This measurement's large horizontal distance from the other data points allows it to dominate the fit. Although the fit itself is not unreasonable, and the error in the liquid nitrogen is comparable in magnitude to the error in the other temperature measurements, the significantly inaccurate values of both absolute zero centigrade and the Boltzmann constant lead us to suspect an important source of systematic error might be affecting the liquid nitrogen measurement. Attempts were made to pinpoint sources of systematic error that could be leading to such a seemingly erroneous liquid nitrogen measurement. 40 voltage measurements were made for V_R and V_S in order to minimize statistical uncertainty, and the resistor was allowed ample time to reach equilibrium temperature in the bath. Efforts were made to ensure the resistor was in contact with liquid nitrogen and not semi-immersed in the gaseous layer at the top of the container, and a glass container was used to mitigate capacitance uncertainty from contact between the Johnson box and the liquid nitrogen container. The measurement was repeated twice - each time yielding similar results. Despite consultation with Dr. Robinson and the JLab Technical Staff, we are not sure what could be causing such a measurement at this time.

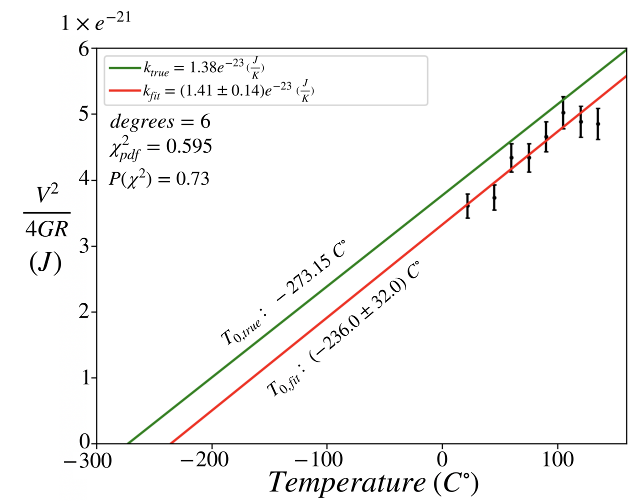

FIG. 5. The same relationship plotted in Fig. 4, excluding the liquid nitrogen measurement. This plot yields predictions much closer to the accepted values.

Excluding the liquid nitrogen measurement yields a good estimate of $k = (1.41 \pm 0.14)e^{-23}\frac{J}{K}$, and a much closer measurement of absolute zero centigrade of $T_0 = (-236.0 \pm 32.0)\ C$. A plot of the fit to this set of points is shown in Fig. 5.

V. CONCLUSIONS

By performing a measurement of thermally dependent Johnson noise, a numerical relationship between the discrete quality of matter and macroscopic reality has been established. The Boltzmann constant was extracted as $k = (1.34 \pm 0.020)e^{-23}\frac{J}{K}$ with all points included, and $k = (1.21 \pm\ 0.018)e^{-23}\frac{J}{K}$ without the dominating 10.02 $kOhm$ and 698.0 $kOhm$ resistors. This can be compared to the accepted value of $k_{true} = 1.38e^{-23}\frac{J}{K}$. The centigrade temperature of absolute zero was extracted to be $T_0 = (-465.0 \pm 37.0)\ C$ with all points, and $T_0 = (-236.0 \pm 32.0)\ C$ without the inclusion of the liquid nitrogen measurement. This can be compared to the accepted value of $T_{true} = -273.15\ C$.

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[1] J. B. Johnson, Physical review **32**, 97 (1928).
[2] H. Nyquist, Physical review **32**, 110 (1928).
[3] . Staff, "Johnson and shot noise lab guide,".
[4] "Agilent 970-series multimeter data sheet," .
[5] "Krohn-hite pass-band filter," .
[6] "Codata boltzmann constant," .