Johnson Noise

Eric Chen, MIT Department of Physics, 11/20/2018

History

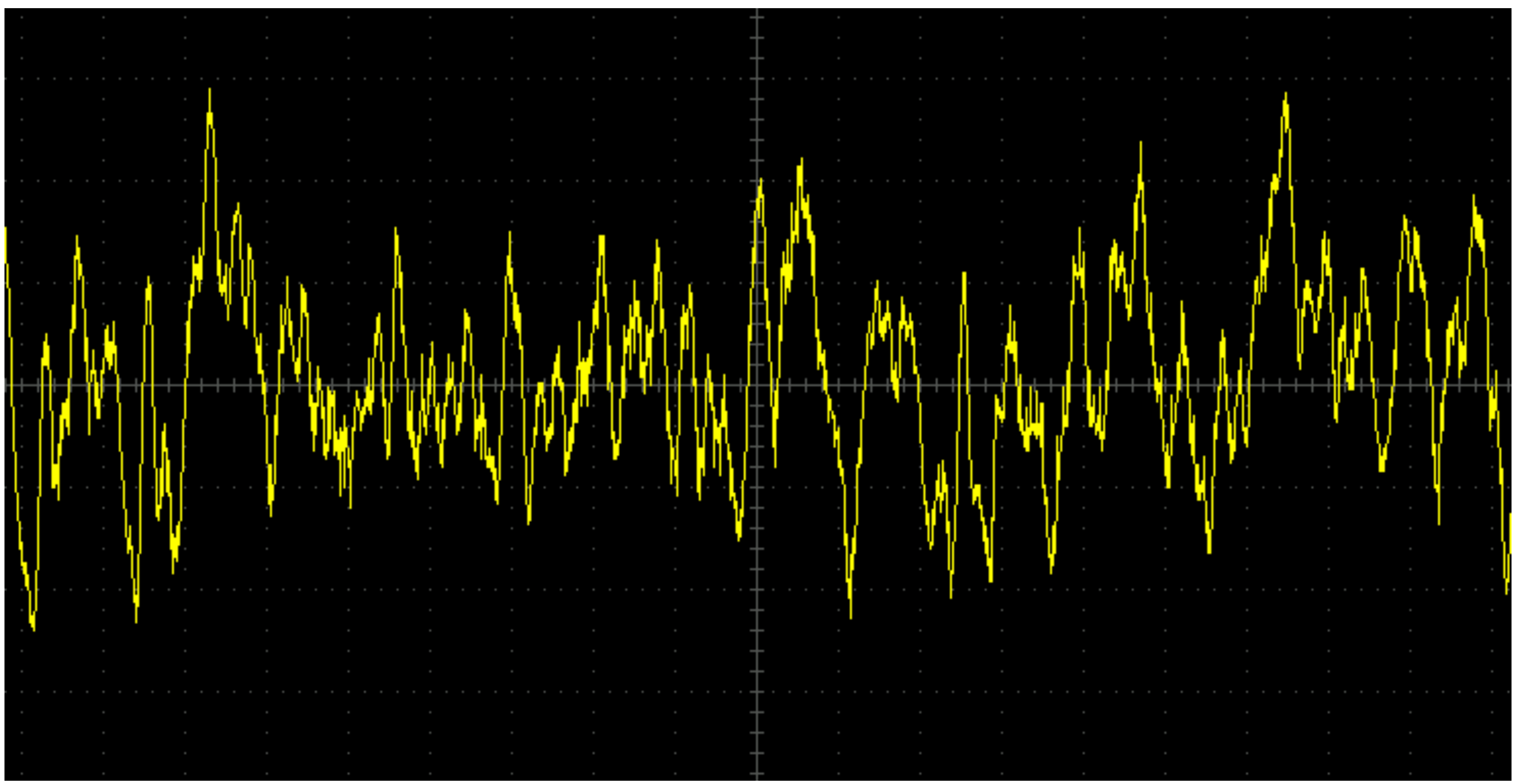
- 1928 Bell Labs: John B. Johnson measures a thermally varying electrical noise experimentally

- Colleague Harry Nyquist publishes a theoretical explanation soon after

Thermal Noise

- Thermal agitation of charge carriers produces fluctuation of voltage across a resistor

Voltage



Time

Interesting Features

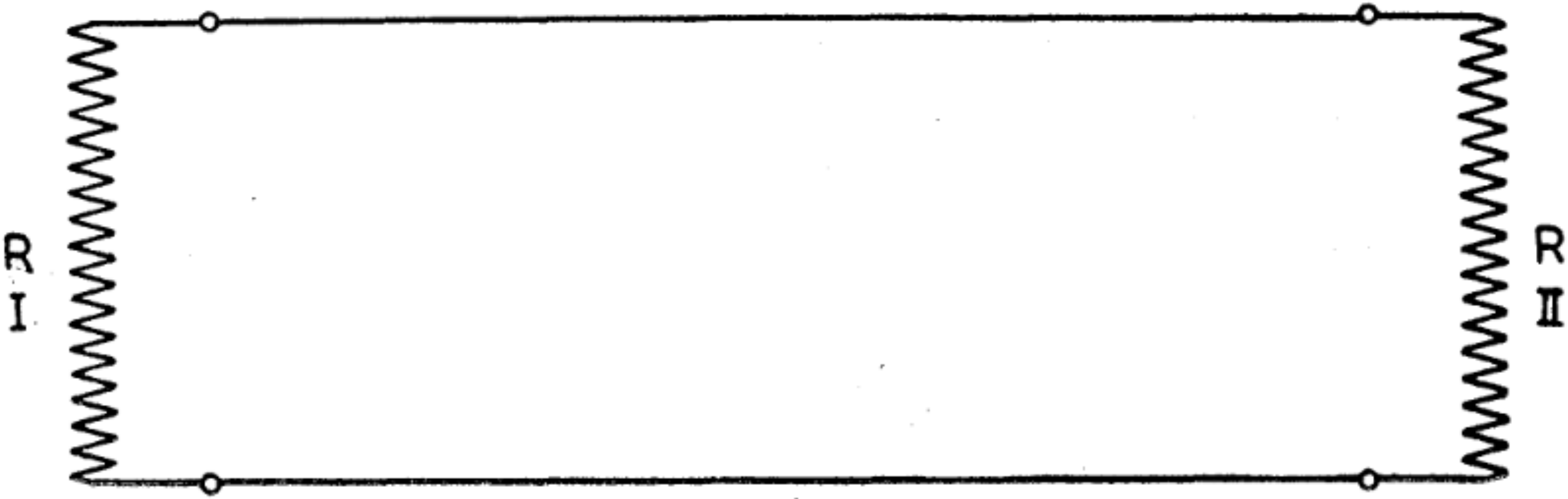
- This noise depends only on the resistance and temperature of the system

- Noise doesn’t depend on what type of charge carrier, or the resistor material

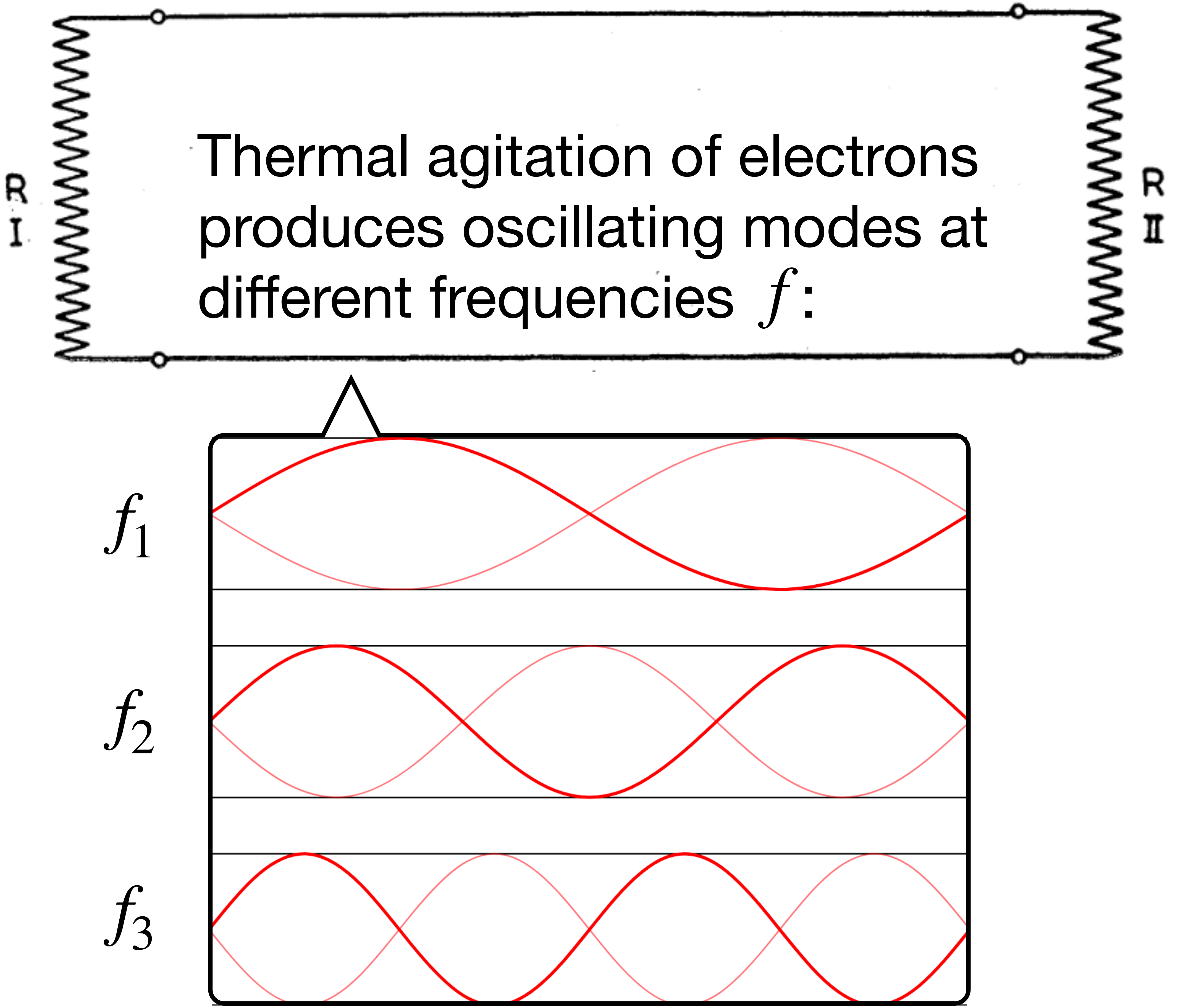
In this Lab:

- We measure Johnson noise for varying resistances and temperatures
- k (Boltzmann's constant) was not known individually
- In this lab, k and centigrade temperature of absolute zero can be extracted from measurements of Johnson noise

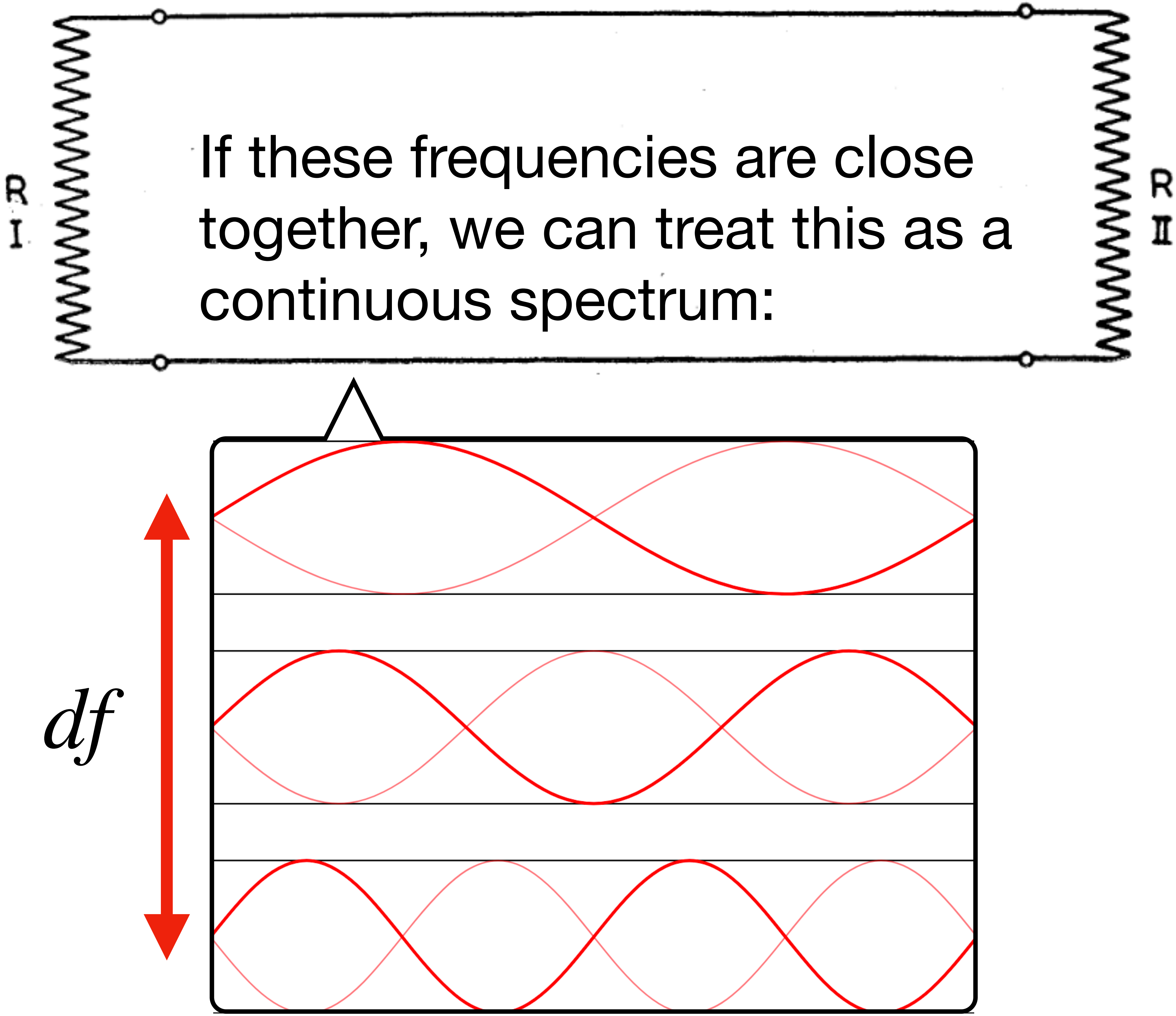
Theory



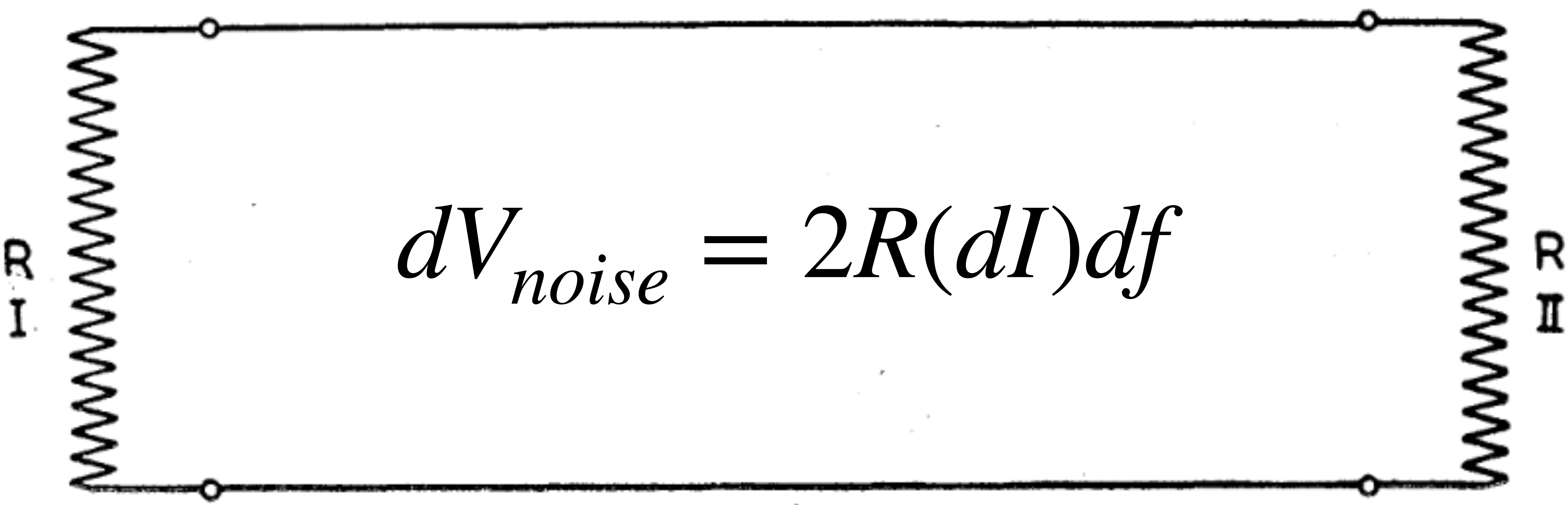
- System temperature T
- Resistances R_I and R_{II}



R I
R II
Thermal agitation of electrons produces oscillating modes at different frequencies f:
f_1
f_2
f_3


If these frequencies are close together, we can treat this as a continuous spectrum:
R I
R II
df

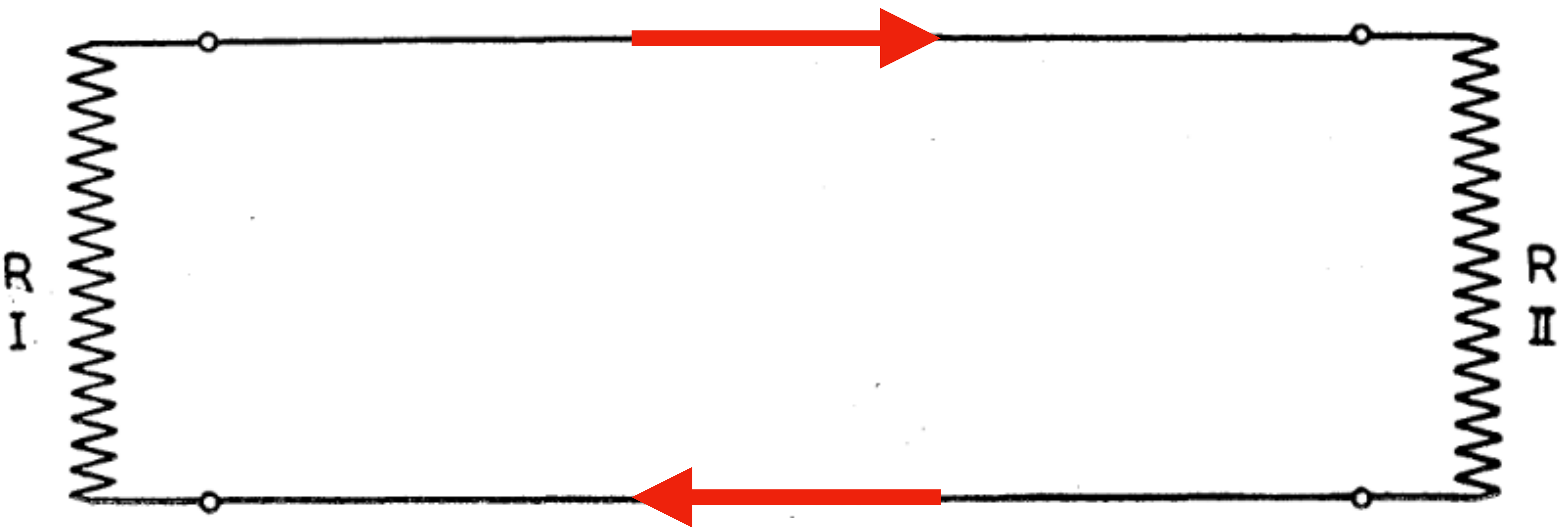
- Each band df contributes a average voltage difference to the overall circuit voltage:



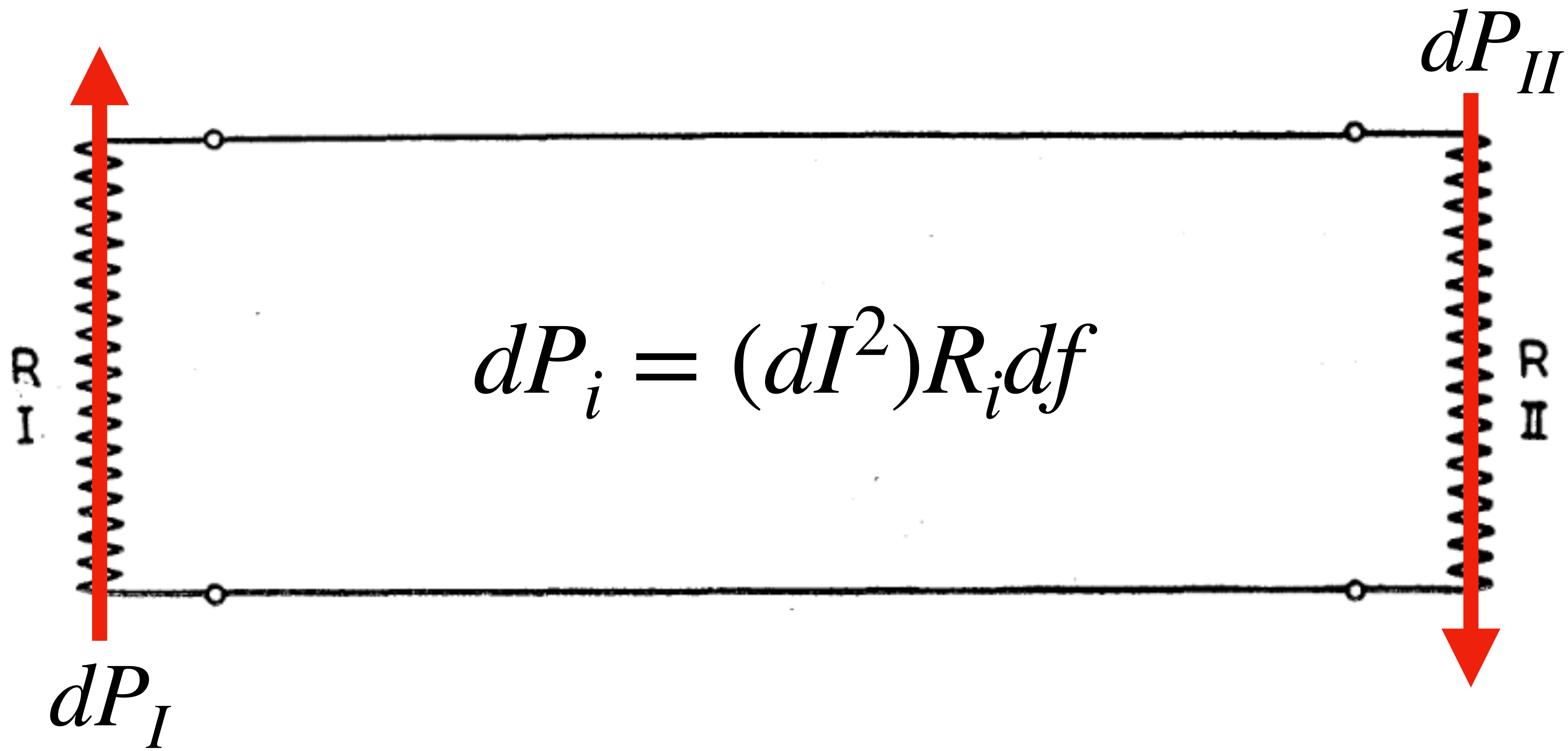
$$dV_{noise} = 2R(dI)df$$

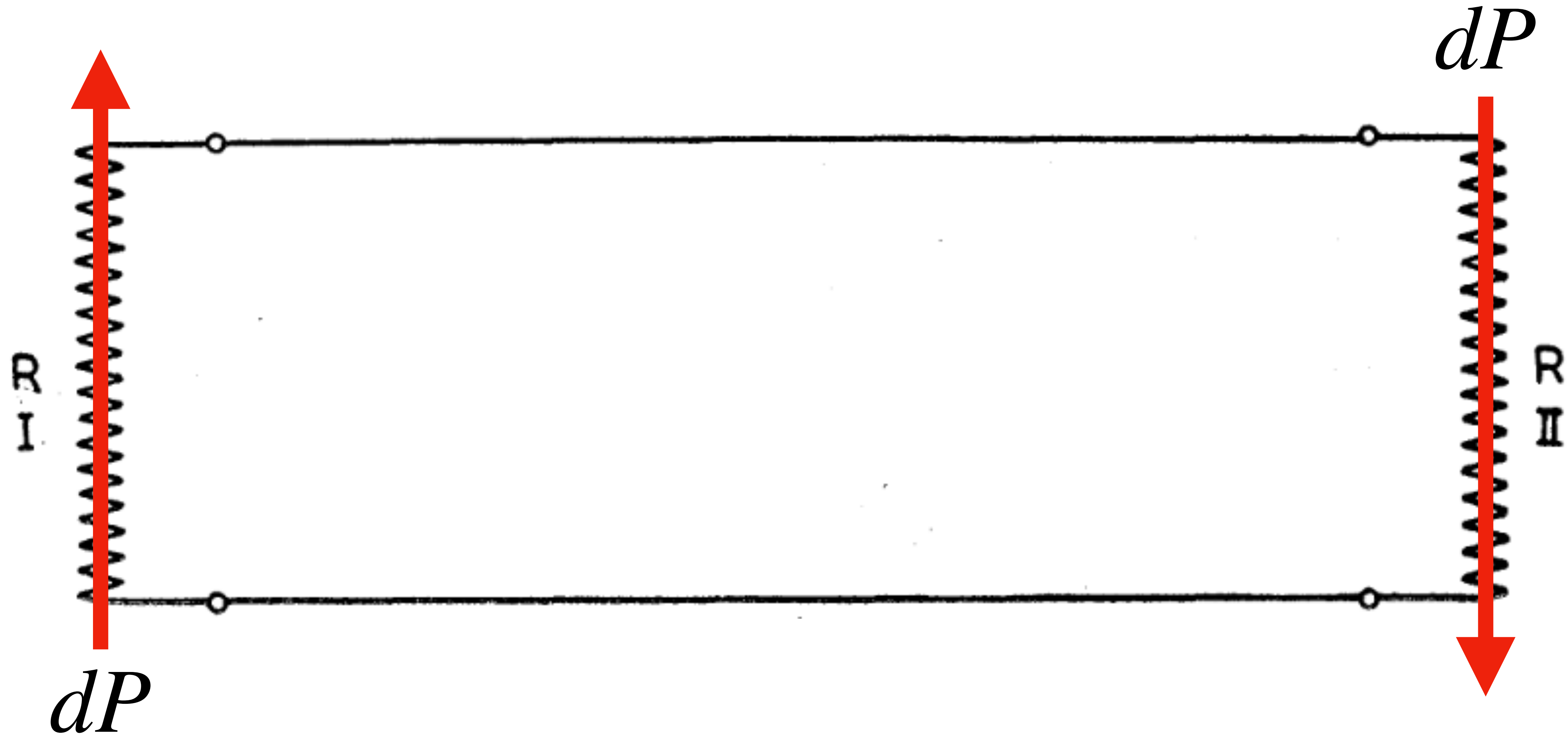
- Each small frequency band of oscillation df contributes an average amount of current:

$$dI = \frac{dV_{noise}}{2R} df$$



- Each band df contributes an amount of average power through a resistor of resistance R_i:

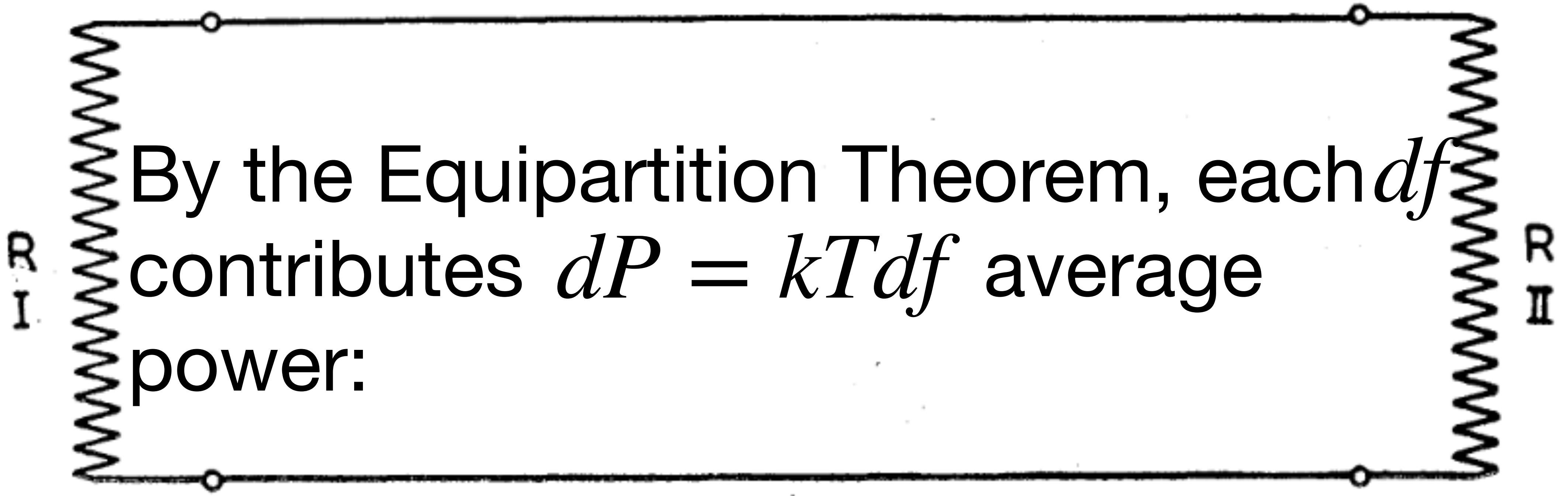


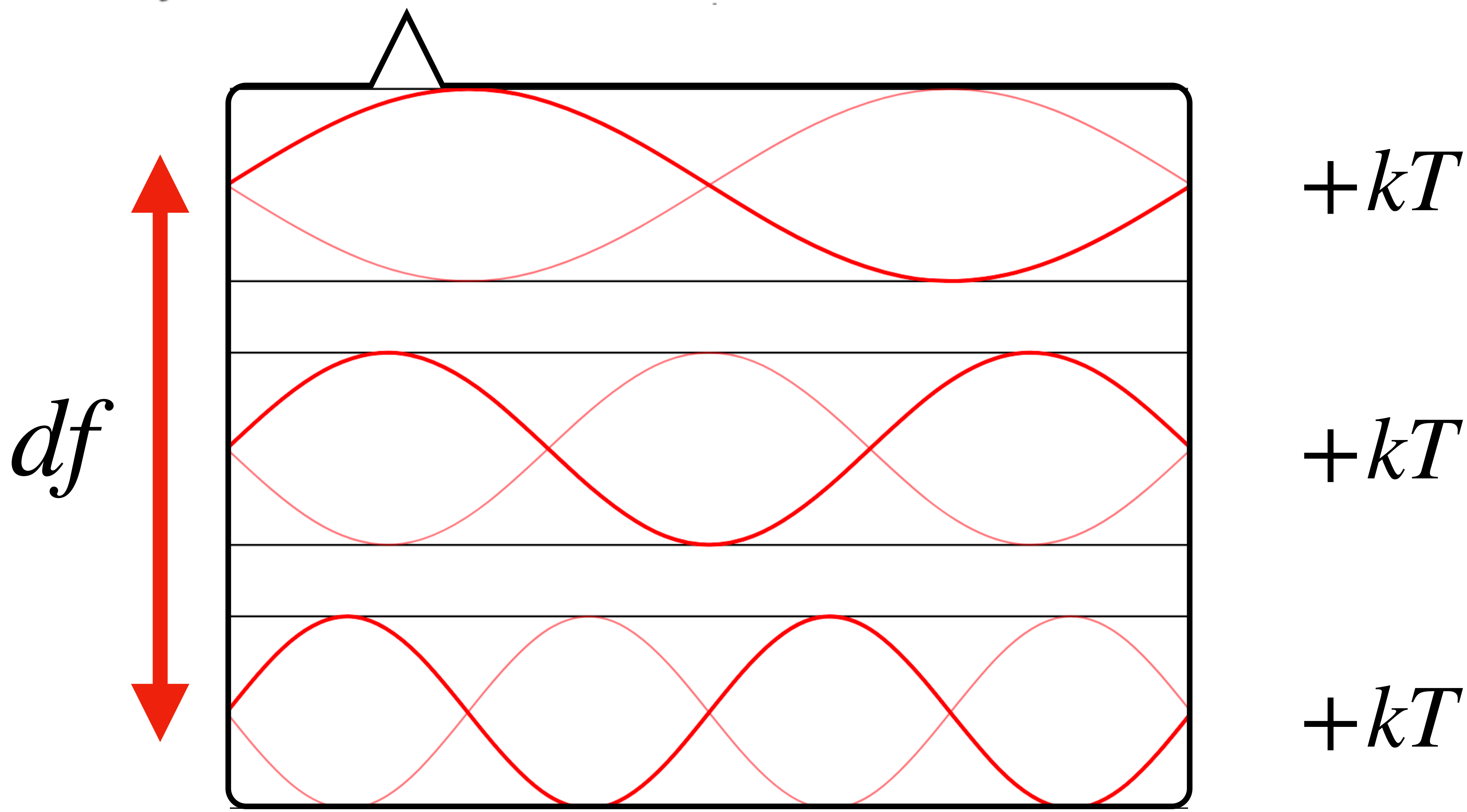


- By the 2nd Law of Thermodynamics, the power passed through each resistor is equal:

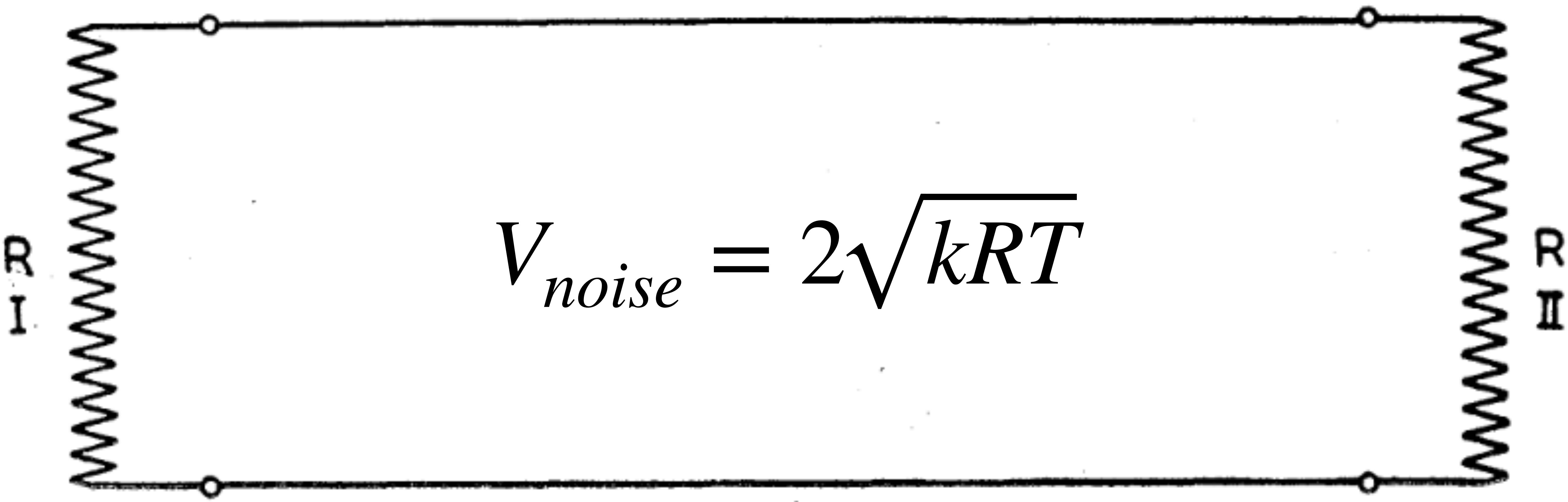
$$dP_I = dP_{II} = dP$$

By the Equipartition Theorem, each df contributes $dP = kTdf$ average power:





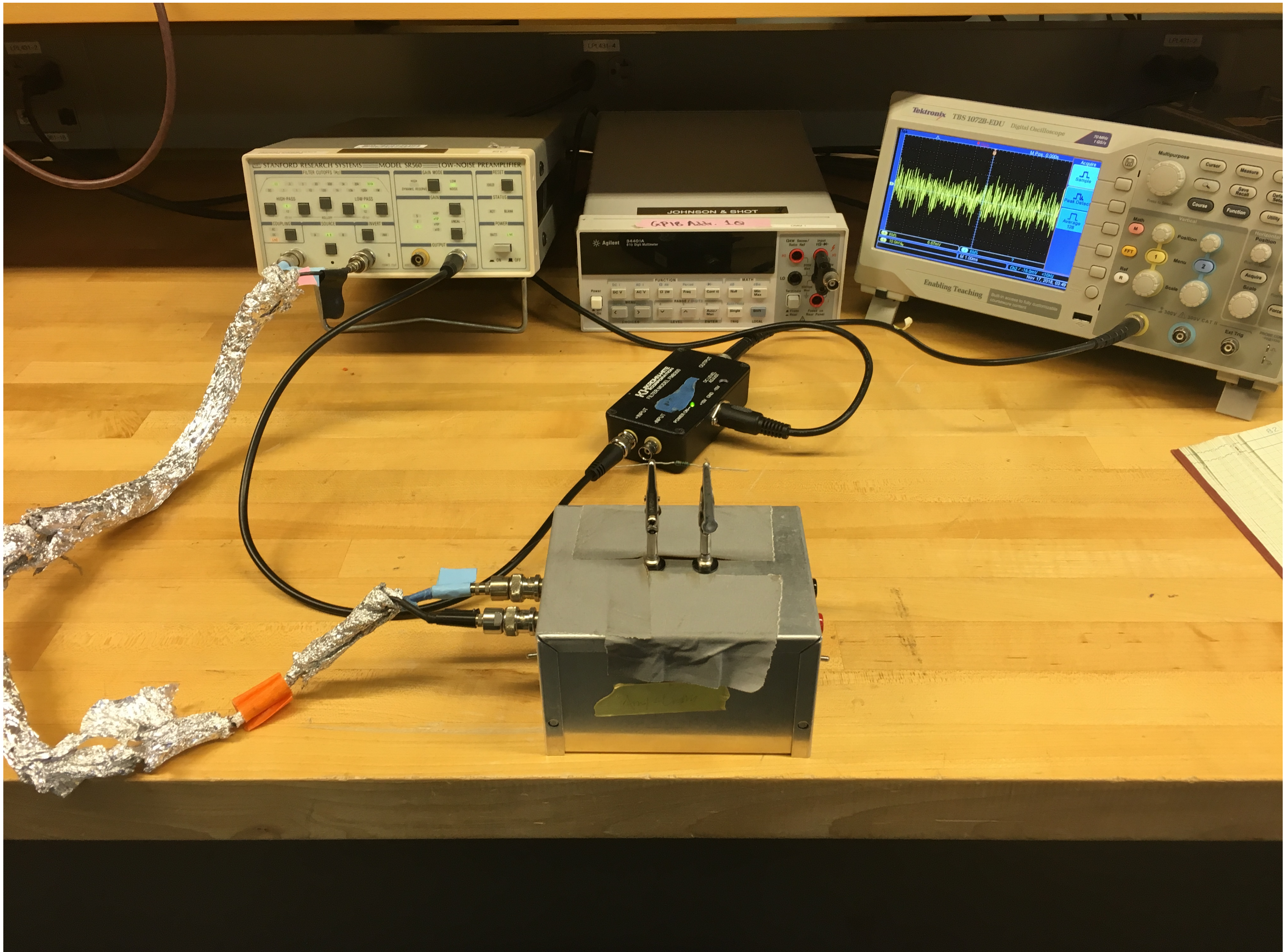
- Using these relationships, the total average voltage contribution over all frequency bands for a system at temperature T can be derived:



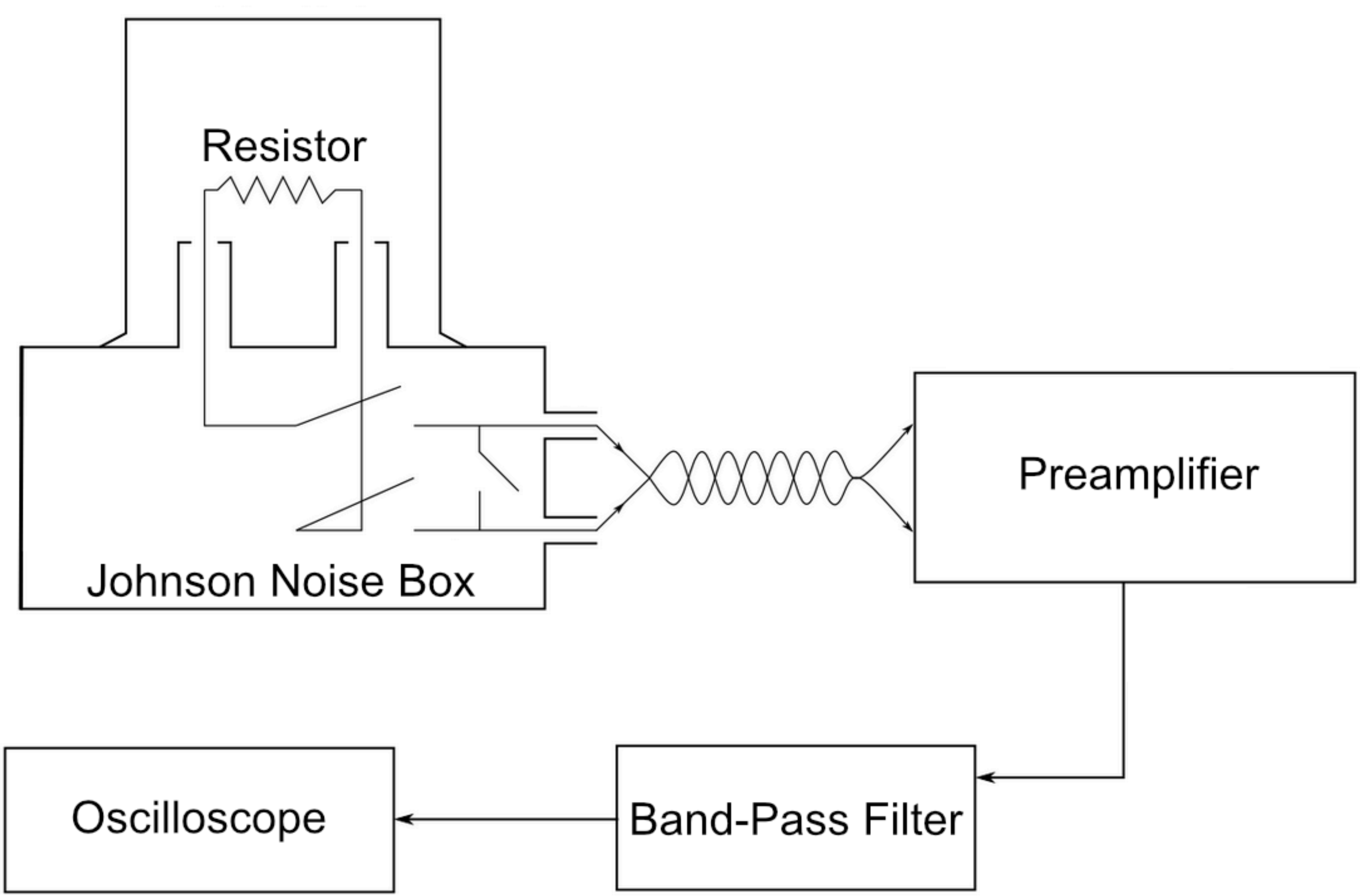
$$V_{noise} = 2\sqrt{kRT}$$

- How do we measure this voltage?

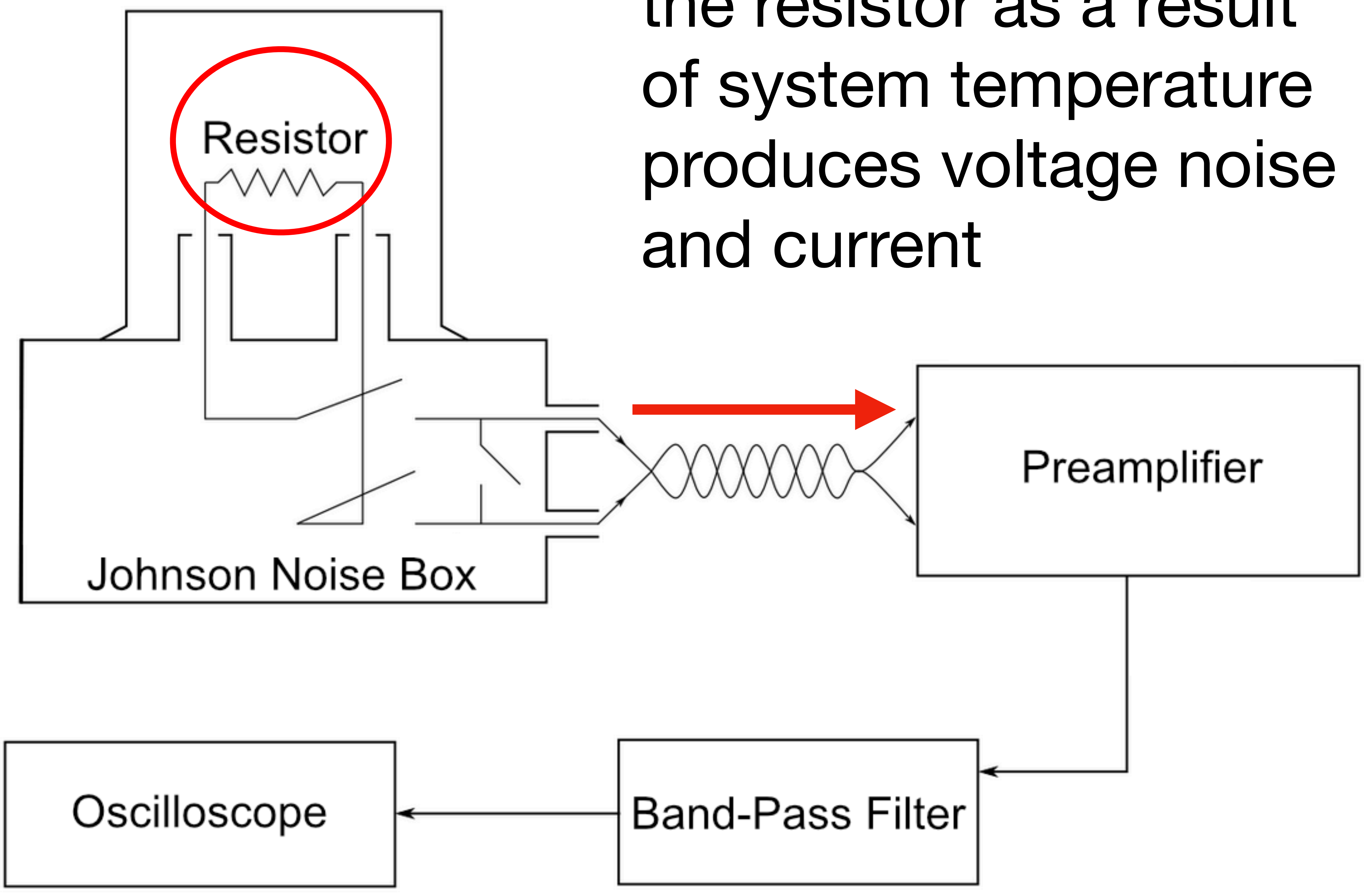
Experimental Setup



Schematic

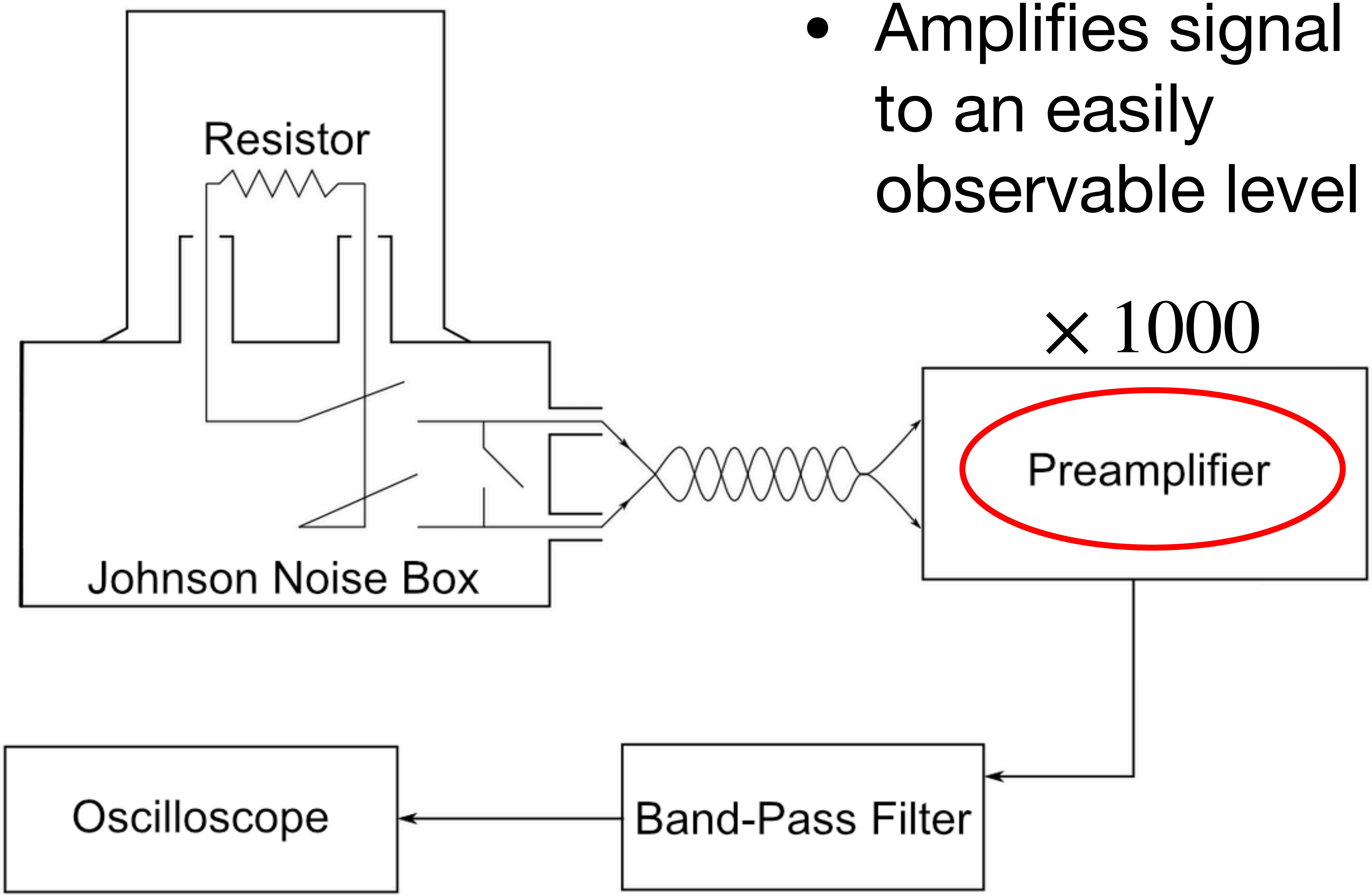

Resistor
Johnson Noise Box
Preamplifier
Band-Pass Filter
Oscilloscope

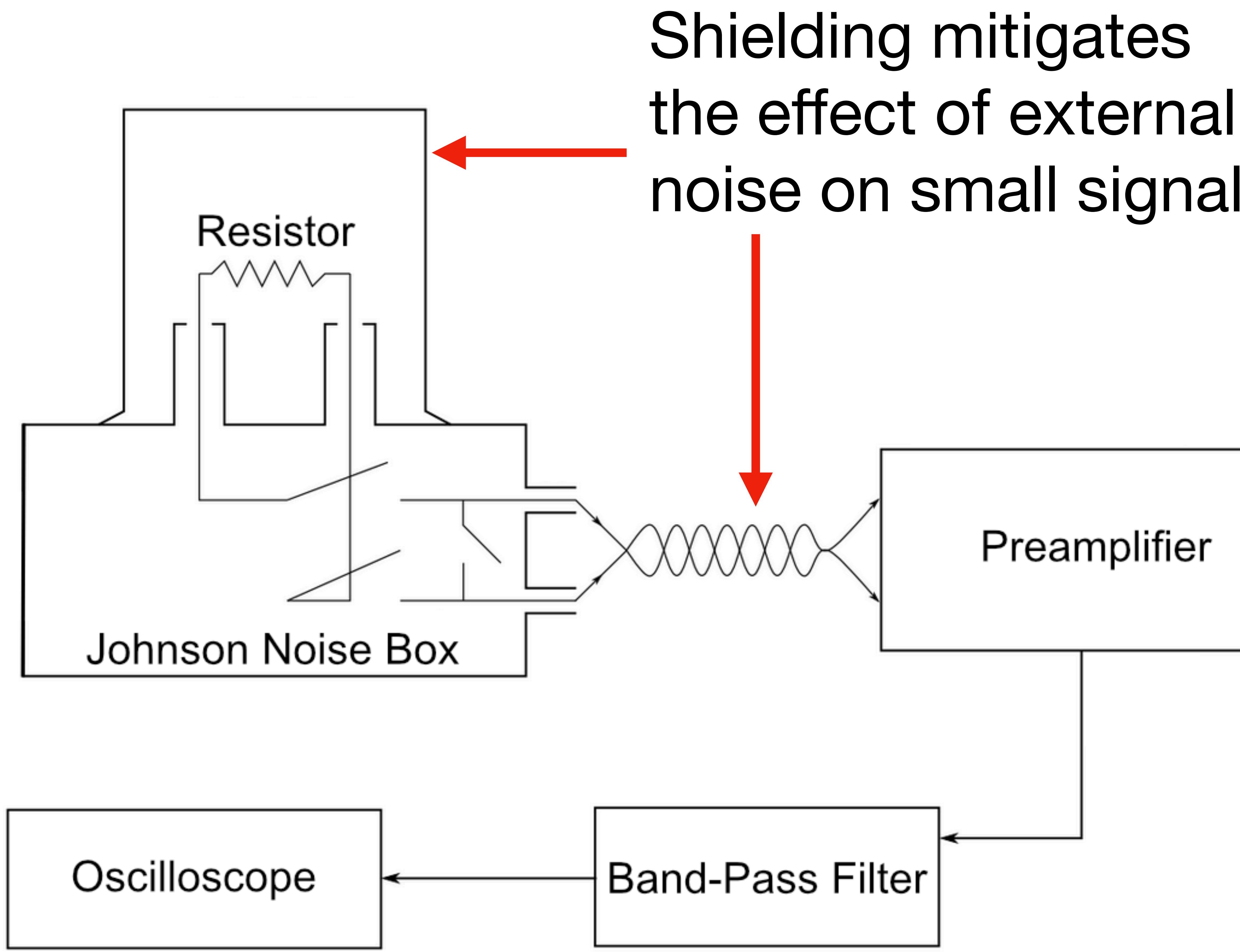
- Thermal excitation of the resistor as a result of system temperature produces voltage noise and current



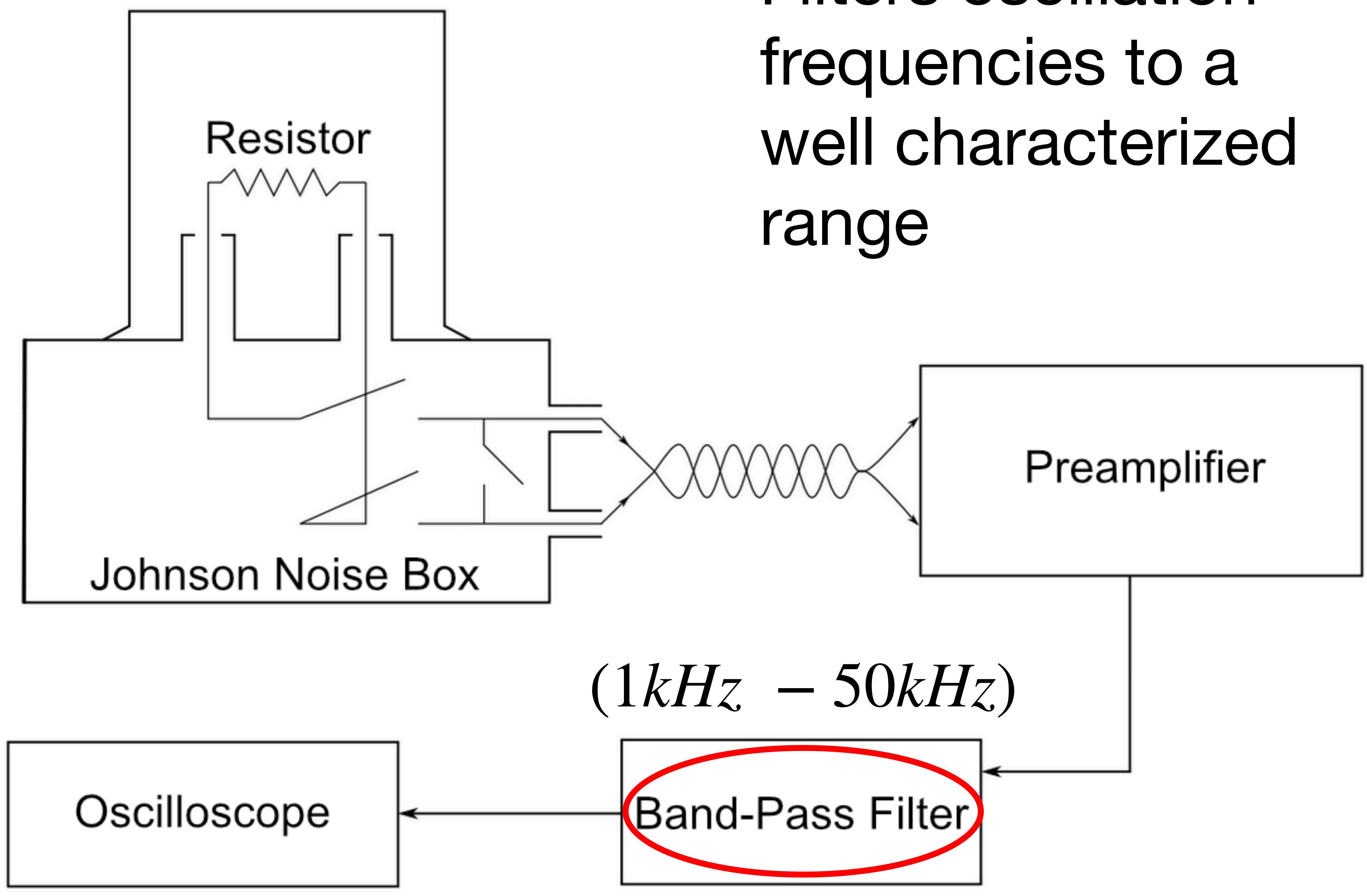
- Amplifies signal to an easily observable level

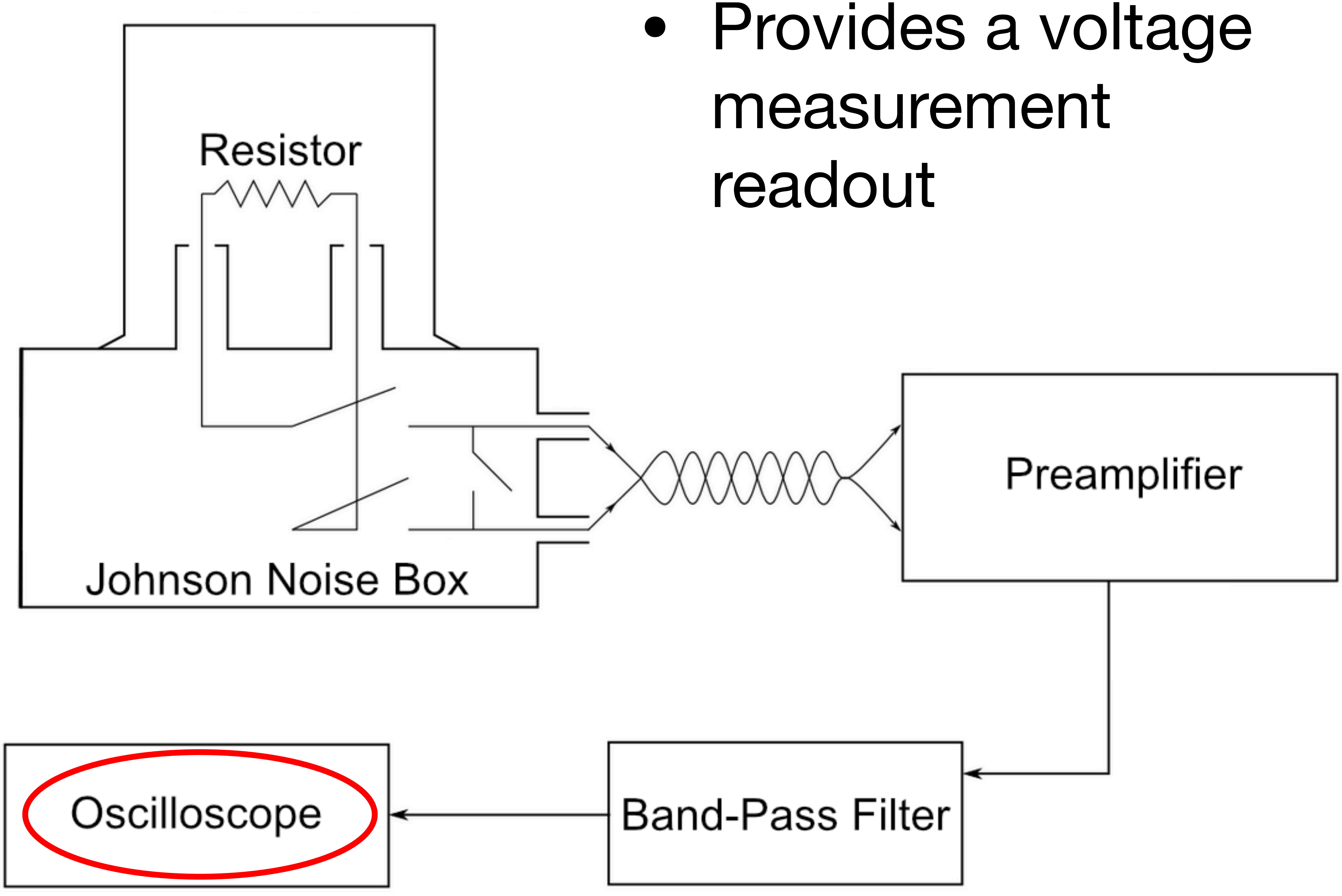
$\times 1000$




Shielding mitigates the effect of external noise on small signal
Resistor
Johnson Noise Box
Preamplifier
Band-Pass Filter
Oscilloscope

- Filters oscillation frequencies to a well characterized range

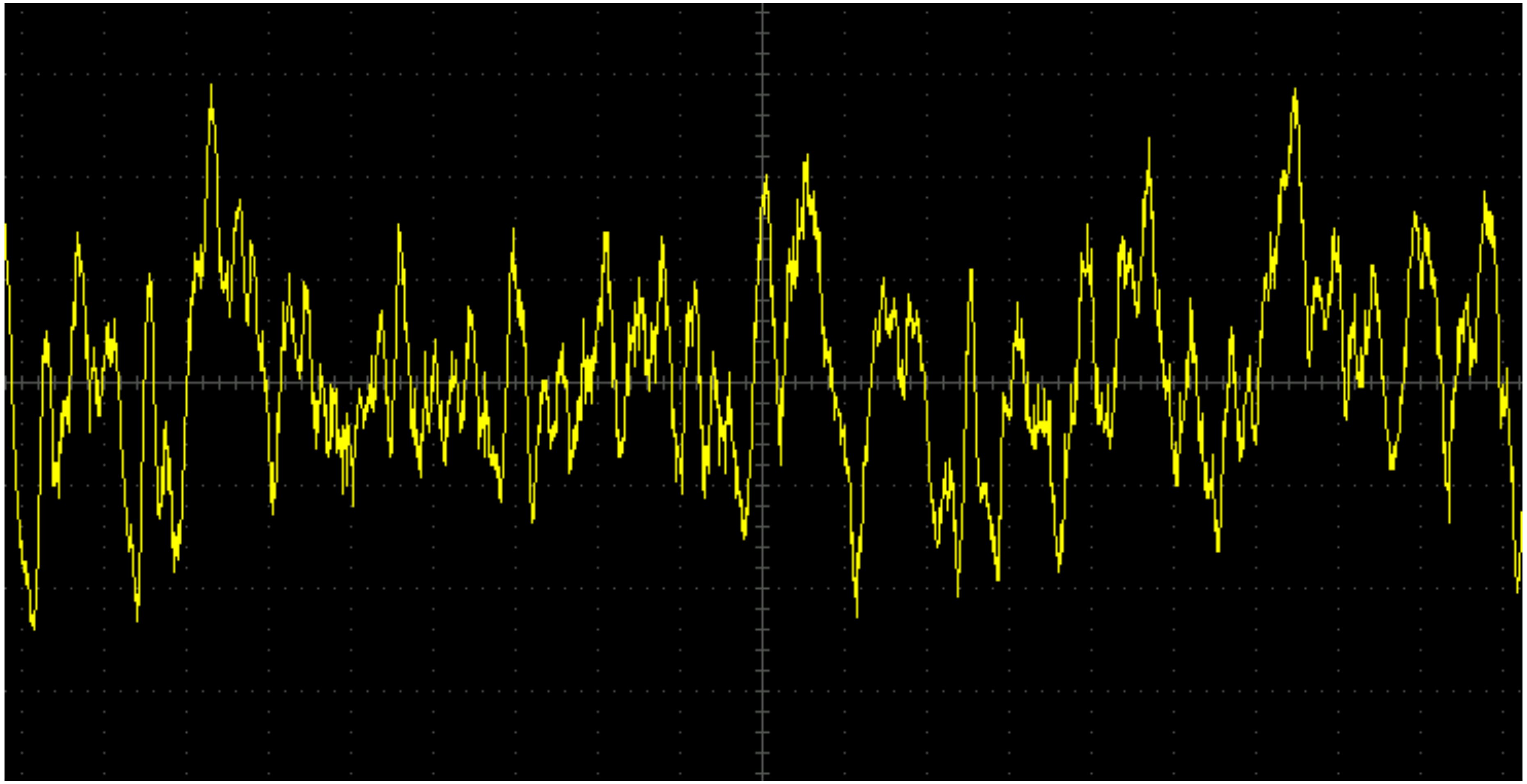




- Provides a voltage measurement readout

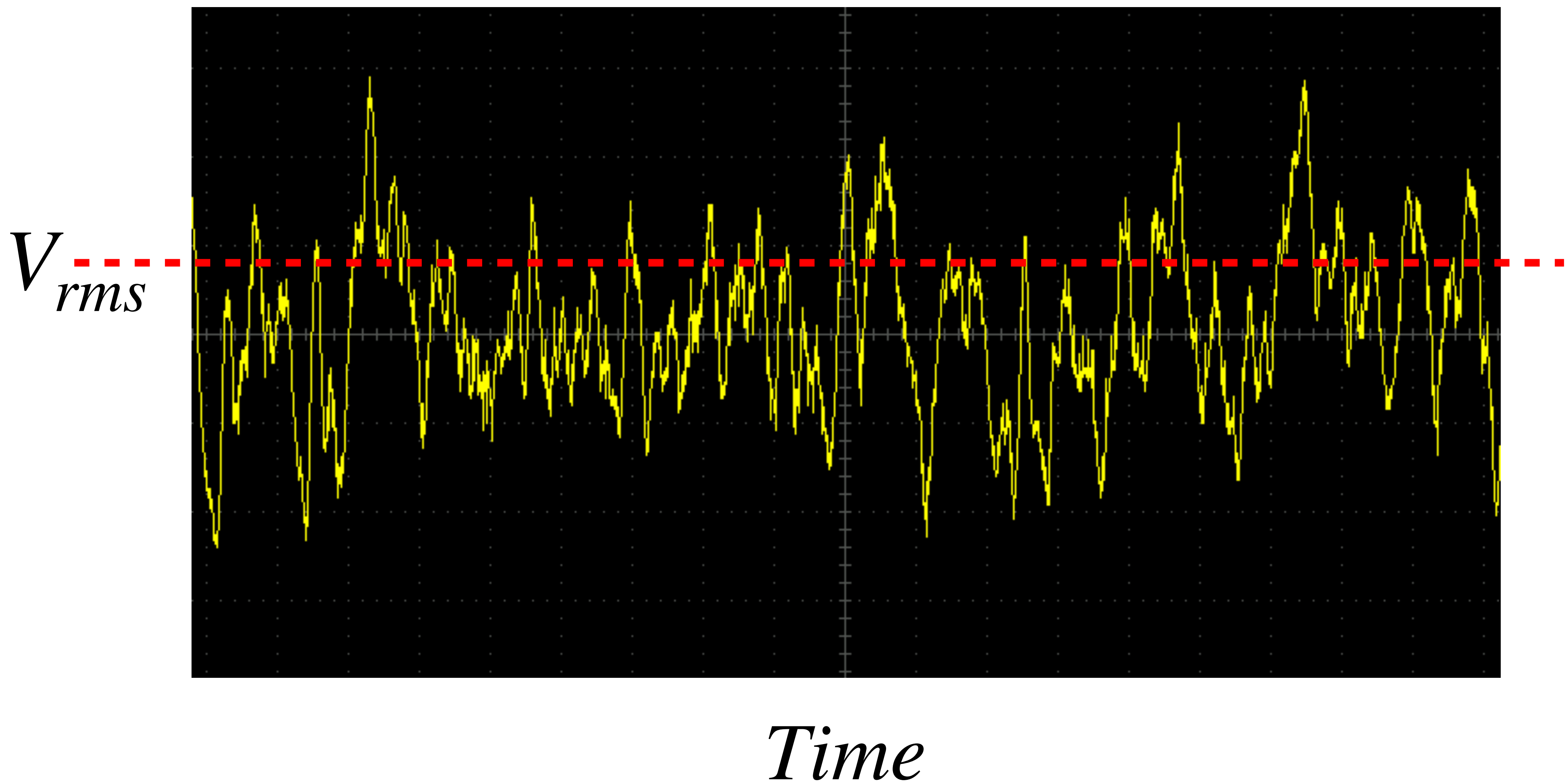
- How do we measure a single voltage for an oscillating noise signal on our oscilloscope?

Voltage



Time

- We measure the root mean square V_{rms} of the signal:



Obtaining Noise from Measurement

- Our setup modifies the noise from the resistor on the Johnson Box
- How do we extract the original noise voltage from our measured voltages?
- Separate amplifier noise
- Determine gain

Amplifier Noise

- Experimental setup contributes electrical noise to the measured value
- How do we isolate the voltage noise contribution from the resistor on the Johnson Box?

- We measure the sum of the noise produced by our resistor and the amplifier V_R

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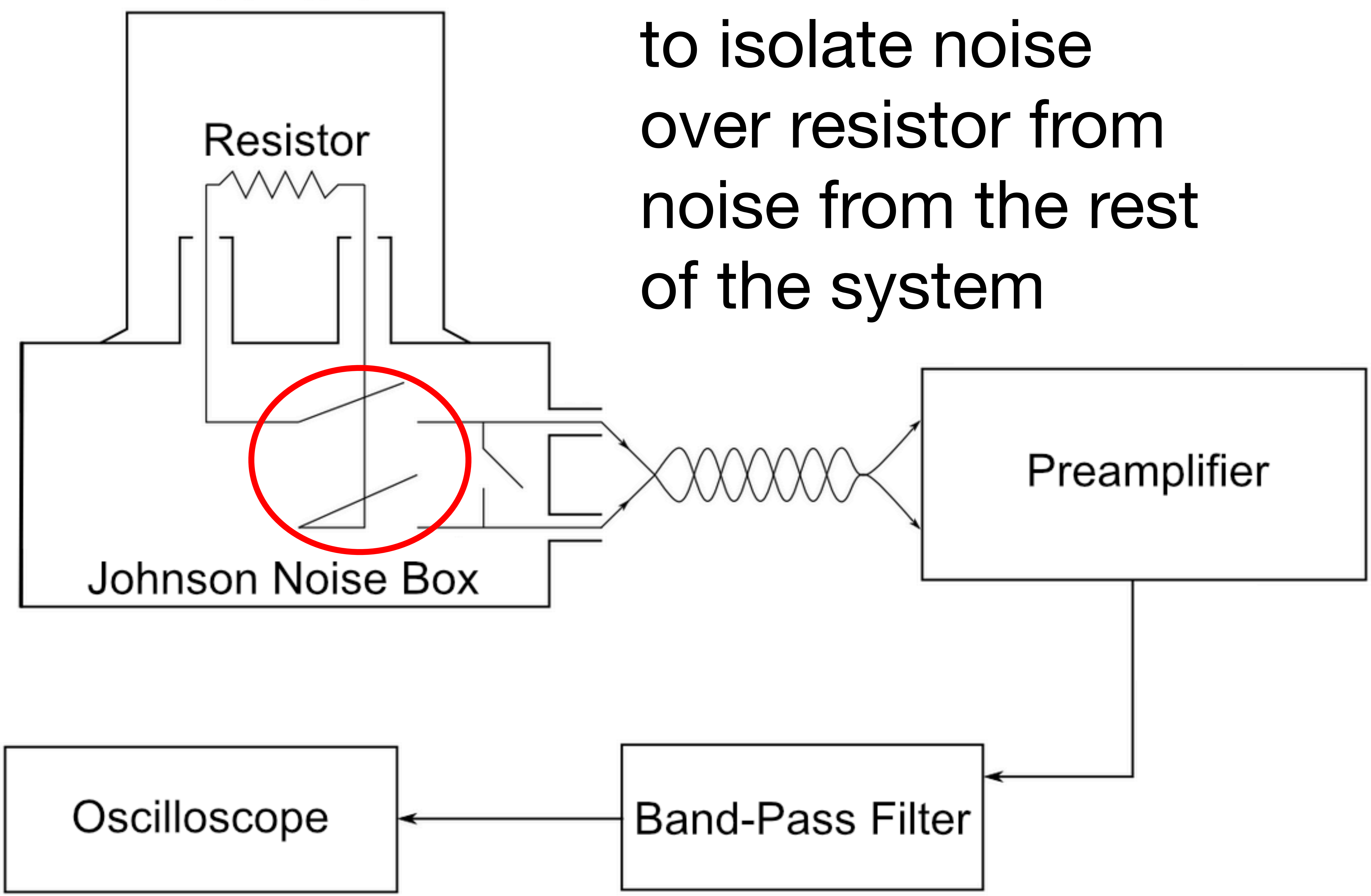
- We measure the noise produced with the resistor shorted out V_S

- Shorting switch mechanism is used to isolate noise over resistor from noise from the rest of the system



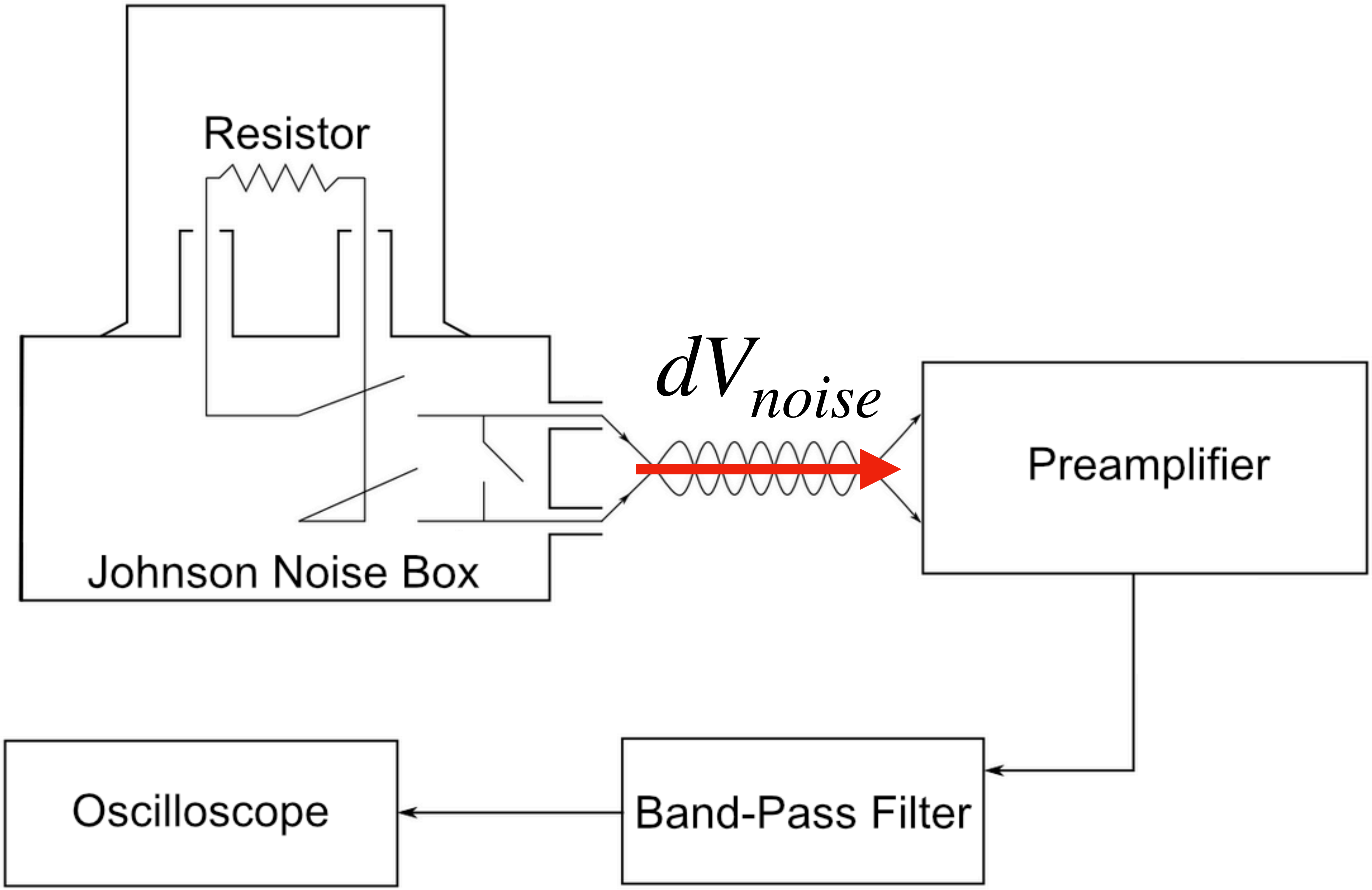
- We measure the sum of the noise produced by our resistor and the amplifier V_R

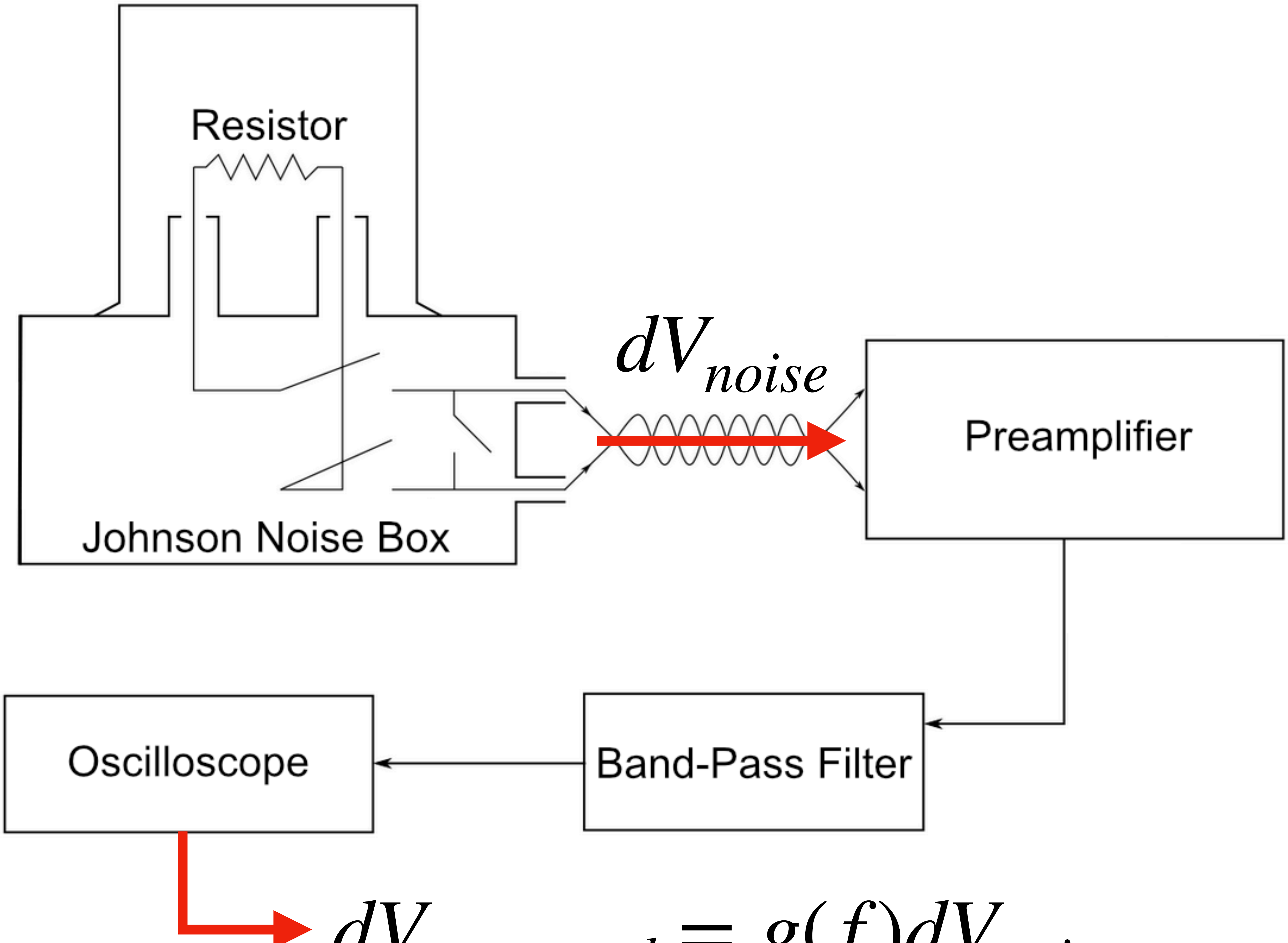
- We measure the noise produced with the resistor shorted out V_S

- We can subtract these independent sources in quadrature to extract the isolated noise of the resistor: $V^2 = V_R^2 - V_S^2$

Gain Calibration

- Amplifier and band-pass filter transform the original average voltage noise produced by a frequency band df by a multiplicative factor


Resistor
dV_{noise}
Preamplifier
Johnson Noise Box
Oscilloscope
Band-Pass Filter

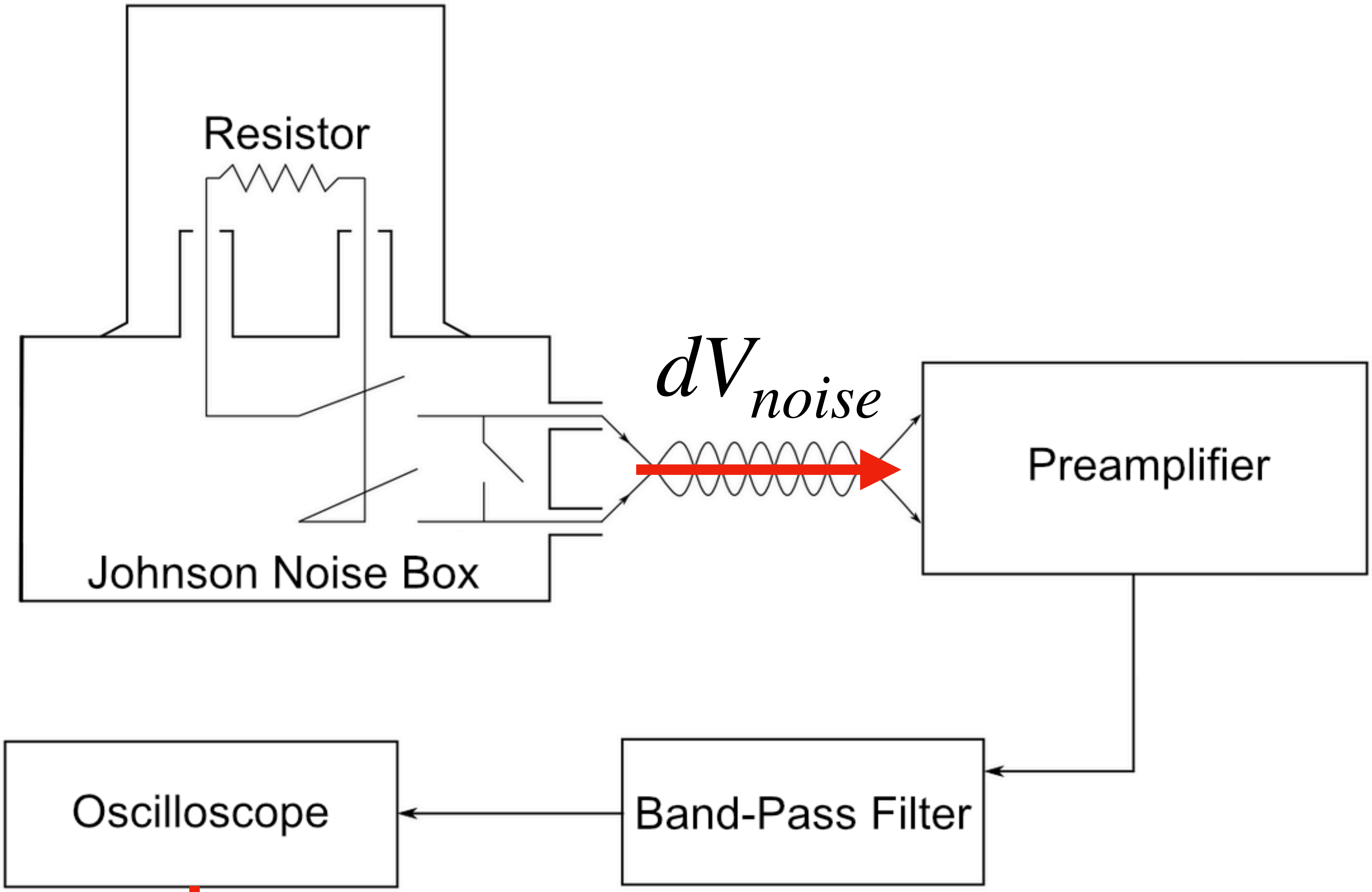


$$dV_{measured} = g(f)dV_{noise}$$

Gain Calibration

- Amplifier and band-pass filter transform the original average voltage noise produced by a frequency band df by a multiplicative factor

- Knowing this factor allows us to extract the raw noise from measured noise



$$dV_{measured} = g(f)dV_{noise}$$

- Plugging in the expression derived for dV_{noise}

$$dV_{measured} = 2\sqrt{kRT}g(f)df$$

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$$dV_{measured} = 2\sqrt{kRT}g(f)df$$

- Integrating over a finite frequency spectrum:

$$V_{measured} = 2\sqrt{kRT}\int_a^b g(f)df$$

- Square our voltage measurement in order to manipulate it in quadrature

$$V_{measured}^2 = 4kRT\int_a^b g^2(f)df$$

- Square our voltage measurement in order to manipulate it in quadrature

$$V^2_{measured} = 4kRT\int_a^b g^2(f)df$$

- Introduce a resistance dependent reducing factor in the gain to model the capacitance of the connecting wires in our setup

$$V^2_{measured} = 4kRT\int_a^b \frac{g^2(f)}{\sqrt{1+(2\pi fRC)^2}}df$$

- So, the effect of amplification and filtering on the raw noise is quantified by defining a gain factor G that differs for each resistor

$$V_{measured}^2 = V_{noise}^2 G$$

- Where:

$$G = \int_a^b \frac{g^2(f)}{1 + (2\pi f RC)^2} df$$

- Factor G relating post and pre amplification noise is then determined for each resistor as follows:

$$G = \int_{0kHz}^{90kHz} \frac{g^2(f)}{1 + (2\pi fRC)^2} df$$

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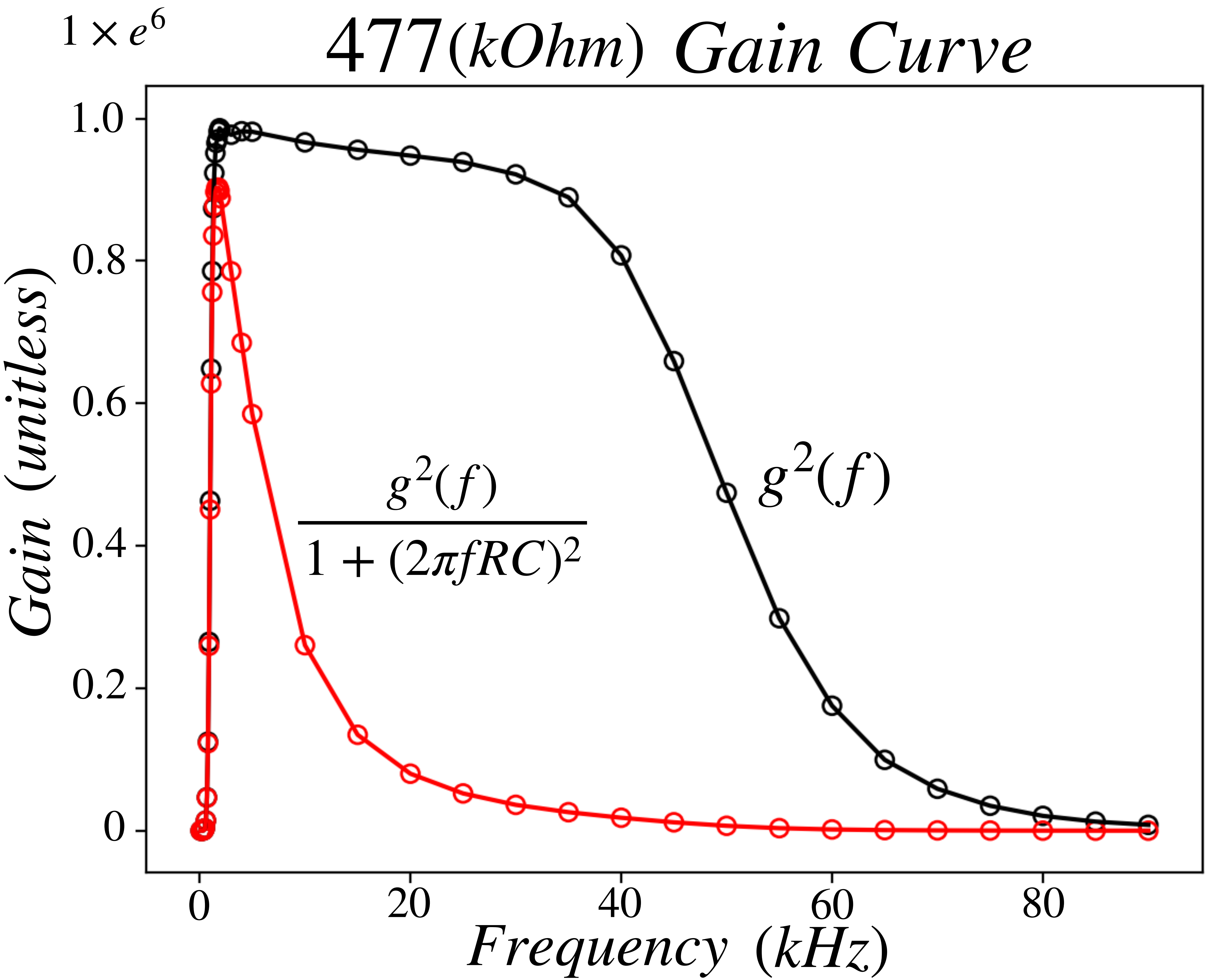
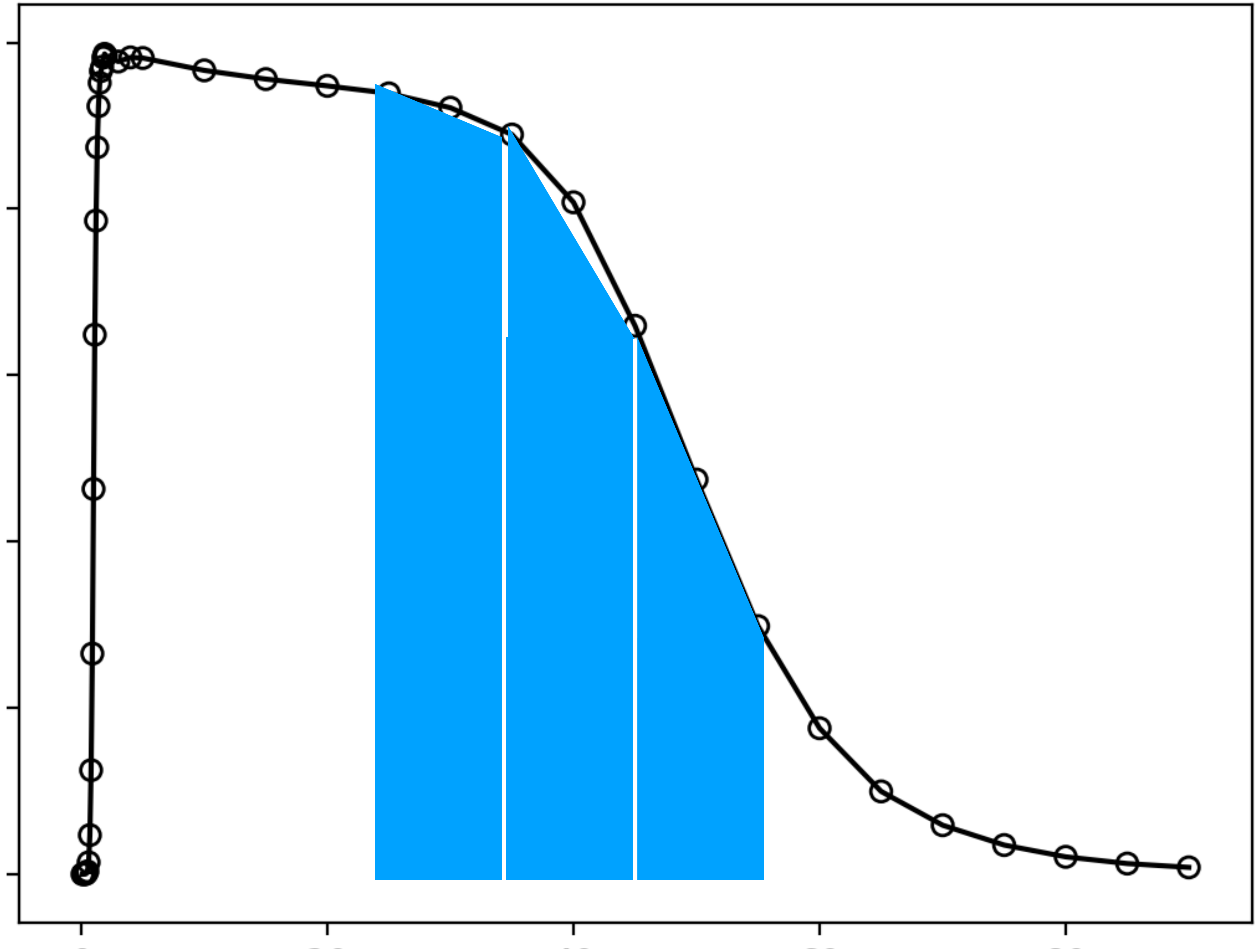
$$G = \int_{0kHz}^{90kHz} \frac{g^2(f)}{1 + (2\pi fRC)^2} df$$

- Value of $g^2(f)$ is determined by measuring the gain of our system without a resistor on the box



$1 \times e^6$
$477(kOhm)$ $Gain\ Curve$
$Gain\ (unitless)$
1.0
0.8
0.6
0.4
0.2
0
$\frac{g^2(f)}{1+(2\pi fRC)^2}$
$g^2(f)$
0
20
40
60
80
$Frequency\ (kHz)$

- Gain integral is calculated using numerical integration methods like the trapezoid rule:



Gain Integral Uncertainty

- Depends on the method of integration used
- We use the Trapezoid rule and Simpson's rule for integration, and compare:

$$\sigma_{G,method} = \frac{G_{trapezoid} - G_{simpson}}{G_{trapezoid}} \approx \pm 0.6\,\%$$

Gain Integral Uncertainty

- Depends on capacitance of the setup
- Parallel wires can produce a capacitance
- We measure two sets of resistor varying data, changing the wiring setup in between

- Taking the ratio of calculated gains for each setup gives an estimate of capacitance fluctuation:

$$\frac{G_{setup1}}{G_{setup2}} \approx \pm 9.0\,\%$$

$$\Big\downarrow$$

$$\sigma_{G,capacitance} \approx \pm 7.5\,\%$$

Noise Measurement Procedure

1) Resistance is measured for resistors in a range of 10 kohm to 1000 kohm

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1) Resistance is measured for resistors in a range of 10 kohm to 1000 kohm

2) RMS Voltages are measured for each resistor both in circuit and shorted out of the circuit

3) Measured voltages subtracted in quadrature are related to pre-amplification noise voltage by calculating G for each resistor

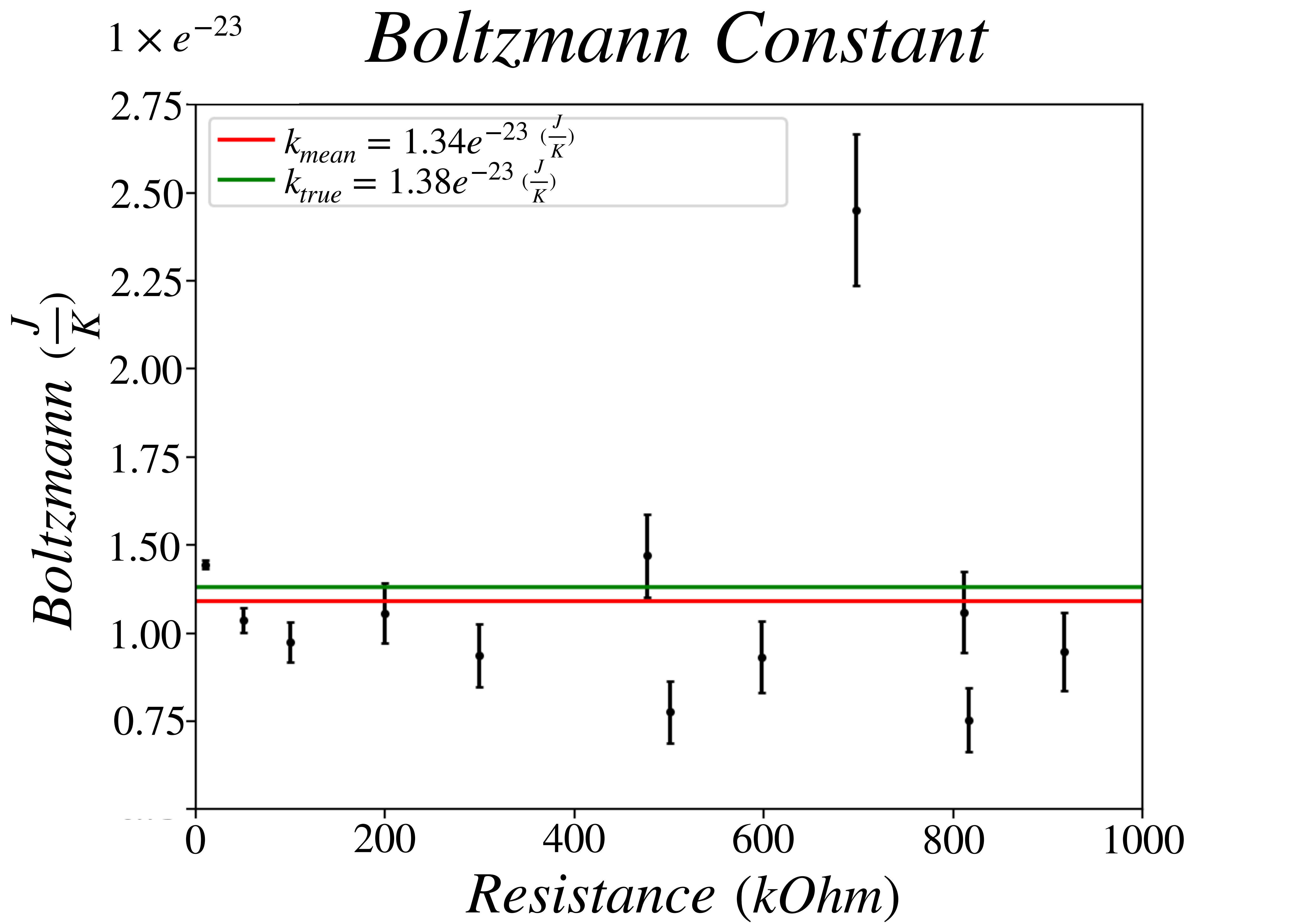
Boltzmann Constant

- Using the derived relationship between measured and original noise for each resistor:

$$V^2 = 4kRTG$$

- Boltzmann Constant k is extracted for each resistor value:

$$k = \frac{V^2}{4TGR}$$



Boltzmann Constant
$1 \times e^{-23}$
$k_{mean} = 1.34e^{-23}$ $(\frac{J}{K})$
$k_{true} = 1.38e^{-23}$ $(\frac{J}{K})$
Boltzmann $(\frac{J}{K})$
2.75
2.50
2.25
2.00
1.75
1.50
1.00
0.75
0
200
400
600
800
1000
Resistance (kOhm)

Uncertainties

- Resistor resistance: σ_R
- Capacitance: $\sigma_{G,capacitance}$
- Gain integral: σ_G
- Squared voltage: σ_{V^2}

Resistance

- Limited by the accuracy of the our multimeter (Hewlett Packard 972A):

Resistance

<table>
<tr><th>Range</th><th>Resolution</th><th>971A</th><th>972A</th><th>973A</th><th>Test Current</th><th>Test Voltage</th></tr>
<tr><td>400 Ω</td><td>100 mΩ</td><td rowspan="5">0.5% + 1</td><td colspan="2" rowspan="4">0.2% + 1</td><td>.8mA</td><td>< 3.2 V</td></tr>
<tr><td>4 kΩ</td><td>1 Ω</td><td><80 μA</td><td rowspan="5"><1.1V</td></tr>
<tr><td>40 kΩ</td><td>10 Ω</td><td><10 μA</td></tr>
<tr><td>400 kΩ</td><td>100 Ω</td><td><1.1 μA</td></tr>
<tr><td>4 MΩ</td><td>1 kΩ</td><td colspan="2">0.5% + 1</td><td rowspan="2"><110 nA</td></tr>
<tr><td>40 MΩ</td><td>10 kΩ</td><td colspan="2">1% + 1</td><td></td></tr>
</table>

Resistance

- Applying this to each resistance based on which range it falls within:

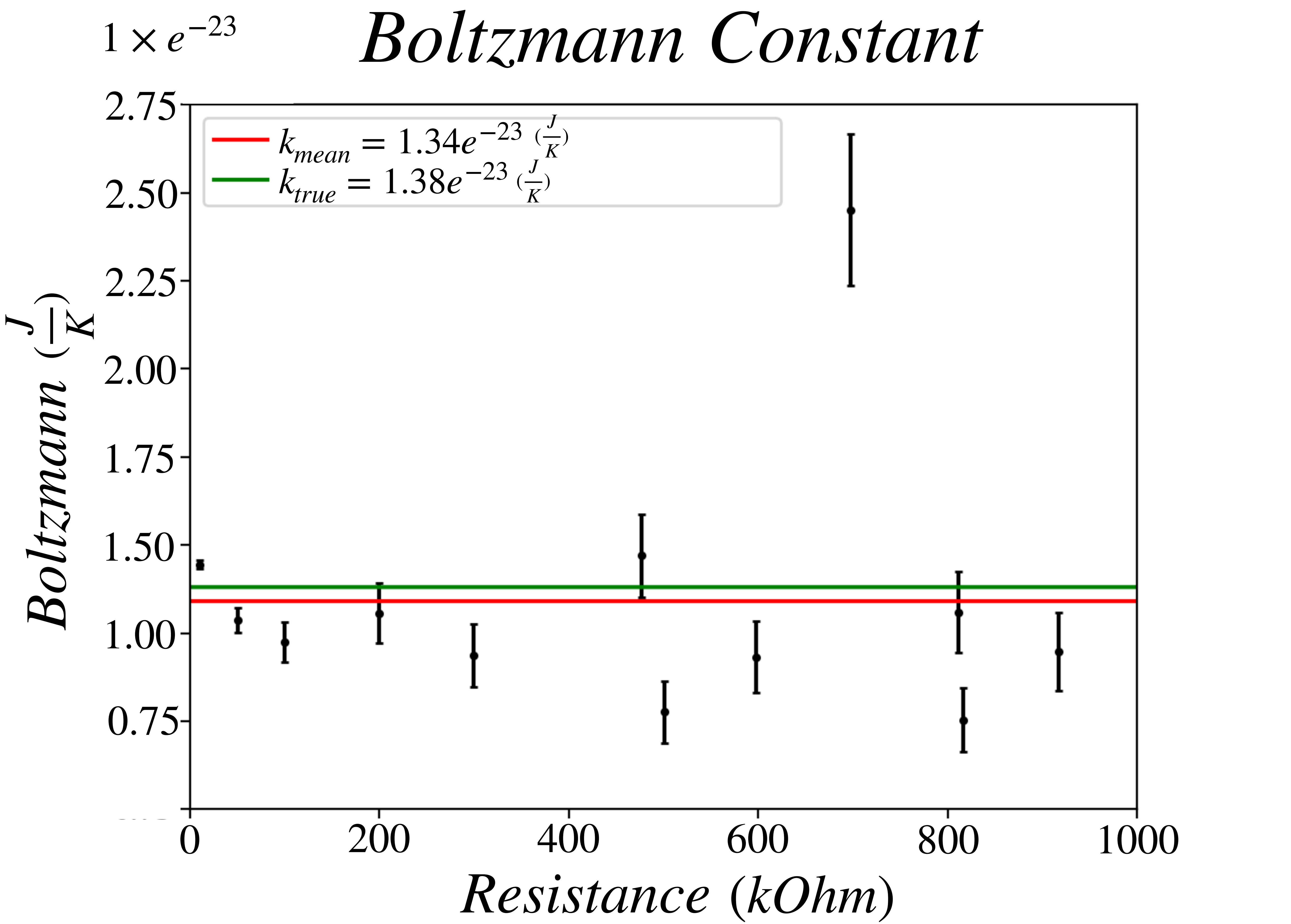
$$\sigma_R \approx \pm 0.4\,\%$$

Gain Integral Uncertainty

- Combining the numerical integration method and capacitance contributions:

$$\sigma_G \approx \pm 7.0\,\%$$

- Uncertainty in capacitance dominates as resistor values get higher


Boltzmann Constant
1 × e^−23
k_mean = 1.34e^−23 (J/K)
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Boltzmann (J/K)
2.75
2.50
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200
400
600
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1000
Resistance (kOhm)

Squared Voltage

- Squared voltage is resistor dependent and is given by:

$$V^2 = V_R^2 - V_S^2$$

- Squared voltage is given by:

$$\sigma_{V^2} = \sqrt{var(V_R^2) + var(V_S^2)} \approx \pm 6.5e^{-7}\,(volts)$$

Boltzmann Error Propagation

- Propagating these sources of error in quadrature through the equation:

$$k = \frac{V^2}{4TGR}$$

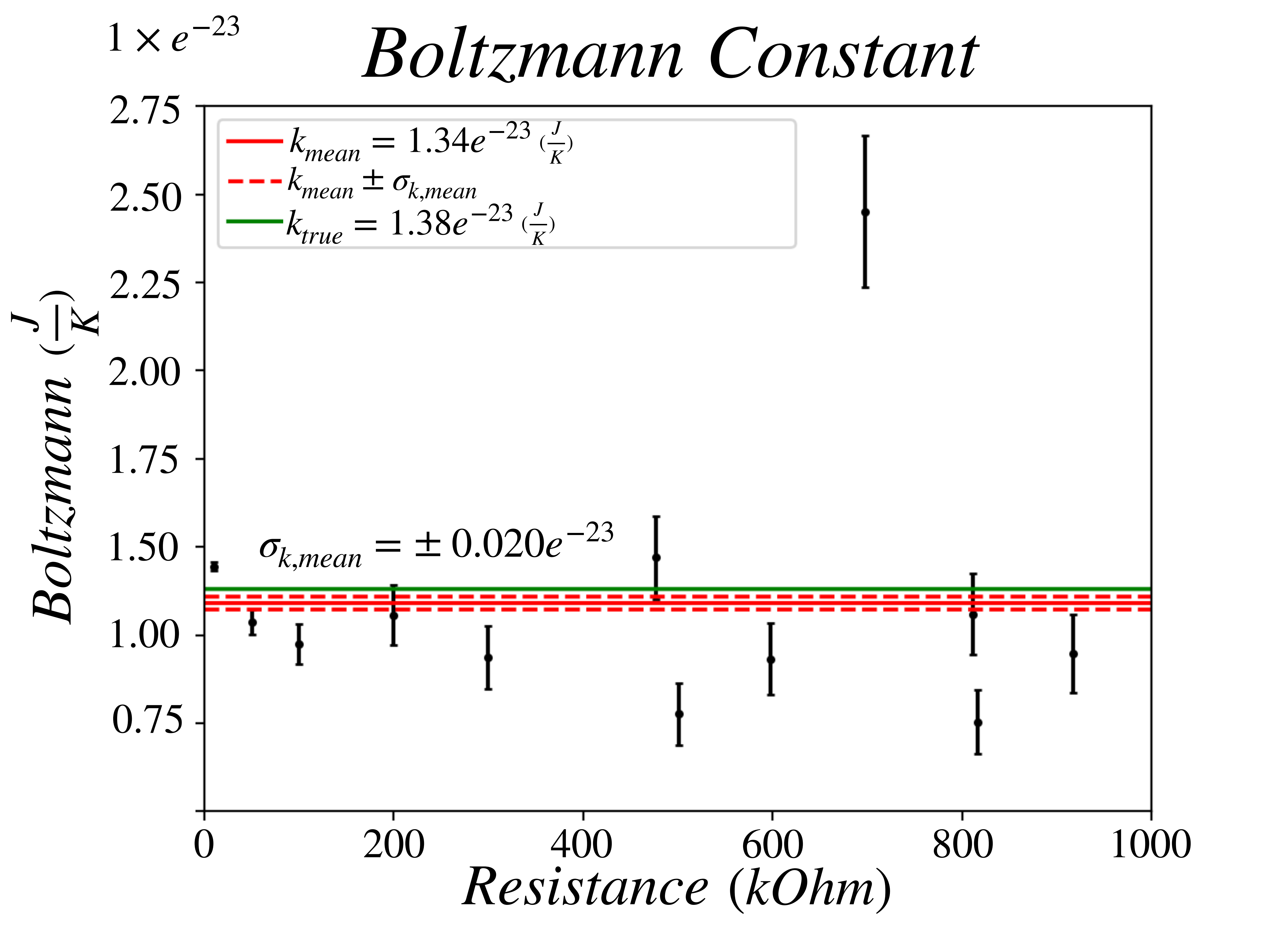
- We obtain values for each resistor:

$$\sigma_k \approx \pm 0.10e^{-23} \left(\frac{J}{K}\right)$$

Uncertainty in Average k

- We use a Monte Carlo simulation to raffle k values within a distribution of width $2\sigma_k$
- We find the average k value for each set of points
- The standard deviation of the raffled average-measured average difference is the uncertainty on our average:

$$\sigma_{k,mean} = \pm 0.018e^{-23} \ (\frac{J}{K})$$



Boltzmann Constant
$1 \times e^{-23}$
$k_{mean} = 1.34e^{-23}$ $(\frac{J}{K})$
$k_{mean} \pm \sigma_{k,mean}$
$k_{true} = 1.38e^{-23}$ $(\frac{J}{K})$
$\sigma_{k,mean} = \pm 0.020e^{-23}$
Boltzmann $(\frac{J}{K})$
Resistance (kOhm)
2.75
2.50
2.25
2.00
1.75
1.50
1.00
0.75
0
200
400
600
800
1000

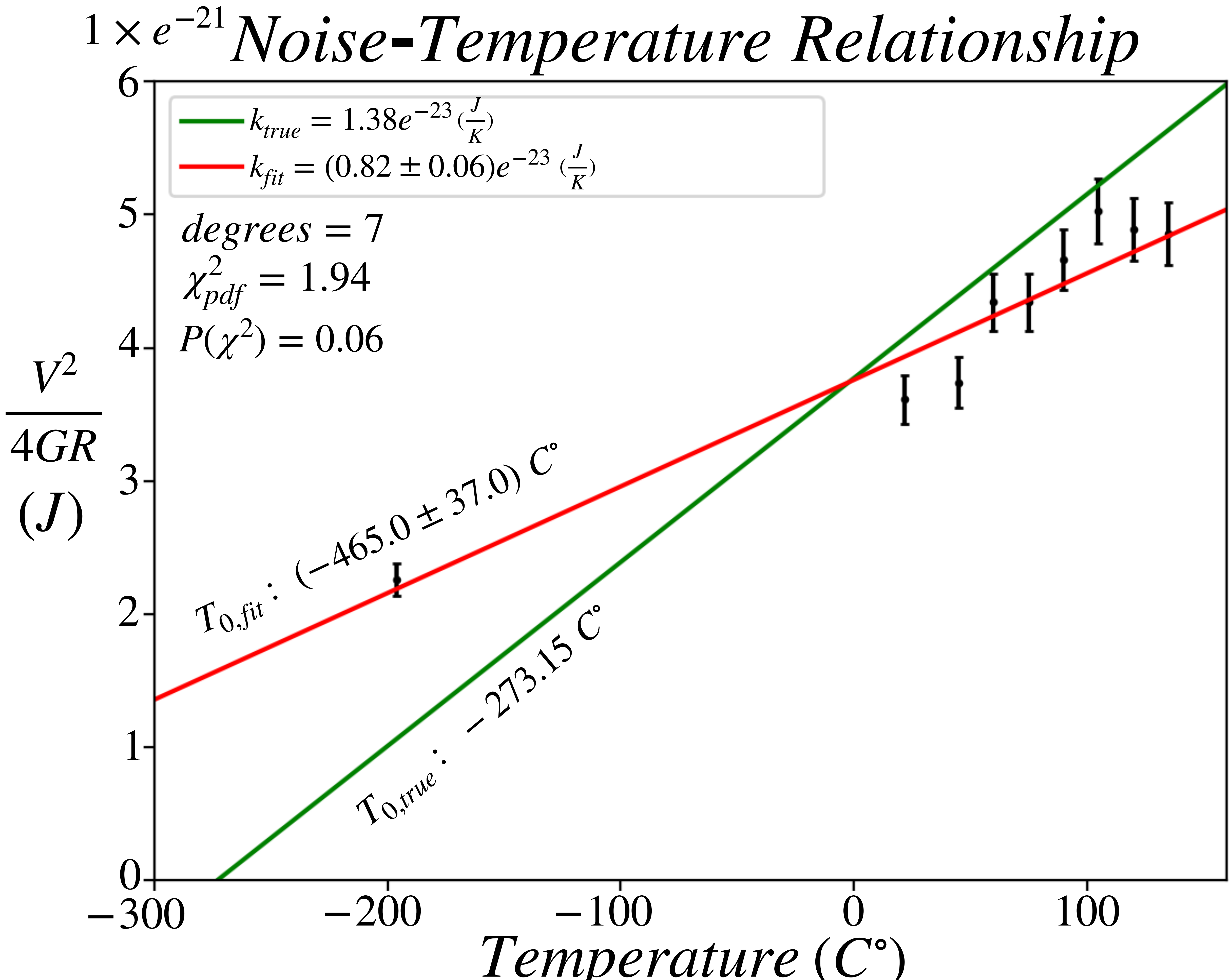
Absolute Zero

- Centigrade temperature of absolute zero can be extracted:

- For a single 100 (kOhm) resistor, noise is measured over a range of temperatures

- We then plot:

$$\frac{V^2}{4GR} \; vs. \; T$$



$1\times e^{-21}$ Noise-Temperature Relationship
$k_{true} = 1.38e^{-23}\,(\frac{J}{K})$
$k_{fit} = (0.82 \pm 0.06)e^{-23}\,(\frac{J}{K})$
$degrees = 7$
$\chi^2_{pdf} = 1.94$
$P(\chi^2) = 0.06$
$\frac{V^2}{4GR}$ (J)
$T_{0,fit}:\ (-465.0 \pm 37.0)\ C^\circ$
$T_{0,true}:\ -273.15\ C^\circ$
0
1
2
3
4
5
6
−300
−200
−100
0
100
Temperature (C°)

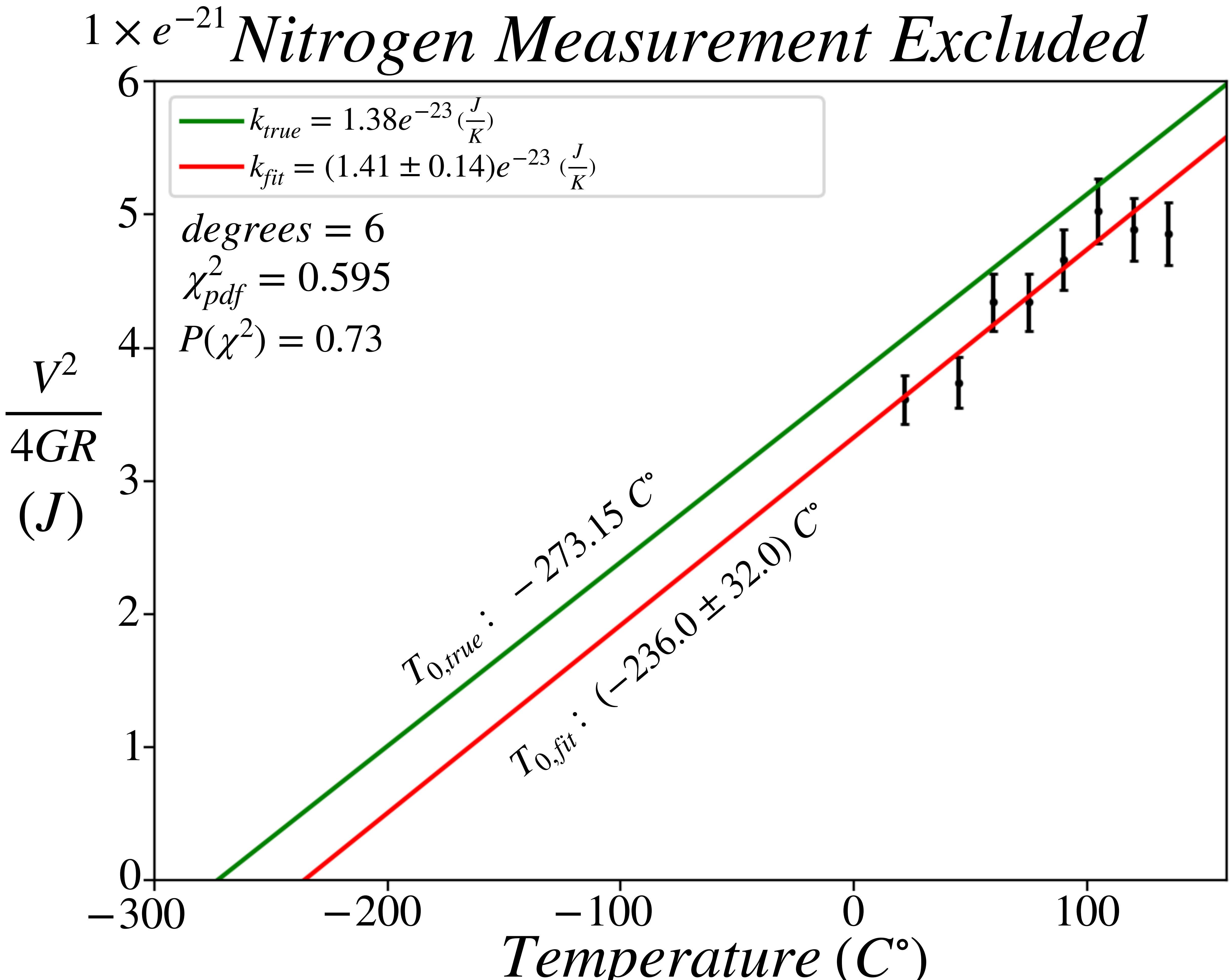
Liquid Nitrogen Data Point

- Follows a slightly different linear relationship than other points

- Seems suspicious - what steps were taken to mitigate systematic uncertainty?

Liquid Nitrogen Data Point

- Resistor was immediately immersed in bath
- Resistor was allowed significant time to come to equilibrium with bath temperature
- Glass container used to mitigate capacitance effect from box-container contact
- 40 V_R and V_S measurements were taken as opposed to 10 for all other temperatures
- Performed the measurement twice - yielded very similar results



$1 \times e^{-21}$ Nitrogen Measurement Excluded
$k_{true} = 1.38e^{-23} (\frac{J}{K})$
$k_{fit} = (1.41 \pm 0.14)e^{-23} (\frac{J}{K})$
$degrees = 6$
$\chi^2_{pdf} = 0.595$
$P(\chi^2) = 0.73$
$T_{0,true}: \; -273.15\ C°$
$T_{0,fit}: \; (-236.0 \pm 32.0)\ C°$
$\frac{V^2}{4GR}$ (J)
Temperature $(C°)$
6
5
4
3
2
1
0
−300
−200
−100
0
100

Absolute Zero Uncertainty

- Using the uncertainty in the linear fit given by the covariance matrix returned, we calculate:

With Nitrogen: $\sigma_{T_0} \approx \pm 37.0\ C^{\circ}$

Without Nitrogen: $\sigma_{T_0} \approx \pm 32.0\ C^{\circ}$

Summary

- From our Johnson noise measurement, we extract the following values for our Boltzmann constant:

$$k_{extracted} = (1.34\ \pm\ 0.020)e^{-23}\ \frac{J}{K}$$

$$k_{accepted} = 1.38e^{-23}\ \frac{J}{K}$$

Summary

- We extract the following values for absolute zero:

$$\text{With Nitrogen: } \begin{cases} \text{Extracted: } (-465.0 \pm 37.0)\ C^\circ \\ \text{Accepted: } -273.15\ C^\circ \end{cases}$$

$$\text{Without Nitrogen: } \begin{cases} \text{Extracted: } (-236.0 \pm 32.0)\ C^\circ \\ \text{Accepted: } -273.15\ C^\circ \end{cases}$$

Applications

- Johnson noise is significant when dealing with high precision electronics application using radio frequency technology

- Cellphones, wifi, radio, and much more

Thank you!