UNDERSTANDING PROGRAM EFFICIENCY: 1

(download slides and .py files and follow along!)

6.0001 LECTURE 10

Today

- Measuring orders of growth of algorithms
- Big "Oh" notation
- Complexity classes

WANT TO UNDERSTAND EFFICIENCY OF PROGRAMS

- computers are fast and getting faster – so maybe efficient programs don't matter?
  - but data sets can be very large (e.g., in 2014, Google served 30,000,000,000,000 pages, covering 100,000,000 GB – how long to search brute force?)
  - thus, simple solutions may simply not scale with size in acceptable manner
- how can we decide which option for program is most efficient?

- separate **time and space efficiency** of a program
- tradeoff between them:
  - can sometimes pre-compute results are stored; then use "lookup" to retrieve (e.g., memoization for Fibonacci)
  - will focus on time efficiency

WANT TO UNDERSTAND EFFICIENCY OF PROGRAMS

Challenges in understanding efficiency of solution to a computational problem:

- a program can be **implemented in many different ways**
- you can solve a problem using only a handful of different **algorithms**
- would like to separate choices of implementation from choices of more abstract algorithm

HOW TO EVALUATE EFFICIENCY OF PROGRAMS

- measure with a **timer**
- **count** the operations
- abstract notion of **order of growth**

will argue that this is the most appropriate way of assessing the impact of choices of algorithm in solving a problem; and in measuring the inherent difficulty in solving a problem

TIMING A PROGRAM

- use time module
- recall that importing means to bring in that class into your own file
- **start** clock
- **call** function
- **stop** clock

```
import time

def c_to_f(c):
    return c*9/5 + 32

t0 = time.clock()
c_to_f(100000)
t1 = time.clock() - t0
Print("t =", t, ":", t1, "s,")
```

TIMING PROGRAMS IS INCONSISTENT

- GOAL: to evaluate different algorithms
- running time **varies between algorithms** ✓
- running time **varies between implementations** ✗
- running time **varies between computers** ✗
- running time is **not predictable** based on small inputs ✗

- time varies for different inputs but cannot really express a relationship between inputs and time ✗

COUNTING OPERATIONS

- assume these steps take **constant time**:
  - mathematical operations
  - comparisons
  - assignments
  - accessing objects in memory
- then count the number of operations executed as function of size of input

```
def c_to_f(c):
    return c*9.0/5 + 32
```

3 ops

```
def mysum(x):
    total = 0
    for i in range(x+1):
        total += i
    return total
```

1 op

loop x times

1 op

2 ops

mysum → 1+3x ops

COUNTING OPERATIONS IS BETTER, BUT STILL…

- GOAL: to evaluate different algorithms
- count **depends on algorithm** ✓
- count **depends on implementations** ✗
- count **independent of computers** ✓
- no clear definition of **which operations** to count ✗

- count varies for different inputs and can come up with a relationship between inputs and the count ✓

STILL NEED A BETTER WAY

- timing and counting **evaluate implementations**
- timing **evaluates machines**

- want to **evaluate algorithm**
- want to **evaluate scalability**
- want to **evaluate in terms of input size**

STILL NEED A BETTER WAY

- Going to focus on idea of counting operations in an algorithm, but not worry about small variations in implementation (e.g., whether we take 3 or 4 primitive operations to execute the steps of a loop)
- Going to focus on how algorithm performs when size of problem gets arbitrarily large
- Want to relate time needed to complete a computation, measured this way, against the size of the input to the problem
- Need to decide what to measure, given that actual number of steps may depend on specifics of trial

NEED TO CHOOSE WHICH INPUT TO USE TO EVALUATE A FUNCTION

- want to express **efficiency in terms of size of input**, so need to decide what your input is
- could be an **integer**
  -- `mysum(x)`
- could be **length of list**
  -- `list_sum(L)`
- **you decide** when multiple parameters to a function
  -- `search_for_elmt(L, e)`

DIFFERENT INPUTS CHANGE HOW THE PROGRAM RUNS

- a function that searches for an element in a list

```
def search_for_elmt(L, e):
    for i in L:
        if i == e:
            return True
    return False
```

- when `e` is **first element** in the list → BEST CASE
- when `e` is **not in list** → WORST CASE
- when **look through about half** of the elements in list → AVERAGE CASE
- want to measure this behavior in a general way

BEST, AVERAGE, WORST CASES

- suppose you are given a list `L` of some length `len(L)`
- **best case**: minimum running time over all possible inputs of a given size, `len(L)`
  - constant for `search_for_elmt`
  - first element in any list
- **average case**: average running time over all possible inputs of a given size, `len(L)`
  - practical measure
- **worst case**: maximum running time over all possible inputs of a given size, `len(L)`
  - linear in length of list for `search_for_elmt`
  - must search entire list and not find it

generally will focus on this case

ORDERS OF GROWTH

Goals:

- want to evaluate program's efficiency when **input is very big**
- want to express the **growth of program's run time** as input size grows
- want to put an **upper bound** on growth – as tight as possible
- do not need to be precise: **"order of" not "exact"** growth
- we will look at **largest factors** in run time (which section of the program will take the longest to run?)
- **thus, generally we want tight upper bound on growth, as function of size of input, in worst case**

MEASURING ORDER OF GROWTH: BIG OH NOTATION

- Big Oh notation measures an **upper bound on the asymptotic growth**, often called order of growth

- **Big Oh or *O()*** is used to describe worst case
  - worst case occurs often and is the bottleneck when a program runs
  - express rate of growth of program relative to the input size
  - evaluate algorithm **NOT** machine or implementation

EXACT STEPS vs O()

```
def fact_iter(n):
    """assumes n an int >= 0"""
    answer = 1
    while n > 1:
        answer *= n
        n -= 1
    return answer
```

answer = answer * n

temp = n-1
n = temp

- computes factorial
- number of steps: $1 + 5n + 1$
- worst case asymptotic complexity: $O(n)$
  - ignore additive constants
  - ignore multiplicative constants

WHAT DOES *O(N)* MEASURE?

- Interested in describing how amount of time needed grows as size of (input to) problem grows
- Thus, given an expression for the number of operations needed to compute an algorithm, want to know asymptotic behavior as size of problem gets large
- Hence, will focus on term that grows most rapidly in a sum of terms
- And will ignore multiplicative constants, since want to know how rapidly time required increases as increase size of input

SIMPLIFICATION EXAMPLES

- drop constants and multiplicative factors
- focus on **dominant terms**

$O(n^2)$: `n`^{2} `+ 2n + 2`

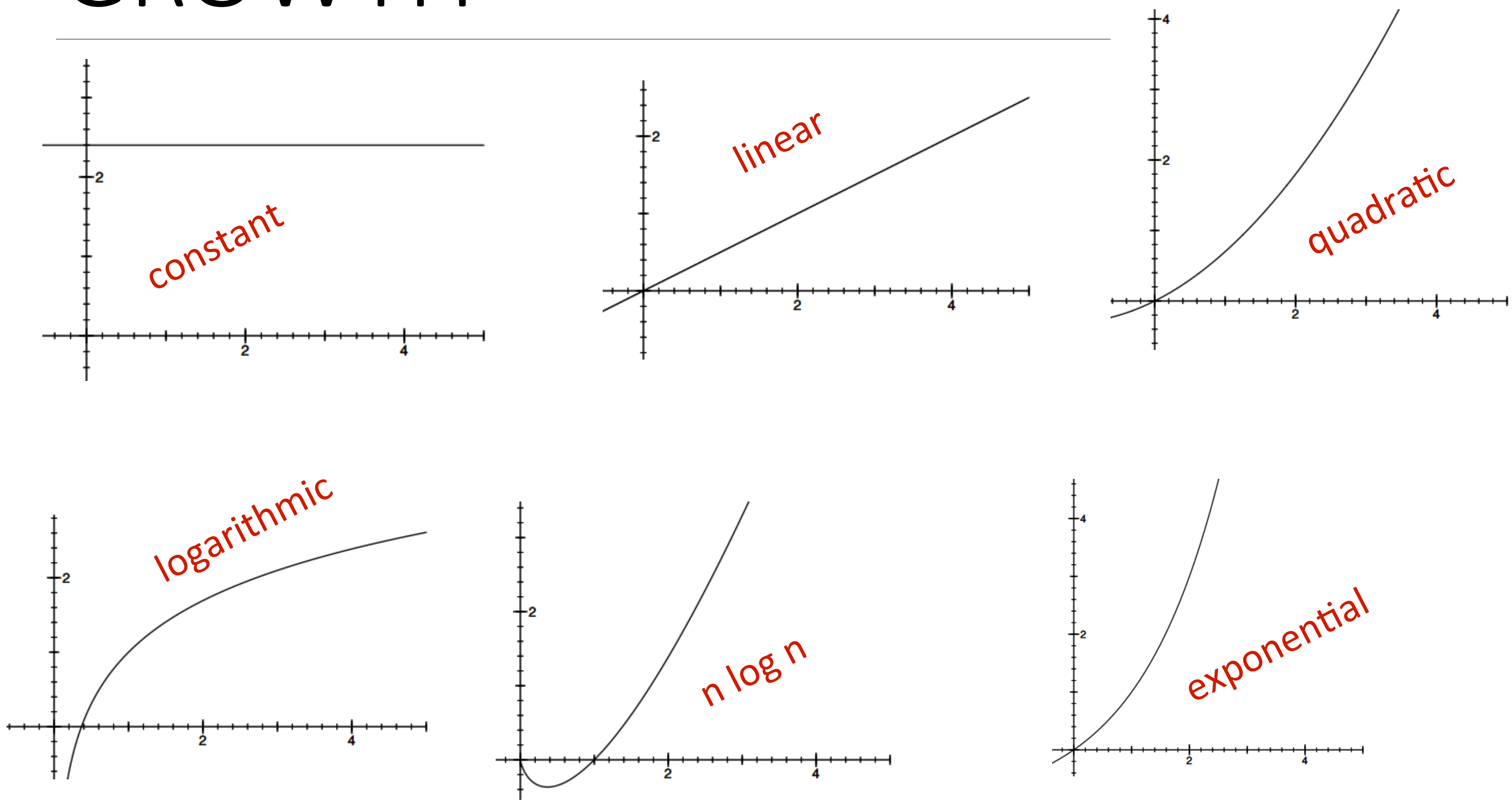
$O(n^2)$: `n`^{2} `+ 100000n + 3`^{1000}

$O(n)$: `log(n) + n + 4`

$O(n \log n)$: `0.0001*n*log(n) + 300n`

$O(3^n)$: `2n`^{30} `+ 3`^{n}

TYPES OF ORDERS OF GROWTH



ANALYZING PROGRAMS AND THEIR COMPLEXITY

- **combine** complexity classes
  - analyze statements inside functions
  - apply some rules, focus on dominant term

Law of Addition for O():

- used with **sequential** statements
- O(f(n)) + O(g(n)) is O(f(n) + g(n))
- for example,

```
for i in range(n):
     print('a')
for j in range(n*n):
     print('b')
```

O(n)

O(n*n)

O(n) + O(n*n)

is O(n) + O(n*n) = O(n+n^2) = O(n^2) because of dominant term

ANALYZING PROGRAMS AND THEIR COMPLEXITY

- **combine** complexity classes
  - analyze statements inside functions
  - apply some rules, focus on dominant term

Law of Multiplication for O():

- used with **nested** statements/loops
- O(f(n)) * O(g(n)) is O(f(n) * g(n))
- for example,

```
for i in range(n):
    for j in range(n):
        print('a')
```

O(n)

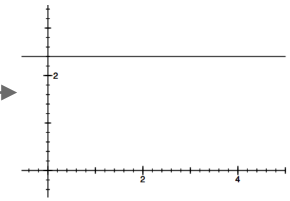
n loops, each O(n) →
O(n)*O(n)

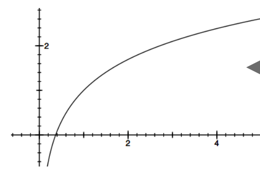
is O(n)*O(n) = O(n*n) = $O(n^2)$ because the outer loop goes n times and the inner loop goes n times for every outer loop iter.

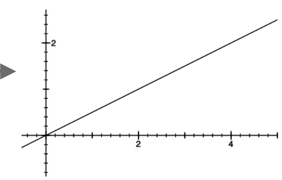
COMPLEXITY CLASSES

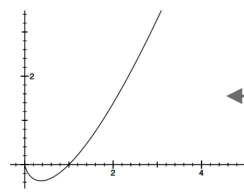
- *O(1)* denotes constant running time
- *O(log n)* denotes logarithmic running time
- *O(n)* denotes linear running time
- *O(n log n)* denotes log-linear running time
- $O(n^c)$ denotes polynomial running time (c is a constant)
- $O(c^n)$ denotes exponential running time (c is a constant being raised to a power based on size of input)

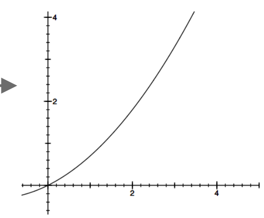
COMPLEXITY CLASSES ORDERED LOW TO HIGH

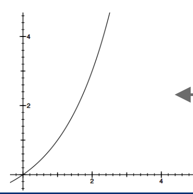
`O(1)` : constant 

`O(log n)` :  logarithmic

`O(n)` : linear 

`O(n log n)` :  loglinear

`O(n^c)` : polynomial 

`O(c^n)` :  exponential

c is a constant

COMPLEXITY GROWTH

CLASS	n=10	= 100	= 1000	= 1000000
O(1)	1	1	1	1
O(log n)	1	2	3	6
O(n)	10	100	1000	1000000
O(n log n)	10	200	3000	6000000
O(n^2)	100	10000	1000000	1000000000000
O(2^n)	1024	12676506 00228229 40149670 3205376	1071508607186267320948425049060 0018105614048117055336074437503 8837035105112493612249319837881 5695858127594672917553146825187 1452856923140435984577574698574 8039345677748242309854210746050 6237114187795418215304647498358 1941267398767559165543946077062 9145711964776865421676604298316 52624386837205668069376	**Good luck!!**

LINEAR COMPLEXITY

- Simple iterative loop algorithms are typically linear in complexity

LINEAR SEARCH ON UNSORTED LIST

```
def linear_search(L, e):
    found = False
    for i in range(len(L)):
        if e == L[i]:
            found = True
    return found
```

speed up a little by returning True here, but speed up doesn't impact worst case

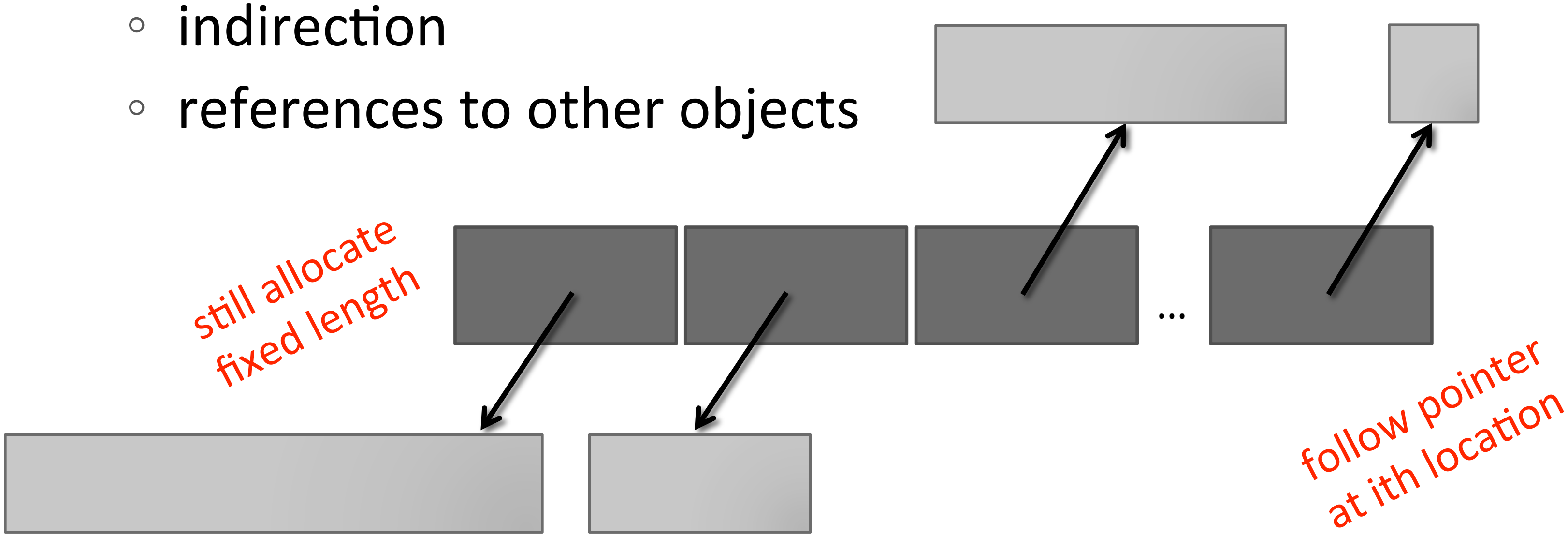
- must look through all elements to decide it's not there
- O(len(L)) for the loop * O(1) to test if e == L[i]
  - O(1 + 4n + 1) = O(4n + 2) = O(n)
- overall complexity is **O(n) – where n is len(L)**

Assumes we can retrieve element of list in constant time

CONSTANT TIME LIST ACCESS

- if list is all ints
  - i^{th} element at
    - base + 4*i

- if list is heterogeneous
  - indirection
  - references to other objects



LINEAR SEARCH ON SORTED LIST

```
def search(L, e):
    for i in range(len(L)):
        if L[i] == e:
            return True
        if L[i] > e:
            return False
    return False
```

- must only look until reach a number greater than e
- $\boxed{\text{O(len(L))}}$ for the loop * O(1) to test if e == L[i]
- overall complexity is **O(n) – where n is len(L)**
- **NOTE:** order of growth is same, though run time may differ for two search methods

worst case will need to look at whole list

LINEAR COMPLEXITY

- searching a list in sequence to see if an element is present
- add characters of a string, assumed to be composed of decimal digits

```
def addDigits(s):
    val = 0
    for c in s:
        val += int(c)
    return val
```

- *O(len(s))*

LINEAR COMPLEXITY

- complexity often depends on number of iterations

```
def fact_iter(n):
    prod = 1
    for i in range(1, n+1):
        prod *= i
    return prod
```

- number of times around loop is n
- number of operations inside loop is a constant (in this case, 3 – set i, multiply, set prod)
  - O(1 + 3n + 1) = O(3n + 2) = O(n)
- overall just *O(n)*

NESTED LOOPS

- simple loops are linear in complexity
- what about loops that have loops within them?

QUADRATIC COMPLEXITY

determine if one list is subset of second, i.e., every element of first, appears in second (assume no duplicates)

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

QUADRATIC COMPLEXITY

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

outer loop executed len(L1) times

each iteration will execute inner loop up to len(L2) times, with constant number of operations

*O(len(L1)*len(L2))*

worst case when L1 and L2 same length, none of elements of L1 in L2

O(len(L1)^2)

QUADRATIC COMPLEXITY

find intersection of two lists, return a list with each element appearing only once

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
            res.append(e)
    return res
```

QUADRATIC COMPLEXITY

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
            res.append(e)
    return res
```

- first nested loop takes *len(L1)*len(L2)* steps
- second loop takes at most *len(L1)* steps
- determining if element in list might take *len(L1)* steps
- if we assume lists are of roughly same length, then
- *O(len(L1)^2)*

O() FOR NESTED LOOPS

```
def g(n):
    """ assume n >= 0 """
    x = 0
    for i in range(n):
        for j in range(n):
            x += 1
    return x
```

- computes n^2 very inefficiently
- when dealing with nested loops, **look at the ranges**
- nested loops, **each iterating n times**
- ***$O(n^2)$***

THIS TIME AND NEXT TIME

- have seen examples of loops, and nested loops
- give rise to linear and quadratic complexity algorithms
- next time, will more carefully examine examples from each of the different complexity classes