Visual Tools for Teaching Propositional Logic

Aránzazu San Ginés

Departamento de Filosofía 1
University of Granada, Spain
aransangines@gmail.com

Abstract. In this paper, I will outline a diagram-based proposal for teaching propositional logic, as well as the reasons that led me to it. The paper is divided into three sections. In the first section I introduce, and try to justify, the hypothesis that processes like thinking, reasoning or speaking are intimately connected with the process of *constructing* what we see. The second section presents a criticism of the didactic ideas underlying the trend of *modern mathematics* in countries like France and the U.S.A. The final section is devoted to the schematic presentation of the specific diagram-based approach.

Keywords: Propositional logic, didactics, abstract thought.

1 The Hypothesis

The language that we usually use in mathematics is riddled with metaphors. In this way, bridges are built from the realm of the abstraction to the realm of the physical. We say, for instance, that a *sequence escapes us* when it diverges, or that a function with a certain type of discontinuity at a point *jumps at this point*. As a result, we succeed in making an essentially abstruse domain, that of mathematics, more accessible. This phenomenon is not exclusive to mathematics; on the contrary, it seems to appear in all types of contexts, scientific and otherwise[1]. And, of course, logic is not an exception. Expressions such as *the argument does not hold up* or *the conclusion follows from the premises* derive from the close link between the users of logic and their physical environment.

The use of this kind of expressions in domains so apparently away from the physical world, just as logic and mathematics seem to be, raises a question about the possible connections between the abstract and the spatial cognitions. Is the use of 'spatial schemas' [2] a mere ruse of logicians or mathematicians to relate cognitive processes that are essentially different, and so to simplify the specific task at hand, or is there something else to say about them? The answer that will be proposed on this paper is that there is certainly a big deal to say. My position intends to be in keeping with the approach of evolutionary theorists; an approach that can be found, to some degree, in the following text [3] (p.371):

> "A fundamental puzzle in the study of the mind is how evolution could have produced a brain capable of intricate specialized achievements like

P. Blackburn et al. (Eds.): TICTTL 2011, LNAI 6680, pp. 215–222, 2011.

> mathematics, science, and art given the total absence of selection pressure for such abstract abilities at any point in history. [...] There are precedents for explaining the emergence of novel capabilities in evolution: old parts can be recruited to new uses."

I therefore start from the hypothesis that the extraordinary human ability for generating abstract thought is a product of the adaptation of those parts of the brain which are responsible for perception and manipulation of space. I will actually go further on this line, and give the mechanisms involved in the construction of the visual a prominent role in the construction of the rational. Hoffman [4] (p.1) describes the human as a "visual virtuoso", a "creative genius for vision". This innate talent has achieved such a degree of sophistication in the course of our evolutionary history, that it seems natural to regard it as the expected support of novel cognitive processes.

However, according to Pylyshyn [5] the mental images reported by the great majority of scientists are no more than the epiphenomenal recreation during reasoning of the (symbolic) laws which seem to govern the world. The laws, rather than the images, would then be part of reasoning. Pylyshyn is especially against the pictorial view of mental images, a view that seems to fly over the Shepard's proposal. In his 1978 paper, Shepard[6] equates in importance logical and analogical processes of thought, where analogical process is understood as "a process in which the intermediate internal states have a natural one-to-one correspondence to appropriate intermediate states in the external world" (p.135). From this claim, it could be concluded that the relationship between the processes of perception and reasoning is limited to the manipulation, during reasoning, of mental objects that are analogous to those that are the result of perception. But, how to explain then the way in which Mozart, for instance, said he imagined his compositions during the creative process? See, according to Pylyshyn [5], p.32, the Mozart's letter reproduced in Ghiselin (1952): "Nor do I hear in my imagination, the parts *successively*, but I hear them, as it were, all at once." If the one-to-one correspondence mentioned by Shepard were exact, then it seems that Mozart should have heard the complete composition in his imagination note by note. Yet he did not. He did not imagine each part, each note, one after the other. Mozart said he imagined each of his compositions all at once, in a process which, to my judgement, resembles the way people *construct* the following picture [4]:

Our 'visual intelligence', as Hoffman calls it, constructs from the four black independent figures above a square which was actually never explicitly drawn (a *subjective surface*). In fact, although we are absolutely sure about what we see, if someone asked us about the sides of the square, curiously enough, it would not be easy to give a precise answer, even though having sides is an essential characteristic of being a square. However, at the same time, we would be perfectly able to outline four appropriate sides to the figure. We have all the information we need to do that, in the same way Mozart counted on the necessary elements to transcribe the composition that he had imagined *all at once*. Both processes seem to have something in common. And it is precisely this similarity what my proposal aims to stress. In this sense, my hypothesis is not only that people reason on the basis of images but, further, that the very processes of thinking, reasoning or speaking are intimately connected with the process of *constructing* what we see.

Now, if we accept that the previous hypothesis is plausible, it should not appear so strange the possibility of producing discovery through the visual.

Giaquinto[7], for instance, argues for the possibility of making discoveries in geometry by visual means. He claims that the experience of visualisation brings about the recovery of certain items that are present in the individual's cognitive state. These items would constitute the pieces in a puzzle which, when fitted together, would enable one to succeed in making discoveries reliably.

My hypothesis differs from Giaquinto's proposal in that, according to my view, the visual contribution is not limited to the activation of items but also concerns the strategy which will eventually enable them to be combined as new beliefs. We can thus see a certain parallel between the mechanisms involved in, respectively, discovery and perceptual construction. Although the two models appear to have different objectives and to function without any apparent connection, they apply very similar rules. In the case of reasoning, these rules could be described as deriving from those governing the construction of perception.

2 A Didactic Plan

If logic has a lot to do with reasoning well, and reasoning is connected to 'vision' as much as I argued for in the previous section, it thus appears natural and even advisable the use, while teaching logic, of didactic resources that involve and exercise actively the visual abilities of the students. This is so because, by making use of those resources, we will ease the process of learning logic, and strengthen the arguing and logical skills of the students.

Making deductions is probably one of the most difficult things to be learnt by the students of logic. They have to learn to deal, with dexterity and harmony, with the rules of inference of specific systems; rules that sometimes do not make any sense to them. Making deductions is certainly not an easy task. Discovering the formulas that can be proved from a sound set of premises is even harder.

Let us suppose that the hypothesis defended along the first section were correct. Then, the use of appropriate diagrams should simplify considerably both deductive and discovery tasks, and at the same time provide adapted training to the kind of mental processes involved. The use of graphic systems would allow the students to strengthen abilities which, in spite of having been proved to be substantial help to the significant progress of science, have been systematically refused and consigned to oblivion for nearly two centuries. Which could be the reason of this contempt for the use of diagrams in science? Let us consider the question for a moment.

According to Mancosu [8] (p.15) "[...] one of the main paradigmatic examples that were used to discredit the role of geometric intuition in analysis [was] Weierstrass' discovery of a continuous nowhere differentiable function." This result seemed to demonstrate that geometrical intuition is deceptive and, therefore, that the most reliable role that can be assigned to diagrams is that of helping in the construction of reasoning, but never that of playing a decisive role in arguments. This is why scientists in general, and mathematicians in particular, decided to relegate the use of diagrams to a merely heuristic role in the 19th and 20th centuries. And this is one of the reasons didactic perspectives that largely reject the informal background of pupils and students have been put into practice for instance in mathematics. In fact, according to Dehaene [9] (p.139), this informal background has been considered "in most math courses [...] as a handicap rather than an asset". The objective pursued was to teach *modern mathematics* to the pupils, or otherwise, to familiarize them, from the very beginning, with a way of doing mathematics consisting in the manipulation of abstract symbols from a solid axiomatic basis. But to achieve this objective, it was necessary that the students made a new fresh start; that they forgot all intuition acquired out of the premises of their school, which is now set up as the great temple of the abstract, formal, reliable and correct knowledge. Few of them would be then the chosen ones that could appreciate the beauty and truth so zealously hidden by mathematics. Therefore, in this sense, the *modern mathematics* proposes a learning plan that, on the one hand scolds pupils for finger counting, and on the other, tries to familiarize them "with the general theoretical principles of numeration before being taught the specifics of our base-10 system." [9] (p.140). Thus, according to Dehaene "believe it or not, some arithmetic textbooks started off by explaining that 3+4 is 2 - in base 5! It is hard to think of a better way to befuddle children's thinking."

Dehaene has not been the only one to react against these modern techniques of teaching mathematics. Other specialists in the last 50 years (for example, Kline [10]) have also considered those to be more a potential risk on the development of the creative abilities of the students than an efficient way to dynamize their argumentative and symbolic skills. In *modern mathematics*, the teacher has the aim to teach the students to see mathematics in the way professionals do. As a consequence, they neglect the fundamental fact that mathematics such as we know them at present have been the result of a long process in which the intuitions today despised played a very important role. But above all, *modern mathematics* appears to forget that human beings have developed these extraordinary abilities which characterize them (thinking logically and mathematically) in the particular physical environment in which they exist. Each new advance should be seen as a valuable ally of our natural capacities, not as a replacement for them. To my mind, only when the syllabi reflect in some degree this desideratum, we will observe some considerable improvement in the education of our students. And maybe then it will be also possible to reduce the feeling of failure and demoralization that tends to overwhelm many of them.

3 A Diagram-Based Proposal for Teaching Propositional Logic

Mine is not at all the only proposal that there has been up to date in relation to the use of diagrams in logic. Hammer [11] (p.129), for example, tells us the following about C.S. Peirce: "From his experience with chemistry and other parts of science, Peirce had become convinced that logic needed a more visually perspicuous notation [...]". Peirce would develop three types of diagrammatic systems associated to propositional logic, logic of predicates, and modal logic, respectively. And Peirce would not be the only one. *Hyperproof* is a more recent example of that. This is a computer program created by J. Barwise and J. Etchemendy for teaching logic in the context of the project *Openproof* at the University of Stanford.

Although it is not the only one, I expect my diagram-based proposal for teaching propositional logic to be the closest to the didactic interest expressed before. With the following proposal I will try to involve, in the deductive and creative process, other brain processes (related to vision) that I have defended before to be specially linked to abstract thought. Let us pass now to briefly describe the system.

The formulas and the rules for derivations for propositional logic will be represented in terms of colored matrices in the following way.

3.1 Formulas

Let p and q be atomic formulas, and let M be a matrix, that is, a square divided into the same number of rows and columns.

- p is represented in M by coloring one entry of the matrix. The color could be any but once is chosen it will identify the atomic formula.
- $\neg p$ is represented in M by coloring one entry with the color used for p and crossing it out: $\cancel{p}$[1]
- $p \rightarrow q$ is represented in M by alternating the colors of p and q in one chosen entry of the matrix. The movement from p to q is represented as being quicker than the movement from q to p.
- $p \leftrightarrow q$ is represented in M by alternating the colors of p and q in one chosen entry of the matrix. The movement from p to q is represented as quick as the movement from q to p.
- $p \wedge q$ is represented in M by representing p and q in different entries of the matrix. The chosen entries are represented close to each other (contiguous entries).
- $p \vee q$ is represented in M using just one entry of the matrix. The entry will be split in two by the diagonal. Each one of the arguments of the disjunction (p and q in this case) will take up a part of the split entry.

[1] I will not use colors in the examples here but the propositional letters itself. We shall understand any propositional letter in the matrix as if the entry was colored.

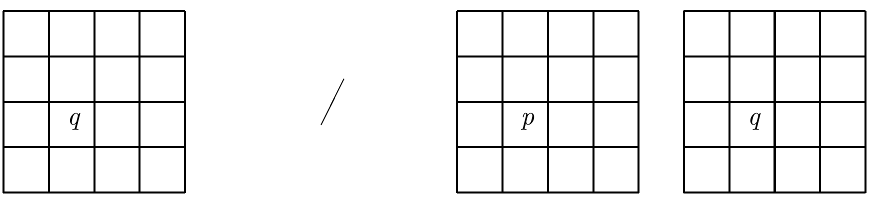
But what what if the arguments were not atomic formulas? For example, in the case that we had formulas such as $\neg\Big(q\vee\big(r\vee\ (p\rightarrow\neg q)\big)\Big)$ or $\big((p\wedge q)\vee r\big)\rightarrow s$?

The first thing to do will be then to draw up the genealogical tree of the specific formulas:

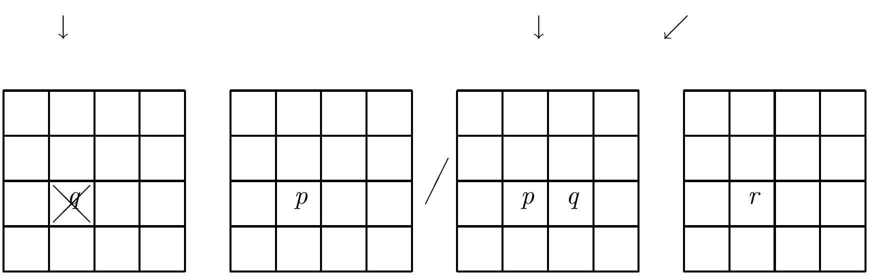
(Step 6)	$\neg\Big(q\vee\big(r\vee(p\rightarrow\neg q)\big)\Big)$	$\big((p\wedge q)\vee r\big)\rightarrow s$	(Step 4')
(Step 5)	$q\vee\big(r\vee(p\rightarrow\neg q)\big)$	$(p\wedge q)\vee r$ $\quad s$	(Step 3')
(Step 4)	q $\quad r\vee(p\rightarrow\neg q)$	$p\wedge q$ $\quad r$	(Step 2')
(Step 3)	r $\quad p\rightarrow\neg q$	p $\quad q$	(Step 1')
(Step 2)	p $\quad \neg q$		
(Step 1)	q		

Secondly, the steps will be represented from down up:

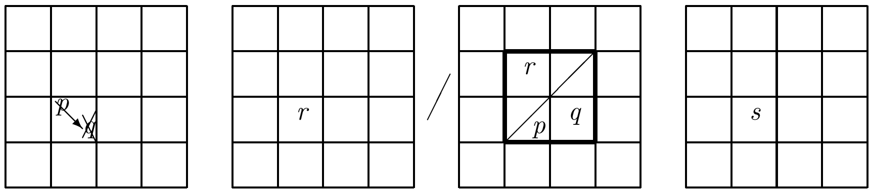
<u>(Step 1) / (Step 1')</u>



<u>(Step 2) / (Step 2')</u>



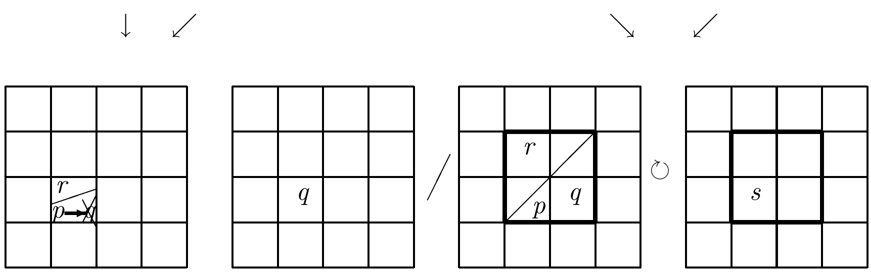
<u>(Step 3) / (Step 3')</u>



I shall make explicit here the $\vee$-rule applied in (Step 3'):

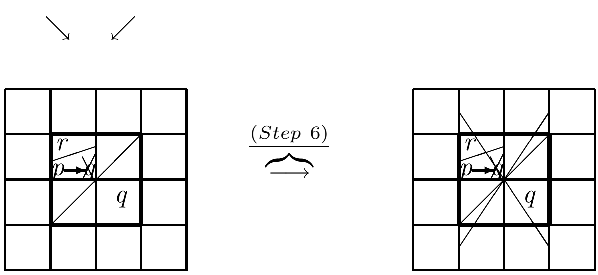
- If each one of the arguments of the disjunction were placed previously (Step 2') in just one (not split) entry of the matrix, then the disjunction will be represented in one single entry. This is not the present case (Step 3'), but will be case in (Step 4): $r \vee (p \rightarrow \neg q)$
- Otherwise, it will be taken the smallest submatrix in which, once it has been split by the diagonal, it will be able to represent each one of the arguments of the disjunction in different parts of the split submatrix. This is the rule we are applying in (Step 3'). The submatrix appears above emphasized.

<u>(Step 4) / (Step 4')</u>



The submatrix in which is represented $(p \wedge q) \vee r$ in (Step 3') will blink in (Step 4') indefinitely from $(p \wedge q) \vee r$ to s.

<u>(Step 5)</u>

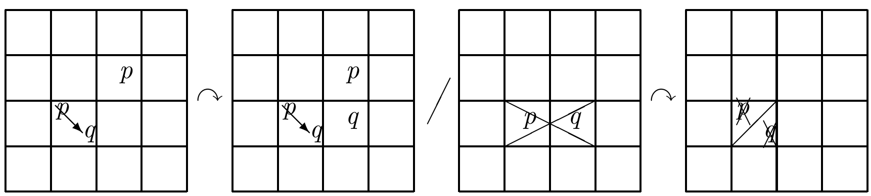


I do not represent the disjunction in (Step 5) in one single entry because one of the arguments of the disjunction $(r \vee (p \rightarrow \neg q))$ was represented, in the previous step (Step 4), in one split entry. As for the negation, (Step 6), it works by crossing the representation of its entire argument out.

3.2 Rules for Derivations

Imagine we wanted to make a deduction in propositional logic by making use of the matrix representation. The first thing to do would be then to represent the premises in a single matrix M. Following that, we would transform M until getting represented in it the conclusion we are aiming for. Thus the question is: what are the rules which allow us to transform a matrix into another? Here is, to finish, a sample of these rules:

$\rightarrow$-*elimination*: / *De Morgan rule*:



Acknowledgements

This work has been partly supported by the research group HUM 432 of the Plan Andaluz de Investigación, Desarrollo e Innovación (PAIDI) and by the project FFI2010-15704 of the Ministerio Español de Ciencia e Innovación, both of which are gratefully acknowledged. I also thank María José Frápolli and the referees.

References

1. Lakoff, G., Johnson, M.: Metaphors we live by. University of Chicago Press, Chicago (1980)
2. Gattis, M. (ed.): Spatial Schemas and Abstract Thought. The MIT Press, Cambridge (2001)
3. Pinker, S.: Learnability and cognition: the acquisition of argument structure. The MIT Press, Cambridge (1989)
4. Hoffman, D.D.: Visual intelligence: how we create what we see. W.W. Norton & Company, New York (2000)
5. Pylyshyn, Z.W.: The Imagery debate: Analogue Media Versus Tacit Knowledge. Psychological Review 88(1), 16–45 (1981)
6. Shepard, R.N.: The Mental Image. American Psychologist, 125–137 (February 1978)
7. Giaquinto, M.: Visual Thinking in Mathematics. An Epistemological Study. Oxford University Press, Oxford (2007)
8. Mancosu, P., Jorgensen, K.F. (eds.): Visualization, explanation and reasoning styles in mathematics. Springer, Heidelberg (2005)
9. Dehaene, S.: The number sense. How The Mind Creates Mathematics. Oxford University Press, Oxford (1997)
10. Kline, M.: El fracaso de la matemática moderna. Siglo XXI de España editores (1984)
11. Hammer, E.: Peircean Graphs for propositional Logic. In: Barwise, J., Allwein, G. (eds.) Logical Reasoning with Diagrams. Oxford University Press, Oxford (1996)