Information-Theoretic Perspective for Teaching Logic

Ángel Nepomuceno-Fernández

Group for Logic, Language and Information,
University of Seville
nepomuce@us.es

Abstract. This work is about using conceptual tools for teaching logic. Our perspective is based on the information theoretic logic studied by J. Corcoran. Argumentation is referred in terms of a new terminology that is introduced from the concept of information, which is taken as primitive. Trying to show that the understanding of several basic concepts of logic could be facilitated, the notion of information content of a proposition is studied, the concept of logical implication is redefined from this informational point of view and the three kinds of inferences are informationally analysed.

Keywords: Information theoretic perspective, informative content of propositions, kinds of argumentation, logical implication.

1 Introduction

In order to teach basic logic sometimes argumentation theory has been taken as a useful starting point. Then the role of ordinary language may be decisive and some conceptual tools should be used. In this work we propose an information-theoretic point of view based on Corcoran's ideas[1] that could facilitate the understanding of several basic concepts of logic.

The concept of information is taken as primitive, though we can consider the entire information of a given domain of investigation and represent that by means of diagrams or in terms of set theory. So, if $\Im$ represents the entire information, $\emptyset$ represents its complementary information, that is to say, the null information. Then, the complementary, union and the interesection of information are defined in the same way as the corresponding set theory notions: for $A \subseteq \Im$,

1. $A \cup \sim A = \Im$
2. $A \cap \sim A = \emptyset$
3. $A \cup \emptyset = A$
4. $A \cap \emptyset = \emptyset$
5. $A \cup \Im = \Im$
6. $A \cap \Im = A$

[1] Presented in [2] by the author. In [7] a panoramic view of this author is given. A first formal study of such point of view is in [5].

P. Blackburn et al. (Eds.): TICTTL 2011, LNAI 6680, pp. 170–177, 2011.

The work is organized as follows. First, argumentation is defined in terms of the new terminology. In the central section the notion of information content of a proposition is studied and the concept of logical implication is redefined from this informational point of view, the classical logical implication as well as the relevant one, then the three kinds of inferences, deduction, abduction and induction, are informationally analysed. Finally, in a last section some concluding remarks are offered.

2 Argumentation in Informational Terms

Teaching argumentation can be conceived as a first step in order to pave the way for teaching logic. Then some aspects of the theory of argumentation, and a set of pertinent notions, can be pointed out. In [3] there are interesting suggestions about how to tackle the problem of studying argumentations in several contexts, which can be transfered to the field of logic itself. Argumentation is the rational activity par excellence, which involves information processes. Then we have to pay attention to propositions. To be precise, the approach to information-theoretic logic in [2] atributes information to propositions, but given a domain of investigation the set of propositions to take into account is not the set of "all" propositions but a restricted one, namely the set of pertinent propositions only. So "the set of propositions" should be understood as "the set of pertinent propositions" relative to a given domain of investigation. This restriction permits us to reject spurious syntactic expressions. For example, "the virtue is green" or "this stone is virtuous" are not pertinent propositions when ethics or geology are the domains of investigation, respectively. Whatever the case may be, propositions should be studied with respect to an inferential context, which is the context regulated by an inferential system, which could be codified in terms of classical logic, modal logic, etc.

Despite the diversity of philosophical conceptions of information[2], every proposition has an information content but it is different from the meaning of the sentence by means of which the proposition is expressed. Intuitively, in Fregean terms, a proposition is an objective thought that can be communicated (in particular by linguistical means), but if a sentence is the vehicle to communicate a proposition, the information content of such proposition cannot be reduced to the semantic value of such sentence. Though the only way to achieve the information content of any proposition is by means of the analysis of the sentence used to express it, the information content of a proposition is not directly expressed by a logical or grammatical form. In fact, several sentences can express the same proposition (active or passive voice, etc.) and though a proposition has a unique form and a unique information content, an information content *per se* does not have a form ([2], p. 116).

An argumentation is a process that culminates in a sequence of propositions, ordered according to certain criteria. In a such sequence it can be distinguished

[2] Most of them in [1].

an initial set of propositions, called *premises*. Other intermediate set of propositions follow the former, the so called *chain of reasons*, and a last proposition called *conclusion*. The (ordered) pair of extremes of an argumentation, namely premises and conclusion, is the *argument* of such argumentation (see [3]). The most important aspect of argumentation, and logic, is how the propositions of the sequence are linked, what are the criteria to consider a sequence of propositions as the result of an argumentation, and to avoid arbitrary sequences. The chain of reasons provides information for accepting the conclusion when the premises have been accepted. In theory of argumentation the emphasis is put on the efficacy of the chain of reasons to convince someone of accepting a conclusion, whereas logical theories study formal aspects of the relation between premises and conclusion.

According to the relation between the terms of the corresponding arguments, several kinds of argumentations can be distinguished. Particularly the three kinds defined by Peirce can be defined from this informational point of view. There are argumentations called

1. *Deduction.* The purpose is to achieve a conclusion whose information content is contained within the information of the premises. When the deduction has been completed, the conclusion may be a new proposition but no new information will have been obtained.
2. *Induction.* The information content of the premises is extended in order to obtain the information content of the conclusion. In this case an increase in information is achieved.
3. *Abduction.* The agent must expand on the information content of the premises until the information of the conclusion becomes included. When an abduction is finished, a deduction can be reconstructed.

3 A Treatment of Information Content

Greek letters π, ρ, σ will be used to name propositions and $Inf(\pi)$, $Inf(\rho)$ and $Inf(\sigma)$ represent the information content of such propositions, respectively. For a set of propositions $\Gamma = \{\pi_1, \pi_2, ..., \pi_n\}$, $n \geq 1$,

$$Inf(\Gamma) = Inf(\pi_1) \cup Inf(\pi_2) \cup ... \cup Inf(\pi_n).$$

Given two propositions π and ρ, a domain of investigation whose entire information is $\Im$ and the class Ω of the pertinent propositions, it can be shown that

1. $Inf(\pi) \subseteq \Im$, provided $\pi \in \Omega$;
2. $Inf(\pi) \cup Inf(\rho) \subseteq \Im$ and $Inf(\pi) \cap Inf(\rho) \subseteq \Im$, provided $\pi, \rho \in \Omega$;
3. $Inf(\pi) = \emptyset$ if π has no information content;
4. If π contains all information, then $Inf(\pi) = \Im$;
5. $\sim Inf(\pi) = \Im - Inf(\pi)$.

In general, for every class of pertinent propositions $\mathit{\Delta}$, we defined

$$Inf(\mathit{\Delta}) = \bigcup_{\pi \in \mathit{\Delta}} Inf(\pi)$$

The concept of *logical implication –logical consequence*, or *entailment relation*– can be defined from the primitive notion of information. According to Corcoran's point of view, an information theoretic approach to logic can be characterized by six rules that relate that logical concept to the notion of information content. For a set of propositions $\mathit{\Gamma}$, propositions π, ρ and σ,the propositional connectives $\neg$ and $\vee$ and the total information $\Im$, the rules that define the (entailment) relation $\models_I$ are

1. $\mathit{\Gamma} \models_I \sigma$ if $Inf(\sigma) \subseteq Inf(\mathit{\Gamma})$;
2. $\mathit{\Gamma} \not\models_I \sigma$ if $Inf(\sigma) \cap Inf(\mathit{\Gamma}) \neq Inf(\sigma)$;
3. If a proposition π is a tautology, then $Inf(\pi) = \emptyset$. In general, for every proposition π, $Inf(\pi \vee \neg\pi) = \emptyset$;
4. If π is a contradiction, $Inf(\pi) = \Im$;
5. $Inf(\pi) \cap Inf(\neg\pi) = \emptyset$ and $Inf(\pi) \cup Inf(\neg\pi) = \Im$;
6. $Inf(\pi \vee \rho) = Inf(\pi) \cap Inf(\rho)$;
7. π and ρ are equivalent[3] if and only if $Inf(\pi) = Inf(\rho)$.

Some facts that describe the information-theoretic approach to logic can be derived from theses rules. Among them we point out the following ones. Validity of any argument can be studied in informational terms. The entailment relation, from an informational point of view, represented by $\models_I$, satifies the following known structural rules[4]:

1. Reflexivity. For every $\rho \in \mathit{\Gamma}$, we have that $\mathit{\Gamma} \models_I \rho$
2. Monotonicity. If $\mathit{\Gamma} \models_I \pi$, then $\mathit{\Gamma}^* \models_I \pi$ for any $\mathit{\Gamma}^*$ such that $\mathit{\Gamma} \subseteq \mathit{\Gamma}^*$
3. Transitivity. If $\mathit{\Gamma} \models_I \pi$ and $\pi \models_I \rho$, then $\mathit{\Gamma} \models_I \rho$

Given a domain of investigation, the total information $\Im$, a set of premises $\mathit{\Gamma}$ and a conclusion π, the argument $\langle \mathit{\Gamma}, \pi \rangle$ is valid if and only if $\mathit{\Gamma}$ entails π from an informational point of view, formally

$$Val(\langle \mathit{\Gamma}, \pi \rangle) \text{ if and only if } \mathit{\Gamma} \models_I \pi,$$

which is equivalent to say that

$$Val(\langle \mathit{\Gamma}, \pi \rangle) \text{ if and only if } Inf(\pi) \cap (\mathit{\Gamma}) = Inf(\pi).$$

Of course, given two arguments $\langle \mathit{\Gamma}, \pi \rangle$ and $\langle \mathit{\Delta}, \rho \rangle$, we can determine when both of them have the same content

$$\langle \mathit{\Gamma}, \pi \rangle = \langle \mathit{\Delta}, \rho \rangle \text{ if and only if } Inf(\mathit{\Gamma}) = Inf(\mathit{\Delta}) \text{ and } Inf(\pi) = Inf(\rho)$$

[3] This rule corresponds to a remark in [2] and it can be added as a seventh rule, since every rule is just proposed as a remark.

[4] Permutation and contraction, since the information content of a proposition has been defined in set-theoretical terms, are trivially verified.

According to this conception, for propositions π, ρ and the entire information $\Im$ of a domain of investigation, it is verified that

1. $Inf(\neg\pi) = \sim Inf(\pi)$;
2. $Inf(\pi) \cap Inf(\neg\pi) = \emptyset$;
3. $Inf(\pi) \cup Inf(\neg\pi) = \Im$;
4. Defining both $\wedge$ and $\rightarrow$ as functions of $\vee$ and $\neg$,
   (a) $Inf(\pi \wedge \rho) = Inf(\pi) \cup Inf(\rho)$;
   (b) $Inf(\pi \rightarrow \rho) = \sim Inf(\pi) \cap Inf(\rho)$.

In [5] a modification in clauses 1 and 2 permits to define relevance conditions. In that case the entailment relation from an informational perspective with such conditions can be represented as $\models_{IR}$. In particular, such two first clauses to define $\models_{IR}$ will be[5]

1'. $\Gamma \models_{IR} \sigma$ if $Inf(\sigma) \subseteq Inf(\Gamma)$ and $Inf(\Gamma) \neq \Im$;
2'. $\Gamma \not\models_{IR} \sigma$ if $Inf(\sigma) \cap Inf(\Gamma) \neq Inf(\sigma)$ and $Inf(\sigma) \neq \emptyset$.

This informational point of view may be epistemologically useful for examining scientific practices as long as characteristics attributed to the logic of relevance are included in the modified clauses. In particular, in $\models_{IR}$, the set of premises can not be contradictory, and no tautology can be a conclusin, since information content of premises must be different from the total information of the domain of investigation and the information content of any conclusion must be different from the null information, respectively.

In order to take advantage of this approach, an informational study of the other argument forms is accessible. Let us see first abduction. Given a background theory –a set of proposition Θ– and a fact expressed with the proposition ρ, another proposition π explains ρ if the conjunction of Θ and π entail (in informational sense) ρ. Formally, the corresponding abductive problem can be represented formally as

$$\Theta \cup \{?\} \models_I \rho \text{ or } \Theta \cup \{?\} \models_{IR} \rho$$

where $\{?\}$ represents the gap to be filled. Abduction is a kind of inference in which the conclusion (of a deduction) and part of the premises (of such potential deduction) are the starting point of the inference and the goal is a hypothesis (the 'conclusion' of the abductive inference, so to speak). It can also be seen as a process of searching for a premise to complete the set of premises of a deduction. For a background theory Θ and a proposition ρ, the main abductive rule permits us to complete Θ in order to deduce such conclusion[6]. From an informational perspective the rule could be expressed as

$$\frac{Inf(\rho) \not\subseteq Inf(\Theta);\ Inf(\rho) \cap Inf(\Theta) = A \neq \emptyset;\ Inf(\pi) = A}{\pi}$$

[5] The other clauses are the same.

[6] This is a refined version of what could be called the Peirce's rule for abduction, accordding to which, for "facts" A and C, when the surprinsing fact C is observed, if A were true, C would be a matter of course, then there is a reason to suspect that A is true (in [6], p.231.)

This rule is sound regardless of whether the perspective is $\models_I$ or $\models_{IR}$. So given the abductive problem $\Theta \cup \{?\} \models_I \rho$ –or $\Theta \cup \{?\} \models_{IR} \rho$, when relevance conditions are taken into account–, the process of obtaining the hypothesis π is an inference: the abductive problem constitutes the set of premises and the solution is the conclusion[7]. This is expressible as $(\Theta, \rho) \gg_{Ab} \pi$.

Unlike $\models_I$, in general if we add relevance conditions some structural rules fail. Let π be a proposition, though $\pi \wedge \neg\pi \models_I \pi$, $\pi \wedge \neg\pi \not\models_{IR} \pi$, and $\pi \models_I \pi \vee \neg\pi$ but $\pi \not\models_{IR} \pi \vee \neg\pi$. Moreover, since $\pi \models_{IR} \pi$ but $\pi \wedge \neg\pi \not\models_{IR} \pi$, monotonicity is not verified. On the other hand, $\gg_{Ab}$ does not verify monotonicity either, since an increase in information in the antecedent of the given rule could block the formulation of the corresponding hypothesis.

Finally, inductive generalization can be seen as an inferential process of increasing information from given information. To illustrate that, suppose as given a number of propositions π_1, ..., π_n. Each one of them says "a_i has the property P". The proposition π says "all subjects (of the given context) have the property P". Then the result of an inductive inference can be represented as

$$\pi_1, \pi_2, ..., \pi_n \sqsupset_I \pi$$

provided that

1. $Inf(\pi_1) \cup Inf(\pi_2) \cup ... \cup Inf(\pi_n) \subseteq Inf(\pi)$
2. $Inf(\pi) \neq \Im$
3. $Inf(\pi_1) \cup Inf(\pi_2) \cup ... \cup Inf(\pi_n) \neq \emptyset$
4. If there is a subject different from a_i, $i \leq n$, $\pi_1 \wedge \pi_2 \wedge ... \wedge \pi_n$ and π cannot be equivalent, that is to say $Inf(\pi_1 \wedge \pi_2 \wedge ... \wedge \pi_n) \neq Inf(\pi)$

This inference relation is not monotonic. Suppose $\pi_1, \pi_2, ..., \pi_n \sqsupset_I \pi$. According to previous clauses, it can be shown that $Inf(\pi_1) \cup Inf(\pi_2) \cup ... \cup Inf(\pi_n) \subseteq Inf(\pi)$, but if $Inf(\rho)$ is added[8], for a certain proposition ρ, then the conclusion may change

$$Inf(\pi_1) \cup ... \cup Inf(\pi_n) \cup Inf(\rho) \nsubseteq Inf(\pi)$$

as a result $\pi_1, \pi_2, ..., \pi_n, \rho \not\sqsupset_I \pi$.

4 Concluding Remarks

The information-theoretic approach to logic is not the only way of approaching our discipline. In [2] another point of view is presented. According to this new

[7] The role of some propositions, it should be noted, changes with respect to the corresponding deduction. In abduction, the background theory and the fact to be explained are premises, then the conclusion is a new proposition to fill the gap, which is a premise (with the background theory) of the deduction that justifies the abductive result, meanwhile such fact is the conclusion of this deduction.

[8] Every proposition that represents a particular case of having the property is now a premise of the inductive inference.

perspective, the model theoretic, set theoretic and substitution theoretic approaches to logic could all be understood as *transformation-theoretic* approach. To have a minimal comparison between the two approaches, a short explanation about this new perspective can be given in the following way. First, let τ be a one-one transformation defined from propositions to propositions. This is a function with domain in the set of (pertinent) propositions Ω and range in the same set, such that

1. For every $\pi \in \Omega$, $\tau(\pi) \in \Omega$
2. For $\pi \in \Omega$, $\tau(\neg\pi) = \neg\tau(\pi)$
3. $\tau(\pi * \rho) = \tau(\pi) * \tau(\rho)$, for $* \in \{\wedge, \vee, \rightarrow\}$
4. For $\Gamma \subseteq \Omega$, $\tau(\Gamma) = \{\tau(\varrho) \in \Omega : \varrho \in \Gamma\}$

Since any (pertinent) proposition is true or false, we can associate every proposition with its truth value, a set of propositions is associated with the value "true" if all its members are true, or with "false" otherwise, in symbols, $\tau(\pi) \equiv \top$, or $\tau(\pi) \equiv \bot$, respectively, for any proposition π, and similarly for sets of propositions. Now, for any set of propositions $\Gamma \subseteq \Omega$ and a proposition $\pi \in \Omega$, in transformation-theoretic terms, $Val(\langle\Gamma, \pi\rangle)$ if and only if no transformation τ carries $\langle\Gamma, \pi\rangle$ onto $\langle\tau(\Gamma), \tau(\pi)\rangle$ such that $\tau(\Gamma) \equiv \top$ and $\tau(\pi) \equiv \bot$, that is to say, no transformation carries the given argument onto an argument with (transformed) true premises and a (transformed) false conclusion.

Given this perspective, for teaching basic logic we can distinguish two ways of using the conceptual tools that have been presented in this paper: as set theory analysis and as a diagrammatical representation of set theoretical relations, namely in

1. Scientific and technical teachings, where the set theory language may be more familiar to many students;
2. Humanities and social sciences, where diagrammatical representations of sets and relations may be more intuitive and accessible to many students.

On the other hand, these methods could be used as a "metatheory" for other developed logics (modal logic, epistemic logic, dynamic epistemic logic, etc.). Whatever the case may be, such methods should be applied in context, since each group of students is different and no program of study can be fixed in advance if the context is not well known.

References

1. Adriaans, P., van Benthem, J. (eds.): Philosophy of Information. Handbook of the Philosophy of Science, vol. 8. Elsevier North-Holland, Amsterdam (2008)
2. Corcoran, J.: Information-theoretic logic. In: Martínez, C., Rivas, U., Villegas-Forero, L. (eds.) Truth in Perspective, pp. 113–135. Ashgate Publisihing Ld. (1998)
3. Corcoran, J.: Argumentations and logic. Argumentation (3), 17–43 (1987)
4. Martínez, C., Falgueras, J.L., Sagüillo, J.M. (eds.): Current Topics in Logic and Analytic Philosophy. Cursos e Congresos da Universidade de Santiago de Compostela (2007)

5. A. Nepomuceno: Information and logic. In: [4], pp. 219–232 (2007)
6. Peirce, C.S.: Pragmatism as the Logic of Abduction. In: The Essential Peirce, vol. 2, Edited by the Peirce Edition Project, pp. 226–241. Indiana University Press, Bloomington (1998)
7. Sagüillo, J.M.: Corcoran the philosopher. In: [4], pp. 246–269 (2007)